Specifying Nodes as Sets of Choices

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SPECIFYING NODES AS SETS OF CHOICES

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Abstract. Osborne and Rubinstein (1994) specify each node in a game tree as a sequence of actions. It is well-known that such actions can be replaced by choices (i.e. agent-specific actions) without loss of generality.

I find that this sequential formulation is redundant in the sense that nodes can be equivalently specified as *sets* of choices. The only cost of doing so is to rule out absent-mindedness. My analysis encompasses both ordered and unordered information sets and both finite and infinite horizons. (This specification of nodes as sets of choices differs from the literature's specification of nodes as sets of outcomes.)

1. Introduction

1.1. Motivation

Osborne and Rubinstein (1994) specify each node in a game tree by the sequence of actions leading to it. For simplicity, assume that each agent (i.e. information set) has its own actions, and let these agent-specific actions be called choices. It is well-known that this entails no loss of generality.

This paper introduces and justifies the idea of formulating each node as a set of choices rather than a sequence of choices. This differs from

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formulating each node as a set of outcomes, as is done by von Neumann and Morgenstern (1944) and Alós-Ferrer and Ritzberger (2005, 2008, 2013). The relation between this paper's choice-set formulation and their discrete outcome-set formulation is explored in Streufert (2015b).

There are circumstances in which this paper's choice-set formulation is especially useful. These circumstances arise when choices or actions are more convenient than outcomes, and sets are more convenient than sequences.

For example, such circumstances arise when Streufert (2015a) sets up to characterize the supports of consistent assessments (Kreps and Wilson (1982)). The critical step is to find an ordering that (a) extends the assessment's infinite-relative-likelihood relation among nodes and (b) has a representation that is additive across the choices leading to each node. Similar steps in the literature have been relatively obscure.

Since each node can now be regarded as a set of choices, this step reduces to finding an ordering that (a) extends a partial ordering among sets of choices and (b) has a representation that is additive across the choices in each set. More abstractly, it reduces to finding an ordering that (a) extends a partial ordering among sets and (b) has an additive representation. That abstract problem was addressed by Kraft, Pratt, and Seidenberg (1959) while laying the foundations of probability theory. Its solution requires nothing more than Farkas' Lemma. Thereby, an aspect of Kreps-Wilson consistency becomes more transparent.

1.2. Overview

Section 2 merely restates the definition of an extensive form that appears in Osborne and Rubinstein (1994). I call my restatement an "OR* choice-sequence form". The purpose of my restatement is to conserve notation. For example, I implicitly specify agents (i.e. information sets) by mimicking an analogous construction in the outcome-set formulation of Alós-Ferrer and Ritzberger (2005). In spite of such notational streamlining, an OR* choice-sequence form is at the full generality of Osborne and Rubinstein (1994). In particular, it admits continuum choice spaces, unordered agents (i.e. information sets), and both finite and infinite horizons.[1]

[1]This paper extends an earlier version (Streufert (2012)) that only admitted finite horizons.

Section 3 then introduces the concept of a "choice-set form" in which the nodes are choice sets rather than choice sequences. This new formulation is specified so as to highlight its many similarities with an $\mathtt{OR}^*$ choice-sequence form.

Section 4 begins with Proposition 4.1, which shows that an $\mathtt{OR}^*$ choice-sequence form has no-absent-mindedness (Piccione and Rubinstein (1997)) iff there is a one-to-one correspondence between the form's choice sequences and the sets of choices that they list. Thus the order explicit in the sequential notation is redundant in any $\mathtt{OR}^*$ choice-sequence form with no-absent-mindedness. This suggests that any such form can be converted into an equivalent choice-set form simply by converting each of its sequences into the set of choices that it lists.

Theorem 1(a) does this. Then the remainder of the theorem shows a number of ways in which the original no-absent-minded $\mathtt{OR}^*$ choice-sequence form is "equivalent" to the derived choice-set form. Part (b) uses Proposition 4.1 to show that there is a one-to-one correspondence between the original choice-sequence nodes and the derived choice-set nodes. Part (c) shows that finite choice-sequence nodes coincide with finite choice-set nodes. Then (d) shows that choices play similar roles in the two formulations, and finally, (e) and (f) naturally translate feasible choices and immediate predecessors from one formulation into the other.

Theorem 2 shows that the conversion process itself is a bijection from the class of no-absent-minded $\mathtt{OR}^*$ choice-sequence forms onto the class of choice-set forms. This one-to-one correspondence further strengthens the notion that no-absent-minded $\mathtt{OR}^*$ choice-sequence forms are "equivalent" to choice-set forms.

Section 5 derives several corollaries for choice-set forms. Corollary 5.1 characterizes the predecessors of a finite node. Corollary 5.2 characterizes the nonterminal nodes. Corollary 5.3 characterizes the predecessors of an infinite node. And finally, Corollary 5.4 shows that every pair of nodes has a greatest common subnode, which in turn implies that the collection of nodes is a sublattice. These natural but nontrivial results are used extensively by Streufert (2015b).

2. Reviewing choice-sequence forms

This section merely restates the definition of an extensive form that appears in Osborne and Rubinstein (1994, page 200). Nodes are specified as sequences, and Osborne (2008, Section 3) credits Rubinstein with the important idea of doing so. Accordingly, I call my restatement an "$\texttt{OR}^*$ choice-sequence form". The purpose of my restatement is to conserve notation.

2.1. Choices and nodes

Let C be an arbitrary set and call a member c of the set C a *choice*. Then let $\bar{s}$ denote an arbitrary sequence in C. In other words, let $\bar{s}$ be a choice sequence. Such an $\bar{s}$ can be an infinite sequence of the form $(\bar{s}_1, \bar{s}_2, ...)$, a nonempty finite sequence of the form $(\bar{s}_1, \bar{s}_2, ...\bar{s}_{L(\bar{s})})$, or the empty sequence $\{\}$. Note that the length of a finite sequence $\bar{s}$ is denoted $L(\bar{s})$, and that the length $L(\{\})$ of the empty sequence $\{\}$ is defined to be zero.

For any nonempty sequence $\bar{s}$, let

$$_1\bar{s}_\ell := (\bar{s}_1, \bar{s}_2, ...\, \bar{s}_\ell)$$

be the sequence consisting of the first $\ell{\geq}1$ elements of $\bar{s}$, where $\ell{\leq}L(\bar{s})$ if $\bar{s}$ is finite. By convention, let $_1\bar{s}_0$ be the empty sequence $\{\}$ regardless of $\bar{s}$. Call $_1\bar{s}_\ell$ a *subsequence* if either (a) $\bar{s}$ is infinite or (b) $\bar{s}$ is finite and $\ell < L(\bar{s})$. Further, for any finite sequence $\bar{s}$, let

$$\bar{s}{\oplus}(c) := (\bar{s}_1, \bar{s}_2, ...\, \bar{s}_{L(\bar{s})}, c)$$

be the concatenation of $\bar{s}$ with the one-element sequence (c). Finally, for any sequence $\bar{s}$, let

$$R(\bar{s}) := \{\ c \mid (\exists\ell)\, \bar{s}_\ell{=}c\ \}$$

be the range of $\bar{s}$. For example, R takes $(\bar{s}_1, \bar{s}_2, ...)$ to $\{\bar{s}_1, \bar{s}_2, ...\}$. Similarly, R takes $(\bar{s}_1, \bar{s}_2, ...\, \bar{s}_{L(\bar{s})})$ to $\{\bar{s}_1, \bar{s}_2, ...\, \bar{s}_{L(\bar{s})}\}$.

An $\texttt{OR}^*$ *choice-sequence preform* is a pair $(C, \bar{N})$ such that

(1a) $\quad \bar{N}$ is a nonempty collection of sequences in C ,

(1b) $$C \subseteq \cup R(\bar{N})\ ,$$

(1c) $$\bar{N}{\smallsetminus}\bar{T} = \{\ \bar{s} \mid (\forall\ell{\geq}1)\, {_1\bar{s}_\ell} \in \bar{T}\ \}\ ,$$

(1d) $$(\forall\bar{t}{\neq}\{\})\ {_1\bar{t}_{L(\bar{t})-1}} \in \bar{T}\ ,\ \text{and}$$

(1e) $$(\forall\bar{t}^1, \bar{t}^2)\ \bar{F}(\bar{t}^1){=}\bar{F}(\bar{t}^2)\ \text{or}\ \bar{F}(\bar{t}^1){\cap}\bar{F}(\bar{t}^2){=}\varnothing\ .$$

where

$$\bar{T} := \{\ \bar{n} \mid \bar{n} \text{ is finite }\} \text{ and} \tag{2}$$

$$\bar{F} := \{\ (\bar{t},c) \mid \bar{t}\oplus(c)\in\bar{T}\ \}\ . \tag{3}$$

Call a member $\bar{n}$ of the set $\bar{N}$ a *node*.[2] Further, call $\bar{F}$ the *feasibility correspondence*. Accordingly, call $\bar{F}(\bar{t})$ the set of choices that are *feasible* at $\bar{t}$.

(1a) states that nodes are choice sequences. It also specifies that there is at least one node. Accordingly, $\{\}$ must be a node by an elementary argument using (1c) and (1d). The smallest $\texttt{OR}^*$ choice-sequence preform is specified by $C = \varnothing$ and $\bar{N} = \{\{\}\}$.[3]

(1b) states that every choice appears in at least one node. This is accomplished by $C \subseteq \cup R(\bar{N})$ since $R(\bar{N}) = \{R(\bar{n})|\bar{n}\in\bar{N}\}$ is the collection of the ranges of all the choice sequences in $\bar{N}$. This assumption entails no loss of generality in applications, for if it were violated, one could simply remove the superfluous choices from C.

(1c) in the $\subseteq$ direction states that all the subsequences of an infinite node are themselves nodes. Conversely, the $\supseteq$ direction states that if all the subsequences of an infinite sequence are nodes, then that infinite sequence is itself a node.

(1d) requires that another node results when the last element of a nonempty finite node is removed. By repeated application, this implies that all the subsequences of a finite node are themselves nodes.

(1e) permits the implicit specification of agents (i.e. information sets). In particular, let

$$\bar{H} := \{\ \bar{F}^{-1}(c) \mid c\ \}\ , \tag{4}$$

and call an element $\bar{h}$ of $\bar{H}$ an *agent*. Thus an agent $\bar{h}$ is specified as a collection of nodes $\bar{F}^{-1}(c)$ from which some choice c is feasible. This implicit specification of agents mimics a similar construction in Alós-Ferrer and Ritzberger (2005, page 791, Definition 7(i)). The following lemma shows how it is related to an explicit specification of agents. I use the implicit specification to conserve notation.

[2] Osborne and Rubinstein (1994) refer to such a sequence as a "history" and denote it by h. I reserve "h" for an agent (i.e. information set).

[3] As a matter of convention, I denote the empty set by $\{\}$ when it is regarded as a node. Elsewhere I denote it by $\varnothing$.

Lemma 2.1. *Suppose that $(C,\bar{N})$ satisfies (1a)–(1d). Derive its $\bar{T}$ (2) and $\bar{F}$ (3).*

(a) Assume (1e) and define $\bar{H}$ by (4). Then

(5a) $$\bar{H} \textit{ partitions } \bar{F}^{-1}(C) ,$$

(5b) $$[(\exists\bar{h})\{\bar{t}^1,\bar{t}^2\}\subseteq\bar{h}] \Rightarrow \bar{F}(\bar{t}^1)=\bar{F}(\bar{t}^2) \text{ , } \textit{and}$$

(5c) $$\bar{h}^1\neq\bar{h}^2 \Rightarrow \bar{F}(\bar{h}^1)\cap\bar{F}(\bar{h}^2)=\varnothing \text{ .}^4$$

(b) Conversely, (1e) holds if there is an $\bar{H}^*$ that satisfies (5). Further, any $\bar{H}^*$ that satisfies (5) equals the $\bar{H}$ defined by (4). (Proof A.3.)

(5a) states that $\bar{H}$ partitions the collection of nonterminal nodes. To conserve notation, I do not define a special symbol for the collection $\bar{F}^{-1}(C) = \{\ \bar{t} \mid \bar{F}(\bar{t})\neq\varnothing\ \}$ of nonterminal nodes.

(5b) states that the same choices are feasible from any two nodes in an agent $\bar{h}$. This condition is standard, and it allows one to interpret $\bar{F}(\bar{h})$ as the set of choices feasible for agent $\bar{h}$.[4]

(5c) states that different agents have different choices. This condition entails no loss of generality in applications because one can always introduce enough choices so that agents never share choices (this is only a matter of notation).

2.2. Players

Let I be an arbitrary set, and call an element i of the set I a *player*. An `OR`* *choice-sequence form* is a pair $((C_i)_i,\bar{N})$ such that

(6a) $$(\cup_i C_i,\bar{N}) \text{ is an } \texttt{OR}^* \text{ choice-sequence preform (1) ,}$$

(6b) $$(\forall i\neq j)\ C_i\cap C_j = \varnothing \text{ , and}$$

(6c) $$(\forall i)(\forall\bar{t})\ \bar{F}(\bar{t})\subseteq C_i \text{ or } \bar{F}(\bar{t})\cap C_i=\varnothing \text{ .}$$

(6a) and (6b) state that the choices in the preform $(\cup_i C_i,\bar{N})$ are allocated to the players i by means of their choice sets C_i. Accordingly, a preform can be understood as a one-player form. To be precise, $(C,\bar{N})$ is a preform iff $((C),\bar{N})$ is a form, provided that $(C_i)_i = (C)$ is taken to mean $I = \{1\}$ and $C_1 = C$.

(6c) permits the implicit assignment of agents and nodes to players. In particular, define $(\bar{H}_i)_i$ by

(7) $$(\forall i)\ \bar{H}_i = \{\ \bar{F}^{-1}(c) \mid c\in C_i\ \} \text{ .}$$

[4]As with any correspondence, the value $\bar{F}(\bar{h})$ of the correspondence $\bar{F}$ at the *set* $\bar{h}$ is defined to be $\{c|(\exists\bar{t}\in\bar{h})c\in\bar{F}(\bar{t})\}$. (5b) implies that $(\forall\bar{t}\in\bar{h})\ \bar{F}(\bar{t}) = \bar{F}(\bar{h})$.

Then $\bar{H}_i$ is the set of agents belonging to player i, and $\cup\bar{H}_i$ is the set of nodes belonging to player i. In the following lemma, a *prepartition* of a set A is a collection of disjoint subsets of A whose union is A. Note that the empty set can be an element of a prepartition.

Lemma 2.2. *Let* $((C_i)_i, \bar{N})$ *be an* $\mathtt{OR}^*$ *choice-sequence form (6) with its* $C{=}\cup_i C_i$, $\bar{F}$ *(3),* $\bar{H}$ *(4), and* $(\bar{H}_i)_i$ *(7). Then*
(a) $\{\bar{H}_i|i\}$ *is a prepartition of* $\bar{H}$ *and*
(b) $\{\cup\bar{H}_i|i\}$ *is a prepartition of* $\bar{F}^{-1}(C)$. *(Proof A.4.)*

The above admits the possibility of a vacuous, as opposed to nonexistent, chance player. Accordingly, one could require that the player set I always contains a chance player i^o, and then set $C_{i^o} = \varnothing$ to model the special case of no randomness. In this special case, one would have (a) $\bar{H}_{i^o} = \varnothing$ (that the chance player has no agents) and (b) $\cup\bar{H}_{i^o} = \varnothing$ (that the chance player has no nodes). This very minor innovation can simplify notation, as in Streufert (2015a, page 38, last paragraph).

Finally, note that an $\mathtt{OR}^*$ choice-sequence game could be specified by augmenting an $\mathtt{OR}^*$ choice-sequence form with (1) chance probabilities and (2) preferences. For (1), one would specify, for each chance agent $\bar{h}{\in}\bar{H}_{i^o}$, a probability measure over $\bar{F}(\bar{h})$. For (2), one would specify, for each nonchance player $i{\in}I{\smallsetminus}\{i^o\}$, a binary relation over lotteries over the set $\bar{N}{\smallsetminus}\bar{F}^{-1}(C)$ of terminal nodes.

2.3. Discussion

An $\mathtt{OR}^*$ choice-sequence form (6) conserves notation by implicitly specifying agents and by implicitly assigning agents and nodes to players.

Nonetheless, an $\mathtt{OR}^*$ choice-sequence form is at the full generality of Osborne and Rubinstein (1994, Definition 200.1). (1) It admits continuum choice spaces. Thus, since a type is a chance choice, it admits continuum type spaces. (2) It admits unordered agents (i.e. information sets). Thus it admits arbitrarily arranged agents that cannot be specified in multistage formulations such as von Neumann and Morgenstern (1944, Sections 9 and 10) and Myerson (1991, page 296). (3) It admits both finite and infinite horizons.

There are also two minor differences. (1) An $\mathtt{OR}^*$ form imposes (5c), which states that each agent has its own choices. As is well-known, this imposes no loss of generality. (2) An $\mathtt{OR}^*$ form allows the chance player, like any other player, to have nonsingleton agents.

Although difference (1) is inconsequential technically, it does correspond to a difference in nomenclature: while an Osborne-Rubinstein form has "actions" a, an $\mathtt{OR}^*$ form has "choices" c. Thus an $\mathtt{OR}^*$ form conforms with the standard nomenclature in the outcome-set literature, where a property like (5c) is implicit (von Neumann and Morgenstern (1944, Sections 9 and 10), Ritzberger (2002, Section 3.2), Alós-Ferrer and Ritzberger (2005, 2008, 2013)).

3. Introducing choice-set forms

This section introduces a new kind of extensive form in which the nodes are sets of choices rather than sequences of choices.

3.1. Choices and nodes

As before, let C be an arbitrary set, and call a member c of the set C a *choice*. A *choice-set preform* is a pair (C, N) such that

(8a) $$N \text{ is a nonempty collection of subsets of } C\ ,$$

(8b) $$C \subseteq \cup N\ ,$$

(8c) $$N \smallsetminus T = \{\ \cup T^* \mid T^* \text{ is an infinite chain in } T\ \}\ ,$$

(8d) $$(\forall t{\neq}\{\})(\exists! c)\ c{\in}t \text{ and } t{\smallsetminus}\{c\}{\in}T\ , \text{ and}$$

(8e) $$(\forall t^1, t^2)\ F(t^1){=}F(t^2) \text{ or } F(t^1){\cap}F(t^2){=}\varnothing\ ,$$

where

(9) $$T := \{\ n \mid n \text{ is finite}\ \} \text{ and}$$

(10) $$F := \{\ (t,c) \mid c{\notin}t \text{ and } t{\cup}\{c\}{\in}T\ \}\ .$$

Call a member n of the set N a *node*, and call F the *feasibility correspondence*.

