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Abstract

We offer a general derivation of axiomatizations for allocation rules, referred to as "duality" and "anti-duality" approach. We show basic properties of duality and anti-duality approach. Using these properties, we can derive axiomatizations of allocation rules by taking (anti-)dual of axioms involved in axiomatizations of their self-(anti-)dual rules. As an illustration, we derive a new axiomatization of the Shapley value for bidding ring problems from using the notion of duality and axioms involved in axiomatizations of the Shapley value for airport problems. As another illustration, we derive a new axiomatization of the nucleolus for bidding ring problems from using the notion of anti-duality and axioms involved in axiomatizations of the nucleolus for airport problems.

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1 Introduction

"Claims problems" are well-known allocation problems in economics. They deal with a situation where the liquidation value of a bankrupt firm has to be allocated between its creditors, but there is not enough to honor the claims of all creditors. The problem is to determine how the creditors should share the liquidation value (O'Neill 1982; Thomson 2003, for a survey of the literature). In claims problems, Thomson and Yeh (2008) introduced operators on the space of division rules and uncover the underlying structure of the space of division rules. The notion of "duality" plays an important role in their analysis. For each claims problem, this notion gives us its dual problem. Intuitively speaking, the dual of a claims problem is to determine how the creditors should abandon some part of their claims. Also, the notion of duality is applied to division rules: Given a division rule for claims problems, its dual rule is the same division rule for their dual problems. A division rule is said to be "self-dual" if the outcome chosen by this division rule and that chosen by its dual rule always coincide with each other.

Analogously to claims problems, one can define "dual solutions" and "self-dual solutions" in cooperative game theory. Oishi et al. (2016) proposed a general approach for axiomatization of solutions for TU games, referred to as "duality" and "anti-duality" approach. The dual of a TU game is well-known in the literature.[1] The anti-dual of a TU game (Oishi and Nakayama 2009) is obtained by multiplying its dual by -1. Using these definitions, given a solution, its "dual" and "anti-dual" are defined. Oishi et al. (2016) applied the notions of (anti-)dual solutions to axioms: Two axioms are (anti-)dual to each other if whenever a solution satisfies one of them, its (anti-)dual satisfies the other. The duality and anti-duality approach allows us to relate some existing axiomatizations of solutions for TU games, and find new ones. However, it is an open question to uncover how the duality and anti-duality approach is applicable to axiomatic analysis of allocation rules for economic problems. The present study provides an answer to this question. In this answer, we offer a general derivation of axiomatizations for allocation rules. This new approach may be useful for axiomatic views in social choice theory and mechanism design theory.

We develop the duality and anti-duality approach toward axiomatic analysis of allocation rules for economic problems. Given a rule on a domain of allocation problems, its (anti-)dual can be defined. A rule is self-(anti-) dual if it is own (anti-)dual. Given an axiom, its (anti-)dual can be defined: Two axioms are (anti-)dual to each other if whenever a rule satisfies one of them, its (anti-)dual satisfies the other. Using these notions, we can derive axiomati-

[1] The notion of "dual games" is well known in the literature on cooperative games. The definition of dual games and their interpretation are stated in Section 2.

zations of allocation rules for economic problems from those of their self-(anti) dual rules. Thus, an axiomatization of allocation rules for some problems, which has not been analyzed in the existing literature, is possible automatically.

First, we show basic properties of (anti-)dual axioms and of (anti-)dual axiomatizations of allocation rules. That is, we verify that an axiom for the (anti-)dual of a rule can be derived from taking the (anti-)dual of an axiom for the original rule. We also verify that the (anti-)dual of a rule can be axiomatized by taking the (anti-)dual of the axioms involved in an axiomatization of the original rule.

Next, we apply the notion of duality and anti-duality approach to "airport problems" and "bidding ring problems". Airport problems are cost sharing problems of an airstrip among airlines (Littlechild and Owen 1973; Thomson 2007, for a survey of the literature). A bidding ring problem (Graham et al. 1990) describes a situation where bidders form a ring in a single-object English auction. The ring reduces or eliminates buyer competition, thereby securing an advantage over the seller. The problem forced by the members of the ring is to share the benefit of their strategy.

