Institute for Economic Studies, Keio University

Keio-IES Discussion Paper Series

コンピュータ近似された動学モデルの収束：非コンパクト空間での証明

高梨 耕作

2018 年 3 月 5 日
DP2018-003
https://ies.keio.ac.jp/publications/9104/

Institute for Economic Studies, Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan
ies-office@adst.keio.ac.jp

5 March, 2018

コンピュータ近似された動学モデルの収束：非コンパクト空間での証明

高梨 耕作

IES Keio DP2018-003
2018 年 3 月 5 日
JEL Classification: C63; C61; C18
キーワード: 動学モデル; コンピュータ近似; 不変分布; 収束レート

【要旨】

解析解をもたないモデルはコンピュテーショナルに近似されたpolicy functionで分析を行う。その際、近似されたpolicy functionの導く近似された不変分布は、policy functionの近似が正確であれば、不変分布もまた、正確な不変分布へ収束するだろうか？この論文では、既に得られている証明を拡張し、仮定されていたコンパクト条件を外す。これにより、正規分布など標準的な分布を、ショックに仮定できるようになる。

高梨 耕作
慶應義塾大学経済学部
〒108-8345
東京都港区三田2-15-45
takanasi@econ.keio.ac.jp

Convergence of Computed Dynamic Models with Unbounded Shock

Kōsaku Takanashi

Faculty of Economics, Keio University

Abstract

The purpose of this paper is to provide the conditions for the convergence of invariant measure obtained from numerical approximations to the exact invariant measure. Santos and Peralta-Alva (2005) have studied the convergence of computed invariant measure of economic models which cannot be solved analytically and must be solved numerically or with some other form of approximation. However, they assume that the state space is compact and therefore, the support of the shock of dynamical system is assumed to be bounded. This paper is to relax the compactness assumption for the convergence of the approximated invariant measure.

1 Introduction

Most dynamic economic models do not have a closed-form analytical solution. Therefore, the researcher rely on the numerical approximation when they evalute the policy function from conditions of economic theory. Model's policy functions which are obtainable are approximated by numerical methods. The attainable invariant measure of economic dynamics is an approximated invariant measure associated with the approximated transition function rather than the exact invariant measure implied by the exact transition function. The purpose of this paper is to provide the conditions for the convergence of invariant measure obtained from numerical simulations to the exact invariant measure.

Santos and Peralta-Alva (2005) have studied the convergence of computed invariant measure of economic models which cannot be solved analytically and must be solved numerically or with some other form of approximation (see also section 12.5 of Stokey and Lucas (1989)). However, they assume that the state space is compact and therefore, the support of the shock of dynamical system is assumed to be bounded. Although this assumption is standard in the numerical literature, but this assumption excludes from the dynamical model the normal distribution (see, Fernandez-Villaverde et al. (2006), Ackergerg et al. (2009)). Normally distrubuted shock in the economic dynamics is important in the empirical studies in which the evaluation of the likelihood is caluculated by the comprehensive use of the Kalman-filter.

Recent works by Stachurski (2002), Nishimura and Stachurski (2005), Kamihigashi (2007), Zhang (2007), Liao and Stachurski (2015) and Kamihigashi and Stachurski (2016) study the asymptotic invariant measure of the stochastic neoclassical growth model without compactness of the shocks and states. The purpose of this paper is to relax the compactness assumption for the convergence of the approximated invariant measure.

The rest of this paper is as follows. Section 2 gives the set-up of dynamical economic models and preliminary of Markov operator. Section 3 present and show our result on the convergence of the invariant measure. Section 4 derive error bounds for these approximations.

2 Model Set-up and Preliminaries

We follow a set of notations and models of Santos and Peralta-Alva (2005). The equilibrium law of motion of the state variables can be specified by a dynamical system of the form

$$s_{t+1} = \varphi\left(s_t, \varepsilon_{t+1}\right), \qquad t = 0, 1, 2, \ldots$$

Here, s_t is a vector of state variables that characterize the evolution of the system. The vector s_t belongs to a mesurable state space $(S, \mathcal{S})$. We endow S with its relative Borel $\sigma-$algebra, which we denote by $\mathcal{S}$. The variable ε is an independent and identically distributed shock which is defined on sample space $(E, \mathcal{E})$. The distribution of the shock ε is given by a stochastic kernel $Q : S \times E \to [0, 1]$; $Q\left(s, A\right)$ is the probability of realizing the event $A \in \mathcal{E}$, given that the current state is $s \in S$.