(8a) states that nodes are choice sets. It also states that there is at least one node. Accordingly, $\{\}$ must be a node by Lemma B.5. The smallest choice-set preform is specified by $C = \varnothing$ and $N = \{\{\}\}$.

(8b) states that every choice appears in at least one node. This assumption entails no loss of generality in applications, for if it were violated, one could simply remove the superfluous choices from C.

(8c) relates infinite nodes to finite nodes. By definition, a *chain* in T is a subcollection $T^* \subseteq T$ such that any two distinct nodes t and t' in

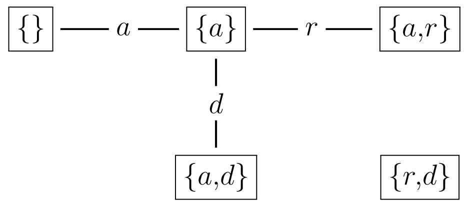

FIGURE 1. $C = \{a, r, d\}$ and $N = \{\{\}, \{a\}, \{a,r\}, \{a,d\}, \{r,d\}\}$. This violates (8d) because $\{r,d\}$ has no last choice.

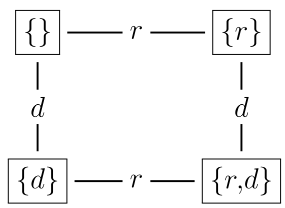

FIGURE 2. $C = \{r, d\}$ and $N = \{\{\}, \{r\}, \{d\}, \{r,d\}\}$. This violates (8d) because $\{r,d\}$ has two last choices.

T^* satisfy $t \subset t'$ or $t \supset t'$. The union of an infinite chain of finite nodes is obviously an infinite set. The $\supseteq$ direction of (8c) further stipulates that such a union must be a node. The $\subseteq$ direction of (8c) requires that every infinite node is the union of at least one chain of finite nodes.[5]

(8d) is discussed in this and the next two paragraphs. To begin, let a *last choice* of a finite node t be any choice $c \in t$ such that $t \setminus \{c\}$ is also a node. In other words, let a last choice of a finite node be any choice in the node whose removal results in another node. (8d) requires that every nonempty finite node has a unique last choice.

Figures 1, 2, and 3 provide three examples. In each case, the figure's caption defines (C, N), and accordingly, the definition of the example is complete without the illustration itself. Each illustration links two nodes with a choice-labelled line exactly when (a) that choice is a last choice of the larger set and (b) the smaller set is the larger set without that choice. The example of Figure 1 violates (8d) because $\{r, d\}$ does not have a last choice. The example of Figure 2 violates (8d) because $\{r, d\}$ has two last choices. Meanwhile, the example of Figure 3 satisfies (8d) because each of its four nonempty nodes has a unique last choice.

[5]Later, Corollary 5.3(b) will show the sense in which this chain is unique.

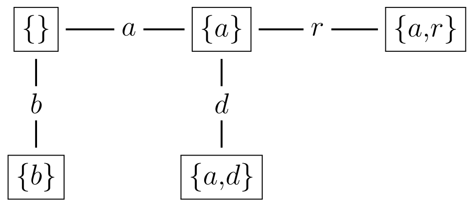

FIGURE 3. $C=\{a,b,r,d\}$ and $N=\{\{\},\{a\},\{a,r\},\{b\},\{a,d\}\}$. This is a choice-set preform (8).

For notational ease, define

(11) $$p := \{\ (t, t{\smallsetminus}\{c\}) \mid c{\in}t \text{ and } t{\smallsetminus}\{c\}{\in}T\ \}\ .$$

By (8d), p is a function from $T{\smallsetminus}\{\{\}\}$ into T. Call p the *immediate predecessor function*.[6] By inspection, the range $p(T{\smallsetminus}\{\{\}\})$ of p equals the domain $F^{-1}(C)$ of F.[7]

(8e) permits the implicit specification of agents. In particular, let

(12) $$H := \{\ F^{-1}(c) \mid c\ \}\ ,$$

and call an element h of H an *agent*. Thus an agent h is specified as a collection of nodes $F^{-1}(c)$ from which some choice c is feasible. This implicit specification of agents is directly analogous to the implicit specification (4) of agents in an $\texttt{OR}^*$ choice-sequence preform. Accordingly, the following lemma is directly analogous to Lemma 2.1, and the discussion of that lemma applies here as well.

Lemma 3.1. *Suppose that (C,N) satisfies (8a)–(8d). Derive its T (9) and F (10).*

(a) Assume (8e) and define H by (12). Then

(13a) $$H \textit{ partitions } F^{-1}(C)\ ,^7$$

(13b) $$[(\exists h)\{t^1,t^2\}{\subseteq}h] \Rightarrow F(t^1){=}F(t^2) \textit{ and}$$

(13c) $$h^1{\neq}h^2 \Rightarrow F(h^1){\cap}F(h^2){=}\varnothing\ .$$

[6]Section 5 will define the concept of one node "preceding" another. Then Corollaries 5.1(b) and 5.3(b) will characterize the predecessors of a node in terms of the immediate predecessor function p.

[7]Section 5 will define the concept of a "terminal" node. Then Corollary 5.2(c) will show that the collection of nonterminal nodes is $F^{-1}(C)$.

(b) Conversely, (8e) holds if there is any H^* that satisfies (13). Further, any H^* that satisfies (13) equals the H defined by (12). (Proof B.3.)

3.2. Players

Let I be an arbitrary set, and call a member i of the set I a *player*. A *choice-set form* is a pair $((C_i)_i, N)$ such that

(14a) $$(\cup_i C_i, N) \text{ is a choice-set preform (8) },$$

(14b) $$(\forall i{\neq}j)\ C_i{\cap}C_j = \varnothing\ ,\ \text{and}$$

(14c) $$(\forall i)(\forall t)\ F(t){\subseteq}C_i \text{ or } F(t){\cap}C_i{=}\varnothing\ .$$

(14a) and (14b) state that the choices in the preform $(\cup_i C_i, N)$ are allocated to the players i according to their choice sets C_i. They are directly analogous to (6a) and (6b) for $\mathtt{OR}^*$ choice-sequence forms.

(14c) permits the implicit assignment of agents and nodes to players. In particular, define $(H_i)_i$ by

(15) $$(\forall i)\ H_i := \{\ F^{-1}(c)\ |\ c{\in}C_i\ \}\ .$$

Then H_i is the set of agents belonging to player i, and $\cup H_i$ is the set of nodes belonging to player i. This specification of $(H_i)_i$ is directly analogous to the specification (7) of $(\bar{H}_i)_i$ in an $\mathtt{OR}^*$ choice-sequence form. Accordingly, the following lemma is directly analogous to Lemma 2.2.

Lemma 3.2. *Let $((C_i)_i, N)$ be a choice-set form (14) with its $C{=}\cup_i C_i$, F (10), H (12), and $(H_i)_i$ (15). Then*

(a) $\{H_i|i\}$ is a prepartition of H and

(b) $\{\cup H_i|i\}$ is a prepartition of $F^{-1}(C)$.[7] (Proof B.4.)

The last two paragraphs of Section 2.2 showed how to augment an $\mathtt{OR}^*$ choice-sequence form with (a) chance probabilities and (b) preferences over terminal nodes. The choice-set analog is straightforward given Section 5's definition of terminal choice-set nodes (in accord with Note 7, the collection of terminal choice-set nodes is $N{\setminus}F^{-1}(C)$).

4. Equivalence

4.1. No-absent-mindedness and the injectivity of $R|_{\bar{N}}$

Let $((C_i)_i, \bar{N})$ be an $\texttt{OR}^*$ choice-sequence form (6) with its $\bar{T}$ (2) and $\bar{H}$ (4). As in Piccione and Rubinstein (1997, page 10), an agent $\bar{h}$ is *absent-minded* if there exist $\bar{t}$ and $0 \leq \ell < L(\bar{t})$ such that $\{{}_1\bar{t}_\ell, \bar{t}\} \subseteq \bar{h}$. In other words, an agent is absent-minded if it contains both a node and one of this node's predecessors. Accordingly, *no-absent-mindedness* is the property that

$$(\forall \bar{h})(\forall \bar{t})(\forall 0{\leq}\ell{<}L(\bar{t}))\ \{{}_1\bar{t}_\ell, \bar{t}\} \not\subseteq \bar{h}\ . \tag{16}$$

No-absent-mindedness is weak. It is strictly weaker than perfect recall, and perfect recall is assumed by many authors including Kreps and Wilson (1982). Specifically, they define perfect recall as the combination of their equations (2.2) and (2.3). Their equation (2.2) is equivalent to no-absent-mindedness (16).[8] Meanwhile, their equation (2.3) imposes the additional assumption that players recall what their own past agents knew and did. This second component of perfect recall plays no role in this paper.

Proposition 4.1. *Suppose* $((C)_i, \bar{N})$ *is an* $\texttt{OR}^*$ *choice-sequence form (6) with its* $\bar{T}$ *(2). Then the following are equivalent.*
(a) $((C_i)_i, \bar{N})$ *has no-absent-mindedness (16).*
(b) $R|_{\bar{T}}$ *is injective.*
(c) $R|_{\bar{N}}$ *is injective. (Proof A.6.)*

Proposition 4.1 relates no-absent-mindedness to the injectivity of $R|_{\bar{N}}$, where R (by its definition in Section 2.1) takes any sequence to its range, and $R|_{\bar{N}}$ is the restriction of R to the form's $\bar{N}$.

First note that $R|_{\bar{N}}$ must be injective when the agents of $((C_i)_i, \bar{N})$ are ordered. To see this, consider any $\bar{n}$. Since a choice determines the agent that plays it, the choices in $R(\bar{n})$ must be played in the order of their agents. Hence the set $R(\bar{n})$ determines the sequence $\bar{n}$.

But Proposition 4.1(a⇒c) goes further. It shows that $R|_{\bar{N}}$ is injective even when the agents of $((C_i)_i, \bar{N})$ are unordered, provided only that no-absent-mindedness holds. For example, consider Figure 4,[9] which

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[8]Their $x{\in}H(x')$ translates to $(\exists\bar{h})\,\{\bar{t},\bar{t}'\}{\subseteq}\bar{h}$ and their $x{\not<}x'$ translates to $(\nexists\ell{\leq}L(\bar{t}'))\,\bar{t}{=}{}_1\bar{t}'_\ell$.

[9]Imagine that Spy 1 and Spy 2 are racing to recover a document from a safe deposit box. En route one spy realizes that if she reaches the box first, she can

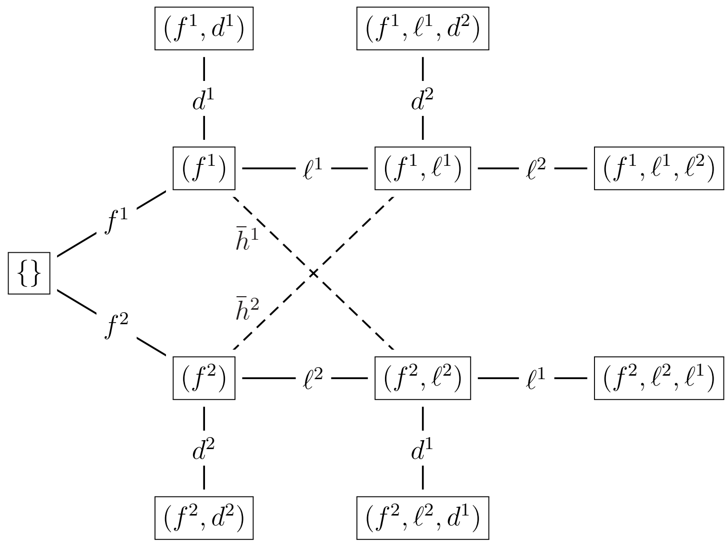

FIGURE 4. An OR* choice-sequence form with no-absent-mindedness. In accord with Proposition 4.1, $R|_{\bar{N}}$ is injective.

replicates the classic example of unordered agents from Kuhn (1953, Figure 1), Gilboa (1997, Figure 2), Ritzberger (1999, Figure 1), and Ritzberger (2002, Figure 3.8). Unordered agents give rise to choices that can be played in different orders. Accordingly, the choices ℓ^1 and ℓ^2 in Figure 4 have been played in different orders at the nodes (f^1, ℓ^1, ℓ^2) and (f^2, ℓ^2, ℓ^1). However, the choices in

$$R((f^1, \ell^1, \ell^2)) = \{f^1, \ell^1, \ell^2\}$$

can only be played in the order (f^1, ℓ^1, ℓ^2), and the choices in

$$R((f^2, \ell^2, \ell^1)) = \{f^2, \ell^1, \ell^2\}$$

can only be played in the order (f^2, ℓ^2, ℓ^1). Intuitively, this happens because the set $\{f^1, \ell^1, \ell^2\}$ contains f^1, and because the set $\{f^2, \ell^1, \ell^2\}$ contains f^2. This suggests that if a form has two choices whose order is not exogenously determined, then any sequence that lists the two choices must also list another choice (or set of choices) that determines

install a bomb that will explode when the other spy reaches the box after her. But then she realizes that the other spy will be thinking the same thing, and hence, if she opens the box when she reaches it, she will find either the document or an exploding bomb. So, she considers destroying the bank without opening the box in hopes of keeping the document from the other spy. Figure 4 specifies this situation. Chance determines whether Spy 1 (f^1) or Spy 2 (f^2) arrives first. Then the two spies either look (ℓ) in the box or destroy (d) the bank.

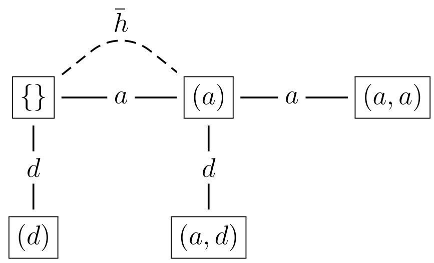

FIGURE 5. An $\texttt{OR}^*$ choice-sequence form with absent-mindedness. In accord with Proposition 4.1, $R|_{\bar{N}}$ is not injective (consider (a) and (a,a)).

their order. Showing that this can be done, whenever there is no-absent-mindedness, is the interesting part of the proposition's proof.

Conversely, Proposition 4.1(a⇐b) shows that no-absent-mindedness is necessary for injectivity. For example, consider Figure 5, which replicates the classic example of absent-mindedness in Piccione and Rubinstein (1997, Figure 1). Here R takes both the sequence (a) and the sequence (a,a) to the set $\{a\}$. Thus, $R|_{\bar{T}}$ is not injective. The proposition's proof shows that something similar happens whenever no-absent-mindedness is violated.

4.2. CONVERTING CHOICE-SEQUENCE FORMS TO CHOICE-SET FORMS

The previous subsection concerned only $\texttt{OR}^*$ choice-sequence forms. Yet Proposition 4.1(a⇒c) showed that the order explicitly specified in choice sequences is redundant whenever no-absent-mindedness holds. This suggests that every no-absent-minded $\texttt{OR}^*$ choice-sequence form can be converted into an "equivalent" choice-set form.

The following theorem does so. In particular, part (a) shows how to convert a no-absent-minded $\texttt{OR}^*$ choice-sequence form into a choice-set form. Then the remaining parts of the theorem describe several senses in which the original choice-sequence form and the new choice-set form are "equivalent" to one another.

Theorem 1. *Suppose $((C_i)_i, \bar{N})$ is an $\texttt{OR}^*$ choice-sequence form (6) with no-absent-mindedness (16). Let $N = R(\bar{N})$. Then*

(a) $((C_i)_i, N)$ is a choice-set form (14).

Further, derive $C{=}\cup_i C_i$, $\bar{T}$ (2), $\bar{F}$ (3), T (9), F (10), and p (11). Then the following hold.

(b) $R|_{\bar{N}}$ *is a bijection from* $\bar{N}$ *onto* N*.*
(c) $R(\bar{T}) = T$*.*
(d) $(\forall \bar{t}, c, \bar{t}^\sharp)\ \bar{t}{\oplus}(c){=}\bar{t}^\sharp \Leftrightarrow (\ c{\notin}R(\bar{t})$ *and* $R(\bar{t}){\cup}\{c\} = R(\bar{t}^\sharp)\ )$*.*
(e) $F = \{\ (R(\bar{t}), c) \mid (\bar{t}, c){\in}\bar{F}\ \}$*.*
(f) $p = \{\ (R(\bar{t}), R({}_1\bar{t}_{L(\bar{t})-1})) \mid \bar{t}{\neq}\{\}\ \}$*. (Proof C.4.)*

Since $N = R(\bar{N})$ by definition, part (b) of the theorem merely restates Proposition 4.1(a⇒c). The remaining conclusions are natural but not trivial.

In particular, part (a) shows that the axioms defining an $\mathtt{OR}^*$ choice-sequence form imply the axioms defining a choice-set form. Part (c) relates the finite choice sequences in $\bar{T}$ to the finite choice sets in T. Then part (d) shows that nodes and choices play parallel roles in the two formulations. Accordingly, concatenation in the context of choice sequences corresponds to union in the context of choice sets. Part (e) relates the feasibility correspondences of the two formulations. And finally, part (f) relates the choice-set immediate-predecessor function p to its choice-sequence counterpart.

4.3. Bijection

Theorem 1(a) allows us to define the operator $\widehat{\mathtt{R}}$ that takes no-absent-minded $\mathtt{OR}^*$ choice-sequence forms to choice-set forms by the rule

$$\widehat{\mathtt{R}}\ :\ ((C_i)_i, \bar{N})\ \mapsto\ ((C_i)_i, R(\bar{N}))\ . \tag{17}$$

Proposition 4.1(a⇐b) shows that the domain of $\widehat{\mathtt{R}}$ cannot be meaningfully extended to include $\mathtt{OR}^*$ choice-sequence forms with absent-mindedness.

The following theorem establishes that $\widehat{\mathtt{R}}$ is a one-to-one correspondence between (1) the class of no-absent-minded $\mathtt{OR}^*$ choice-sequence forms and (2) the class of choice-set forms. This result and Theorem 1(b–f) both support the claim that no-absent-minded $\mathtt{OR}^*$ choice-sequence forms are "equivalent" to choice-set forms.

Theorem 2. $\widehat{\mathtt{R}}$ *(17) is a bijection from the class of* $\mathtt{OR}^*$ *choice-sequence forms (6) with no-absent-mindedness (16) onto the class of choice-set forms (14). (Proof D.3.)*

Since Theorem 1(a) already showed that $\widehat{\mathtt{R}}$ maps into the class of choice-set forms, the proof of Theorem 2 must only show that $\widehat{\mathtt{R}}$ is injective and surjective.

