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# 1 Introduction

Modern macroeconomic models represented by dynamic stochastic general equilibrium (DSGE) models are built on the premise that prices and allocations are uniquely determined by fundamental elements, such as technology, preferences, policy disturbances, etc. Such uniqueness of economic outcomes is referred to as model determinacy and plays a central role in macroeconomic modeling. At the same time, it is well known that particular constraints must be imposed on economic activity to ensure determinacy; otherwise, numerous prices and allocations emerge corresponding to a set of fundamental variables, which leads to a state of so-called indeterminacy.

When the economy faces indeterminacy, multiple equilibria arise and economic outcomes can be affected by nonfundamental elements. These nonfundamental elements are often expressed as "sunspots" or "animal spirits" and have been incorporated into economic models. How these nonfundamental elements affect economic activity has been an active research area since the seminal paper of Cass and Shell (1983), in which the authors show that sunspots matter for real allocations under certain conditions.

One of the typical examples where a particular constraint must be imposed to ensure determinacy is the central bank's commitment to fight inflation, which is widely known as the Taylor principle in the standard New Keynesian framework. The common wisdom that the nominal interest rate must respond more than one for one when the inflation rate deviates from the central bank's target is viewed as playing an essential role in stabilizing the economy. When the Taylor principle is violated, the economy suffers indeterminacy, and an infinite number of equilibrium paths converging to the steady state arise.

In recent years, most advanced economies have been confined to a situation where the Taylor principle is violated, namely the liquidity trap. As the interest rate is stuck at the zero lower bound (ZLB), central banks have been unable to respond to changes in the inflation rate for a substantially extended period. Although many central banks have departed from conventional monetary policy that manipulates short-term interest rates and have adopted unconventional monetary policies, there is little room for such policies to react to inflation substantially. Such a state allows economic agents to form inflation expectations inconsistent with the central bank's long-run target and leads to indeterminacy.

The existence of the ZLB, at the same time, is known to generate nonlinearity in economic agents' behavior. As the central bank cannot lower the interest rate below the ZLB, a decrease in the inflation rate is associated with an increase in the real interest rate. As discussed by Fernández-Villaverde et al. (2015), a rise in the real interest rate aggravates the economic outcome by putting further downward pressure on output and inflation in a nonlinear manner at the ZLB; thus appropriately modeling the nonlinearity arising from the ZLB is considered to be important.

Although liquidity traps are the source of both indeterminacy and nonlinearity, there has been little investigation of the dynamics of a nonlinear indeterminate system. The aim of this paper is to fill this gap regarding these two key aspects of liquidity traps. To this end, we present a novel methodology to derive a nonlinear solution of an indeterminate DSGE model in which the decision rules are affected by sunspot shocks. We consider sunspot shocks in a locally indeterminate nonlinear system and show that the nonlinear solutions can be derived by incorporating an auxiliary equation and an auxiliary variable proposed by the recent work of Bianchi and Nicolò (forthcoming) into the projection method.

As an application of our newly developed solution method, we first consider a simple case in which the Taylor principle is not satisfied because of a passive monetary policy. We find that the intuition in the linear model carries over to the nonlinear model and the advantages of solving the model nonlinearly are limited. This is because it is extremely rare for the ZLB to be reached when monetary policy is passive, and such infrequent cases are not considered in deriving the nonlinear solutions. Therefore, the similarity of the two solutions can be attributed to the fact that the model is almost linear when the ZLB is not binding.

We then solve the model around the expectations-driven liquidity trap—a liquidity trap that arises because of the ZLB constraint on the nominal interest rate and the de-anchoring of economic agents' expectations—and show that nonlinearity plays a significant role in the model dynamics. Nonlinearity emerges because the ZLB ceases to bind once the inflation rate increases because of a temporary rise in inflation expectations. These findings provide important insights into monetary policy conduct because inflation and consumption may temporarily increase to a level that lifts the interest rate above zero even if agents believe that the economy converges to a deflationary state in the long run.

Although extant studies have explored nonlinearity and indeterminacy arising in liquidity traps separately, none have succeeded in combining these two important elements. This paper is the first to combine these two elements and derive a nonlinear solution that allows nonfundamental sunspot shocks to affect prices and allocations. Therefore, this study contributes significantly to the literature by linking these two key elements of liquidity traps; as such, it can be related to two different strands of the literature. The first strand involves studies that focus on the effects of sunspot shocks on economic activity when the model exhibits indeterminacy. The seminal paper of Cass and Shell (1983) studies how sunspots play a role in equilibrium allocation in both static and dynamic models and shows the conditions under which sunspots matter. Farmer and Guo (1994) study a model with an aggregate technology that is subject to increasing returns and show that investors' "animal spirits" can generate business cycle fluctuations. The recent work by Farmer (2019) provides a comprehensive survey of models featuring indeterminacy and sunspots.

As this study develops a method to derive nonlinear solutions of an indeterminate model, it is closely related to studies that explore methods to solve and estimate indeterminate models. The pioneering work of Lubik and Schorfheide (2004) presents a methodology to solve and estimate an indeterminate model and applies it to US data. Farmer et al. (2015) propose a detailed methodology to solve linear indeterminate models and show how it could be applied to existing software packages. More recently, Bianchi and Nicolò (forthcoming) propose a novel methodology to solve linear indeterminate models by introducing auxiliary equations and variables and apply their new methodology to a DSGE model with bubbles in the setup of Galí (forthcoming). They find that the US data support the presence of two degrees of indeterminacy, implying that the central bank was not reacting strongly enough to the bubble component.

The second strand comprises studies focusing on liquidity traps. Among liquidity traps arising from different causes, expectations-driven liquidity traps, which were first investigated in depth in the seminal paper of Benhabib et al. (2001), have attracted considerable attention from both empirical and theoretical perspectives. On the empirical side, Aruoba et al. (2018) investigate whether the US and Japan have transitioned to a deflationary regime using a nonlinear DSGE model and suggest that Japan is likely to have moved to such a regime in the late 1990s, while it is unlikely for the US. As we discuss later, Aruoba et al. (2018) select a particular solution and abstract from indeterminacy arising because of the ZLB. Hirose (2020) adopts the method proposed by Bianchi and Nicolò (forthcoming) and estimates a linear indeterminate DSGE model around an expectations-driven liquidity trap using Japanese data.

On the theoretical side, recent studies have emphasized how fiscal policies can be implemented to deal with expectations-driven liquidity traps. Studies such as Benhabib et al. (2002), Schmidt (2016), and Tamanyu (2019) focus on the use of fiscal policies to prevent expectations-driven liquidity traps. Other recent studies, such as Bilbiie (2018) and Nakata and Schmidt (2019) compare how monetary and fiscal policies can be implemented to confront expectations-driven liquidity traps.

As our findings highlight the importance of considering nonlinearity when the economy is in a liquidity trap, they can be related to the recent literature that investigates how model dynamics are affected by the existence of the ZLB. Fernández-Villaverde et al. (2015) argue for the importance of explicitly considering nonlinearities in a model that faces the ZLB and derive nonlinear decision rules using projection methods. Richter and Throckmorton (2015) show that a tradeoff exists between the numerical convergence of a particular solution algorithm and the expected frequency and average duration of the ZLB events. Atkinson et al. (2020) compare the difference between a full nonlinear solution and a piecewise linear solution and find that there is a large practical advantage in using the latter.

The remainder of this paper is organized as follows. In Section 2, the details of the model are provided. In Section 3, by considering passive monetary policy as an example, we present the new methodology to derive nonlinear solutions for indeterminate models. In Section 4, we apply our method to the expectations-driven liquidity trap and explore the model dynamics. Section 5 concludes.

# 2 The model

As the most basic model suffices to explore the key aspects of the model dynamics under indeterminacy, this study builds on a canonical small-scale New Keynesian DSGE model. The model consists of three equilibrium equations: the downward sloping demand equation derived from the representative household's optimization problem, the upward sloping supply equation derived from the firm's optimization problem, and the monetary policy rule constrained by the ZLB.