Given a random dynamical system, one should be able to define a transition probability on the state space in the following way. Define the transition probability function as

$$P\left(s, A\right) = Q\left(\left\{\varepsilon \mid \varphi\left(s, \varepsilon\right) \in A\right\}\right).$$

We define a transition function $P : S \times \mathcal{S} \to [0,1]$ by

$$P(s, A) = Q\left(s, \varphi^{-1}(A)\right).$$

Let $B(S)$ be the set of all bounded S-mesurable real valued functions on S with sup norm $|f| = \sup_S |f(s)|$. Markov operator associated with P is defined as

$$\begin{aligned} Tf(s) &\triangleq \int f(t) P(s, dt) \\ &= \int f(\varphi(s, \varepsilon)) Q(s, d\varepsilon) \end{aligned} \tag{2.1}$$

For any given initial condition μ_0 on $\mathbb{S}$, the evolution of future probabilities, $\{\mu_n\}$, can be specified by the following operator T^* that takes the space

$$\mu_{n+1} = (T^* \mu_n)(A) = \int P(s, A) \mu_n(ds)$$

for all A in $\mathbb{S}$ and $n \geq 0$. The adjoint $T^*$of T is defined by the formula

$$(T^* \mu)(A) = \int P(t, A) \mu(dt).$$

We maintain the following basic assumptions:

Assumption 1. *The sets S and E are both locally compact and σ-compact space.*

Remark 2. Locally compact means that for each points $x \in S$, there is some compact subspace C of S that contains a neighborhood of $x \in S$. σ-compact is a contable union of compact spaces. Note that the space $\mathbb{R}^d$ is both locally compact and σ-compact. Both locally compact and σ-compact space can be written as an increasing union of countably many open sets each of which has compact closere. In Santos and Peralta-Alva (2005), they impose the compactness assumption on the satate S and E. We relax this restriction to the non-compact case which allow to use a whole Euclidian state $S = \mathbb{R}^d$ and the normal distribution.

Recall that the probability measure P is called tight if for all $\varepsilon > 0$ there is a compact set $K \subset \mathcal{S}$ such that $P(K) \geqq 1 - \varepsilon$. Note that by the theorem below any probability measures on the complete separable metric space is tight.

Theorem 3. *[Ulam's theorem (Theorem 7.1.4. in Dudley (2002))]*
On any complete separable metric space, any finite Borel measure is tight.

Assumption 4. *There exist a point $s_0 \in S$ such that for any point $s \in S$, any neighborhood U of s_0 and any integer $k \geq 1$, $P^{nk}(s, U) > 0$.*

From assumption 4, the Markov operator T^* has unique fixed point μ_0: $T^* \mu_0 = \mu_0$, see Futia (1982)(Section 3.2).

Assumption 5. *Function $\varphi : S \times E \to S$ is jointly measurable. Moreover, for every continuous function $f : S \to \mathbb{R}$,*

$$\int f(\varphi(s_j, \varepsilon)) Q(d\varepsilon) \to \int f(\varphi(s, \varepsilon)) Q(d\varepsilon) \;\; as \; s_j \to s.$$

Assumption 5 is the same one of the Assumption 2 in Santos and Peralta-Alva (2005), page1942.

In most case, researcher dose not know the exact form of transition equations φ. He only access to numerical approximation to the transition equations φ_j with index j. The index j indicate the approximation and imply that as j goes to infinity the approximation φ_j connverge to their exact values (the metric of convergence is defined later). Every numerical approximation φ_j satisfies the above assumptions 1, 4 and 5. And every numerical approximation φ_j define the transition probability P_j on $(S, \mathcal{S})$.

3 Convergence of Invariant Distribution

Now let us recall the convergence of probability measures on S. The state space S is separable, then we can introduce a metric D in the space of probability measures on S in such a way that $\lim_n D(\mu_n, \mu) = 0$ if and only if μ_n converges in the law to μ. Especially, a metric we take is the Fortet-Mourier metric (Dudley (2002), Section11.3):

$$D(\mu_n, \mu) = \sup_{f \in BL(S)} \left| \int_S f(s)\, d\mu_n - \int_S f(s)\, d\mu \right| \tag{3.1}$$

where the supremum $\sup_{f \in BL(S)}$ is taken over all bounded Lipschitz continuous functions defined on S: $BL(S)$. $BL(S)$ can be relaxed to infinitely continuously differentialbe functions on S: $C^{\infty}(S)$ by using morifier argument.