The "Shapley rule" is a mapping on some domain of allocation problems that associates with each problem in the domain the "Shapley value" of the corresponding TU game. The Shapley value (Shapley 1953) is the most important single-valued solution of TU games with economic applications. For instance, Chun et al. (2012) investigated several axiomatizations of the Shapley rule for airport problems. Applying duality approach to these axioms involved in axiomatizations of the Shapley rule for airport problems, we present a new axiomatization of the Shapley rule for bidding ring problems. In the axiomatization, we obtain a new axiom, referred to as "first-agent transfer agreement equivalence". This property requires that the outcome chosen by a rule should be invariant even if "transfer agreement" between a buyer with the smallest valuation and any bidding ring in the other buyers is made. Transfer agreement means that a buyer with the smallest valuation and any bidding ring in the other buyers agree upon that some part of his profits is transferred from him to the bidding ring. The first-agent transfer agreement equivalence is dual to first-agent airport consistency involved in an axiomatization of the Shapley rule for airport problems appearing in Chun et al. (2012). The other axioms of the Shapley rule for bidding ring problems (i.e. reasonableness, equal share lower bound, and individual monotonicity) are self-duals to those for airport problems appearing in Chun et al. (2012), respectively.

The "nucleolus rule" is a mapping on some domain of allocation problems that associates with each problem in the domain the "nucleolus" of the corresponding TU game. The nucleolus (Schmeidler 1969) is another important single-valued solution of TU games with economic applications. For instance, Hwang and Yeh (2012) and Yeh (2004) axiomatized the nucleolus rule for airport problems. Applying anti-duality approach to these axioms involved in axiomatizations of the nucleolus rule for airport problems, we present a new axiomatization of the nucleolus for bidding ring problems. In the axiomatization, we obtain a new axiom, referred to as "last-agent secret agreement equivalence". This property requires that the outcome chosen by a rule should be invariant even if "secret agreement" between a buyer with the largest valuation and any bidding ring in the other buyers is made. Secret agreement means that a buyer with the largest valuation and any bidding ring in the other buyers agree upon that he cooperates with the bidding ring and obtains his reward that is the outcome chosen by a rule. The last-agent secret agreement equivalence is anti-dual to last-agent airport consistency involved in an axiomatization of the nucleolus rule for airport problems appearing in Yeh (2004). The other axioms of the nucleolus rule for bidding ring problems (i.e. equal treatment of equals, and last-agent additivity) are self-anti-duals to those for airport problems appearing in Yeh (2004), respectively.

The rest of this paper is organized as follows. In Section 2, we explain the notions of duality and anti-duality in cooperative game theory. In Section 3, we introduce the notions of duality and anti-duality for allocation rules, and show basic properties of these notions. In Section 4, using the duality approach, we axiomatize the Shapley rule for bidding ring problems. Also, using the anti-duality approach, we axiomatize the nucleolus rule for bidding ring problems. In Section 5, we discuss a generalization of duality and anti-duality approach.

2 Preliminaries

There is a universe of potential agents, denoted $\mathcal{I} \subseteq \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.[2] Let $\mathcal{N}$ be the class of non-empty and finite subsets of $\mathcal{I}$, and $N \in \mathcal{N}$. A **coalitional game with transferable utility for N** (a **TU game for N**, for short) is a function $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. A set $S \in 2^N$ is called a **coalition**. For all $S \in 2^N$, $v(S)$ represents what coalition S can achieve on its own. Let $\boldsymbol{\mathcal{V}^N}$ be the **class of TU games for N**, and $\mathcal{V} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{V}^N$.

Let $\mathbb{R}^N$ denote the Cartesian product of $|N|$ copies of $\mathbb{R}$, indexed by the members of N. A **payoff vector** for N is an element x of $\mathbb{R}^N$. For all $x \in \mathbb{R}^N$ and all $S \in 2^N$, let $x_S = (x_i)_{i\in S}$.

A **solution**, denoted $\boldsymbol{\varphi}$, is a mapping defined on some domain of games that associates with each game in the domain a non-empty set of payoff vectors. A solution is **single-valued** if it associates with each game in its domain a unique payoff vector.

[2] We use $\subseteq$ for weak set inclusion, and $\subset$ for strict set inclusion.

Given a game v for N, the **dual of $\boldsymbol{v}$**, denoted $\boldsymbol{v^d}$, is defined by setting, for all $S \subseteq N$,

$$v^d(S) \equiv v(N) - v(N \backslash S).$$

The number $v^d(S)$ is the amount that the complementary coalition $N \backslash S$ cannot prevent S from obtaining.