To model price stickiness, we introduce price adjustment costs à la Rotemberg (1982). As Rotemberg (1982) pricing does not require an additional state variable, it is preferred in the studies concerned with nonlinear solution methods. In the following subsections, we provide the details of the model.

## 2.1 Household

There is a representative household that gains utility from consumption and disutility from labor supply. The household maximizes expected lifetime utility by choice of consumption $c_t$, labor supply $l_t$, and bond holdings $b_t$ given prices and subject to a budget constraint as follows:

$$\max_{\{c_{t+s},l_{t+s},b_{t+s}\}_{s=0}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Big[\frac{c_{t+s}^{1-\sigma}-1}{1-\sigma} - \frac{l_{t+s}^{\eta+1}-1}{\eta+1}\Big], \tag{1}$$

$$\text{s.t.} \quad c_t + \frac{b_t}{R_t} = w_t l_t + \frac{b_{t-1}}{\Pi_t} + d_t. \tag{2}$$

$R_t$ and $\Pi_t$ are the gross nominal interest rate and the gross inflation rate respectively. $w_t$ is the real wage and $d_t$ is a dividend from intermediate goods firms. From the first-order conditions, we can derive the Euler equation and the wage equation as follows:

$$c_t^{-\sigma} = \beta R_t \mathbb{E}_t \Big[\frac{c_{t+1}^{-\sigma}}{\Pi_{t+1}}\Big], \tag{3}$$

$$\frac{c_t^{-\sigma}}{l_t^{\eta}} = \frac{1}{w_t}. \tag{4}$$

## 2.2 Firms

There are two types of firms in the economy: a continuum of intermediate goods producers and a final goods producer. The final goods producer uses intermediate goods as the only input and has CES production technology. The final goods producer is perfectly competitive and takes both output and input prices as given. The static profit maximization problem is given as follows:

$$\max_{\{y_t,y_{i,t}\}} P_t y_t - \int_0^1 P_{i,t} y_{i,t} di, \tag{5}$$

$$\text{s.t.} \quad y_t = \Big(\int_0^1 y_{i,t}^{\frac{\theta-1}{\theta}} di\Big)^{\frac{\theta}{\theta-1}}. \tag{6}$$

Perfect competition drives final good producers' profits to zero. From the first-order conditions, we can derive the demand for intermediate goods and the associated price index:

$$y_{i,t} = \Big(\frac{P_{i,t}}{P_{t+s}}\Big)^{-\theta} y_t, \tag{7}$$

$$P_t = \Big(\int_0^1 P_{i,t}^{1-\theta} di\Big)^{\frac{1}{1-\theta}}. \tag{8}$$

There is a continuum of intermediate goods producers indexed by $i$. They are monopolistically competitive and incur quadratic price adjustment costs as in Rotemberg (1982). Each producer uses labor as an input in production. Firm $i$ chooses optimal price $P_{i,t}$ and labor input $l_{i,t}$ given the current aggregate output $y_t$ and aggregate price level $P_t$. It maximizes the present value of discounted dividends $d_{i,t}$ according to the following optimization problem:

$$\max_{\{y_{i,t+s}, P_{i,t+s}, l_{i,t+s}\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} Q_{t+s} d_{i,t+s}, \tag{9}$$

$$\text{s.t.} \quad d_{i,t+s} = \frac{P_{i,t+s}}{P_{t+s}} y_{i,t+s} - w_{t+s} l_{i,t+s} - \frac{\psi}{2}\Big(\frac{P_{i,t+s}}{P_{i,t+s-1}} - \Pi^*\Big)^2 y_{t+s}, \tag{10}$$

$$y_{i,t+s} = A_t l_{i,t+s}, \tag{11}$$

$$y_{i,t+s} = \Big(\frac{P_{i,t+s}}{P_{t+s}}\Big)^{-\theta} y_{t+s}, \tag{12}$$

where the real stochastic discount factor is defined as

$$Q_{t+s} \equiv \beta^s c_{t+s}^{-\sigma}. \tag{13}$$

Productivity is determined exogenously as

$$A_t = A_{t-1}^{\rho_a} \exp(\varepsilon_{a,t}), \qquad i.i.d.\ \varepsilon_{a,t} \sim N(0, \sigma_a^2). \tag{14}$$

Combining the first-order conditions and imposing symmetry across firms, we derive the following Phillips curve:

$$\psi(\Pi_t - \Pi^*)\Pi_t - \theta \frac{w_t}{A_t} + \theta - 1 = \beta \mathbb{E}_t\Big[\Big(\frac{c_t}{c_{t+1}}\Big)^{\sigma}\Big(\frac{y_{t+1}}{y_t}\Big)\psi(\Pi_{t+1} - \Pi^*)\Pi_{t+1}\Big]. \tag{15}$$

The aggregate production function and dividend payouts are

$$y_t = A_t l_t, \tag{16}$$

$$d_t = y_t - w_t l_t - \frac{\psi}{2}(\Pi_t - \Pi^*)^2 y_t. \tag{17}$$

## 2.3 Central bank

The central bank sets the interest rate following the standard Taylor rule where the net nominal interest rate is bounded below by zero as follows:

$$R_t = \max\Big[1, \frac{\Pi^*}{\beta}\Big(\frac{\Pi_t}{\Pi^*}\Big)^{\phi_\pi}\exp(\varepsilon_{m,t})\Big], \quad i.i.d.\ \varepsilon_{m,t} \sim N(0, \sigma_m^2). \tag{18}$$

We abstract from government spending for simplicity. Thus $b_t = 0$ holds for all $t$ from Ricardian equivalence.

## 2.4 Equilibrium conditions

The resource constraint of the economy is derived by combining equations (2) and (17) as follows:

$$c_t + \frac{\psi}{2}(\Pi_t - \Pi^*)^2 y_t = y_t. \tag{19}$$

Equations (3), (4), (15), (16), (18), and the resource constraint (19) are the equilibrium conditions. The nonlinear equilibrium conditions can be summarized as the following two equations:

$$1 = \max\Big[1, \frac{\Pi^*}{\beta}\Big(\frac{\Pi_t}{\Pi^*}\Big)^{\phi_\pi}\exp(\varepsilon_{m,t})\Big]\beta\mathbb{E}_t\Big[\frac{1}{\Pi_{t+1}}\Big(\frac{\{1-\psi(\Pi_{t+1}-\Pi^*)^2/2\}y_{t+1}}{\{1-\psi(\Pi_t-\Pi^*)^2/2\}y_t}\Big)^{-\sigma}\Big], \tag{20}$$

$$\Big(\frac{y_t}{A_t}\Big)^{\eta}\Big[\Big\{1-\frac{\psi}{2}(\Pi_t-\Pi^*)^2\Big\}y_t\Big]^{\sigma} - \frac{\theta-1}{\theta} = \frac{\psi}{\theta}(\Pi_t-\Pi^*)\Pi_t$$
$$-\frac{\psi}{\theta}\beta?\mathbb{E}_t\Big[\Big(\frac{\{1-\psi(\Pi_{t+1}-\Pi^*)^2/2\}y_{t+1}}{\{1-\psi(\Pi_t-\Pi^*)^2/2\}y_t}\Big)^{-\sigma}(\Pi_{t+1}-\Pi^*)\Pi_{t+1}\frac{y_{t+1}}{y_t}\Big]. \tag{21}$$

Let us denote the targeted steady state as TSS. The steady state values in the TSS can be derived as follows:

$$\Pi_{TSS} = \Pi^*, \tag{22}$$

$$R_{TSS} = \frac{\Pi^*}{\beta}, \tag{23}$$

$$y_{TSS} = c_{TSS} = \left(\frac{\theta - 1}{\theta}\right)^{\frac{1}{\sigma+\eta}}. \tag{24}$$

When the Taylor principle is satisfied ($\phi_\pi > 1$), there is another steady state that exhibits deflation, which we call the unintended steady state (USS). The steady state values in the USS are as follows:

$$\Pi_{USS} = \beta, \tag{25}$$

$$R_{USS} = 1, \tag{26}$$

$$y_{USS} = \left[1 - \frac{\psi}{2}(\Pi^* - \beta)^2\right]^{-\frac{\sigma}{\sigma+\eta}} \left[\frac{\theta - 1}{\theta} - \frac{\psi\beta}{\theta}(1-\beta)(\Pi^* - \beta)\right]^{\frac{1}{\sigma+\eta}}, \tag{27}$$

$$c_{USS} = \left[1 - \frac{\psi}{2}(\Pi^* - \beta)^2\right]^{\frac{\eta}{\sigma+\eta}} \left[\frac{\theta - 1}{\theta} - \frac{\psi\beta}{\theta}(1-\beta)(\Pi^* - \beta)\right]^{\frac{1}{\sigma+\eta}}. \tag{28}$$

It is clear that the consumption level in the USS is lower than that in the TSS ($c_{USS} < c_{TSS}$) because there is a loss from the price adjustment cost in the USS.