Lemma 6. $\lim_n D(\mu, \mu_n) = 0$ *if and only if*

$$\lim_{n \to \infty} \sup_{f \in C^{\infty}(S)} \left| \int_S f(s)\, d\mu_n - \int_S f(s)\, d\mu \right| = 0.$$

Proof. "Only if" part is derived from $C^{\infty} \subset BL$. We show the "if" part. For any $f \in BL(S)$ and $\varepsilon > 0$, following molifier technique (see, Feller (1968)), there exists $u \in C^{\infty}(S)$ such that $\sup_{s \in S} |f(s) - u(s)| < \varepsilon$. By triangular inequality, we have

$$\begin{aligned}
&\sup_{f\in BL(S)}\left|\int_S f(s)\,d\mu_n-\int_S f(s)\,d\mu\right| \\
&\leq \sup_{f\in BL(S)}\left|\int_S f(s)\,d\mu_n-\int_S u(s)\,d\mu_n\right|+\sup_{u\in C^\infty(S)}\left|\int_S u(s)\,d\mu_n-\int_S u(s)\,d\mu\right| \\
&\quad+\sup_{f\in BL(S)}\left|\int_S u(s)\,d\mu-\int_S f(s)\,d\mu\right|
\end{aligned}$$

In the right-hand side of the inequality, the first and third term are smaller than ε. The second term is given in the definition. □

Note that by Assumptions 2 and 3, each φ_n defines the associated pair (P_j, T_j) : Markov operator associated with φ_j is defined as

$$\begin{aligned}
T_j f(s) &\triangleq \int f(t)\,P_j(s,dt) \\
&= \int f(\varphi_j(s,\varepsilon))\,Q(s,d\varepsilon)
\end{aligned} \tag{3.2}$$

The adjoint T_j^* of T_j is as

$$\begin{aligned}
\langle T_j f,\mu_j\rangle &= \int\int f(\varphi_j(s,\varepsilon))\,Q(s,d\varepsilon)\,d\mu_j(s) \\
\left\langle f,T_j^*\mu_j\right\rangle &= \int\int f(\varphi_j(s,\varepsilon))\,Q(s,d\varepsilon)\,d\mu_j(s)
\end{aligned}$$

Moreover there always exists an invariant distribution $\mu_j^*=T_j^*\mu_j^*$.

Proposition 7. *Suppose assumptions 1,4 and 5 are satisfied for each approximated models φ_i and φ_0. Then, a sufficient condition for a sequence of the measure μ_j*

associated with T_j converge to μ_0 associated with T_0 in the sense of (3.1) is the strong convergence of T_j to T_0.

Proof. Storong convergence of the sequence of operators means that

$$\begin{aligned}\sup_{s\in S}\|Tf(s)-T_jf(s)\| &= \sup_{s\in S}\left|\int f(\varphi(s,\varepsilon))\,Q(s,d\varepsilon)-\int f(\varphi_j(s,\varepsilon))\,Q(s,d\varepsilon)\right| \\ &= \sup_{s\in S}\left|\int\{f(\varphi(s,\varepsilon))-f(\varphi_j(s,\varepsilon))\}\,Q(s,d\varepsilon)\right| \\ &\to 0\end{aligned}$$

for all $f\in C^2(S)$. This simply means that

$$\lim_{j\to\infty}\mathbb{E}[f(\varphi_j(s))]=\mathbb{E}[f(\varphi(s))],\ {}^{\forall}f\in C^2(S).$$

Let f belong to $\mathcal{A}$. Then, for any two T and T_j, and corrensponding invariant measures μ and μ_j, we have

$$|\langle Tf,\hat{\mu}\rangle-\langle T_jf,\mu_j\rangle|\le|\langle Tf,\mu\rangle-\langle Tf,\mu_j\rangle|+|\langle Tf,\mu_j\rangle-\langle T_jf,\mu_j\rangle|. \quad (3.3)$$

It follows from Proholov's Theorem that $\{\mu_j\}$ has a weakly convergent subsequence. Let $\{\mu_{j_k}\}$ be such a subsequence, and let $\hat{\mu}$ be its limit. Then for the first term in (3.3)

$$|\langle Tf,\hat{\mu}\rangle-\langle Tf,\mu_{j_k}\rangle|\to 0.$$

Next, we show the equality $\hat{\mu} = \mu_0$. We have

$$\begin{aligned}|\langle f, \hat{\mu}\rangle - \langle T_0 f, \hat{\mu}\rangle| \leq & |\langle f, \hat{\mu}\rangle - \langle f, \mu_{j_k}\rangle| + |\langle f, \mu_{j_k}\rangle - \langle T_0 f, \mu_{j_k}\rangle| \\ & + |\langle T_0 f, \mu_{j_k}\rangle - \langle T_0 f, \hat{\mu}\rangle| \,.\end{aligned}$$