The proof of the injectivity of $\widehat{\mathtt{R}}$ resembles the proof of the injectivity of $R|_{\bar{N}}$ in Proposition 4.1(a⇒c). However, the argument here is at a deeper level. There, an $\mathtt{OR}^*$ choice-sequence form $((C_i)_i, \bar{N})$ was taken as given, and it was shown that R cannot take two choice sequences in $\bar{N}$ to the same choice set. In contrast, it is shown here that $\widehat{\mathtt{R}}$ cannot take two $\mathtt{OR}^*$ choice-sequence forms $((C_i^1)_i, \bar{N}^1)$ and $((C_i^2)_i, \bar{N}^2)$ to the same choice-set form. So, at this deeper level as well, the order explicitly specified in sequences is redundant.

To prove the surjectivity of $\widehat{\mathtt{R}}$, I take an arbitrary choice-set form and convert it into an $\mathtt{OR}^*$ choice-sequence form. The proof's most difficult step shows that the infinite choice-set nodes correspond to the infinite choice-sequence nodes. Another step shows that the axioms defining a choice-set form imply the axioms defining an $\mathtt{OR}^*$ choice-sequence form. Throughout the entire proof, I lean heavily on Lemma B.6, which shows that all choice-set forms implicitly satisfy a property that resembles Lemma A.5(a)'s characterization of no-absent-mindedness for $\mathtt{OR}^*$ choice-sequence forms. This resemblance further reinforces the notion that choice-set forms are "equivalent" to no-absent-minded $\mathtt{OR}^*$ choice-sequence forms.

5. COROLLARIES

This section contains several results about choice-set forms. Although these results are natural, they are nontrivial. All are proved as corollaries of Theorems 1 and 2, and they appear here in the order in which they are proved in Appendix E. All are used extensively by Streufert (2015b).

Consider a choice-set form $((C_i)_i, N)$. Say that one node n^1 *precedes* another node n^2 if $n^1 \subset n^2$. Equivalently, say that n^2 *succeeds* n^1. Corollary 5.1(b) characterizes the predecessors of a finite node. Part (a) is a useful intermediate result.

Corollary 5.1. *Suppose $((C_i)_i, N)$ is a choice-set form (14) with its T (9) and p (11). Then the following hold.*[10]

(a) $t^\flat \subset t$ iff both $|t^\flat| < |t|$ and $t^\flat = p^{|t|-|t^\flat|}(t)$.

(b) $\{t^\flat | t^\flat \subset t\}$ is the chain $\{p^j(t) \,|\, |t| \geq j \geq 1\}$. (Proof E.1.)

[10] I use the superscript $^\flat$ to suggest a predecessor, and the superscript $^\sharp$ to suggest a successor.

Let a *terminal* node be a node without successors. Corollary 5.2(c) characterizes the nonterminal nodes. In particular, it shows that the collection of nonterminal nodes equals the collection of finite nodes with nonempty feasible sets. The latter is expressed as $F^{-1}(C)$, which is the domain of the feasibility correspondence F. Parts (a) and (b) are useful intermediate results. In particular, part (b) shows that one infinite node cannot be included within another infinite node.

Corollary 5.2. *Suppose* $((C_i)_i, N)$ *is a choice-set form (14) with its* $C{=}\cup_i C_i$, T *(9), and* F *(10). Then the following hold.*
(a) If $t{\subset}n^\sharp$ *then* $F(t){\cap}n^\sharp{\neq}\varnothing$.
(b) $\{\ n \mid (\exists n^\sharp)\, n{\subset}n^\sharp\ \} \subseteq T$.
(c) $\{\ n \mid (\exists n^\sharp)\, n{\subset}n^\sharp\ \} = F^{-1}(C)$. *(Proof E.2.)*

Corollary 5.3(b) characterizes the predecessors of an infinite node. It does so by part (a), which shows that every infinite node is uniquely associated with an infinite sequence of finite nodes.

Corollary 5.3. *Suppose* $((C_i)_i, N)$ *is a choice-set form (14) with its* T *(9) and* p *(11). Take any* $n \notin T$. *Then the following hold.*
(a) There exists a unique $(t^k)_{k\geq 0}$ *such that*

$$n{=}\cup\{t^k|k\}\ ,\ t^0{=}\{\}\ ,\ \text{and } (\forall k{\geq}1)\, p(t^k){=}t^{k-1}\ .$$

(b) $\{n^\flat|n^\flat{\subset}n\}$ *is the infinite chain* $\{t^k|k\}$, *where* $(t^k)_{k\geq 0}$ *is defined in part (a). (Proof E.3.)*

Finally, for any choice-set form $((C_i)_i, N)$, define the binary operator $\wedge$ on N by

$$(18) \qquad (\forall n^1, n^2)\ n^1{\wedge}n^2 := \max\{\ m \mid m{\subseteq}n^1{\cap}n^2\ \}\ ,$$

where m denotes an arbitrary member of N. Corollary 5.4(a) shows that $\wedge$ is well-defined, and thus the partially ordered set $(N, \subseteq)$ is a sublattice with meet $\wedge$. Further, Corollary 5.4(b) shows that the meet of any two distinct nodes is finite.

Corollary 5.4. *Suppose* $((C_i)_i, N)$ *is a choice-set form (14). Derive* T *(9) and* $\wedge$ *(18). Take any* n^1 *and* n^2. *Then*
(a) $n^1{\wedge}n^2$ *is well-defined, and*
(b) $n^1{\neq}n^2$ *implies* $n^1{\wedge}n^2{\in}T$. *(Proof E.6.)*

Appendix A. For choice-sequence forms only

A.1. The implicit specification of agents[11]

Lemma A.1. *Suppose* $(C,\bar{N})$ *satisfies (1a)–(1d) and derive its* $\bar{F}$ *(3). Then* $(\forall c)\ \bar{F}^{-1}(c) \neq \varnothing$.

Proof. Derive $\bar{T}$ (2). Take any c. By (1b), there exists $\bar{n}$ such that $c \in R(\bar{n})$. Either (a) $\bar{n} \in \bar{T}$ and $\bar{n}$ itself is a $\bar{t}$ such that $c \in R(\bar{t})$, or (b) $\bar{n} \notin \bar{T}$ and (1c) implies the existence of a $\bar{t}$ such that $c \in R(\bar{t})$. Hence in either case, there exist $\bar{t}$ and ℓ such that $\bar{t}_\ell = c$. Thus ${}_1\bar{t}_{\ell-1} \oplus (c) = {}_1\bar{t}_\ell$, which implies that $({}_1\bar{t}_{\ell-1}, c) \in \bar{F}$. □

Lemma A.2. *If* $\bar{F} \subseteq \bar{T}\times C$*, the following are equivalent.*
(a) $(\forall c,c')\ \bar{F}^{-1}(c){=}\bar{F}^{-1}(c')$ *or* $\bar{F}^{-1}(c){\cap}\bar{F}^{-1}(c'){=}\varnothing$.
(b) $(\forall \bar{t},\bar{t}')\ \bar{F}(\bar{t}){=}\bar{F}(\bar{t}')$ *or* $\bar{F}(\bar{t}){\cap}\bar{F}(\bar{t}'){=}\varnothing$.

Proof. By inspection, the following seven statements are equivalent.

$$(\exists c,c')\ \bar{F}^{-1}(c) \neq \bar{F}^{-1}(c') \text{ and } \bar{F}^{-1}(c){\cap}\bar{F}^{-1}(c') \neq \varnothing. \tag{19a}$$
$$(\exists c^1,c^2)\ \bar{F}^{-1}(c^2){\setminus}\bar{F}^{-1}(c^1) \neq \varnothing \text{ and } \bar{F}^{-1}(c^2){\cap}\bar{F}^{-1}(c^1) \neq \varnothing.$$
$$(\exists c^1,c^2,\bar{t}^1,\bar{t}^2)\ \bar{t}^1{\in}\bar{F}^{-1}(c^2),\ \bar{t}^1{\notin}\bar{F}^{-1}(c^1),\ \bar{t}^2{\in}\bar{F}^{-1}(c^2), \text{ and } \bar{t}^2{\in}\bar{F}^{-1}(c^1).$$
$$(\exists c^1,c^2,\bar{t}^1,\bar{t}^2)\ (\bar{t}^1,c^1){\notin}\bar{F} \text{ and } \{(\bar{t}^1,c^2),(\bar{t}^2,c^1),(\bar{t}^2,c^2)\}{\subseteq}\bar{F}.$$
$$(\exists c^1,c^2,\bar{t}^1,\bar{t}^2)\ c^1{\in}\bar{F}(\bar{t}^2),\ c^1{\notin}\bar{F}(\bar{t}^1),\ c^2{\in}\bar{F}(\bar{t}^2), \text{ and } c^2{\in}\bar{F}(\bar{t}^1).$$
$$(\exists \bar{t}^1,\bar{t}^2)\ \bar{F}(\bar{t}^2){\setminus}\bar{F}(\bar{t}^1) \neq \varnothing \text{ and } \bar{F}(\bar{t}^2){\cap}\bar{F}(\bar{t}^1) \neq \varnothing.$$
$$(\exists \bar{t},\bar{t}')\ \bar{F}(\bar{t}) \neq \bar{F}(\bar{t}') \text{ and } \bar{F}(\bar{t}){\cap}\bar{F}(\bar{t}') \neq \varnothing. \tag{19b}$$

(19a) is the negation of (a), and (19b) is the negation of (b). □

Proof A.3 (for Lemma 2.1)**.**

(a). To prove (5a), I must show that $\bar{H}$ is a partition of $\bar{F}^{-1}(C)$. First, by the definition of $\bar{H}$,

$$\cup\bar{H} = \cup\{\bar{F}^{-1}(c)|c\} = \bar{F}^{-1}(C)\ .$$

Second, (1e) and Lemma A.2(b⇒a) imply that

$$(\forall c,c')\ \bar{F}^{-1}(c){=}\bar{F}^{-1}(c') \text{ or } \bar{F}^{-1}(c){\cap}\bar{F}^{-1}(c'){=}\varnothing\ .$$

[11] The agents of `OR`* choice-sequence forms are needed for the definition of no-absent-mindedness (16). However, Lemma A.5 will characterize no-absent-mindedness without the use of agents. Thereafter, the agents of `OR`* choice-sequence forms play no role in the appendices. (The agents of *choice-set* forms appear briefly in Appendix B.1.)

Thus by the definition of $\bar{H}$, the members of $\bar{H}$ are disjoint. Third, by Lemma A.1 and the definition of $\bar{H}$, each member of $\bar{H}$ is nonempty.

To prove (5b), suppose $\bar{t}^1$, $\bar{t}^2$, and $\bar{h}$ satisfy $\{\bar{t}^1,\bar{t}^2\} \subseteq \bar{h}$. By the definition of $\bar{H}$, there exists some c such that $\bar{h} = \bar{F}^{-1}(c)$. Thus by the next-to-last sentence, $\{\bar{t}^1,\bar{t}^2\} \subseteq \bar{F}^{-1}(c)$. Hence $c \in \bar{F}(\bar{t}^1) \cap \bar{F}(\bar{t}^2)$. Thus (1e) implies $\bar{F}(\bar{t}^1) = \bar{F}(\bar{t}^2)$.

To prove the contrapositive of (5c), suppose $\bar{F}(\bar{h}^1) \cap \bar{F}(\bar{h}^2) \neq \varnothing$. Then there exists $\bar{t}^1$, $\bar{t}^2$, and c^* such that

$$\bar{t}^1 \in \bar{h}^1 \ , \ \bar{t}^2 \in \bar{h}^2 \ , \tag{20a}$$

$$c^* \in \bar{F}(\bar{t}^1) \text{ and } c^* \in \bar{F}(\bar{t}^2) \ . \tag{20b}$$

(20b) implies that $\{\bar{t}^1,\bar{t}^2\} \subseteq \bar{F}^{-1}(c^*)$. Thus by the definition of $\bar{H}$, $\bar{t}^1$ and $\bar{t}^2$ belong to a common $\bar{h}$. This, (20a), and the already proved fact (5a) that $\bar{H}$ is a partition, imply that both $\bar{h}^1$ and $\bar{h}^2$ equal $\bar{h}$. Thus $\bar{h}^1 = \bar{h}^2$.

(b). Assume $\bar{H}^*$ satisfies (5).

This paragraph shows that $(C,\bar{N})$ satisfies (1e). Accordingly, take any $\bar{t}$ and $\bar{t}'$. Either there exists an $\bar{h} \in \bar{H}^*$ containing both $\bar{t}$ and $\bar{t}'$ or there does not. In the first case, (5b) for $\bar{H}^*$ implies that $\bar{F}(\bar{t}) = \bar{F}(\bar{t}')$. In the second case, (5a) for $\bar{H}^*$ implies the existence of $\{\bar{h},\bar{h}'\} \subseteq \bar{H}^*$ such that $\bar{h} \neq \bar{h}'$, $\bar{h} \ni \bar{t}$, and $\bar{h}' \ni \bar{t}'$. Here $\bar{h} \neq \bar{h}'$ and (5c) for $\bar{H}^*$ imply $\bar{F}(\bar{h}) \cap \bar{F}(\bar{h}') = \varnothing$. Thus $\bar{h} \ni \bar{t}$ and $\bar{h}' \ni \bar{t}'$ imply $\bar{F}(\bar{t}) \cap \bar{F}(\bar{t}') = \varnothing$.

Since the previous paragraph established (1e), part (a) implies that the $\bar{H}$ defined by (4) satisfies (5). Similarly, the $\bar{H}^*$ assumed by this part (b) satisfies (5) by definition. Thus we can show that $\bar{H}^* = \bar{H}$ by showing that no more than one partition of $\bar{F}^{-1}(C)$ can satisfy (5b) and (5c).

Accordingly, suppose that $\bar{H}^1$ and $\bar{H}^2$ are two distinct partitions of $\bar{F}^{-1}(C)$ that satisfy (5b) and (5c). Then $\bar{H}^1 \setminus \bar{H}^2$ or $\bar{H}^2 \setminus \bar{H}^1$ is nonempty. Without loss of generality, assume the former and take $\bar{h}^1 \in \bar{H}^1 \setminus \bar{H}^2$. Since $\bar{H}^1$ is a partition, $\bar{h}^1$ is nonempty and thus we may take $\bar{t} \in \bar{h}^1$. Further, since $\bar{H}^2$ is a partition, there is an $\bar{h}^2 \in \bar{H}^2$ such that $\bar{t} \in \bar{h}^2$. Since $\bar{h}^1 \in \bar{H}^1 \setminus \bar{H}^2$, it must be that $\bar{h}^1 \neq \bar{h}^2$. Also note that $\bar{t} \in \bar{h}^1 \cap \bar{h}^2$ by the definitions of $\bar{t}$ and $\bar{h}^2$.

This paragraph shows that having $\bar{h}^1 \neq \bar{h}^2$ and $\bar{t} \in \bar{h}^1 \cap \bar{h}^2$ leads to a contradiction. Since $\bar{h}^1 \neq \bar{h}^2$, it must be that $\bar{h}^1 \setminus \bar{h}^2$ or $\bar{h}^2 \setminus \bar{h}^1$ is nonempty. Without loss of generality, suppose the former and take $\bar{t}^1 \in \bar{h}^1 \setminus \bar{h}^2$.

Since (a) $\bar{t}{\in}\bar{h}^2$, (b) $\bar{t}^1{\notin}\bar{h}^2$, and (c) $\bar{H}^2$ is a partition, $\bar{t}$ and $\bar{t}^1$ belong to different members of the partition $\bar{H}^2$. Hence (5c) for $\bar{H}^2$ implies that $\bar{F}(\bar{t}){\cap}\bar{F}(\bar{t}^1){=}\varnothing$. Yet, since $\bar{h}^1$ contains both $\bar{t}$ and $\bar{t}^1$, (5b) for $\bar{H}^1$ implies that $\bar{F}(\bar{t}){=}\bar{F}(\bar{t}^1)$. Further, $\bar{F}(\bar{t})$ is nonempty since $\bar{t}{\in}\bar{h}^1{\subseteq}\bar{F}^{-1}(C)$, where the set inclusion holds because $\bar{H}^1$ partitions $\bar{F}^{-1}(C)$. The last three sentences are logically inconsistent. □

Proof A.4 (for Lemma 2.2)**.**

(a). Note

$$\begin{aligned}
&\cup\{\bar{H}_i|i\}\\
&= \cup\{\ \{\bar{F}^{-1}(c)|c{\in}C_i\}\ |i\}\\
&= \{\bar{F}^{-1}(c)|c{\in}\cup_i C_i\}\\
&= \{\bar{F}^{-1}(c)|c\}\\
&= \bar{H}\ ,
\end{aligned}$$

where the first equality holds by the definition of $(\bar{H}_i)_i$, the third holds by the definition of C, and the fourth is the definition of $\bar{H}$.

It remains to be shown that the members of $(\bar{H}_i)_i$ are disjoint. Accordingly, suppose that $\bar{H}_{i^1}{\cap}\bar{H}_{i^2} \neq \varnothing$. Then by the definition of $(\bar{H}_i)_i$, there exists $c^1 \in C_{i^1}$ and $c^2 \in C_{i^2}$ such that $\bar{F}^{-1}(c^1) = \bar{F}^{-1}(c^2)$. Thus by Lemma A.1, there exists $\bar{t}$ such that

$$(\bar{t}, c^1) \in \bar{F} \text{ and} \tag{21a}$$

$$(\bar{t}, c^2) \in \bar{F}\ . \tag{21b}$$

(21a), the definition of c^1, and (6c) together imply $\bar{F}(\bar{t}) \subseteq C_{i^1}$. Similarly, (21b), the definition of c^2, and (6c) imply $\bar{F}(\bar{t}) \subseteq C_{i^2}$. Since $\bar{F}(\bar{t}) \neq \varnothing$ by (21a), the last two sentences imply $C_{i^1}{\cap}C_{i^2} \neq \varnothing$. This violates (6b).

(b). Note

$$\begin{aligned}
&\cup\{\cup\bar{H}_i|i\}\\
&= \cup\{\cup\{\bar{F}^{-1}(c)|c{\in}C_i\}|i\}\\
&= \cup\{\bar{F}^{-1}(C_i)|i\}\\
&= \bar{F}^{-1}(\cup_i C_i)\\
&= \bar{F}^{-1}(C)\ ,
\end{aligned}$$

where the first equality holds by the definition of $(\bar{H}_i)_i$, and the last holds by the definition of C.