## 2.5 Calibration

It is assumed that the model period corresponds to a quarter. The discount factor is set to $\beta = 0.99$, which yields an annual real interest rate of four percent. We set the elasticity of intertemporal substitution to $\sigma = 1$ and the Frisch elasticity of labor to $\eta = 1$, which yield log utility and linear disutility, respectively. The elasticity of substitution between intermediate goods is set to $\theta = 6$, which yields a markup of 20 percent. The price adjustment cost is set to $\psi = 58$, which is chosen to match the price stickiness of $\omega = 0.75$ under Calvo (1983) price stickiness.[1] The parameters regarding the stochastic processes are set to $\rho_a = 0.9$ and $\sigma_a = 0.0025$ for productivity shocks, $\sigma_m = 0.001$ for monetary policy shocks, and $\sigma_\nu = 0.0025$ for sunspot shocks.

The target net inflation rate is set equal to zero, a stable price level. As for the

[1] In a linearized model with zero steady state inflation, either assuming Rotemberg (1982) or Calvo (1983) price stickiness yields identical Phillips curves when the parameter is chosen to satisfy $\psi = \omega(\theta-1)/[(1-\omega)(1-\beta\omega)]$.

Taylor coefficient, we set to $\phi_\pi = 1.5$ for the active case and $\phi_\pi = 0.5$ for the passive case.

# 3 Indeterminacy arising from passive monetary policy

This section presents the methodology to derive a nonlinear solution of an indeterminate model. As an application of the method, we first explore a case where the Taylor coefficient of the interest rate rule is smaller than one ($\phi_\pi = 0.5$) and therefore exhibits indeterminacy.

We first investigate the properties of the solution of the linear indeterminate model using the stylized three-equation model. Then, following the intuition obtained in the linear case, we present how to derive nonlinear solutions of an indeterminate model and apply the method to the case of passive monetary policy.

## 3.1 Decision rules of linear indeterminate models: the case of the minimal state variable (MSV)

Let us begin our analysis by first investigating the dynamics of the linear model. By log-linearizing the equilibrium conditions (3), (15), and (18) around the TSS, we can obtain the stylized three-equation model as follows:

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \sigma^{-1}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}), \tag{29}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \hat{y}_t, \tag{30}$$

$$\hat{R}_t = \phi_\pi \hat{\pi}_t + \varepsilon_{m,t}, \tag{31}$$

where $\kappa \equiv (\theta - 1)(\sigma + \eta)/\psi$. Variables with hats denote the log deviation from the TSS. Monetary policy shock ($\varepsilon_{R,t}$) is included as an exogenous disturbance.

When monetary policy is passive ($\phi_\pi < 1$) and the Taylor principle is not satisfied, the model exhibits indeterminacy. In this case, techniques proposed by Blanchard and Kahn (1980) are not applicable to derive the decision rules because the number of stable roots does not equal the number of eigenvalues outside the unit circle.[2]

[2]Functions that solve a set of equilibrium conditions and map state variables onto control variables are often called policy functions. In this study, we call such functions decision rules of the economic

Even if the Blanchard and Kahn (1980) conditions are not satisfied, we can obtain a particular solution based on the MSV criteria as discussed by McCallum (1999). As the model is linearized around the steady state and the monetary policy shock is the only exogenous shock, we can conjecture that the MSV decision rules are linear functions of the exogenous process in the following form:

$$\hat{y}_t = A_0 + A_1 \varepsilon_{m,t}, \tag{32}$$

$$\hat{\pi}_t = B_0 + B_1 \varepsilon_{m,t}. \tag{33}$$

Substituting the above conjecture into the equilibrium conditions, we can derive the MSV decision rules as follows:

$$\hat{y}_t = -(\sigma + \kappa\phi_\pi)^{-1} \varepsilon_{m,t}, \tag{34}$$

$$\hat{\pi}_t = -\kappa(\sigma + \kappa\phi_\pi)^{-1} \varepsilon_{m,t}. \tag{35}$$

The decision rules expressed by equations (34) and (35) are one particular solution to the equilibrium conditions given by (29)–(31).

Figure 1 shows the impulse responses of output, inflation, and interest rate to a monetary policy shock in the case of the MSV solution. Both output and inflation rate respond negatively to a positive monetary policy shock because the real interest rate increases in response to monetary tightening and the household decreases its consumption. All the variables respond simultaneously to the shock and we do not observe any persistence in the dynamics.

As we have derived the decision rules by the so-called "guess and verify" method, the MSV decision rules solve the equilibrium conditions as if the system were determinate. The intuition of the MSV decision rules is that although nonfundamental sunspot shocks can potentially affect prices and allocations, agents coordinate to respond only against fundamental shocks. This MSV solution is often adopted in existing studies on expectations-driven liquidity traps because the researcher can work with fewer variables, which simplifies the analysis. As we will see in the next subsection, however, a larger set of solutions arises when the system is indeterminate.

agents following Fernández-Villaverde et al. (2016).

## 3.2 Decision rules of linear indeterminate models: the case with sunspots

Let us derive the decision rules that allow sunspot shocks to affect prices and allocations. Existing studies such as Lubik and Schorfheide (2003) and Farmer et al. (2015) propose methods to derive a complete set of solutions of linear indeterminate models. Along with the above studies, the recent work by Bianchi and Nicolò (forthcoming) proposes a solution method that introduces an auxiliary variable $\hat{\omega}_t$ and converts an indeterminate system to a determinate system. For the case of the three-equation model, the auxiliary equation is introduced as follows:

$$\hat{\omega}_t = \frac{1}{\alpha}\hat{\omega}_{t-1} + \nu_t - \eta_t, \tag{36}$$

$$\text{where} \quad \hat{\pi}_t = \mathbb{E}_{t-1}\hat{\pi}_t + \eta_t, \tag{37}$$

where $\nu_t$ is a sunspot shock and $\eta_t$ is an expectational error. In this study, it is assumed that $\nu_t$ is white noise and is individually, identically, and normally distributed with mean zero and standard deviation of $\sigma_\nu$.[3]

As proposed by Bianchi and Nicolò (forthcoming), the model can be converted to a determinate system when the parameter satisfies $0 < \alpha < 1$. In this case, $\hat{\omega}_t = 0$ must hold for all $t$ for a unique solution to exist, as $\hat{\omega}_t$ follows an explosive path.

Current inflation is affected by inflation expectations in the previous period $\mathbb{E}_{t-1}\hat{\pi}_t$ as well as the sunspot shock $\nu_t$. The solution of the model can be derived as

$$\begin{pmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \mathbb{E}_t\hat{\pi}_{t+1} \end{pmatrix} = G\mathbb{E}_{t-1}\hat{\pi}_t + H \begin{bmatrix} \varepsilon_{m,t} \\ \nu_t \end{bmatrix}, \tag{38}$$

where $G$ and $H$ are matrices of parameters defined as

$$G = \begin{pmatrix} -\frac{a_2}{2\kappa} \\ 1 \\ \frac{a_1}{2\beta} \end{pmatrix}, \quad H = \begin{pmatrix} -\frac{2\beta\sigma^{-1}}{a_3} & \frac{2\kappa\sigma^{-1}(1-\beta\phi_\pi)-a_2}{a_3\kappa} \\ 0 & 1 \\ \frac{2\kappa\sigma^{-1}}{a_3} & -\frac{2(1+\kappa\phi_\pi\sigma^{-1})}{a_3} \end{pmatrix},$$

[3]Sunspot shocks are often allowed to be correlated with other fundamental shocks. Empirical results in Hirose (2020) show a significant positive correlation between sunspot shocks and investment adjustment costs and price markup shocks.

with $a_1 = (\beta - b_1 + \kappa\sigma^{-1} + 1)$, $a_2 = (a_1 - 2)$, $a_3 = (a_1 + 2b_1)$, $b_1 = [(1 + \beta + \kappa\sigma^{-1})^2 - 4\beta(1 + \kappa\phi_\pi\sigma^{-1})]^{-\frac{1}{2}}$, respectively.