Since Tf and $T_0 f$ are continuous and $\{\mu_{j_k}\}$ convergence weakly to $\hat{\mu}$, the first and third terms on the right approach to zero as goes infinity. For second term,

$$\begin{aligned}|\langle f, \mu_{j_k}\rangle - \langle T_0 f, \mu_{j_k}\rangle| &= \left|\left\langle f, T_{j_k}^{*} \mu_{j_k}\right\rangle - \langle T_0 f, \mu_{j_k}\rangle\right| \\ &= |\langle T_{j_k} f, \mu_{j_k}\rangle - \langle T_0 f, \mu_{j_k}\rangle| \\ &\leq \|T_{j_k} f - T_0 f\|\end{aligned}$$

□

Theorem 8. *Stone-Cech compactification (Theorem 38.2 of Munkres (2000)) Let S be a completely regular space. Then there exists a compactification $\beta(S)$ of S having the property that every bounded continuous function $f : S \to \mathbb{R}$ extends uniquely to a continuous function of $\beta(S)$ into $\mathbb{R}$.*

Since S is a completely regular, it has Stone-Cech compactification $\beta(S)$ which is S is a dense subspace of a compact space $\beta(S)$ satisfying the property that each bounded continuous function $f : S \to \mathbb{R}$ has a continuous extension $g : \beta(S) \to \mathbb{R}$. We endow the metric in the space of functions defined on locally compact and σ-compact space S. For any two vector-value functions φ and $\hat{\varphi}$ let

$d(\cdot,\cdot)$ be

$$d(\varphi,\hat{\varphi}) = \max_{f\in C^{\infty}(\beta(S))} \max_{s\in\beta(S)} \left[\int |f(\varphi(s,\varepsilon)) - f(\hat{\varphi}(s,\varepsilon))| \, Q(d\varepsilon) \right]. \tag{3.4}$$

In this section, convergence of the sequence of functions $\{\varphi_n\}$ is in this notion of distances. This metric can accommodate the noncontinuous functions φ and $\hat{\varphi}$. Although we will impose continuous differentiability on φ for the convergence of the approximated likelihood studied in the next section, the metric $d(\cdot,\cdot)$ is sufficient to guarantee the convergence of invariant distribution.

Remark 9. In Santos and Peralta-Alva (2005), they endow the metric in the space of functions φ and $\hat{\varphi}$ as

$$\max_{s\in S} \left[\int \|\varphi(s,\varepsilon) - \hat{\varphi}(s,\varepsilon)\| \, Q(d\varepsilon) \right]$$

where $\|\cdot\|$ is the max norm in $\mathbb{R}^l$. This metric works only in the compactness assumption on S. The metric (3.4) is weaker than this metric and extend to the non-compact state space.

Theorem 10. *Let $\{\varphi_j\}$ be a sequence of functions that converge to φ in the sense of $d(\cdot,\cdot)$ in (3.4). Let $\{\mu_j^*\}$ be a sequence of probabilities on $\mathbb{S}$ associated to $\{\varphi_j\}$ such that $\mu_j^* = T_j^*\mu_j^*$ for each j. Under Assumptions 1 and 4, if μ^* is a weak limit point of $\{\mu_j^*\}$, then $\mu^* = T^*\mu^*$.*

This theorem asserts the bilinear convergence of $T_j^*\mu_j^*$ to $T^*\mu^*$ in the weak topology.

Proof. The topology of weak convergence can be defined by the metric on the probability measure space

$$d(\mu,\nu)=\sup_{f\in\mathcal{A}}\left\{\left|\int f(s)\,\mu(ds)-\int f(s)\,\nu(ds)\right|\right\}$$

where $\mathcal{A}$ is the space of Lipschitz function on S with constant $L\leq 1$ and such that $-1\leq f\leq 1$.

$$\begin{aligned}\|Tf(s)-T_jf(s)\| &= \left|\int f(\varphi(s,\varepsilon))\,Q(d\varepsilon)-\int f(\varphi_j(s,\varepsilon))\,Q(d\varepsilon)\right| \\ &= \left|\int f(\varphi(s,\varepsilon))-f(\varphi_j(s,\varepsilon))\,Q(d\varepsilon)\right|\end{aligned}$$