It remains to be shown that the members of $\{\cup\bar{H}_i|i\}$ are disjoint. Accordingly, suppose that $(\cup\bar{H}_{i^1})\cap(\cup\bar{H}_{i^2}) \neq \varnothing$. Then by the definition of $(\bar{H}_i)_i$

$$(\cup\{\bar{F}^{-1}(c^1)|c^1{\in}C_{i^1}\}) \cap (\cup\{\bar{F}^{-1}(c^2)|c^2{\in}C_{i^2}\}) \neq \varnothing \ .$$

Thus there exists $\bar{t}$, $c^1{\in}C_{i^1}$, and $c^2{\in}C_{i^2}$, such that $\bar{t} \in \bar{F}^{-1}(c^1){\cup}\bar{F}^{-1}(c^2)$. Hence

$$(\bar{t}, c^1) \in \bar{F} \text{ and} \tag{22a}$$

$$(\bar{t}, c^2) \in \bar{F} \ . \tag{22b}$$

(22a), the definition of c^1, and (6c) together imply $\bar{F}(\bar{t}) \subseteq C_{i^1}$. Similarly, (22b), the definition of c^2, and (6c) imply $\bar{F}(\bar{t}) \subseteq C_{i^2}$. Since $\bar{F}(\bar{t}) \neq \varnothing$ by (22a), the last two sentences imply $C_{i^1}{\cap}C_{i^2} \neq \varnothing$. This violates (6b). □

A.2. Preliminary results for theorems

Lemma A.5. *Suppose $((C_i)_i, \bar{N})$ is an* OR* *choice-sequence form (6) with its $\bar{T}$ (2) and $\bar{F}$ (3). Then each of the following is equivalent to no-absent-mindedness (16).*

(a) $(\forall\bar{t})(\forall\bar{n})\ |\{\,\ell\,|\,\bar{n}_\ell{\in}\bar{F}(\bar{t})\,\}| \leq 1$.

(b) $(\forall\bar{t})\ |R(\bar{t})| = L(\bar{t})$.

Proof. Define $\bar{H}$ by (4). Since the negation of no-absent-mindedness is (23), since the negation of (a) is (24), and since the negation of (b) is (25) because $|R(\bar{t})| > L(\bar{t})$ is inconceivable, it suffices to show the equivalence of

$$(\exists\bar{h})(\exists\bar{t})(\exists 0{\leq}k{<}L(\bar{t}))\ \{{}_1\bar{t}_k, \bar{t}\}{\subseteq}\bar{h} \ , \tag{23}$$

$$(\exists\bar{t})(\exists\bar{n})\ |\{\,\ell\,|\,\bar{n}_\ell{\in}\bar{F}(\bar{t})\,\}| \geq 2 \ , \text{ and} \tag{24}$$

$$(\exists\bar{t})\ |R(\bar{t})| < L(\bar{t}) \ . \tag{25}$$

(23) implies (24). Let $\bar{h}$, $\bar{t}$, and $0 \leq k < L(\bar{t})$ be such that $\{{}_1\bar{t}_k, \bar{t}\} \subseteq \bar{h}$. Since $k < L(\bar{t})$, $\bar{t}_{k+1}$ exists and satisfies

$$\bar{t}_{k+1} \in \bar{F}({}_1\bar{t}_k) \ . \tag{26}$$

Thus, by $\{{}_1\bar{t}_k, \bar{t}\} \subseteq \bar{h}$, and by (5b) of Lemma 2.1(a), we have $\bar{t}_{k+1} \in \bar{F}(\bar{t})$. Thus we may construct $\bar{n} = \bar{t} \oplus (\bar{t}_{k+1})$. Then

$$|\{\, \ell \,|\, \bar{n}_\ell \in \bar{F}({}_1\bar{t}_k) \,\}| \;\geq\; |\{\, \ell \,|\, \bar{n}_\ell = \bar{t}_{k+1} \,\}| \;\geq\; |\{\, k{+}1,\, L(\bar{t}){+}1 \,\}| \;=\; 2 \ ,$$

where the first inequality holds by (26), the second inequality holds by the construction of $\bar{n}$, and the equality holds by $k < L(\bar{t})$.

(24) implies (25). Let $\bar{t}$ and $\bar{n}$ such that $|\{\ \ell \mid \bar{n}_\ell \in \bar{F}(\bar{t})\ \}| \geq 2$. Then there exist k and ℓ such that $k < \ell$ and $\{\bar{n}_k, \bar{n}_\ell\} \subseteq \bar{F}(\bar{t})$. Thus

$$\bar{t} \in \bar{F}^{-1}(\bar{n}_k) \cap \bar{F}^{-1}(\bar{n}_\ell) \ . \tag{27}$$

By the definition of $\bar{H}$, both of these inverse images are agents. Hence by (27), and by (5a) of Lemma 2.1(a),

$$\bar{F}^{-1}(\bar{n}_k) = \bar{F}^{-1}(\bar{n}_\ell)$$

Thus, since ${}_1\bar{n}_{\ell-1} \in \bar{F}^{-1}(\bar{n}_\ell)$, we have ${}_1\bar{n}_{\ell-1} \in \bar{F}^{-1}(\bar{n}_k)$. In other words, we have $\bar{n}_k \in \bar{F}({}_1\bar{n}_{\ell-1})$. Hence we may construct $\bar{t}^* = {}_1\bar{n}_{\ell-1} \oplus (\bar{n}_k)$. Since $k < \ell$, $\bar{t}^*_k$ is well-defined and equals $\bar{n}_k$. Since both $\bar{t}^*_k$ and $\bar{t}^*_\ell$ equal $\bar{n}_k$, $|R(\bar{t}^*)| < L(\bar{t}^*)$.

(25) implies (23). Let $\bar{t}$ be such that $|R(\bar{t})| < L(\bar{t})$. Then there are k and ℓ such that $1 \leq k < \ell$ and $\bar{t}_k = \bar{t}_\ell$. Since ${}_1\bar{t}_{k-1} \in \bar{F}^{-1}(\bar{t}_k)$, since ${}_1\bar{t}_{\ell-1} \in \bar{F}^{-1}(\bar{t}_\ell)$, and since $\bar{t}_k = \bar{t}_\ell$, we have

$$\{{}_1\bar{t}_{k-1}, {}_1\bar{t}_{\ell-1}\} \subseteq \bar{F}^{-1}(\bar{t}_k) \ . \tag{28}$$

By the definition of $\bar{H}$, let $\bar{h} = \bar{F}^{-1}(\bar{t}_k)$. Then (28) implies

$$\{{}_1\bar{t}_{k-1}, {}_1\bar{t}_{\ell-1}\} \subseteq \bar{h} \ . \tag{29}$$

Further, let $\bar{t}^* = {}_1\bar{t}_{\ell-1}$. Then $1 \leq k < \ell$ and (29) imply

$$0 \leq k{-}1 < \ell{-}1 = L(\bar{t}^*) \text{ and}$$
$$\{{}_1\bar{t}^*_{k-1}, \bar{t}^*\} \subseteq \bar{h} \ .$$

□

Proof A.6 (for Proposition 4.1)**.** It suffices to prove the equivalence of

$$((C_i)_i, \bar{N}) \text{ has absent-mindedness} \ , \tag{30}$$

$$(\exists \bar{t}^1, \bar{t}^2)\ \bar{t}^1 {\neq} \bar{t}^2 \text{ and } R(\bar{t}^1) {=} R(\bar{t}^2) \ , \text{ and} \tag{31}$$

$$(\exists \bar{n}^1, \bar{n}^2)\ \bar{n}^1 {\neq} \bar{n}^2 \text{ and } R(\bar{n}^1) {=} R(\bar{n}^2) \ . \tag{32}$$

(30) implies (31). Suppose $((C_i)_i, \bar{N})$ has absent-mindedness. Then Lemma A.5(b) implies the existence of a sequence $\bar{t}$ such that $|R(\bar{t})| < L(\bar{t})$. Thus there exist indices $1 \le k < \ell \le L(\bar{t})$ such that $\bar{t}_k = \bar{t}_\ell$. Hence $R({}_1\bar{t}_{\ell-1}) = R({}_1\bar{t}_\ell)$.

(31) implies (32). This is obvious since $\bar{T} \subseteq \bar{N}$.

(32) implies (30). Assume that $\bar{n}^1$ and $\bar{n}^2$ are distinct elements of $\bar{N}$ such that $R(\bar{n}^1) = R(\bar{n}^2)$. Define

$$\begin{aligned} K^{\neq} = \{\ 1{\le}\ell \mid\ & \bar{n}^1_\ell \neq \bar{n}^2_\ell, \\ & \ell{\le}L(\bar{n}^1) \text{ if } \bar{n}^1 \text{ is finite}, \\ & \ell{\le}L(\bar{n}^2) \text{ if } \bar{n}^2 \text{ is finite } \}\ . \end{aligned}$$

On the one hand, suppose $K^{\neq}$ is empty. Then the distinctness of $\bar{n}^1$ and $\bar{n}^2$ implies that one is a subsequence (Section 2.1) of the other. Without loss of generality, suppose $\bar{n}^1$ is a subsequence of $\bar{n}^2$. Hence $\bar{n}^1$ is finite,

$$\bar{n}^1 = {}_1\bar{n}^2_{L(\bar{n}^1)}\ , \text{ and} \tag{33a}$$

$$(\ L(\bar{n}^1){<}L(\bar{n}^2) \text{ or } \bar{n}^2 \text{ is infinite })\ . \tag{33b}$$

By (33b), $\bar{n}^2_{L(\bar{n}^1)+1}$ exists and is an element of $R(\bar{n}^2)$. Thus since $R(\bar{n}^1) = R(\bar{n}^2)$ by assumption, there exists some $k \le L(\bar{n}^1)$ such that $\bar{n}^1_k = \bar{n}^2_{L(\bar{n}^1)+1}$. Thus by (33a), $\bar{n}^2_k = \bar{n}^2_{L(\bar{n}^1)+1}$. So, since $k \le L(\bar{n}^1)$, both the k-th component and the last component of ${}_1\bar{n}^2_{L(\bar{n}^1)+1}$ equal n^2_k. Hence $|R({}_1\bar{n}^2_{L(\bar{n}^1)+1})| < L({}_1\bar{n}^2_{L(\bar{n}^1)+1})$. This inequality implies absent-mindedness by Lemma A.5(b).

On the other hand, suppose $K^{\neq}$ is nonempty. Define $k = \min K^{\neq}$. Then

$$\bar{n}^1_k \neq \bar{n}^2_k \text{ and} \tag{34a}$$

$${}_1\bar{n}^1_{k-1} = {}_1\bar{n}^2_{k-1}\ . \tag{34b}$$

Since $\bar{n}^1_k \in \bar{F}({}_1\bar{n}^1_{k-1})$ and $\bar{n}^2_k \in \bar{F}({}_1\bar{n}^2_{k-1})$, (34b) implies

$$\{\bar{n}^1_k, \bar{n}^2_k\} \subseteq \bar{F}({}_1\bar{n}^1_{k-1})\ . \tag{35}$$

Further, (34a) and $R(\bar{n}^1) = R(\bar{n}^2)$ imply the existence of some $k' \neq k$ such that $\bar{n}^1_{k'} = \bar{n}^2_k$. Thus by (35),

$$\{\bar{n}^1_k, \bar{n}^1_{k'}\} \subseteq \bar{F}({}_1\bar{n}^1_{k-1})\ .$$

This and $k' \neq k$ imply absent-mindedness by Lemma A.5(a) at $\bar{t} = {}_1\bar{n}^1_{k-1}$. □

Lemma A.7 (The "zipper" lemma).[12] *Suppose* $((C_i)_i, \bar{N})$ *is an* $\mathtt{OR}^*$ *choice-sequence form (6) with no-absent-mindedness (16). Derive its* $\bar{T}$ *by (2). Then*[10]

$$(\forall \bar{t}^\flat, \bar{t}) \;\; R(\bar{t}^\flat) \subseteq R(\bar{t}) \;\Rightarrow\; [\; L(\bar{t}^\flat) \leq L(\bar{t}) \textit{ and } \bar{t}^\flat = {}_1\bar{t}_{L(\bar{t}^\flat)} \;] \;.$$

Proof. Take any $\bar{t}^\flat$ and $\bar{t}$ such that $R(\bar{t}^\flat) \subseteq R(\bar{t})$. By Lemma A.5(b), by $R(\bar{t}^\flat) \subseteq R(\bar{t})$, and by Lemma A.5(b) again, we have

$$L(\bar{t}^\flat) = |R(\bar{t}^\flat)| \leq |R(\bar{t})| = L(\bar{t}) \;.$$

This is the first of the lemma's two conclusions. The next two paragraphs will show by induction on $\ell \in \{1, 2, ... L(\bar{t}^\flat)\}$ that $(\forall \ell \leq L(\bar{t}^\flat))$ ${}_1\bar{t}^\flat_\ell = {}_1\bar{t}_\ell$.

For the initial step at $n = 1$, suppose that $\bar{t}^\flat_1 \neq \bar{t}_1$. Note that $\{\bar{t}^\flat_1, \bar{t}_1\} \subseteq \bar{F}(\{\})$. Since $R(\bar{t}^\flat) \subseteq R(\bar{t})$, it must be that $\bar{t}^\flat_1 \in R(\bar{t})$, and hence there exists a $k > 1$ such that $\bar{t}^\flat_1 = \bar{t}_k$. The last two sentences imply that there exists a $k > 1$ such that $\{\bar{t}_k, \bar{t}_1\} \subseteq \bar{F}(\{\})$. By Lemma A.5(a) at its $\bar{n}$ equal to the $\bar{t}$ here, this contradicts no-absent-mindedness.

For the inductive step at $\ell \in \{2, 3, ...L(\bar{t}^\flat)\}$, assume that ${}_1\bar{t}^\flat_{\ell-1} = {}_1\bar{t}_{\ell-1}$ and suppose that $\bar{t}^\flat_\ell \neq \bar{t}_\ell$. The equality implies that $\{\bar{t}^\flat_\ell, \bar{t}_\ell\} \subseteq \bar{F}({}_1\bar{t}_{\ell-1})$. Also, since $R(\bar{t}^\flat) \subseteq R(\bar{t})$, it must be that $\bar{t}^\flat_\ell \in R(\bar{t})$, and hence there exists a $k \neq \ell$ such that $\bar{t}^\flat_\ell = \bar{t}_k$. The last two sentences imply that there exists a $k \neq \ell$ such that $\{\bar{t}_k, \bar{t}_\ell\} \subseteq \bar{F}({}_1\bar{t}_{\ell-1})$. By Lemma A.5(a) at its $\bar{n}$ equal to the $\bar{t}$ here, this contradicts no-absent-mindedness.

Therefore $(\forall \ell \leq L(\bar{t}^\flat))$ ${}_1\bar{t}^\flat_\ell = {}_1\bar{t}_\ell$. In particular, at $\ell = L(\bar{t}^\flat)$, we have ${}_1\bar{t}^\flat_{L(\bar{t}^\flat)} = {}_1\bar{t}_{L(\bar{t}^\flat)}$. The left-hand side is $\bar{t}^\flat$. □

APPENDIX B. FOR CHOICE-SET FORMS ONLY

B.1. THE IMPLICIT SPECIFICATION OF AGENTS

This Appendix B.1 can stand alone, without referring to any other appendix. Further, no appendix refers to it.

Lemma B.1. *Suppose that* (C, N) *is a choice-set preform (8) with its* F *(10). Then* $(\forall c)\; F^{-1}(c) \neq \varnothing$.

[12] The lemma's two sequences are like the two sides of an unusual zipper whose sides may have different lengths. The lemma's inductive proof starts with the sequences' first choices and works its way up. Lemma E.4(a,c) extends this lemma to accommodate infinite sequences.

Proof. Derive T (9) and p (11). Then take any c. By (8b), there exists n such that $c \in n$. Further, there exists t such that $c \in t$ because either (a) $n \in T$ and n itself is a t such that $c \in t$, or (b) $n \notin T$ and (8c) implies the existence of a t such that $c \in t$. Hence by the definition of p, there exists a j such that $|t| \geq j \geq 1$, $c \notin p^j(t)$, and $p^j(t) \cap \{c\} = p^{j-1}(t)$. Hence $(p^j(t), c) \in F$. □

Lemma B.2. *If $F \subseteq T \times C$, the following are equivalent.*
(a) $(\forall c, c')$ $F^{-1}(c) = F^{-1}(c')$ *or* $F^{-1}(c) \cap F^{-1}(c') = \varnothing$.
(b) $(\forall t, t')$ $F(t) = F(t')$ *or* $F(t) \cap F(t') = \varnothing$.

Proof. This proof is directly analogous to that of Lemma A.2. Replace $\bar{T}$ with T, and $\bar{F}$ with F. □

Proof B.3 (for Lemma 3.1)**.**
This proof is directly analogous to Proof A.3 for Lemma 2.1. Replace $\bar{T}$ with T, $\bar{F}$ with F, $\bar{H}$ with H, (1) with (8), (4) with (12), (5) with (13), Lemma A.1 with Lemma B.1, and Lemma A.2 with Lemma B.2. □

Proof B.4 (for Lemma 3.2)**.**
This proof is directly analogous to Proof A.4 for Lemma 2.2. Replace $\bar{T}$ with T, $\bar{F}$ with F, $\bar{H}$ with H, (6) with (14), and Lemma A.1 with Lemma B.1. □

B.2. Preliminary results for theorems

Lemma B.5. *If (C, N) is a choice-set preform (8), then $\{\} \in N$.*

Proof. Define T by (9). By (8a), there is an n. Thus there is a t because (1) if $n \in T$ then n itself is a t, and (2) if $n \notin T$ then (8c) implies the existence of a $t \subset n$. If $t = \{\}$, we are done because $T \subseteq N$ by the definition of N. If not,

$$\{\} = p^{|t|}(t) \in T \subseteq N \ ,$$

where the equality and set membership both follow from (8d) and the definition of p, and where the set inclusion follows from the definition of T. □

Lemma B.6. *Suppose that* (C,N) *is a choice-set preform (8) and derive its* T *(9) and* F *(10). Then* $(\forall t)(\forall n)\ |F(t)\cap n| \leq 1$.[13]

Proof. Take any t^o and n^o, and suppose that c^1 and c^2 are distinct elements of $F(t^o)\cap n^o$.

This paragraph defines t such that $\{c^1,c^2\} \subseteq t$. On the one hand, if $n^o \in T$, let $t = n^o$. On the other hand, if $n^o \notin T$, (8c) implies the existence of an infinite chain T^* such that $\cup T^* = n^o$. Since $\{c^1,c^2\}$ is finite, there is a $t \in T^*$ such that $\{c^1,c^2\} \subseteq t$.