The key feature of the decision rules provided by equation (38) is that when the original system described by equations (29)–(31) is indeterminate, an additional variable $\mathbb{E}_{t-1}\hat{\pi}_t$ enters as a state variable. The sunspot shock $\nu_t$ captures the temporary deviation in inflation expectations from the fundamentals, which leads to a multiplicity of equilibria.

Figure 2 displays the impulse responses of the variables to a positive monetary policy shock for the sunspot case.[4] Two major differences from the MSV case are worth noting. First, current inflation $\hat{\pi}_t$ does not respond to a monetary policy shock $\varepsilon_{R,t}$ on impact (T=1), which can be confirmed from the zero loading in the matrix $H$. As the current inflation rate $\hat{\pi}_t$ is predetermined by $\mathbb{E}_{t-1}\hat{\pi}_t$ in the previous period, the fundamental shock itself does not affect the inflation rate on impact.[5] Second, output decreases in response to a positive monetary policy shock, while inflation increases with a lag. This contrasts with the results in the MSV case, where the inflation rate declines in response to a positive monetary policy shock.

To investigate the role of productivity shocks, we can derive the decision rules by replacing equation (30) by the following equation:

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \kappa\hat{y}_t - \frac{(\theta-1)(1+\eta)}{\psi}a_t. \tag{39}$$

The impulse responses of the variables to a positive productivity shock are shown in Figure 3 (a). Both inflation rate and output increase in response to a rise in productivity. This is in sharp contrast to the standard determinate case, in which the Taylor principle is satisfied and inflation declines in response to a positive productivity shock. In the indeterminate case, inflation increases because the monetary policy does not respond sufficiently to the inflation rate and the real interest rate decreases in response to an increase in productivity. This induces the household to further increase and overshoot consumption. In addition, inflation is predetermined and does not respond on impact

[4]The impulse responses are computed using Dynare.

[5]Whether the current inflation rate responds to the fundamental shock depends on how the indeterminacy is modeled. We can model the indeterminacy by allowing current consumption to depend on the previous period's expectations as $\hat{c}_t = \mathbb{E}_{t-1}\hat{c}_t + \eta_t$ instead. Under certain conditions, we can show that both inflation and consumption indeterminacy yield identical results for the linear model. However, it is natural to assume that inflation expectations temporarily deviate from the central bank's target and fluctuate according to sunspot shocks.

but only with lags. Therefore, its response is hump-shaped and rises only gradually, reaching a peak after several periods.

Figure 3 (b) shows the impulse responses to a positive sunspot shock, which increases economic agents' inflation expectations exogenously. We can observe that all variables respond on impact, while the dynamics differ from the stylized determinate models: even though the shock itself is transitory, its impact is persistent. In addition, because the increase in the nominal interest rate is not sufficient to lower the real interest rate, both output and the inflation rate increase because of an increase in household consumption.

## 3.3 Decision rules of nonlinear indeterminate models

Let us now consider the nonlinear solutions. When the system is determinate, current prices and allocations can be pinned down uniquely by fundamental state variables including the exogenous processes as

$$y_t = f^y(X_t), \tag{40}$$

$$\Pi_t = f^\pi(X_t), \tag{41}$$

where $X_t$ denotes the vector of fundamental state variables.

When the system is indeterminate, however, $X_t$ is not sufficient to pin down current $y_t$ and $\Pi_t$ uniquely. For example, different prices and allocations may exist depending on nonfundamental variables in addition to the fundamental state variables:

$$y_t = f^y(Y_t, X_t), \tag{42}$$

$$\Pi_t = f^\pi(Y_t, X_t), \tag{43}$$

where $Y_t$ is a vector of nonfundamental variables.[6]

To derive nonlinear solutions of the indeterminate model, in addition to the nonlinear equilibrium conditions (20) and (21), we introduce an auxiliary variable $\omega_t$ and an auxiliary equation as follows:

$$\omega_t = \omega_{t-1}^{1/\alpha} \exp(\nu_t) \exp(-\eta_t), \tag{44}$$

$$\text{where} \quad \Pi_t = \left(\mathbb{E}_{t-1}\Pi_t\right) \exp(\eta_t). \tag{45}$$

[6] $X_t$ and $Y_t$ can include past realizations of each state variable.

When the parameter is chosen to satisfy $0 < \alpha < 1$, the system has a unique solution if and only if $\omega_t = 1$ holds for all $t$, which corresponds to the case of $\hat{\omega}_t = 0$ in the linear case. Otherwise, equation (44) follows either an explosive path or converges to zero, which leads to a violation of the transversality condition. Substituting $\omega_t = 1$ for all periods, the auxiliary equation can be rearranged as

$$\Pi_t = \left(\mathbb{E}_{t-1}\Pi_t\right)\exp(\nu_t). \tag{46}$$

Let us consider two exogenous processes, productivity $A_t$ and sunspot shock $\nu_t$ on the inflation expectations. Next-period inflation expectations can be considered to be an individual state variable, therefore we introduce a new auxiliary variable $\Phi_t$ to denote $\mathbb{E}_t\Pi_{t+1}$. Inflation expectations in the previous period are a predetermined variable. The auxiliary equation can be expressed as

$$\Pi_t = \Phi_{t-1}\exp(\nu_t). \tag{47}$$

When the model is indeterminate, the nonlinear decision rules that solve the equilibrium conditions (20), (21), and (47) can be expressed in a general form as follows:

$$\Pi_t = f^{\pi}(\Phi_{t-1}, \nu_t, A_t), \tag{48}$$

$$c_t = f^{c}(\Phi_{t-1}, \nu_t, A_t), \tag{49}$$

$$\Phi_t = f^{\Phi}(\Phi_{t-1}, \nu_t, A_t), \tag{50}$$

where $\Phi_{t-1}$ and $\nu_t$ are included in $Y_t$ and $A_t$ is included in $X_t$. The above decision rules are analogous to the linear rules summarized in equation (38). Note that the inflation expectations in the previous period $\Phi_{t-1}$ enter the decision rules as the nonfundamental predetermined variable.

When we approximate decision rules numerically, the choice of the variable is often crucial to obtain solutions efficiently. On applying the projection method, we consider an auxiliary variable $\mathcal{E}_t \equiv \beta\mathbb{E}_t\left[c_{t+1}^{-\sigma}/\Pi_{t+1}\right]$ instead of deriving the decision rule for consumption $c_t$. This is because consumption $c_t$ is known to exhibit kinks when the ZLB binds, making it difficult to approximate the decision rules. $\mathcal{E}_t$, on the other hand, is smooth because the kink is smoothed out by the expectations operator. Therefore, the decision rules we approximate in this study are (48), (50), and

$$\mathcal{E}_t = f^{\mathcal{E}}(\Phi_{t-1}, \nu_t, A_t). \tag{51}$$

We derive the decision rules numerically by applying the projection method. In this study, we choose Chebychev polynomials as the basis function and use Smolyak sparse grids.[7] The details of the methodology to apply Smolyak sparse girds are provided in Judd et al. (2014).