Since $f\in C^{\infty}$, there exists a constant K such that

$$\left|\int [f(\varphi(s,\varepsilon))-f(\varphi_j(s,\varepsilon))]\,Q(d\varepsilon)\right|\leq Kd(\varphi,\varphi_j).$$

□

4 Error Bounds

In numerical applications, computations must stop in finite time, and hence it is often desirable to bound the size of the approximation error. Santos and Peralta-Alva (2005) gives a bound the size of the approximation error under compactness assumption. In this section, we relax this compactness assumption. To begin the discussion, we introduce the notion of a compactness of Markov operator. The Markov operator T is *compact* if the image $T(bX)$ has compact closure in X, where $bX=\{x\in X|\,\|x\|\leq 1\}$. The Markov operator T is *quasi-compact* if there is a unique compact operator L and an integer n such that

$$\sup_{x \in bX} \|T^n x - Lx\| < 1.$$

If the above quasi-compactness is satisfied, one can obtain the convergence of the sequence of operators $\{T^n\}$ to the invariant probability at a geometric rate. The following theorem gives this result.

Theorem 11. *[Yosida and Kakutani (1941), p.204, Corollary]*
Let T be a stable, quasi-compact Markov operator defined by (2.1) satisfying 1, 4, 5. Then, there exist constants C, $\varepsilon > 0$ such that

$$\sup_{s \in \beta(S)} \|T^n f(s) - T^* f(s)\| \leq \frac{C}{(1+\varepsilon)^n}.$$

The following theorem bounds the approximation error between the expected values of f over the true invariant mesure μ^* and the approximate invariant mesure $\hat{\mu}^*$ of $\hat{\varphi}$.

Proposition 12. *Let f be a Lipschitz function with constant L. Suppose we have numerical approximation $\hat{\varphi}$ with $d(\hat{\varphi}, \varphi) \leq \delta$ for some $\delta > 0$. Then*

$$\left| \int f(s) \mu^*(ds) - \int f(s) \hat{\mu}^*(ds) \right| \leq \frac{Ld(\hat{\varphi}, \varphi)}{\varepsilon}$$

where μ^* is the unique invariant measure of the exact φ, and $\hat{\mu}^*$ is an unique invariant measure of $\hat{\varphi}$.

Proof. Denote $s_n(s_0)$ as

$$\underbrace{\varphi\left(\varphi\left(\varphi\cdots\left(\varphi\left(s_0,\varepsilon_1\right),\varepsilon_2\right)\right)\right)}_{\text{n times}}.$$

And denote $\hat{s}_n(s_0)$ as

$$\underbrace{\hat{\varphi}\left(\hat{\varphi}\left(\hat{\varphi}\cdots\left(\hat{\varphi}\left(s_0,\varepsilon_1\right),\varepsilon_2\right)\right)\right)}_{\text{n times}}.$$

Since f is a Lipschitz function with constant L, we have

$$\begin{aligned}
&\left|\mathbb{E}\left[f\left(s_n\left(s_0\right)\right)\right]-\mathbb{E}\left[f\left(\hat{s}_n\left(s_0\right)\right)\right]\right| \\
=&\left|\mathbb{E}\left[f\left(\varphi\left(s_{n-1}\left(s_0\right),\varepsilon_n\right)\right)\right]-\mathbb{E}\left[f\left(\hat{\varphi}\left(\hat{s}_{n-1}\left(s_0\right),\varepsilon_n\right)\right)\right]\right| \\
\leq&\left|\mathbb{E}\left[f\left(\varphi\left(s_{n-1}\left(s_0\right),\varepsilon_n\right)\right)\right]-\mathbb{E}\left[f\left(\varphi\left(\hat{s}_{n-1}\left(s_0\right),\varepsilon_n\right)\right)\right]\right| \\
&+\left|\mathbb{E}\left[f\left(\varphi\left(s_{n-1}\left(s_0\right),\varepsilon_n\right)\right)\right]-\mathbb{E}\left[f\left(\hat{\varphi}\left(\hat{s}_{n-1}\left(s_0\right),\varepsilon_n\right)\right)\right]\right| \\
\leq& L\left(1-\varepsilon\right)\mathbb{E}\left\|s_{n-1}\left(s_0\right)-\hat{s}_{n-1}\left(s_0\right)\right\|+Ld\left(\hat{\varphi},\varphi\right).
\end{aligned}$$