Now define

$$t^2 := \min\{\ p^j(t)\ |\ |t|{\geq}j{\geq}0 \text{ and } \{c^1,c^2\}{\subseteq}p^j(t)\ \}\ ,$$

where $p^0(t) := t$ (the set contains at least $p^0(t)$ by the definition of t). Note that $t^2{\smallsetminus}p(t^2)$ is a singleton containing either c^1 or c^2. Without loss of generality, assume $t^2{\smallsetminus}p(t^2) = \{c^2\}$. Note that

(36a) $$c^1 \in p(t^2) \text{ and}$$

(36b) $$c^2 \in F(p(t^2))\ .$$

By the first paragraph and (36b), c^2 is an element of both $F(t^o)$ and $F(p(t^2))$. Thus by (8e), $F(t^o) = F(p(t^2))$. Thus since $c^1 \in F(t^o)$ by the first paragraph, $c^1 \in F(p(t^2))$. This, (36a), and the definition of F contradict one another. □

Lemma B.7. *Suppose that* (C,N) *satisfies (8a) and derive its* T *by (9). Then for any* $s \subseteq C$,

$$(\exists T^*)\ T^* \textit{ is an infinite chain in } T \textit{ and } \cup T^*{=}s\ .$$
$$\Leftrightarrow\ \ (\exists (t^k)_{k\geq 1})\ (\forall k) t^k{\subset}t^{k+1} \textit{ and } \cup_k t^k{=}s\ .$$

Proof. The $\Leftarrow$ direction is proved by setting $T^* = \{t^k|k\}$.

To prove the $\Rightarrow$ direction, take any s and assume T^* is an infinite chain in T such that $\cup T^*{=}s$. By the definition of T, T^* is a chain of finite sets. Thus any nonempty subcollection of T^* has a minimum. Accordingly, define $(t^k)_{k\geq 1}$ recursively by $t^1 = \min T^*$ and $(\forall k{\geq}2)\ t^k = \min T^*{\smallsetminus}\{t^1,t^2,...t^{k-1}\}$. Note that

(37) $$(\forall k{\geq}1)\ t^k \subset t^{k+1}\ .$$

[13] The property here resembles that of Lemma A.5(a). Accordingly, the property here can be loosely regarded as the "no-absent-mindedness" that is implicit in a choice-set form.

Thus it remains to show $\cup_k t^k = s$. Note that $\cup_k t^k \subseteq \cup T^* = s$, because the set inclusion holds by $(\forall k)\ t^k \in T^*$ and because the equality holds by assumption. Conversely, the next two paragraphs show $s \subseteq \cup_k t^k$.

This paragraph shows by induction that

$$(\forall k{\geq}1)\ k{-}1 \leq |t^k|\ . \tag{38}$$

The initial step at k=1 is $0 \leq |t^1|$, which holds trivially. Now take any $k{\geq}1$ and assume $k{-}1 \leq |t^k|$. Then

$$k\ =\ (k{-}1){+}1 \leq |t^k|{+}1 \leq |t^{k+1}|\ ,$$

where the first inequality holds by the inductive hypothesis and the second inequality holds by (37).

Finally take any $c \in s$. Since $s = \cup T^*$ by assumption, there exists some $u \in T^*$ such that $c \in u$. Because T^* is a chain and $(\forall k)\ t^k{\in}T^*$, either $u \subseteq t^{|u|+1}$ or $u \supset t^{|u|+1}$. The latter would imply $|u|\ >\ |t^{|u|+1}|$, which is equivalent to $k{-}1 > |t^k|$ for $k = |u|{+}1$. Since this would contradict (38), it must be that $u \subseteq t^{|u|+1}$. Hence $c \in u \subseteq t^{|u|+1} \subseteq \cup_k t^k$. □

APPENDIX C. FOR THEOREM 1

Lemma C.1.[14] *Take any $(C_i)_i$. Suppose*

(i) $\bar{N}$ is a nonempty collection of sequences in C,

(ii) N is a nonempty collection of subsets of C,

(iii) $R|_{\bar{T}}$ is a bijection from $\bar{T}$ onto T, and

(iv) $(\forall \bar{t}, c, \bar{t}^\sharp)\ \bar{t}{\oplus}(c){=}\bar{t}^\sharp \Leftrightarrow (\ c{\notin}R(\bar{t})$ and $R(\bar{t}){\cup}\{c\}{=}R(\bar{t}^\sharp)\ )$,

where C is defined by $\cup_i C_i$, $\bar{T}$ by (2), and T by (9). Then the following hold.

(a) $F = \{\ (R(\bar{t}), c)\ |\ (\bar{t}, c){\in}\bar{F}\ \}$, where $\bar{F}$ is defined by (3) and F is defined by (10).

(b) $p = \{\ (R(\bar{t}), R({}_1\bar{t}_{L(\bar{t})-1}))\ |\ \bar{t}{\neq}\{\}\ \}$, where p is defined by (11).

(c) (1e) is equivalent to (8e).

(d) (6c) is equivalent to (14c).

Proof. (a). Take any $\bar{t}$ and any c. Then

$$\begin{aligned} & (R(\bar{t}), c) \in F \\ \Leftrightarrow\ & c{\notin}R(\bar{t}) \text{ and } R(\bar{t}){\cup}\{c\}{\in}T \end{aligned}$$

[14] This lemma has an unusual appearance because it is used in both Appendix C and Appendix D.

$$\begin{aligned}
&\Leftrightarrow\ (\exists t^\sharp)\ c{\notin}R(\bar t) \text{ and } R(\bar t){\cup}\{c\}{=}t^\sharp\\
&\Leftrightarrow\ (\exists \bar t^\sharp)\ c{\notin}R(\bar t) \text{ and } R(\bar t){\cup}\{c\}{=}R(\bar t^\sharp)\\
&\Leftrightarrow\ (\exists \bar t^\sharp)\ \bar t{\oplus}(c) = \bar t^\sharp\\
&\Leftrightarrow\ \bar t{\oplus}(c) \in \bar T\\
&\Leftrightarrow\ (\bar t, c) \in \bar F\ ,
\end{aligned}$$

where the first equivalence is the definition of F, the third follows from (iii), the fourth follows from (iv), and the sixth follows from the definition of $\bar F$.

(b). I argue

$$\begin{aligned}
p &= \{\ (t, t{\smallsetminus}\{c\}) \mid c{\in}t \text{ and } t{\smallsetminus}\{c\}{\in}T\ \}\\
&= \{\ (t, t{\smallsetminus}\{c\}) \mid (\exists t^\flat)\ c{\in}t \text{ and } t{\smallsetminus}\{c\}{=}t^\flat\ \}\\
&= \{\ (t, t^\flat) \mid (\exists c)\ c{\in}t \text{ and } t{\smallsetminus}\{c\}{=}t^\flat\ \}\\
&= \{\ (t, t^\flat) \mid (\exists c)\ c{\notin}t^\flat \text{ and } t^\flat{\cup}\{c\}{=}t\ \}\\
&= \{\ (R(\bar t), R(\bar t^\flat)) \mid (\exists c)\ c{\notin}R(\bar t^\flat) \text{ and } R(\bar t^\flat){\cup}\{c\}{=}R(\bar t)\ \}\\
&= \{\ (R(\bar t), R(\bar t^\flat)) \mid (\exists c)\ \bar t^\flat{\oplus}(c){=}\bar t\ \}\\
&= \{\ (R(\bar t), R(\bar t^\flat)) \mid \bar t{\neq}\{\} \text{ and } \bar t^\flat{=}{}_1\bar t_{L(\bar t)-1}\ \}\\
&= \{\ (R(\bar t), R({}_1\bar t_{L(\bar t)-1})) \mid \bar t{\neq}\{\}\ \}\ .
\end{aligned}$$

The first equality is the definition of p. The fifth equality holds by (iii). The sixth equality holds by (iv).

(c). I argue that

$$\begin{aligned}
&\text{(1e)}\\
\Leftrightarrow\ &(\forall \bar t^1, \bar t^2)\ \bar F(\bar t^1){=}\bar F(\bar t^2) \text{ or } \bar F(\bar t^1){\cap}\bar F(\bar t^2){\neq}\varnothing\\
\Leftrightarrow\ &(\forall \bar t^1, \bar t^2)\ F{\circ}R(\bar t^1){=}F{\circ}R(\bar t^2) \text{ or } F{\circ}R(\bar t^1){\cap}F{\circ}R(\bar t^2){\neq}\varnothing\\
\Leftrightarrow\ &(\forall t^1, t^2)\ F(t^1){=}F(t^2) \text{ or } F(t^1){\cap}F(t^2){\neq}\varnothing\\
\Leftrightarrow\ &\text{(8e)}\ .
\end{aligned}$$

The second equivalence holds by part (a), and the third equivalence holds by (iii).

(d). Take any i. I argue that

$$\begin{aligned}
&\text{(6c) at } i\\
\Leftrightarrow\ &(\forall \bar t)\ \bar F(\bar t){\subseteq}C_i \text{ or } \bar F(\bar t){\cap}C_i{=}\varnothing
\end{aligned}$$

$$\Leftrightarrow (\forall\bar{t})\ F{\circ}R(\bar{t}){\subseteq}C_i \text{ or } F{\circ}R(\bar{t}){\cap}C_i{=}\varnothing$$
$$\Leftrightarrow (\forall t)\ F(t){\subseteq}C_i \text{ or } F(t){\cap}C_i{=}\varnothing$$
$$\Leftrightarrow \text{(14c) at } i\ .$$

The second equivalence holds by part (a), and the third equivalence holds by (iii). □

Lemma C.2. *Suppose $((C_i)_i, \bar{N})$ is an $\mathtt{OR}^*$ choice-sequence form (6) with no-absent-mindedness (16). Let $N = R(\bar{N})$. Further, derive $C{=}\cup_i C_i$, $\bar{T}$ (2), $\bar{F}$ (3), T (9), F (10), and p (11). Then the following hold.*

(a) N is a nonempty collection of subsets of C.
(b) $R|_{\bar{N}}$ is a bijection from $\bar{N}$ onto N.
(c) $R(\bar{T}) = T$.
(d) $(\forall\bar{t}, c, \bar{t}^\sharp)\ \bar{t}{\oplus}(c){=}\bar{t}^\sharp \Leftrightarrow (\ c{\notin}R(\bar{t})$ and $R(\bar{t}){\cup}\{c\} = R(\bar{t}^\sharp)\ )$.
(e) $F = \{\ (R(\bar{t}), c)\ |\ (\bar{t}, c){\in}\bar{F}\ \}$.
(f) $p = \{\ (R(\bar{t}), R({}_1\bar{t}_{L(\bar{t})-1}))\ |\ \bar{t}{\neq}\{\}\ \}$.

Proof. (a). N is a nonempty collection of subsets of C because (1) $N = R(\bar{N})$ by definition and (2) $\bar{N}$ is a nonempty collection of sequences in C by (1a).

(b). Since $N = R(\bar{N})$ by definition, $R|_{\bar{N}}$ is onto N. Injectivity follows from Proposition 4.1(a⇒c).

(c). To show the $\subseteq$ direction, take any $\bar{t}$. By the definition of N, $R(\bar{t}) \in N$. Further, $|R(\bar{t})| \leq L(\bar{t})$. By the last two sentences, $R(\bar{t}) \in T$.

To show the $\supseteq$ direction, take any t. By the definition of N, there exists $\bar{n}$ such that $R(\bar{n}) = t$. It remains to show that $\bar{n} \in \bar{T}$. Accordingly, suppose $\bar{n} \notin \bar{T}$. Then (1c) would imply $(\forall\ell{\geq}1)\ {}_1\bar{n}_\ell \in \bar{T}$. Hence

$$(\forall\ell{\geq}1)\ |R(\bar{n})| \geq |R({}_1\bar{n}_\ell)| = L({}_1\bar{n}_\ell) = \ell\ , \tag{39}$$

where the first equality holds by Lemma A.5(b). (39) implies that $|R(\bar{n})|$ is infinite. This contradicts that (a) $|t|$ is finite by the definition of T and (b) $|R(\bar{n})| = |t|$ by the definition of $\bar{n}$.

(d). This paragraph shows

$$(\forall\bar{t}^\flat, c, \bar{t})\ \ \bar{t}^\flat{\oplus}(c){=}\bar{t}\ \Rightarrow\ c{\notin}R(\bar{t}^\flat) \text{ and } R(\bar{t}^\flat){\cup}\{c\}{=}R(\bar{t})\ .$$

Accordingly, take any $\bar{t}^\flat$, c, and $\bar{t}$ such that $\bar{t}^\flat{\oplus}(c) = \bar{t}$. Note that $\bar{t}^\flat{\oplus}(c) = \bar{t}$ implies that $R(\bar{t}^\flat){\cup}\{c\} = R(\bar{t}^\flat{\oplus}(c)) = R(\bar{t})$, which is the second fact to be derived. Also note that

$$|R(\bar{t}^\flat)|+1 = L(\bar{t}^\flat)+1 = L(\bar{t}) = |R(\bar{t})|$$

by Lemma A.5(b), by $\bar{t}^\flat\oplus(c) = \bar{t}$, and by Lemma A.5(b) again. This and $\bar{t}^\flat\oplus(c) = \bar{t}$ yield $c \notin R(\bar{t}^\flat)$, which is the first fact to be derived.

Conversely, this paragraph shows

$$(\forall \bar{t}^\flat, c, \bar{t}) \;\; \bar{t}^\flat\oplus(c){=}\bar{t} \;\Leftarrow\; c{\notin}R(\bar{t}^\flat) \text{ and } R(\bar{t}^\flat){\cup}\{c\}{=}R(\bar{t}) \;.$$

Accordingly, take any $\bar{t}^\flat$, c, and $\bar{t}$ such that $c \notin R(\bar{t}^\flat)$ and $R(\bar{t}^\flat){\cup}\{c\} = R(\bar{t})$. Note

$$L(\bar{t}^\flat)+1 = |R(\bar{t}^\flat)|+1 = |R(\bar{t})| = L(\bar{t}).$$

by Lemma A.5(b), by the assumption of the previous sentence, and by Lemma A.5(b) again. So, trivially, $L(\bar{t}^\flat) = L(\bar{t}){-}1$. Since $R(\bar{t}^\flat) \subseteq R(\bar{t}^\flat){\cup}\{c\} = R(\bar{t})$, the "zipper" Lemma A.7 shows that $\bar{t}^\flat = {}_1\bar{t}_{L(\bar{t}^\flat)}$. So the last two sentences yield $\bar{t}^\flat = {}_1\bar{t}_{L(\bar{t})-1}$. Therefore, since $\{c\} = R(\bar{t}){\sim}R(\bar{t}^\flat)$ by assumption, it must be that $\bar{t}_{L(\bar{t})} = c$. The last two sentences yield $\bar{t}^\flat\oplus(c) = \bar{t}$.

(e,f). This paragraph argues that the assumptions of Lemma C.1 hold. (1a) implies (i). Part (a) implies (ii). Parts (b) and (c) imply (iii). And finally, part (d) implies (iv).

Consequently, Lemma C.1(a) implies part (e), and Lemma C.1(b) implies part (f). □

Lemma C.3. *Suppose $((C_i)_i, \bar{N})$ is an* `OR`* *choice-sequence form (6) with no-absent-mindedness (16). Let $N = \bar{N}$. Then the following hold.*

(a) $(\cup_i C_i, N)$ is a choice-set preform (8).

(b) $((C_i)_i, N)$ is a choice-set form (14).

Proof. (a). (8a). This follows from Lemma C.2(a).

(8b). $C \subseteq \cup R(\bar{N}) = \cup N$ because the set inclusion holds by (1b) and because the equality holds by the definition of N.

(8c). To prove the $\subseteq$ direction, take any $n \notin T$. By the definition of N, we may define $\bar{n}$ to satisfy $n = R(\bar{n})$. Since $n \notin T$, $\bar{n} \notin \bar{T}$ simply because a finite sequence cannot have an infinite range. Hence, by (1c), we may define $T^* = \{R({}_1\bar{n}_\ell)|\ell{\geq}1\}$. $T^* {\subseteq} T$ simply because finite sequences have finite ranges. Further, T^* is an infinite chain because

$(\forall \ell)\ |R({}_1\bar{n}_\ell)| = \ell$ by Lemma A.5(b). Finally,

$$n = R(\bar{n}) = R(\cup\{{}_1\bar{n}_\ell|\ell\}) = \cup\{R({}_1\bar{n}_\ell)|\ell\} = \cup T^*\ ,$$

where the first equality is the definition of $\bar{n}$, the second equality follows from $\bar{n}$ being an infinite sequence, the third equality holds by manipulation, and the last equality follows from the definition of T^*.

To prove the $\supseteq$ direction, let T^* be an infinite chain in T. Because T^* is an infinite chain, $\cup T^*$ must be an infinite set. Thus, it remains to be shown that $\cup T^* \in N$.

By Lemma B.7, there exists $(t^k)_{k\geq 1}$ such that $(\forall k)\ t^k \subset t^{k+1}$ and $\cup_k t^k = \cup T^*$. Then by Lemma C.2(c), there exists $(\bar{t}^k)_{k\geq 1}$ such that $(\forall k)\ t^k = R(\bar{t}^k)$. The last two sentences imply $(\forall k)\ R(\bar{t}^k) \subset R(\bar{t}^{k+1})$. This strict set inclusion has two implications. First, by two applications of Lemma A.5(b), we have

$$(40) \qquad (\forall k)\ L(\bar{t}^k) = |R(\bar{t}^k)| < |R(\bar{t}^{k+1})| = L(\bar{t}^{k+1})\ .$$

Second, by the zipper Lemma A.7, we have

$$(41) \qquad (\forall k)\ \bar{t}^k = {}_1\bar{t}^{k+1}_{L(t^k)}\ .$$

(40) and (41) together imply that $\cup_k \bar{t}^k$ is an infinite sequence. (It need not be the case that $(\forall k)\ L(\bar{t}^k)+1 = L(\bar{t}^{k+1})$.)

For notational ease, define $\bar{s} = \cup_k \bar{t}^k$. The remainder of this paragraph shows

$$(42) \qquad (\forall \ell\geq 1)\ {}_1\bar{s}_\ell \in \bar{T}\ .$$

Take any ℓ. By (40), (41), and the definition of $\bar{s}$, there exists some k such that ${}_1\bar{s}_\ell = {}_1\bar{t}^k_\ell$. Hence $L(\bar{t}^k)-\ell$ applications of (1d) yield that ${}_1\bar{s}_\ell = {}_1\bar{t}^k_\ell$ is in $\bar{T}$.

To conclude, this paragraph argues

$$\cup T^* = \cup_k t^k = \cup_k R(\bar{t}^k) = R(\cup_k \bar{t}^k) = R(\bar{s}) \in N\ .$$

The first equality holds by the definition of $(t^k)_k$. The second equality holds by the definition of $(\bar{t}^k)_k$. The third equality holds by manipulation. The fourth equality holds by the definition of $\bar{s}$. To see the set membership, note that (42) and assumption (1c) imply that $\bar{s} \in \bar{N}$. Thus $R(\bar{s}) \in N$ by the definition of N.