When we use Chebychev polynomials as the basis function, we must choose the domain of the approximation because the variables must be standardized within the range of $[-1, 1]$. In this study, we choose the range to cover three standard deviations of the stationary distribution of each exogenous variable. In a standard model with active monetary policy, three standard deviations are large enough to include an occasion where the ZLB binds. However, because we assume the monetary policy to be passive, such an occasion does not occur within the range of three standard deviations; for example, when the Taylor coefficient is set to $\phi_\pi = 0.5$, the inflation rate must decline three times as much as in the case of $\phi_\pi = 1.5$ for the ZLB to bind. Therefore, although the solution is derived from the nonlinear equilibrium conditions with the ZLB, it actually never binds. As we will confirm later, this leads to the similar results between the linear and nonlinear solutions.

Let us provide a brief sketch of the solution algorithm. The numerically approximated decision rules can be expressed as

$$\Pi_t = \hat{f}^{\pi}(\Phi_{t-1}, \nu_t, A_t|\theta^{\pi}), \tag{52}$$

$$\mathcal{E}_t = \hat{f}^{\mathcal{E}}(\Phi_{t-1}, \nu_t, A_t|\theta^{\mathcal{E}}), \tag{53}$$

$$\Phi_t = \hat{f}^{\Phi}(\Phi_{t-1}, \nu_t, A_t|\theta^{\Phi}), \tag{54}$$

where the hat shows that the functions are approximations. $\theta$ denotes the coefficients

[7]In this paper, we choose the degree of approximation using Smolyak sparse grids of $\mu = 2$.

of the basis functions. Let us define the approximation residuals as

$$\mathcal{R}_t^{\mathcal{E}} \equiv \mathcal{E}_t - \beta \mathbb{E}_t \Big[ \frac{c_{t+1}^{-\sigma}}{\Pi_{t+1}} \Big], \tag{55}$$

$$\mathcal{R}_t^{\pi} \equiv \Big[ \psi(\Pi_t - \Pi^*)\Pi_t - \theta w_t + \theta - 1 \Big] - \beta \mathbb{E}_t \Big[ \Big( \frac{c_t}{c_{t+1}} \Big)^{\sigma} \frac{y_{t+1}}{y_t} \psi(\Pi_{t+1} - \Pi^*)\Pi_{t+1} \Big], \tag{56}$$

$$\mathcal{R}_t^{\Phi} \equiv \Phi_t - \mathbb{E}_t \Pi_{t+1}. \tag{57}$$

Using a Newton–Raphson-type of optimization algorithm, parameters $\hat{\theta}^{\pi}$, $\hat{\theta}^{\mathcal{E}}$, and $\hat{\theta}^{\Phi}$ solve an optimization problem that sets the residuals to $\mathcal{R}_t^{\mathcal{E}} = 0$, $\mathcal{R}_t^{\pi} = 0$, and $\mathcal{R}_t^{\Phi} = 0$. To calculate the expectations, exogenous shocks are discretized using Gauss–Hermite approximation. Further details of the solution algorithm can be found in Fernández-Villaverde et al. (2016).

## 3.4 Comparison between linear and nonlinear decision rules

Once we obtain the decision rules numerically, we can investigate the dynamics of the model. Figure 4 shows the impulse response of the variables to productivity shock and sunspot shock, respectively. The impulse responses show similar dynamics to the linear case: positive productivity and sunspot shocks increase both inflation rate and output, while the shape of the responses differs between the two shocks.

It is known that there are several differences between linear and nonlinear decision rules. For example, the linear solution is derived around the deterministic steady state, thus it cannot capture the effects arising from uncertainty, while such effects are included in nonlinear solutions. Another difference is that the quadratic price adjustment cost of Rotemberg (1982) is always zero in the linear model, while it becomes positive in the nonlinear model.

These effects are generally relatively small in the neighborhood of the steady state, and linear approximation is known to perform effectively. These points can be confirmed by comparing Figure 3 and Figure 4: the dynamics of the impulse responses for linear and nonlinear models exhibit very similar results. Thus the reason why there is little difference between the two methods is because the model is almost linear when the ZLB is not binding. Therefore, for the case of indeterminacy arising from passive monetary policy, the practical gain from applying nonlinear methods is limited.

Note that if we consider a shock large enough to force the central bank to lower its interest rate down to the ZLB, linear and nonlinear solutions can differ significantly.

However, as we discussed earlier, such circumstances are extremely rare events when the Taylor principle is not satisfied. We do not consider such extreme cases in this paper because liquidity traps are relatively infrequent events even in the case of active monetary policy.[8]

# 4 Indeterminacy arising in the expectations-driven liquidity trap

In the previous section, we investigated the characteristics of indeterminate models in which the Taylor coefficient is set lower than one. In this section, we assume that the Taylor principle is satisfied and explore the model dynamics when the economy is trapped in the expectations-driven liquidity trap.

## 4.1 Indeterminacy described in Benhabib et al. (2001)

While the central focus of this study is on local indeterminacy, the existing literature has investigated two different types of indeterminacy: local and global indeterminacy. The term local indeterminacy is associated with the existence of multiple equilibrium paths from *different initial conditions* converging toward a single steady state or a stationary balanced growth path. The term global indeterminacy, however, concerns the existence of multiple equilibrium paths from a given initial condition converging toward *different steady states or convergence paths*.[9]

The main finding of Benhabib et al. (2001) is that a wide class of models with nominal prices exhibits global indeterminacy when the nominal interest rate is determined by the Taylor rule and bounded below by the ZLB. They show that in addition to the TSS, the equilibrium path may converge either to the USS or to a limit cycle around the TSS. It is further discussed that whether the model exhibits a limit cycle depends on the parameterization, and their paper mainly focuses on the case of equilibrium paths converging to the USS. In this study, we choose the USS and solve the model nonlinearly, allowing sunspot disturbances to affect prices and allocations.

[8] Fernández-Villaverde et al. (2015) find that the economy is at the ZLB during 5.53 percent of quarters with similar calibration to this study.

[9] For detailed discussions on local and global indeterminacy, see Brito and Venditti (2010) and Antoci et al. (2011), for example.

## 4.2 Nonlinear decision rules

As we discussed in Section 2, there are two steady states that solve the household's and firm's optimization problem: the TSS and the USS. To characterize the decision rules uniquely, one must choose which steady state is reached when all stochastic elements are shut down and set equal to zero.

We introduce an additional state variable $s_t$ and define that if $s_t = T$, the economy is in the "targeted regime," in which the economy converges to the TSS, while if $s_t = U$, the economy is in the "unintended regime," in which the economy converges to the USS. In this study, we assume that $s_t$ is fixed to either $T$ or $U$ for all periods and does not change over time.[10]

The choice of the regime is often attributed to agents' expectations on the state of the economy in the long run. If agents form an optimistic view on the future economy, inflation converges to the central bank's target. However, if agents form a pessimistic view, the central bank fails to achieve its goal, and the inflation rate converges to a negative value.

The model characterized by equilibrium conditions (20) and (21) is determinate around the TSS, thus the policy functions can be expressed as

$$y_t = f^y(X_t | s_t = T), \tag{58}$$

$$\Pi_t = f^\pi(X_t | s_t = T). \tag{59}$$

However, the model is indeterminate around the USS, therefore one natural candidate of the decision rules is that output and inflation rate are affected by inflation expectations and sunspot shocks as follows:

$$y_t = f^y(Y_t, X_t | s_t = U), \tag{60}$$

$$\Pi_t = f^\pi(Y_t, X_t | s_t = U). \tag{61}$$

Note that the MSV decision rules can be derived for the nonlinear case as well. We assume that the economic agents respond only to fundamental elements and do not respond to nonfundamental elements. In this case, the decision rules can be expressed

[10] Several recent studies that analyze expectations-driven liquidity traps assume Markov regime-switching between the two regimes. In many cases, the targeted regime is assumed to be absorbing to obtain closed-form solutions.

in the following form:

$$y_t = f^y(X_t | s_t = U), \tag{62}$$

$$\Pi_t = f^\pi(X_t | s_t = U). \tag{63}$$

For example, Aruoba et al. (2018) select a particular solution that depends only on fundamental elements and derive decision rules in the expectations-driven liquidity trap assuming the above functional form.

Note that when we derive decision rules of an indeterminate model, the solution may not be unique. That is, there can be multiple pairs of solutions that take the functional form of (60) and (61). Therefore, the solution presented in the following subsections should be viewed as a particular solution of the indeterminate model rather than a unique solution.