And by the same argument above, we get

$$L\left(1-\varepsilon\right)\mathbb{E}\left\|s_{n-1}\left(s_0\right)-\hat{s}_{n-1}\left(s_0\right)\right\| \leq L\left(1-\varepsilon\right)^2\mathbb{E}\left\|s_{n-2}\left(s_0\right)-\hat{s}_{n-2}\left(s_0\right)\right\|+L\left(1-\varepsilon\right)^2 d\left(\hat{\varphi},\varphi\right).$$

Iterating this we obtain

$$\left|\mathbb{E}\left[f\left(s_n\left(s_0\right)\right)\right]-\mathbb{E}\left[f\left(\hat{s}_n\left(s_0\right)\right)\right]\right| \leq \frac{Ld\left(\hat{\varphi},\varphi\right)}{\varepsilon}.$$

Integrate by an invariant measure $\hat{\mu}^*$ of $\hat{\varphi}$, then

$$\left|\int \mathbb{E}\left[f\left(s_n\left(s_0\right)\right)\right] \hat{\mu}^*\left(ds_0\right) - \int \mathbb{E}\left[f\left(\hat{s}_n\left(s_0\right)\right)\right] \hat{\mu}^*\left(ds_0\right)\right| \leq \frac{Ld\left(\hat{\varphi}, \varphi\right)}{\varepsilon}.$$

Note that the second term on the left-hand side is equal to $\int f\left(s\right) \hat{\mu}^*\left(ds\right)$. From the theorem 11, for every s_0, $\mathbb{E}\left[f\left(s_n\left(s_0\right)\right)\right]$ converges uniformly on $\beta\left(S\right)$ to $\int f\left(s\right) \mu^*\left(ds\right)$. Finally, we obtain

$$\left|\int f\left(s\right) \mu^*\left(ds\right) - \int f\left(s\right) \hat{\mu}^*\left(ds\right)\right| \leq \frac{Ld\left(\hat{\varphi}, \varphi\right)}{\varepsilon}.$$

□

Remark 13. The quasi-compact operators enjoy a very useful property : the theorem 11. Furthermore, quasi-compact operators are easily recognized; in fact, we shall find that ”most” operators are quasi-compact (see Futia (1982)). In our circumstance, the assumption 4 guarantees the quasi-compactness of the Markovian operator.

References

ACKERGERG, D., J. GEWEKE, AND J. HAHN (2009): “Comments on ”Convergence properties of the likelihood of computed dynamic models”,” *Econometrica*, 77, 2009–2017.

DUDLEY, R. M. (2002): “Real analysis and probability, volume 74 of Cambridge Studies in Advanced Mathematics,” .

FELLER, W. (1968): *An introduction to probability theory and its applications*, vol. 2, John Wiley & Sons.

FERNANDEZ-VILLAVERDE, J., J. RUBIO-RAMIREZ, AND M. SANTOS (2006): "Convergence Properties of the Likelihood of Computed Dynamic Models," *Econometrica*, 74, 93–119.

FUTIA, C. (1982): "Invariant distributions and the limiting behavior of Markovian economic models," *Econometrica*, 50, 377–408.

KAMIHIGASHI, T. (2007): "Stochastic optimal growth with bounded or unbounded utility and with bounded or unbounded shocks," *Journal of Mathematical Economics*, 43, 477–500.

KAMIHIGASHI, T. AND J. STACHURSKI (2016): "Seeking ergodicity in dynamic economies," *Journal of Economic Theory*, 163, 900–924.

LIAO, Y. AND J. STACHURSKI (2015): "Simulation-based density estimation for time series using covariate data," *Journal of Business & Economic Statistics*, 33, 595–606.

MUNKRES, J. R. (2000): *Topology*, Prentice Hall.

NISHIMURA, K. AND J. STACHURSKI (2005): "Stability of Optimal growth models: a new approach," *Journal of Economic Theory*, 122, 100–118.

SANTOS, M. AND A. PERALTA-ALVA (2005): "Accuracy of simulations for stochastic dynamic models," *Econometrica*, 73, 1939–1976.

STACHURSKI, J. (2002): "Stochastic Optimal growth with unbounded shock," *Journal of Economic Theory*, 106, 40–65.

STOKEY, N. AND R. LUCAS (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press.

YOSIDA, K. AND S. KAKUTANI (1941): "Operator-theoretical treatment of Markoff's process and mean ergodic theorem," *Annals of Mathematics*, 188–228.

ZHANG, Y. (2007): "Stochastic optimal growth with a non-compact state space," *Journal of Mathematical Economics*, 43, 115–129.