(8d). Take any $t \neq \varnothing$. Note that $(R|_{\bar{T}})^{-1}(t)$ is a well-defined sequence in $\bar{T}$ by Lemma C.2(b,c).

This paragraph argues that

$$
\begin{aligned}
(\forall c) \quad & c = [(R|_{\bar{T}})^{-1}(t)]_{L((R|_{\bar{T}})^{-1}(t))} \\
\Leftrightarrow \quad & (\exists \bar{t}^\flat)\ \bar{t}^\flat \oplus (c) = (R|_{\bar{T}})^{-1}(t) \\
\Leftrightarrow \quad & (\exists t^\flat)\ (R|_{\bar{T}})^{-1}(t^\flat) \oplus (c) = (R|_{\bar{T}})^{-1}(t) \\
\Leftrightarrow \quad & (\exists t^\flat)\ c \notin R \circ (R|_{\bar{T}})^{-1}(t^\flat) \text{ and } R \circ (R|_{\bar{T}})^{-1}(t^\flat) \cup \{c\} = R \circ (R|_{\bar{T}})^{-1}(t) \\
\Leftrightarrow \quad & (\exists t^\flat)\ c \notin t^\flat \text{ and } t^\flat \cup \{c\} = t \\
\Leftrightarrow \quad & (\exists t^\flat)\ c \in t \text{ and } t \setminus \{c\} = t^\flat \\
\Leftrightarrow \quad & c \in t \text{ and } t \setminus \{c\} \in T\ .
\end{aligned}
$$

The first equivalence holds by inspection. The second equivalence holds by Lemma C.2(b,c). The third equivalence holds by Lemma C.2(d) at $\bar{t} = (R|_{\bar{T}})^{-1}(t^\flat)$ and $\bar{t}^\sharp = (R|_{\bar{T}})^{-1}(t)$. The remaining equivalences hold by manipulation.

The previous paragraph has established that the last elements of the sequence $(R|_{\bar{T}})^{-1}(t)$ are identical to the last choices of the set t. Since the sequence $(R|_{\bar{T}})^{-1}(t)$ belongs to $\bar{T} \setminus \{\{\}\}$ by $t \neq \{\}$, the sequence has a unique last element. Thus by the last two sentences, the set t has a unique last choice.

(8e). This paragraph argues that the assumptions of Lemma C.1 hold. (1a) implies (i). Lemma C.2(a) implies (ii). Lemma C.2(b–c) imply (iii). And finally, Lemma C.2(d) implies (iv).

Thus by Lemma C.1(c), (8e) is equivalent to assumption (1e).

(b). (14a). This is identical to part (a).

(14b). This is identical to assumption (6b).

(14c). The assumptions of Lemma C.1 hold by the first paragraph in the above argument for (8e). Thus by Lemma C.1(d), (14c) is equivalent to assumption (6c). □

Proof C.4 (for Theorem 1)**.**

Part (a) follows from Lemma C.3(b). The remaining parts of the theorem follow from Lemma C.2(b–f). □

Appendix D. For Theorem 2

Lemma D.1. $\widehat{\mathtt{R}}$ *(17) is injective.*

Proof. Suppose (a) that $((C_i^1)_i, \bar{N}^1)$ and $((C_i^2)_i, \bar{N}^2)$ are two $\mathtt{OR}^*$ choice-sequence forms (6) with no-absent-mindedness (16), and (b) that $\widehat{\mathtt{R}}$ takes them both to the choice-set form (14) $((C_i)_i, N)$. Then by the definition of $\widehat{\mathtt{R}}$ we have

$$(C_i^1)_i = (C_i^2)_i = (C_i)_i \text{ and} \tag{43a}$$

$$R(\bar{N}^1) = R(\bar{N}^2) = N\ . \tag{43b}$$

Since (43a) assures $(C_i^1)_i = (C_i^2)_i$, it remains to show that $\bar{N}^1 = \bar{N}^2$.

Given (43a), I will henceforth replace $(C_i^1)_i$ and $(C_i^2)_i$ with $(C_i)_i$. Derive from $((C_i)_i, \bar{N}^1)$ its $\bar{T}^1$ (2) and $\bar{F}^1$ (3). Derive from $((C_i)_i, \bar{N}^2)$ its $\bar{T}^2$ (2) and $\bar{F}^2$ (3). Derive from $((C_i)_i, N)$ its T (9) and F (10). Set $C = \cup_i C_i$. Note that the conclusions of Theorem 1 are available for both $((C_i), \bar{N}^1)$ and $((C_i)_i, \bar{N}^2)$.

Suppose $\bar{N}^1 \neq \bar{N}^2$. By Theorem 1(b) and (43b), $R|_{\bar{N}^1}$ is a bijection from $\bar{N}^1$ onto N. Similarly, $R|_{\bar{N}^2}$ is a bijection from $\bar{N}^2$ onto N. Thus $\bar{N}^1 \neq \bar{N}^2$ implies the existence of distinct $\bar{n}^1$ and $\bar{n}^2$ such that $R(\bar{n}^1) = R(\bar{n}^2)$. Define

$$\begin{aligned} K^{\neq} = \{\ 1{\leq}\ell \mid\ & \bar{n}^1_\ell \neq \bar{n}^2_\ell, \\ & \ell{\leq}L(\bar{n}^1) \text{ if } \bar{n}^1 \text{ is finite}, \\ & \ell{\leq}L(\bar{n}^2) \text{ if } \bar{n}^2 \text{ is finite }\}\ . \end{aligned}$$

On the one hand, suppose $K^{\neq}$ is empty. Then the distinctness of $\bar{n}^1$ and $\bar{n}^2$ implies that one is a subsequence (Section 2.1) of the other. Without loss of generality, suppose $\bar{n}^1$ is a subsequence of $\bar{n}^2$. Hence $\bar{n}^1$ is finite,

$$\bar{n}^1 = {}_1\bar{n}^2_{L(\bar{n}^1)}\ , \text{ and} \tag{44a}$$

$$(\ L(\bar{n}^1){<}L(\bar{n}^2) \text{ or } \bar{n}^2 \text{ is infinite }) . \tag{44b}$$

By (44b), $\bar{n}^2_{L(\bar{n}^1)+1}$ is a well-defined element of $R(\bar{n}^2)$. Thus since $R(\bar{n}^1) = R(\bar{n}^2)$, there exists some $k \leq L(\bar{n}^1)$ such that $\bar{n}^1_k = \bar{n}^2_{L(\bar{n}^1)+1}$. Thus by (44a), $\bar{n}^2_k = \bar{n}^2_{L(\bar{n}^1)+1}$. Hence, since $k \leq L(\bar{n}^1)$, both the k-th component and the last component of ${}_1\bar{n}^2_{L(\bar{n}^1)+1}$ equal $\bar{n}^2_k$. So $|R({}_1\bar{n}^2_{L(\bar{n}^1)+1})| < L({}_1\bar{n}^2_{L(\bar{n}^1)+1})$. This inequality contradicts the no-absent-mindedness of $((C_i)_i, \bar{N}^2)$ by Lemma A.5(b).

On the other hand, suppose $K^{\neq}$ is nonempty. Let $k = \min K^{\neq}$. Then

(45a) $$\bar{n}^1_k \neq \bar{n}^2_k \text{ and}$$

(45b) $$_1\bar{n}^1_{k-1} = {}_1\bar{n}^2_{k-1} \ ,$$

where the last equality holds even if $k = 1$ because then both $_1\bar{n}^1_0$ and $_1\bar{n}^2_0$ equal $\{\}$. Note

(46a) $$\bar{n}^1_k \in \bar{F}^1(_1\bar{n}^1_{k-1}) \text{ and}$$

(46b) $$\bar{n}^2_k \in \bar{F}^2(_1\bar{n}^2_{k-1})$$

($\bar{F}^1$ and $\bar{F}^2$ need not be equal).

Let $t = R(_1\bar{n}^1_{k-1})$. By (46a), by Theorem 1(e) for $((C_i)_i, \bar{N}^1)$, and by the definition of t, we have

$$\bar{n}^1_k \in \bar{F}^1(_1\bar{n}^1_{k-1}) = F(R(_1\bar{n}^1_{k-1})) = F(t) \ .$$

Similarly, by (46b), by Theorem 1(e) for $((C_i)_i, \bar{N}^2)$, by (45b), and by the definition of t, we have

$$\bar{n}^2_k \in \bar{F}^2(_1\bar{n}^2_{k-1}) = F(R(_1\bar{n}^2_{k-1})) = F(R(_1\bar{n}^1_{k-1})) = F(t) \ .$$

These two together imply $\{\bar{n}^1_k, \bar{n}^2_k\} \subseteq F(t)$.

Now let $n = R(\bar{n}^1)$. Since $R(\bar{n}^1) = R(\bar{n}^2)$, we also have that $n = R(\bar{n}^2)$. Thus $\{\bar{n}^1_k, \bar{n}^2_k\} \subseteq n$. This and the last sentence of the previous paragraph imply that $\{\bar{n}^1_k, \bar{n}^2_k\} \subseteq F(t) \cap n$.

Yet $((C_i)_i, N)$ is a choice-set form by definition. Thus Lemma B.6 implies that $|F(t) \cap n| \leq 1$. This contradicts (45a) and the conclusion of the last paragraph. □

Lemma D.2. $\widehat{\mathtt{R}}$ *(17) is onto the class of choice-set forms (14).*

Proof. Suppose that $((C_i)_i, N)$ is a choice-set form. I will construct an $\bar{N}$ such that (a) $((C_i)_i, \bar{N})$ is an $\mathtt{OR}^*$ choice-sequence form (6) with no-absent-mindedness (16) and (b) $R(\bar{N}) = N$. By the definition of $\widehat{\mathtt{R}}$ this suffices to show that $\widehat{\mathtt{R}}$ takes $((C_i)_i, \bar{N})$ to $((C_i)_i, N)$.

Step 1 will construct $\bar{N}$. Steps 5 and 6 will derive (a) and (b). Steps 2–4 will provide intermediate results.

Step 1: Definition of $\bar{N}$.

First, I derive some objects from $((C_i)_i, N)$. As usual, define C by $\cup_i C_i$, T by (9), F by (10), and p by (11). Further, (8d) implies the existence of a function $c_*{:}T \smallsetminus \{\{\}\} \rightarrow C$ that takes each nonempty t to its unique last choice $c_*(t)$. Note that $p(t)\cup\{c_*(t)\} = t$ for any nonempty t.

Second, define $(T_k)_{k\geq 0}$ by $T_k = \{\ t \mid |t|{=}k\ \}$. By the definition of T,

$$\cup_k T_k = T\ . \tag{47}$$

Also, since $\{\} \in N$ by Lemma B.5,

$$T_0 = \{\{\}\}\ . \tag{48}$$

Third, I define a sequence $(Q_k)_{k\geq 0}$ of functions in which each function Q_k maps each set $t \in T_k$ to some finite sequence in C. I do this recursively. To begin, recall $T_0 = \{\{\}\}$ by (48) and define the one-element function Q_0 by

$$Q_0(\{\}) := \{\}\ . \tag{49}$$

Thus the empty set $t = \{\}$ is mapped to the empty sequence $\{\}$. Then, for any $k{\geq}1$, use Q_{k-1} to define Q_k at each $t \in T_k$ by

$$Q_k(t) := Q_{k-1}(p(t)){\oplus}(c_*(t))\ . \tag{50}$$

Since $t \in T_k$ implies $p(t) \in T_{k-1}$, each $Q_{k-1}(p(t))$ is well-defined.

Finally, define

$$\bar{N} := \cup_k Q_k(T_k) \cup \{\ \bar{s} \mid (\forall \ell{\geq}1)\,{}_1\bar{s}_\ell{\in}\cup_k Q_k(T_k)\ \}\ , \tag{51}$$

where $\bar{s}$ denotes an arbitrary sequence in C. From $((C_i)_i, \bar{N})$ derive $\bar{T}$ by (2) and, for later use, derive $\bar{F}$ by (3). Since every value of every Q_k is a finite sequence, the definition of $\bar{N}$ implies

$$\bar{T} = \cup_k Q_k(T_k) \text{ and} \tag{52a}$$

$$\bar{N}{\smallsetminus}\bar{T} = \{\ \bar{s} \mid (\forall \ell{\geq}1)\,{}_1\bar{s}_\ell{\in}\bar{T}\ \}\ . \tag{52b}$$

Step 2: An intermediate result (54) about $(Q_k(T_k))_k$.

This paragraph shows by induction that

$$(\forall k)(\forall t{\in}T_k)\ L(Q_k(t)) = k\ . \tag{53}$$

This holds at $k = 0$ because $T_0 = \{\{\}\}$ by (48) and because $L(Q_0(\{\}))$ $= L(\{\}) = 0$ by (49). Further, it holds at any $k{\geq}1$ if it holds at $k{-}1$ because

$$\begin{aligned}(\forall t{\in}T_k)\ L(\,Q_k(t)\,) &= L(\,Q_{k-1}(p(t)) \oplus (c_*(t))\,)\\ &= L(\,Q_{k-1}(p(t))\,) + 1\end{aligned}$$

$$= (k{-}1) + 1$$
$$= k \ ,$$

where the first equality holds by the definition (50) of Q_k, and the third by the inductive hypothesis.

I now argue from the previous paragraph that

(54) $$(\forall k)\ \{\ \bar{t} \mid L(\bar{t}){=}k\ \} = Q_k(T_k)\ .$$

The $\supseteq$ direction follows from (53) at k. To show the $\subseteq$ direction, take any $\bar{t}$ such that $L(\bar{t}) = k$. By (52a), there exists a k' such that $\bar{t} \in Q_{k'}(T_{k'})$. Thus (53) at k' implies $L(\bar{t}) = k'$. Therefore, $k = k'$ by the last and the third-to-last sentences, and hence $\bar{t} \in Q_k(T_k)$ by the second-to-last sentence.

Step 3: An intermediate result (57) showing $R|_{\bar{T}}$ is a bijection.

This paragraph shows by induction that

(55) $$(\forall k)(\forall t{\in}T_k)\ R(Q_k(t)) = t\ .$$

This holds at $k{=}0$ because $T_0 = \{\{\}\}$ by (48) and because $R(Q_0(\{\})) = R(\{\}) = \{\}$ by (49). Further, it holds at $k{\geq}1$ if it holds at $k{-}1$ because

$$\begin{aligned}(\forall t{\in}T_k)\ R(\,Q_k(t)\,) &= R\big(\,Q_{k-1}(p(t)) \oplus (c_*(t))\,\big)\\ &= R\big(\,Q_{k-1}(p(t))\,\big) \cup \{c_*(t)\}\\ &= p(t) \cup \{c_*(t)\}\\ &= t\ ,\end{aligned}$$

where the first equality holds by the definition (50) of Q_k, and the third holds by the inductive hypothesis.

By (55) we have that

(56) $$(\forall k)\ Q_k = (R|_{Q_k(T_k)})^{-1} \text{ is a bijection from } T_k \text{ onto } Q_k(T_k)\ .$$

By the definition of $(T_k)_k$, the members of $\{T_k|k\}$ are disjoint. Further, by (54), the members of $\{Q_k(T_k)|k\}$ are disjoint. Thus (56) implies that

$$\cup_k Q_k = (R|_{\cup_k Q_k(T_k)})^{-1} \text{ is a bijection from } \cup_k T_k \text{ onto } \cup_k Q_k(T_k)\ .$$

Hence, since $\cup_k T_k = T$ by (47) and since $\cup_k Q_k(T_k) = \bar{T}$ by (52a),

$$\cup_k Q_k = (R|_{\bar{T}})^{-1} \text{ is a bijection from } T \text{ onto } \bar{T}\ .$$

Therefore

(57) $$R|_{\bar{T}} = (\cup_k Q_k)^{-1} \text{ is a bijection from } \bar{T} \text{ onto } T\ .$$

Step 4: An intermediate result (61) about concatenation.

First, this paragraph argues

$$(58)\qquad \begin{aligned} &(\forall k{\geq}1)(\forall t^\flat{\in}T_{k-1})(\forall c)(\forall t{\in}T_k) \\ &\quad Q_{k-1}(t^\flat){\oplus}(c) = Q_k(t) \\ \Leftrightarrow&\quad Q_{k-1}(t^\flat){\oplus}(c) = Q_{k-1}(p(t)){\oplus}(c_*(t)) \\ \Leftrightarrow&\quad Q_{k-1}(t^\flat){=}Q_{k-1}(p(t)) \text{ and } c{=}c_*(t) \\ \Leftrightarrow&\quad t^\flat{=}p(t) \text{ and } c{=}c_*(t) \\ \Leftrightarrow&\quad c{\notin}t^\flat \text{ and } t^\flat{\cup}\{c\} = t\ . \end{aligned}$$

The first equivalence holds by the definition (50) of Q_k. The second equivalence holds by breaking the vector equality into two components. The third equivalence holds by applying R and (57) to the first equality. The fourth equivalence holds by the definitions of p and c_*.

Next, this paragraph argues

$$(59)\qquad \begin{aligned} &(\forall t^\flat, c, t)\quad (\cup_k Q_k)(t^\flat){\oplus}(c) = (\cup_k Q_k)(t) \\ &\qquad\Leftrightarrow\ c{\notin}t^\flat \text{ and } t^\flat{\cup}\{c\}{=}t\ . \end{aligned}$$

Take any $t^\flat$, c, and t. Assume the left-hand side. By the definition of $(Q_k)_k$ and the disjointness of their domains, $(\cup_k Q_k)(t^\flat) = Q_{|t^\flat|}(t^\flat)$ and $(\cup_k Q_k)(t) = Q_{|t|}(t)$. Thus the left-hand side implies

$$(60)\qquad Q_{|t^\flat|}(t^\flat){\oplus}(c) = Q_{|t|}(t)\ .$$

Two applications of (54) imply $L(Q_{|t^\flat|}(t^\flat)) = |t^\flat|$ and $L(Q_{|t|}(t)) = |t|$. Thus (60) implies $|t|{\geq}1$ and $|t^\flat|{=}|t|{-}1$. Hence (60) and (58) at $k{=}|t|$ imply the right-hand side. Conversely, assume the right-hand side. The right-hand side implies $|t|{\geq}1$ and $|t^\flat|{=}|t|{-}1$. Hence the right-hand side and (58) at $k{=}|t|$ imply (60). (60) immediately implies the left-hand side.