## 4.3 Dynamics of the stochastic model

Let us consider a stochastic model where the inflation expectations fluctuate according to sunspot shocks. We focus solely on the sunspot shocks and abstract from the rest of the fundamental shocks for computational simplicity. The decision rules are derived numerically by the projection method.

Decision rules for consumption and inflation rate are shown in Figure 5. Both rules are computed by taking different values for $\Phi_{t-1}$, while keeping other variables fixed at their steady state values. We can confirm that higher inflation expectations generate higher realized inflation and consumption, which is similar to the results in the case of passive monetary policy. Consumption, however, starts to decline as inflation expectations exceed a certain threshold. This is because of the household's endogenous behavior; once the ZLB ceases to bind, increasing consumption and creating inflationary pressure induce the central bank to raise the interest rate, which leads to an increase in the real rate. Under such circumstances, it is suboptimal for the household to further increase consumption as there are no changes in the fundamentals such as productivity. Therefore, it becomes optimal for the household to refrain from increasing consumption once the monetary policy becomes active.

By combining the diagrams of Figure 5 and substituting out $\Phi_{t-1}$, we can depict the convergence path for $c_t$ and $\Pi_t$ corresponding to different realizations of $\Phi_{t-1}$. Figure 6 depicts the convergence path, which shows a strong nonlinearity in the area where the

ZLB does not bind.

Figure 7 shows the impulse responses of variables to a two-standard-deviation sunspot shock. All variables react positively on impact, and the nominal interest rate escapes from the ZLB as the inflation rate increases; the nominal interest rate is positive for three periods with the rise in the inflation rate. Even though the sunspot shock $\nu_t$ is white noise and transitory, the dynamics of the variables are persistent.

## 4.4 Dynamics of the deterministic model

To evaluate our results of the stochastic model and confirm that our results are not an artifact arising from computational methods, it is worth investigating the dynamics of the deterministic case in the expectations-driven liquidity trap.

Figure 8 shows the equilibrium path converging to the USS in the deterministic setup. A small perturbation from the TSS, shown by "$\times$," leads to a de-anchoring of inflation expectations and converges to the USS, depicted by "+." This convergence path is similar to the path in the stochastic case shown in Figure 6. The area in grey shows the region where the ZLB binds, and we can observe that the equilibrium path starts to kink once it escapes from the area and the ZLB ceases to bind.[11] The inflation rate continues to increase, while consumption gradually starts to decline in this area. The mechanism by which such a curve emerges is similar to the stochastic case; because increasing consumption creates further inflationary pressure and induces the central bank to increase the interest rate, the household refrains endogenously from increasing consumption.

While the comparison between the deterministic and stochastic models shows that our methodology provides persuasive decision rules, some limitations are worth noting. As we fix our solution space to a certain domain when we approximate decision rules using Chebychev polynomials, solutions may not be accurate when the economy is far away from the USS. Especially in the expectations-driven liquidity trap, there exist multiple prices and allocations corresponding to a certain inflation rate $\Pi_t$. For example, in Figure 8, there are more than two equilibrium prices and allocations that satisfy the equilibrium conditions with $\Pi_t = 1$. Therefore, not only the expectations on inflation but also expectations on consumption, for example, are further needed to

[11] As shown by Benhabib et al. (2001), the deterministic model is indeed globally indeterminate; because there are only jump variables in the model, the economy can jump to the TSS or on the trajectory converging to the USS regardless of the past realization of the variables.

pin down current consumption and inflation. As such, the decision rules derived by the projection method with a particular basis function should be regarded as a nonlinear approximation that holds in a relatively limited area around the steady state.

Another limitation is that the solution we investigated in this study is one particular form of solution that incorporates sunspot shocks. As the model is nonlinear, the solution may not be unique and alternative solutions may exist. However, a nonlinear solution that allows inflation expectations to deviate from the fundamentals is intuitive; thus, it is regarded as a natural candidate of the solution of indeterminate models.

### 4.5 The role of nonlinearity in indeterminate models

We have confirmed that the nonlinear solution plays a key role in capturing the characteristic dynamics around the expectations-driven liquidity trap. However, nonlinearity itself is often regarded as of second-order importance depending on the focus of the study.

Atkinson et al. (2020), for example, show that there is a large practical advantage in using a piecewise linear solution compared with a full nonlinear solution.[12] Such results reflect the fact that the major nonlinearity arises from a kink created by the occasionally binding nature of the ZLB.

However, the nonlinearity arising in the expectations-driven liquidity trap is not a simple kink; the nonlinearity appears in a smooth and continuous manner, which can be seen from the curvature in Figure 6. In such a case, the piecewise linear solution cannot appropriately approximate the decision rules. This fact strongly encourages the use of nonlinear methods to derive solutions of indeterminate models, especially in the expectations-driven liquidity traps.

## 5 Conclusion

In this study, we proposed a novel methodology to derive nonlinear solutions of an indeterminate model. We first applied the method to the case of passive monetary policy and found that linear and nonlinear decision rules exhibit similar dynamics, indicating that the practical gains from applying the nonlinear method is limited in

[12] The authors apply the software package OccBin to implement the piecewise linear solution. Details of OccBin are provided by Guerrieri and Iacoviello (2015).

the most basic setup. We then applied the method to the case of an expectations-driven liquidity trap and found that nonlinearity plays a significant role in the model dynamics. These findings suggest the importance of considering both indeterminacy and nonlinearity when investigating the dynamics in a liquidity trap.

An important question that remains unanswered in this study is whether other solutions of the indeterminate model exist. As the solution presented in this study is one particular form that incorporates sunspot shocks with de-anchored inflation expectations, other forms of solutions may exist. Therefore, investigation of a more general set of solutions of nonlinear indeterminate models remains a challenging yet important direction for future work.

This paper focused mainly on the technical aspect of the indeterminacy in DSGE models by presenting a solution in the expectations-driven liquidity trap. However, whether the de-anchoring of inflation expectations is likely to be true in many advanced economies—and if so, how much it has affected the real economic outcomes—remains an important empirical question. Further investigation is left for future work to address such questions.

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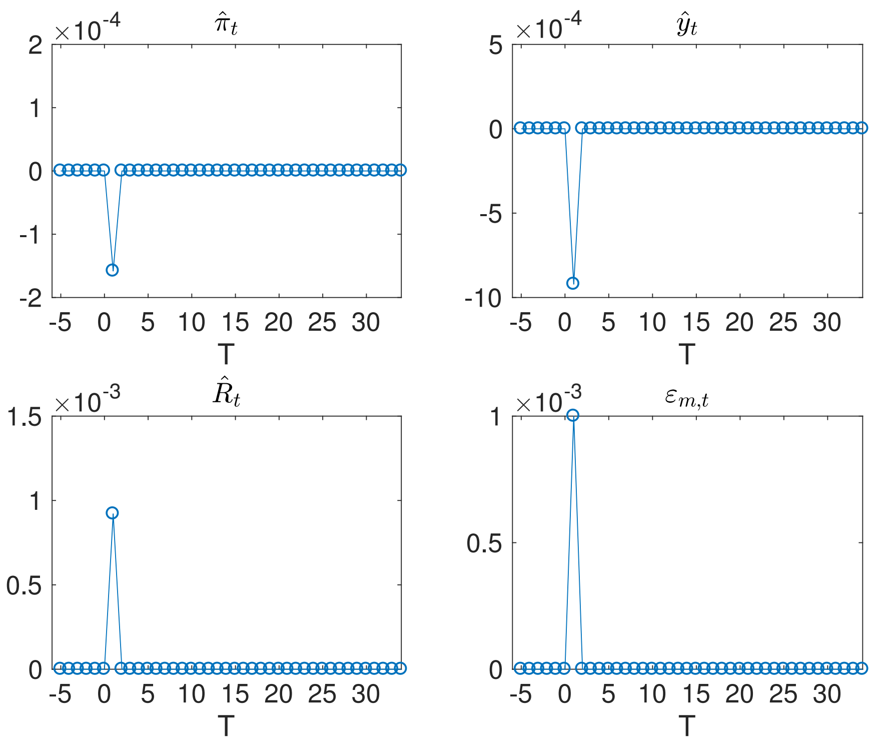



Figure 1: Impulse responses to a monetary policy shock (linear MSV case).

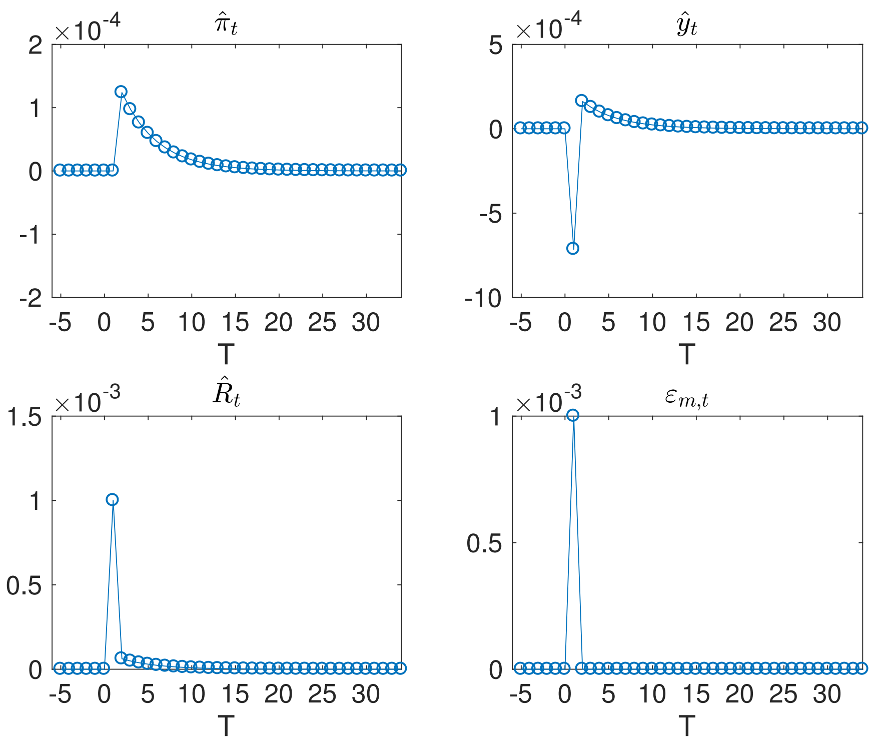



Figure 2: Impulse responses to a monetary policy shock (linear sunspot case).

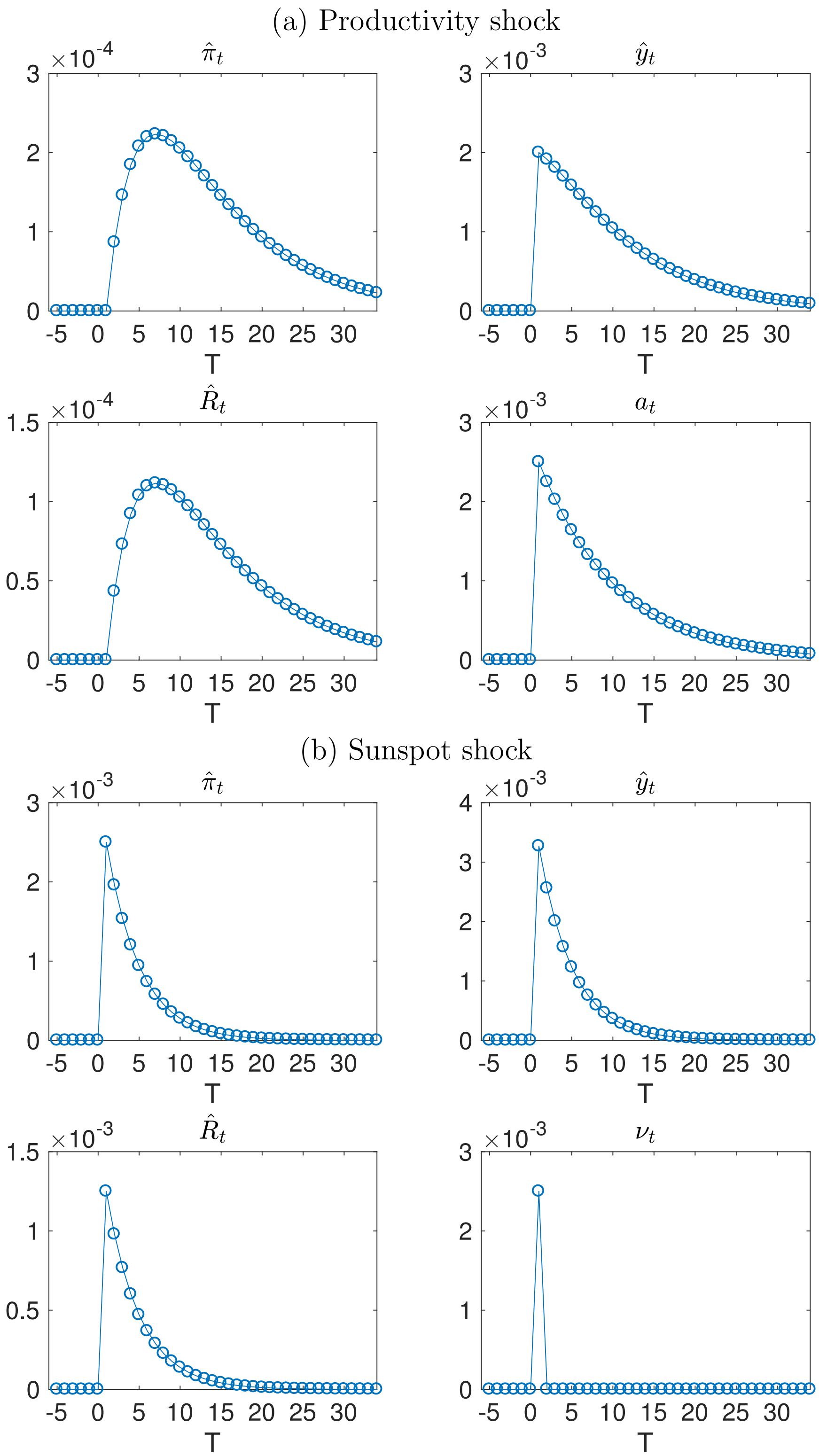



Figure 3: Impulse responses to different shocks (linear sunspot case).