Finally, this paragraph argues

$$(61)\qquad \begin{aligned} &(\forall \bar{t}^\flat, c, \bar{t})\quad \bar{t}^\flat{\oplus}(c) = \bar{t} \\ &\qquad\Leftrightarrow\ c{\notin}R(\bar{t}^\flat) \text{ and } R(\bar{t}^\flat){\cup}\{c\} = R(\bar{t})\ . \end{aligned}$$

Accordingly, take any $\bar{t}^\flat$, c, and $\bar{t}$. Then define $t^\flat = R(\bar{t}^\flat)$ and $t = R(\bar{t})$. I argue

$$\begin{aligned} &\bar{t}^\flat{\oplus}(c) = \bar{t} \\ \Leftrightarrow&\quad (\cup_k Q_k)(t^\flat){\oplus}(c) = (\cup_k Q_k)(t) \end{aligned}$$

$$\Leftrightarrow \ c \notin t^\flat \text{ and } t^\flat \cup \{c\} = t$$
$$\Leftrightarrow \ c \notin R(\bar{t}^\flat) \text{ and } R(\bar{t}^\flat) \cup \{c\} = R(\bar{t}) \ .$$

The first equivalence holds by (57) and the definitions of $t^\flat$ and t. The second equivalence holds by (59). The third equivalence holds by the definitions of $t^\flat$ and t.

Step 5: Proving $((C_i)_i, \bar{N})$ is an OR* *choice-sequence form with no-absent-mindedness.*

By definition, $((C_i)_i, \bar{N})$ is an OR* choice-sequence form (6) iff $(C, \bar{N})$ is an OR* choice-sequence preform (1) and $((C_i)_i, \bar{N})$ satisfies (6b–c). Accordingly, I will show (1a–e), (6b–c), and no-absent-mindedness.

(1a). Return to the definition (51) of $\bar{N}$. Since every $Q_k(T_k)$ is a collection of sequences in C, $\bar{N}$ is also a collection of sequences in C. Further, $\bar{N}$ is nonempty because $Q_0(T_0)$ is nonempty by (48) and (49).

(1b). To show that $C \subseteq \cup R(\bar{N})$, take any c. By (8b), there is an n such that $c \in n$. Thus, either by $n \in T$, or by $n \notin T$ and (8c), there is a t such that $c \in t$. By (57), there exists $\bar{t}$ such that $t = R(\bar{t})$. Then

$$c \in t = R(\bar{t}) \subseteq \cup R(\bar{T}) \subseteq \cup R(\bar{N}) \ ,$$

where the set membership follows from the definition of t, the equality follows from the definition of $\bar{t}$, and the last inclusion follows from the definition of $\bar{T}$.

(1c). This has been established by (52b).

(1d). Take any $\bar{t} \in \bar{T} \smallsetminus \{\{\}\}$. By (54), $\bar{t} \in Q_{L(\bar{t})}(T_{L(\bar{t})})$. Thus there exists $t \in T_{L(\bar{t})}$ such that $\bar{t} = Q_{L(\bar{t})}(t)$. Since $\bar{t} \neq \{\}$ by assumption, $L(\bar{t}) \geq 1$. By the last two sentences and the definition (50) of $Q_{L(\bar{t})-1}$, we have

$$\bar{t} = Q_{L(\bar{t})}(t) = Q_{L(\bar{t})-1}(p(t)) \oplus (c_*(t)) \ . \tag{62}$$

I then argue

$${}_1\bar{t}_{L(\bar{t})-1} = Q_{L(\bar{t})-1}(p(t)) \in Q_{L(\bar{t})-1}(T_{L(\bar{t})-1}) \subseteq \bar{T} \ .$$

The first equality is the initial component of (62). The set membership follows from $p(t) \in T_{L(\bar{t})-1}$, which follows from $t \in T_{L(\bar{t})}$. And finally, the set inclusion follows from (52a).

(1e). This paragraph shows that the assumptions of Lemma C.1 are satisfied. The previously derived (1a) implies (i). The assumed (8a) implies (ii). (57) implies (iii). (61) implies (iv).

Thus by Lemma C.1(c), (1e) is equivalent to the assumed (8e).

(6b). This is identical to the assumed (14b).

(6c). The third-to-last paragraph showed that the assumptions of Lemma C.1 are satisfied. Thus by Lemma C.1(d), (6c) is equivalent to the assumed (14c).

No-absent-mindedness. I have just shown that $((C_i)_i, \bar{N})$ is an $\texttt{OR}^*$ choice-sequence form. Further, $R|_{\bar{T}}$ is injective by (57). Thus Proposition 4.1(a⇐b) implies no-absent-mindedness.

Step 6: Proving $R(\bar{N}) = N$.

$R(\bar{N}) \subseteq N$. By (57) and the definition of T, $R(\bar{T}) = T \subseteq N$. Thus it suffices to show that $R(\bar{N}\smallsetminus\bar{T}) \subseteq N$. Accordingly, take any $\bar{n} \in \bar{N}\smallsetminus\bar{T}$.

Define $T^* = \{R({}_1\bar{n}_\ell)|\ell{\geq}1\}$. The remainder of this paragraph argues that T^* is an infinite chain in T. (1) Every $R({}_1\bar{n}_\ell)$ is in T, because every ${}_1\bar{n}_\ell$ is in $\bar{T}$ by the derived (1c), and because $R(\bar{T}) = T$ by (57). Hence $T^* \subseteq T$. (2) Note that $(\forall \ell)\ R({}_1\bar{n}_\ell) \subseteq R({}_1\bar{n}_{\ell+1})$ simply because ${}_1\bar{n}_\ell \subseteq {}_1\bar{n}_{\ell+1}$. Hence T^* is a chain. (3) By Lemma A.5(b) and the derived no-absent-mindedness, we have $(\forall \ell)\ |R({}_1\bar{n}_\ell)| = \ell$. This implies that T^* is infinite.

I argue

$$R(\bar{n}) = R(\cup\{{}_1\bar{n}_\ell|\ell\}) = \cup\{R({}_1\bar{n}_\ell)|\ell\} = \cup T^* \in N\smallsetminus T \subseteq N.$$

The first equality holds because $\bar{n}$ is an infinite sequence. The second quality holds by manipulation. The third equality holds by the definition of T^*. The set membership follows from assumption (8c) and the last paragraph's result that T^* is an infinite chain in T.

$R(\bar{N}) \supseteq N$. By (57) and the definition of $\bar{T}$, $T = R(\bar{T}) \subseteq R(\bar{N})$. Thus it suffices to show that $N\smallsetminus T \subseteq R(\bar{N})$. Accordingly, take any $n \in N\smallsetminus T$.

This paragraph derives an infinite sequence $\bar{s}$ from the set n. By the assumed (8c), there exists an infinite chain $T^* \subseteq T$ such that $\cup T^* = n$. Thus by Lemma B.7, there exists $(t^k)_{k\geq 1}$ such that $(\forall k)\ t^k \subset t^{k+1}$ and $\cup_k t^k = n$. Then by (57), there exists $(\bar{t}^k)_{k\geq 1}$ such that $(\forall k)\ R(\bar{t}^k) = t^k$. The last two sentences imply $(\forall k)\ R(\bar{t}^k) \subset R(\bar{t}^{k+1})$. This strict inclusion has two implications. First, by two applications of Lemma A.5(b) and the derived no-absent-mindedness, we have

$$(\forall k)\ L(\bar{t}^k) = |R(\bar{t}^k)| < |R(\bar{t}^{k+1})| = L(\bar{t}^{k+1})\ . \tag{63}$$

Second, by the zipper Lemma A.7, we have

$$(\forall k)\ \bar{t}^k = \bar{t}^{k+1}_{L(t^k)}\ . \tag{64}$$

(63) and (64) together imply that $\cup_k \bar{t}^k$ is an infinite sequence. Accordingly, define $\bar{s} = \cup_k \bar{t}^k$. (It need not be the case that $(\forall k)\ L(\bar{t}^k){+}1 = L(\bar{t}^{k+1})$.)

This paragraph argues

$$\bar{s} \in \bar{N} \smallsetminus \bar{T}\ . \tag{65}$$

By the derived (1c), it suffices to show that

$$(\forall \ell{\geq}1)\ {}_1\bar{s}_\ell \in \bar{T}\ .$$

Accordingly, take any ℓ. By (63), (64), and the definition of $\bar{s}$, there exists some k such that ${}_1\bar{s}_\ell = {}_1\bar{t}^k_\ell$. This ${}_1\bar{t}^k_\ell$ is in $\bar{T}$ by $L(\bar{t}^k){-}\ell$ applications of the derived (1d).

This paragraph argues

$$R(\bar{s}) = R(\cup_k \bar{t}^k) = \cup_k R(\bar{t}^k) = \cup_k t^k = n\ . \tag{66}$$

The first equality follows from the definition of $\bar{s}$. The second equality follows by manipulation. The third equality follows from the definition of $(\bar{t}^k)_k$. The final equality follows from the definition of $(t^k)_k$.

Finally,

$$n = R(\bar{s}) \in R(\bar{N} \smallsetminus \bar{T}) \subseteq R(\bar{N})\ ,$$

where the equality is (66) and the set membership follows from (65). □

Proof D.3 (for Theorem 2)**.**

By Theorem 1(a), $\widehat{\mathtt{R}}$ is a function, from the class of $\mathtt{OR}^*$ choice-sequence forms with no-absent-mindedness, into the class of choice-set forms. By Lemma D.1, it is injective. By Lemma D.2, it is surjective. □

Appendix E. For Corollaries

Proof E.1 (for Corollary 5.1)**.**

(a). $\Leftarrow$. This follows immediately from the definition of p.

$\Rightarrow$. By Theorem 2 and the definition (17) of $\widehat{\mathtt{R}}$, there exists an $\mathtt{OR}^*$ choice-sequence form (6) $((C_i)_i, \bar{N})$, with no-absent-mindedness (16), such that $R(\bar{N}) = N$. Derive $\bar{T}$ by (2).

Now suppose $t^\flat \subset t$. By Theorem 1(c), there exists $\bar{t}^\flat$ and $\bar{t}$ such that $R(\bar{t}^\flat) = t^\flat$ and $R(\bar{t}) = t$. Note

$$R(\bar{t}^\flat) = t^\flat \subset t = R(\bar{t}) \ . \tag{67}$$

I argue

$$\bar{t}^\flat = {}_1\bar{t}_{L(\bar{t}^\flat)} = {}_1\bar{t}_{|R(\bar{t}^\flat)|} = {}_1\bar{t}_{|t^\flat|} \ . \tag{68}$$

The first equality holds by (67) and the zipper Lemma A.7. The second equality holds by Lemma A.5(b). The third equality holds by the definition of $\bar{t}^\flat$.

Since $t^\flat \subset t$, $|t^\flat| < |t|$. Thus (68) and $|t|{-}|t^\flat|$ applications of Theorem 1(f) imply that $t^\flat = p^{|t|-|t^\flat|}(t)$.

(b). Take any t. Note that $(\forall t'{\neq}\{\})\ p(t') \subset t'$ by the definition of p. Thus $\{p^j(t)\,|\,|t|{\geq}j{\geq}1\}$ is both a chain and a subcollection of $\{t^\flat|t^\flat{\subset}t\}$. It remains to be shown that, for all $t^\flat$, $t^\flat \subset t$ implies the existence of a j satisfying both $t^\flat{=}p^j(t)$ and $|t|\geq j \geq 1$. Accordingly, take some $t^\flat$ such that $t^\flat \subset t$. By part (a), $t^\flat = p^j(t)$ for j set equal to $|t|{-}|t^\flat| \geq 1$. Further, $|t^\flat| \geq 0$ implies $j \leq |t|$. □

Proof E.2 (for Corollary 5.2)**.**

Derive p by (11).

(a). This paragraph defines $t^\sharp$ such that $t \subset t^\sharp \subseteq n^\sharp$. On the one hand, if $n^\sharp \in T$, let $t^\sharp = n^\sharp$. On the other hand, suppose $n^\sharp \notin T$. Then by (8c) there exists an infinite chain T^* such that $\cup T^* = n^\sharp$. Thus since t is a finite subset of $n^\sharp$, there is some $t^\sharp \in T^*$ such that $t \subset t^\sharp$.

By Corollary 5.1(b), there exists a $j{\geq}1$ such that $t = p^j(t^\sharp)$. Thus there exists a c such that $c \in F(t)$ and $t{\cup}\{c\} = p^{j-1}(t^\sharp)$ (where $p^0(t^\sharp)$ is defined to be $t^\sharp$). Since the second of these statements implies $c \in t^\sharp$, the two statements together imply $c \in F(t){\cap}t^\sharp$. Finally, since $t^\sharp \subseteq n^\sharp$ by the definition of $t^\sharp$, we have $c \in F(t){\cap}n^\sharp$.

(b). Suppose $n \subset n^\sharp$. If $n^\sharp \in T$, then $n \in T$ follows immediately from the definition of T. Accordingly, suppose $n^\sharp \notin T$. Then by (8c), there exists an infinite chain T^* such that $n^\sharp = \cup T^*$.

Since $n \subset n^\sharp$, we may take $c^\sharp \in n^\sharp \smallsetminus n$. Since $n^\sharp = \cup T^*$, we may then take some $t^\sharp \in T^*$ such that $c^\sharp \in t^\sharp$. For future purposes, define $p^0(t^\sharp) = t^\sharp$ and note that $p^0(t^\sharp) \not\subseteq n$ by the ends of the last two sentences.

The set $\{\ p^j(t^\sharp) \subseteq n \mid |t^\sharp| \leq j \leq 1\ \}$ is a nonempty finite chain because $p^{|t^\sharp|}(t^\sharp) = \varnothing$. Thus we may let $p^j(t^\sharp)$ be its maximum. If j=1, the last paragraph showed $p^0(t^\sharp) \not\subseteq n$. If $j \geq 2$, the definition of j implies that $p^{j-1}(t^\sharp) \not\subseteq n$. Thus in either contingency we have

$$p^j(t^\sharp) \subseteq n \text{ and} \tag{69a}$$

$$p^{j-1}(t^\sharp) \not\subseteq n\ . \tag{69b}$$

The definition of p allows us to define c^j as the sole element of $p^{j-1}(t^\sharp) \smallsetminus p^j(t^\sharp)$ (if j=1, this c^j happens to be the $c^\sharp$ from the second paragraph). Thus (69) implies

$$c^j \notin n\ . \tag{70}$$

Further the definition of j and definition of F imply

$$c^j \in F(p^j(t^\sharp))\ . \tag{71}$$

And finally, the definition of c^j, the definition of p, the definition of $t^\sharp$, and the definition of T^* imply

$$c^j \in p^{j-1}(t^\sharp) \subseteq t^\sharp \subseteq \cup T^* = n^\sharp\ . \tag{72}$$

This paragraph shows that $p^j(t^\sharp) = n$. By (69a), I need only rule out $p^j(t^\sharp) \subset n$. Accordingly, suppose $p^j(t^\sharp) \subset n$. By part (a), this implies the existence of a $c \in F(p^j(t^\sharp))$ such that $c \in n$. Since $c^j \notin n$ by (70), it must be that $c \neq c^j$. Thus c and c^j are distinct elements of $F(p^j(t^\sharp))$ by (71) and the definition of c. Hence they cannot both belong to the same node by Lemma B.6. Thus, since $c^j \in n^\sharp$ by (72), we have $c \notin n^\sharp$. But this contradicts $c \in n \subset n^\sharp$, which must hold by the definition of c and the initial assumption that $n \subset n^\sharp$.

Since $n = p^j(t^\sharp)$ by the previous paragraph, n is an element of T.

(c). Suppose $(\exists n^\sharp)\ n \subset n^\sharp$. By part (b), $n \in T$. Thus by part (a) at $t = n$, $F(n) \cap n^\sharp \neq \varnothing$. This implies that $F(n) \neq \varnothing$, or in other words, that $n \in F^{-1}(C)$.

Conversely, suppose $t \in F^{-1}(C)$ (the notation t is not restrictive since $F^{-1}(C) \subseteq T$). Then there exists $c \in F(t)$ and we may set $n^\sharp = t\cup\{c\}$. □

Proof E.3 (for Corollary 5.3)**.**

By Theorem 2 and the definition (17) of $\widehat{\mathtt{R}}$, there exists an $\mathtt{OR}^*$ choice-sequence form (6) $((C_i)_i, \bar{N})$, with no-absent-mindedness (16), such that $R(\bar{N}) = N$. Derive $\bar{T}$ by (2).

(a). Since $R(\bar{N}) = N$, there exists $\bar{n}$ such that $R(\bar{n}) = n$. By (1c), $({}_1\bar{n}_k)_{k\geq 0}$ is a sequence in $\bar{T}$. Thus by Theorem 1(c), we may define $(t^k)_{k\geq 0}$ by

$$(\forall k)\ t^k = R({}_1\bar{n}_k)\ . \tag{73}$$

Note that

$$t^0 = R({}_1\bar{n}_0) = \{\}\ ,$$

where the first equality follows from the definition of t^0, and the second holds because ${}_1\bar{n}_0$ was defined to be $\{\}$. Further,

$$(\forall k{\geq}1)\ p(t^k) = p(R({}_1\bar{n}_k)) = R({}_1\bar{n}_{k-1}) = t^{k-1}\ ,$$

where the first and third equalities follow from the definition of $(t^k)_k$, and the second equality follows from Theorem 1(f). Finally, note that

$$n = R(\bar{n}) = R(\cup\{{}_1\bar{n}_k|k\}) = \cup\{R({}_1\bar{n}_k)|k\} = \cup\{t^k|k\}\ ,$$

where the first equality is the definition of $\bar{n}$, the next two equalities hold by manipulation, and the last equality holds by the definition of $(t^k)_{k\geq 1}$. The equations of the last three sentences establish that $(t^k)_{k\geq 1}$ satisfies the three equations of part (a). Thus existence has been established.

To show uniqueness, suppose that $(s^k)_{k\geq 0}$ is *any* sequence of sets in T that satisfies the three equations of part (a). Then

$$n{=}\cup\{s^k|k\},\ s^0{=}\{\},\ \text{and}\ (\forall k{\geq}1)\, p(s^k){=}s^{k-1}\ . \tag{74}$$

The last two of these equalities and the definition of p together imply that $|s^0| = 0$ and $(\forall k{\geq}1)\ |s^k| = |s^{k-1}|{+}1$. Thus $(\forall k)\ |s^k| = k$.

By Theorem 1(c), we may define $(\bar{s}^k)_{k\geq 0}$ in $\bar{T}$ by $(\forall k)\ s^k = R(\bar{s}^k)$. The last two sentences imply, among other things, that $(\forall k)\ |R(\bar{s}^k)| =$

k. By the no-absent-mindedness of $((C_i)_i, \bar{N})$ and Lemma A.5(b), the last sentence implies

$$(75) \qquad (\forall k)\ L(\bar{s}^k) = k\ .$$

For use in the next paragraph, note

$$(76) \qquad (\forall k)\ p(R(\bar{s}^{k+1})) = R({}_1\bar{s}^{k+1}_{L(\bar{s}^{k+1})-1}) = R({}_1\bar{s}^{k+1}_k)\ ,$$

where the first equality follows from Theorem 1(f) at $\bar{t} = \bar{s}^{k+1}$, and the second equality follows from (75) at k+1.