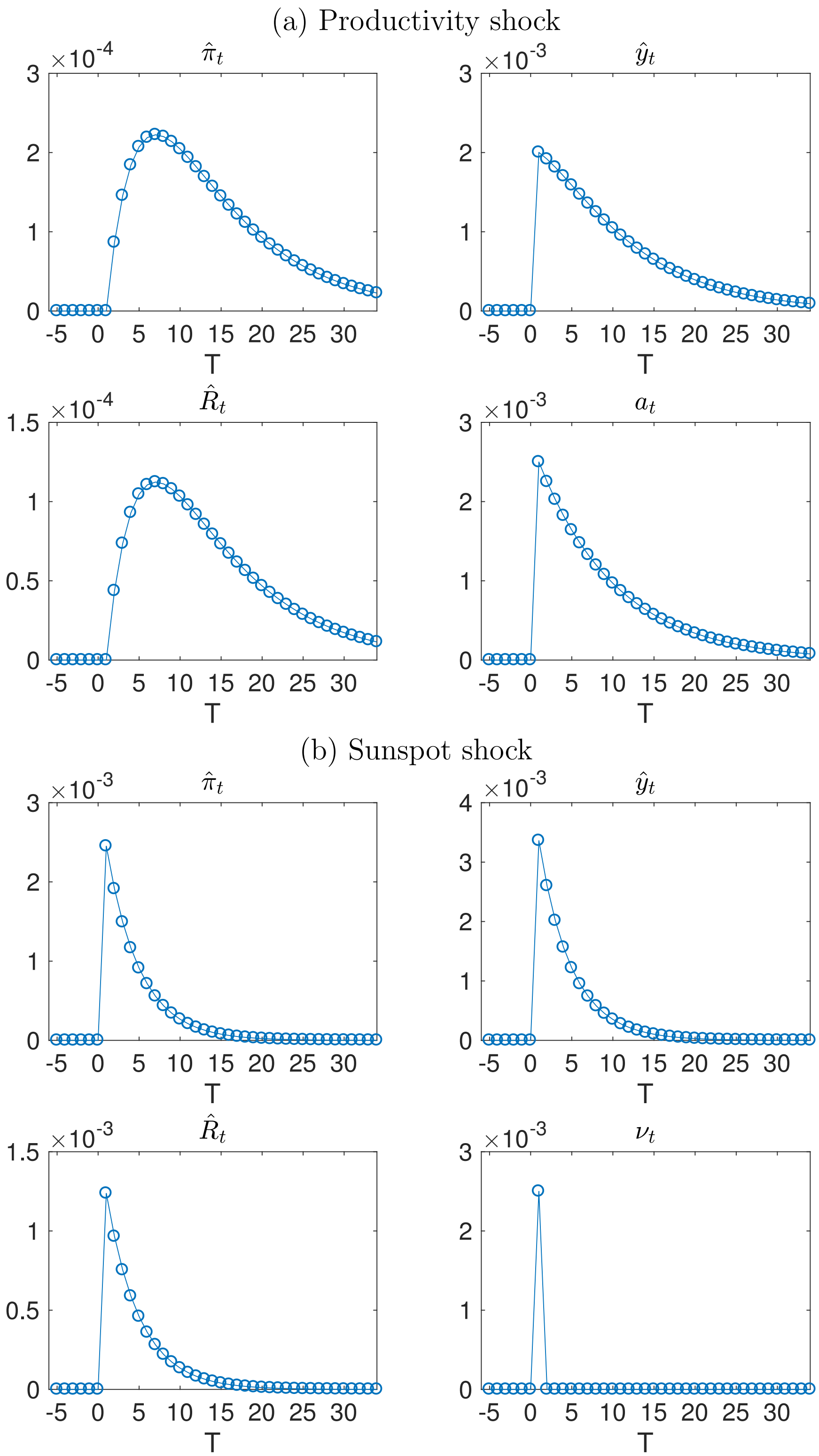
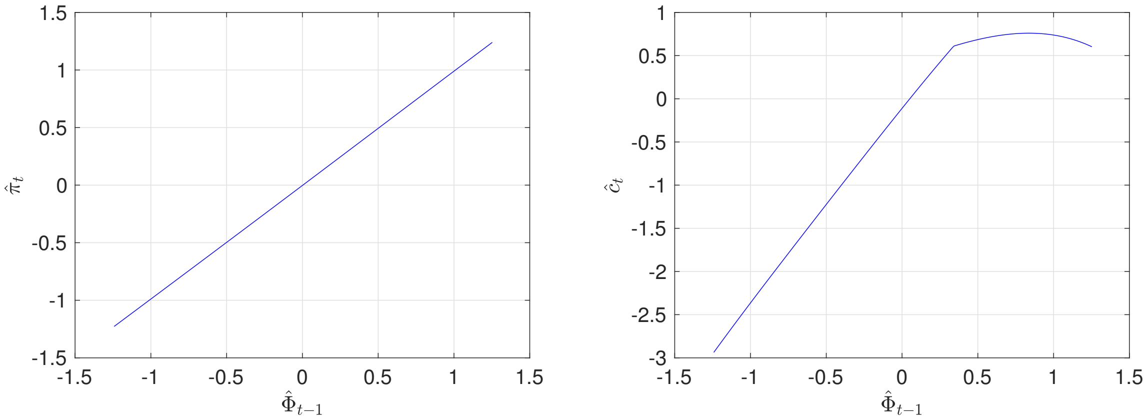



Figure 4: Impulse responses to different shocks (nonlinear sunspot case).




(a) Inflation (b) Consumption

Figure 5: Decision rules for inflation and consumption.

Note: Variables expressed as percentage deviations from the deterministic USS.

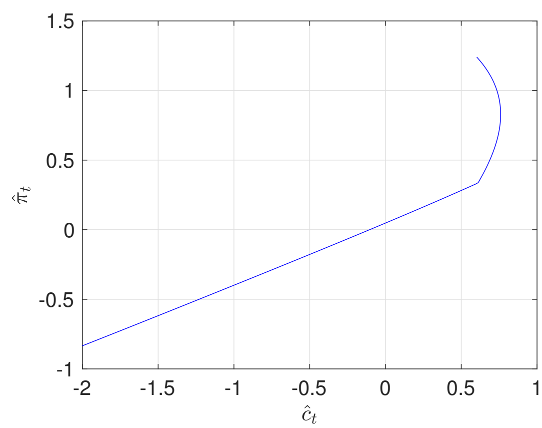



Figure 6: Convergence path corresponding to different $\Phi_{t-1}$.

Note: Variables expressed as percentage deviations from the deterministic USS.

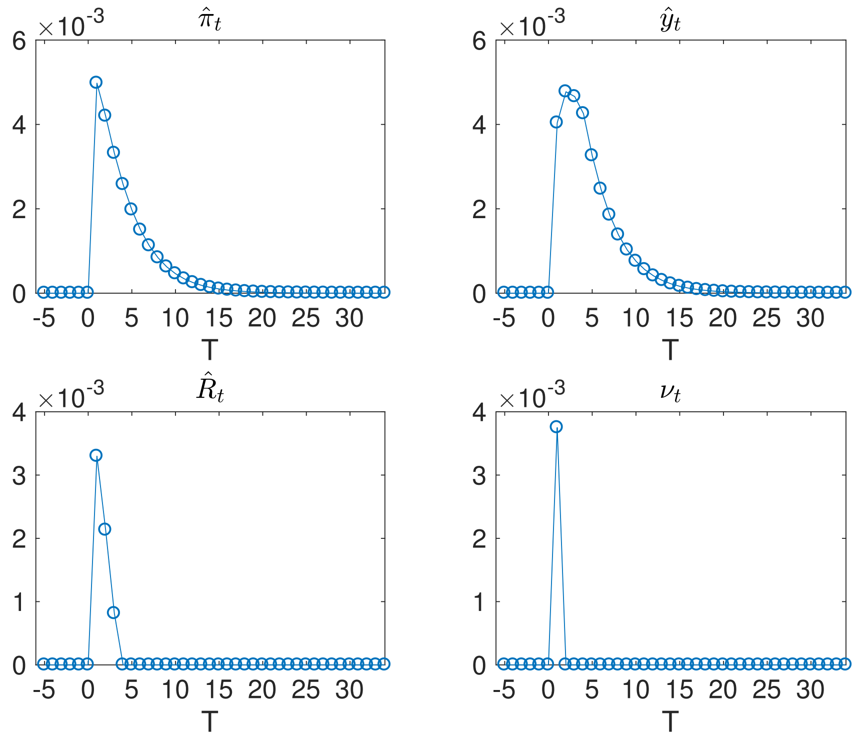



Figure 7: Impulse responses to a large sunspot shock.

Note: Variables expressed as log deviations from the stochastic USS.

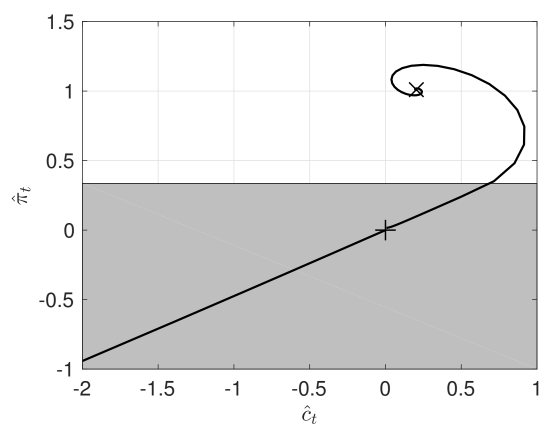



Figure 8: Equilibrium path converging to the deterministic USS.

Note: Variables with hats are measured as deviations from the deterministic USS. "×" and "+" denote the TSS and the USS, respectively.