This paragraph argues

$$(77) \qquad (\forall k)\ \bar{s}^k = R|_{\bar{N}}^{-1}(s^k) = R|_{\bar{N}}^{-1}(p(s^{k+1})) \\ = R|_{\bar{N}}^{-1}(p(R(\bar{s}^{k+1}))) = R|_{\bar{N}}^{-1}(R({}_1\bar{s}^{k+1}_k)) = {}_1\bar{s}^{k+1}_k\ .$$

The first equality follows from the definition of $\bar{s}^k$ and Theorem 1(b). The second equality follows from the last statement in (74). The third equality follows from the definition of $\bar{s}^{k+1}$. The fourth equality follows from (76). The last equality holds if ${}_1\bar{s}^{k+1}_k \in \bar{N}$, and this set membership holds because (1) $\bar{s}^{k+1} \in \bar{T}$ by its definition, (2) $L(\bar{s}^{k+1})=k+1$ by (75), and thus (3) ${}_1\bar{s}^{k+1}_k \in \bar{T}$ by (1d).

(77) implies that $\cup\{\bar{s}^k|k\}$ is a sequence. Denote it $\bar{m}$ (so as to distinguish it from the $\bar{n}$ defined in the first paragraph of this proof of part (a)). Since $(\forall k)\ L(\bar{s}^k) = k$ by (75), we have

$$(78) \qquad (\forall k)\ {}_1\bar{m}_k = \bar{s}^k\ .$$

Thus the definition of $\bar{s}^k$ implies that $(\forall k)\ {}_1\bar{m}_k \in \bar{T}$. Hence (1c) implies that $\bar{m} \in \bar{N}$.

Notice that

$$R(\bar{m}) = R(\cup\{\bar{s}^k|k\}) = \cup\{R(\bar{s}^k)|k\} = \cup\{s^k|k\} = n\ ,$$

where the first equality holds by the definition of $\bar{m}$, the second equality holds by manipulation, the third equality holds by the definition of $(\bar{s}^k)_k$, and the final equality holds by the first statement in (74). Thus, since $R(\bar{n})$ also equals n by the definition of $\bar{n}$, Theorem 1(b) implies that $\bar{m} = \bar{n}$. So

$$(\forall k)\ s^k = R(\bar{s}^k) = R({}_1\bar{m}_k) = R({}_1\bar{n}_k) = t^k\ ,$$

where the first equality holds by the definition of $\bar{s}^k$, the second equality holds by (78), the third equality holds by the last sentence, and the last equality is the definition (73) of t^k. Therefore, since $(s^k)_{k\geq 0}$

was assumed to be any sequence in T solving the three conditions of part (a), $(t^k)_{k\geq 0}$ is the only such sequence.

(b). Recall the very first paragraph of this proof, which began before the proof of part (a). The statement of part (b) uses part (a) to define $(t^k)_{k\geq 0}$ as the unique sequence of sets that satisfies

$$n{=}\cup\{t^k|k\}\ ,\ t^0{=}\{\}\ ,\ \text{and}\ (\forall k{\geq}1)\ p(t^k){=}t^{k-1}\ . \tag{79}$$

By the last two of these three equations, $\{t^k|k\}$ is an infinite chain. Thus it remains to be shown that

$$\{n^\flat|n^\flat{\subset}n\} = \{t^k|k\}\ .$$

To see the $\supseteq$ direction, take any t^k. Then $t^k \subseteq n$ by the first equality in the definition (79) of $(t^k)_k$. Further, $t^k \subset n$ since t^k is finite and n is infinite by assumption.

To see the $\subseteq$ direction, take any $n^\flat \subset n$. Note that $n^\flat \in T$ by Corollary 5.2(b). This and the next paragraph will incorporate $n^\flat$ into an infinite sequence $(s^k)_{k\geq 0}$ in T. First, define $(s^k)_{k=0}^{|n^\flat|}$ by $s^k = p^{|n^\flat|-k}(n^\flat)$, where $p^0(n^\flat)$ is set equal to $n^\flat$. It follows immediately that

$$s^0 = \{\}\ ,\ (\forall 1{\leq}k{\leq}|n^\flat|)\ p(s^k) = s^{k-1}\ , \tag{80a}$$

$$\text{and}\ (\forall 0{\leq}k{\leq}|n^\flat|)\ s^k \subset n\ . \tag{80b}$$

Second, define $(s^k)_{k\geq|n^\flat|+1}$ recursively as follows. [1] Take $k{\geq}|n^\flat|{+}1$ and assume

$$s^{k-1} \subset n\ . \tag{81}$$

[2] By Lemma 5.2(a), take $c^k \in F(s^{k-1}){\cap}n$, where F is defined by (10). [3] Since $c^k \in F(s^{k-1})$, define $s^k = s^{k-1}{\cup}\{c^k\}$. Note that

$$p(s^k) = s^{k-1}\ . \tag{82}$$

[4] Also note that $s^k = s^{k-1}{\cup}\{c^k\} \subseteq n$ since $s^{k-1} \subset n$ by (81) and since $c^k \in n$ by the definition of c^k. This implies

$$s^k \subset n \tag{83}$$

because s^k is finite by construction and n is infinite by assumption. This four-step recursion can be initiated at $k{=}|n^\flat|{+}1$ because (81) at $k{=}|n^\flat|{+}1$ is (80b) at $k{=}|n^\flat|$. The recursion can be sustained because

(81) at k is (83) at $k{-}1$. Finally, by (82) and (83) within this recursion, we have

(84a) $$(\forall k{\geq}|n^\flat|{+}1)\ p(s^k) = s^{k-1}$$

(84b) $$\text{and } (\forall k{\geq}|n^\flat|{+}1)\ s^k \subset n\ .$$

Equations (80a) and (84a) imply that $\{s^k|k\}$ is an infinite chain in T. Thus by (8c), $\cup_k s^k$ is an infinite member of N. Meanwhile, (80b) and (84b) imply that $\cup_k s^k \subseteq n$. The last two sentences and Corollary 5.2(b) imply that $\cup_k s^k = n$.

The conclusion of the last paragraph, (80a), and (84a) imply that

$$n{=}\cup_k s^k\ ,\ s^0{=}\{\}\ ,\ \text{and } (\forall k{\geq}1)\, p(s^k){=}s^{k-1}\ .$$

Thus, by the uniqueness in the definition of $(t^k)_k$ at the start of this proof of part (b), $(s^k)_k = (t^k)_k$. So

$$n^\flat = s^{|n^\flat|} = t^{|n^\flat|}\ ,$$

where the first equality is the definition of $s^{|n^\flat|}$ and the second equality follows from the previous sentence. Consequently, $n^\flat \in \{t^k|k\}$. □

Lemma E.4. [15] *Suppose that* $((C_i)_i, \bar{N})$ *is an* `OR`* *choice-sequence form (6) with no-absent-mindedness (16). Derive its* $\bar{T}$ *(2). Then if* $R(\bar{n}^\flat) \subset R(\bar{n})$*, the following hold.*

(a) $\bar{n}^\flat \in \bar{T}$.

(b) If $\bar{n} \in \bar{T}$*, then* $L(\bar{n}^\flat) < L(\bar{n})$ *and* $\bar{n}^\flat = {}_1\bar{n}_{L(\bar{n}^\flat)}$.

(c) If $\bar{n} \notin \bar{T}$*, then* $\bar{n}^\flat = {}_1\bar{n}_{L(\bar{n}^\flat)}$.

Proof. (a). Let $N = R(\bar{N})$. By Theorem 1(a), $((C_i)_i, N)$ is a choice-set form. Note that $R(\bar{n}^\flat)$ and $R(\bar{n})$ are nodes in N. Thus, the assumption $R(\bar{n}^\flat) \subset R(\bar{n})$ and Corollary 5.2(b) imply that $R(\bar{n}^\flat) \in T$, where T is defined by (9). Hence $\bar{n}^\flat \in \bar{T}$ by Theorem 1(b,c).

[15] Lemmata E.4 and E.5 are needed by Proof E.6. Incidentally, they also appear to be new results for choice-sequence forms. They derive conclusions from no-absent-mindedness. For example, conclusion (a) of Lemma E.4 rules out the existence of an infinite sequence whose every choice is also made along a distinct second infinite sequence. [This implies that no sequence (in $\bar{N}$) is an infinite "subsequence" of another sequence (in $\bar{N}$), where in this sentence only, the term "subsequence" is used as it would be in topology rather than as it was defined in Section 2.1.] Conclusions (b) and (c) go a step further. They state that if every choice of a sequence is also made along a distinct second sequence, then the first sequence is a (finite) "subsequence" of the second in the precise sense of Section 2.1.

(b). Suppose $\bar{n} \in \bar{T}$. By part (a), $\bar{n}^\flat \in \bar{T}$. Since both sequences are finite, two applications of Lemma A.5(b) and the assumption that $R(\bar{n}^\flat) \subset R(\bar{n})$ imply that

$$L(\bar{n}^\flat) = |R(\bar{n}^\flat)| < |R(\bar{n})| = L(\bar{n}) \ .$$

Further, the zipper Lemma A.7 implies $\bar{n}^\flat = {}_1\bar{n}_{L(\bar{n}^\flat)}$.

(c). Suppose $\bar{n} \notin \bar{T}$. Since $R(\bar{n}^\flat) \subset R(\bar{n})$ by assumption and since $R(\bar{n}^\flat)$ is finite by part (a), there is some k such that $R(\bar{n}^\flat) \subseteq R({}_1\bar{n}_k)$. By part (a), $\bar{n}^\flat \in \bar{T}$. By (1c), ${}_1\bar{n}_k \in \bar{T}$. The last three sentences and the zipper Lemma A.7 imply

$$L(\bar{n}^\flat) \leq L({}_1\bar{n}_k) \text{ and} \tag{85a}$$

$$\bar{n}^\flat = {}_1({}_1\bar{n}_k)_{L(\bar{n}^\flat)} \ . \tag{85b}$$

Since $L({}_1\bar{n}_k) = k$, (85a) implies $L(\bar{n}^\flat) \leq k$. Thus (85b) simplifies to $\bar{n}^\flat = {}_1\bar{n}_{L(\bar{n}^\flat)}$. □

Lemma E.5. *Suppose that $((C_i)_i, \bar{N})$ is an* OR* *choice-sequence form (6) with no-absent-mindedness (16). Take any distinct $\bar{n}^1$ and $\bar{n}^2$, and let*

$$K := \{\ k{\geq}0 \mid k{\leq}L(\bar{n}^1) \text{ if } \bar{n}^1 \text{ is finite and } k{\leq}L(\bar{n}^2) \text{ if } \bar{n}^2 \text{ is finite}\ \} \ .$$

Then for any $k^*{\in}K$,

$$k^* = max\{\ k{\in}K \mid {}_1\bar{n}^1_k = {}_1\bar{n}^2_k\ \}$$
$$\Leftrightarrow\ R({}_1\bar{n}^1_{k^*}) = max\{\ R(\bar{m}) \mid R(\bar{m}) \subseteq R(\bar{n}^1) \cap R(\bar{n}^2)\ \} \ ,$$

where $\bar{m}$ is an arbitrary element of $\bar{N}$.

Proof.

$\Rightarrow$ *Direction.* Suppose

$$k^* = \max\{\ k{\in}K \mid {}_1\bar{n}^1_k{=}{}_1\bar{n}^2_k\ \} \ . \tag{86}$$

Since $k^*{\in}K$, both $R({}_1\bar{n}^1_{k^*})$ and $R({}_1\bar{n}^2_{k^*})$ are well-defined. Clearly $R({}_1\bar{n}^1_{k^*}) \subseteq R(\bar{n}^1)$. Also, by (86), $R({}_1\bar{n}^1_{k^*}) = R({}_1\bar{n}^2_{k^*}) \subseteq R(\bar{n}^2)$. Thus by the last two sentences, $R({}_1\bar{n}^1_{k^*}) \subseteq R(\bar{n}^1) \cap R(\bar{n}^2)$.

Hence it remains to be shown that

$$(\forall \bar{m})\ \ R(\bar{m}){\subseteq}R(\bar{n}^1){\cap}R(\bar{n}^2)\ \Rightarrow\ R(\bar{m}){\subseteq}R({}_1\bar{n}^1_{k^*}) \ .$$

Accordingly, take any $\bar{m}$ such that $R(\bar{m}){\subseteq}R(\bar{n}^1){\cap}R(\bar{n}^2)$. If $R(\bar{m})$ was equal both $R(\bar{n}^1)$ and $R(\bar{n}^2)$, Proposition 4.1 would imply that $\bar{m}$ was equal to both $\bar{n}^1$ and $\bar{n}^2$, which would imply $\bar{n}^1{=}\bar{n}^2$ in contradiction to the assumption that $\bar{n}^1$ and $\bar{n}^2$ are distinct. Accordingly, assume without loss of generality that $R(\bar{m}){\subset}R(\bar{n}^1)$ and $R(\bar{m}){\subseteq}R(\bar{n}^2)$.

By $R(\bar{m}){\subset}R(\bar{n}^1)$ and Lemma E.4(a,b,c), $\bar{m}$ is finite and

$$\bar{m} = {}_1\bar{n}^1_{L(\bar{m})} \ . \tag{87}$$

The remainder of the paragraph shows

$$\bar{m} = {}_1\bar{n}^2_{L(\bar{m})} \ . \tag{88}$$

Either $R(\bar{m}){=}R(\bar{n}^2)$ or $R(\bar{m}){\subset}R(\bar{n}^2)$. In the first case, Proposition 4.1 implies $\bar{m}{=}\bar{n}^2$. Thus (88) holds because $\bar{m}$ is finite by the first sentence of this paragraph. In the second case, (88) holds by Lemma E.4(b,c).

By (87) and (88), ${}_1\bar{n}^1_{L(\bar{m})} = {}_1\bar{n}^2_{L(\bar{m})}$. Thus by (86), $L(\bar{m}) \leq k^*$. So by (87) and the previous sentence,

$$R(\bar{m}) = R({}_1\bar{n}^1_{L(\bar{m})}) \subseteq R({}_1\bar{n}^1_{k^*}) \ .$$

$\Leftarrow$ *Direction.* Suppose that $k^*{\in}K$ satisfies

$$R({}_1\bar{n}^1_{k^*}) = \max\{\ R(\bar{m}) \mid R(\bar{m}){\subseteq}R(\bar{n}^1){\cap}R(\bar{n}^2)\ \} \ . \tag{89}$$

Then $R({}_1\bar{n}^1_{k^*}) \subseteq R(\bar{n}^2)$. On the one hand, suppose $R({}_1\bar{n}^1_{k^*}) = R(\bar{n}^2)$. Then Proposition 4.1 implies ${}_1\bar{n}^1_{k^*}{=}\bar{n}^2$, which immediately implies

$${}_1\bar{n}^1_{k^*} = {}_1\bar{n}^2_{k^*} \ . \tag{90}$$

On the other hand, suppose $R({}_1\bar{n}^1_{k^*}) \subset R(\bar{n}^2)$. Then Lemma E.4(b,c) implies ${}_1\bar{n}^1_{k^*} = {}_1\bar{n}^2_{L({}_1\bar{n}^1_{k^*})}$, which implies (90) because $L({}_1\bar{n}^1_{k^*}) = k^*$.

Since (90) implies $k^* \in \{\ k{\in}K \mid {}_1\bar{n}^1_k{=}{}_1\bar{n}^2_k\ \}$, it remains to show

$$(\forall k{\in}K)\ \ {}_1\bar{n}^1_k{=}{}_1\bar{n}^2_k \ \Rightarrow\ k{\leq}k^* \ .$$

Accordingly, take any $k{\in}K$ such that ${}_1\bar{n}^1_k{=}{}_1\bar{n}^2_k$. Then

$$R({}_1\bar{n}^1_k) \subseteq R(\bar{n}^1){\cap}R(\bar{n}^2) \ .$$

By (89), this implies $R({}_1\bar{n}^1_k) \subseteq R({}_1\bar{n}^1_{k^*})$, which implies $|R({}_1\bar{n}^1_k)| \leq |R({}_1\bar{n}^1_{k^*})|$. By Lemma A.5(b), this is equivalent to $L({}_1\bar{n}^1_k) \leq L({}_1\bar{n}^1_{k^*})$, which is equivalent to $k \leq k^*$. □

Proof E.6 (for Corollary 5.4)**.**

By Theorem 2 and the definition (17) of $\widehat{\mathtt{R}}$, there exists an $\mathtt{OR}^*$ choice-sequence form (6) $((C_i)_i, \bar{N})$, with no-absent-mindedness (16), such that $R(\bar{N}) = N$. Derive $\bar{T}$ by (2).

Take any n^1 and n^2 in N. If $n^1 = n^2$, then both parts of the corollary hold vacuously. Accordingly, assume $n^1 \neq n^2$. By the definition of $\bar{N}$, we may let $\bar{n}^1$ and $\bar{n}^2$ be such that $n^1{=}R(\bar{n}^1)$ and $n^2{=}R(\bar{n}^2)$. Because $n^1 \neq n^2$, Theorem 1(b) implies $\bar{n}^1 \neq \bar{n}^2$.

Define the set K as in Lemma E.5. Since $\bar{n}^1 \neq \bar{n}^2$,

$$k^* := \max\{\ k{\in}K \mid {}_1\bar{n}^1_k{=}{}_1\bar{n}^2_k\ \}\ .$$

is well-defined. By Lemma E.5,

$$R({}_1\bar{n}^1_{k^*}) = \max\{\ R(\bar{m}) \mid R(\bar{m}){\subseteq}R(\bar{n}^1){\cap}R(\bar{n}^2)\ \}\ ,$$

where $\bar{m}$ is an arbitrary member of $\bar{N}$. Thus by the definition of $\bar{N}$,

$$R({}_1\bar{n}^1_{k^*}) = \max\{\ m \mid m{\subseteq}R(\bar{n}^1){\cap}R(\bar{n}^2)\ \}\ .$$

where m is an arbitrary member of N. Thus

$$R({}_1\bar{n}^1_{k^*}) = \max\{\ m \mid m{\subseteq}n^1{\cap}n^2\ \} = n^1{\wedge}n^2\ , \tag{91}$$

where the first equality holds by the previous sentence and the definitions of $\bar{n}^1$ and $\bar{n}^2$, and the second equality holds by the definition of $\wedge$. This establishes part (a).

Further, since ${}_1\bar{n}^1_{k^*} \in \bar{T}$, Theorem 1(c) implies that $R({}_1\bar{n}^1_{k^*}) \in T$. Hence by (91), $n^1{\wedge}n^2 \in T$. This establishes part (b). □

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