Let $\mathcal{V}$ be a class of games such that if $v \in \mathcal{V}$, then $v^d \in \mathcal{V}$. Given a solution φ on $\mathcal{V}$, the **dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^d}$, is defined by setting, for all $v \in \mathcal{V}$,

$$\varphi^d(v) \equiv \varphi(v^d).$$

A solution φ on $\mathcal{V}$ is **self-dual** if for all $v \in \mathcal{V}$, $\varphi(v) = \varphi^d(v)$.

An **axiom** of a solution is a property that should be satisfied by the solution. **Two axioms are dual to each other** if whenever a solution satisfies one of them, the dual of this solution satisfies the other. That is, two axioms A and A' are dual to each other if for all solutions that satisfy A, it holds that their duals satisfy A', and conversely, for all solutions that satisfy A', it holds that their duals satisfy A. **An axiom is self-dual** if it is its own dual.

Given a game v for N, the **anti-dual of $\boldsymbol{v}$**, denoted $\boldsymbol{v^{ad}}$, is defined by setting, for all $S \subseteq N$,

$$v^{ad}(S) \equiv -v^d(S).$$

Let $\mathcal{V}$ be a class of games such that if $v \in \mathcal{V}$, then $v^{ad} \in \mathcal{V}$. Given a solution φ on $\mathcal{V}$, the **anti-dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^{ad}}$, is defined by setting, for all $v \in \mathcal{V}$,

$$\varphi^{ad}(v) \equiv -\varphi(v^{ad}).$$

A solution φ on $\mathcal{V}$ is **self-anti-dual** if for all $v \in \mathcal{V}$, $\varphi(v) = \varphi^{ad}(v)$. **Two axioms are anti-dual to each other** if whenever a solution satisfies one of them, the anti-dual of this solution satisfies the other. That is, two axioms A and A' are anti-dual to each other if for all solutions that satisfy A, it holds that their anti-duals satisfy A', and conversely, for all solutions that satisfy A', it holds that their anti-duals satisfy A. **An axiom is self-anti-dual** if it is its own anti-dual.

Finally, we introduce well-known solutions for coalitional games. The **Shapley value** (Shapley 1953) is defined as the following single-valued solution: for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $i \in N$,

$$Sh_i(v) \equiv \sum_{\substack{S \subseteq N \\ S \not\ni i}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \big[v(S \cup \{i\}) - v(S)\big].$$

Given $N \in \mathcal{N}$ and $v \in \mathcal{V}^N$, let $I(v)$ be the set of vectors $x \in \mathbb{R}^N$ such that for all $i \in N$, $x_i \geq v(\{i\})$, and $\sum_N x_i = v(N)$. Let $\mathcal{V}^N$ be a class of games such that for all $v \in \mathcal{V}^N$, $I(v) \neq \emptyset$. For all $x \in I(v)$, let $e(v,x) \in \mathbb{R}^{2^N}$ be defined by setting, for all $S \subseteq N$, $e_S(v,x) \equiv v(S) - \sum_S x_i$. For all $z \in \mathbb{R}^{2^N}$, $\theta(z) \in \mathbb{R}^{2^N}$ is defined by rearranging the coordinates of z in non-increasing order. For all $z \in \mathbb{R}^{2^N}$, $\boldsymbol{z}$ **is lexicographically smaller than** $\boldsymbol{z'}$ if $\theta_1(z) < \theta_1(z')$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) < \theta_2(z')]$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) = \theta_2(z')$ and $\theta_3(z) < \theta_3(z')]$, and so on. The **nucleolus** (Schmeidler 1969) is defined as follows: for all $N \in \mathcal{N}$, and all $v \in \mathcal{V}^N$,

$$Nu(v) \equiv \left\{ x \in I(v) \;\middle|\; \begin{array}{c} \text{For all } y \in I(v) \backslash \{x\},\ e(v,x) \text{ is} \\ \text{lexicographically smaller than } e(v,y) \end{array} \right\}.$$

The nucleolus is a *single-valued* solution.

3 Duality and anti-duality for allocation rules, and their basic properties

In this section, we introduce the notions of duality and anti-duality for allocation rules. We also show properties of (anti-)dual axioms, and (anti-)dual axiomatizations of allocation rules.

3.1 Duality and anti-duality for allocation rules

An **allocation problem for** $\boldsymbol{N}$ is a pair (N, p), where $N \in \mathcal{N}$ is a finite non-empty set of agents and $p = (p_i)_{i \in N}$ is a profile of parameters for N. For each $i \in N$, the parameter p_i is the benefit or the cost experienced by agent $i \in N$ when engaging in some economic activity. Let $\mathcal{P}$ be the set of all allocation problems on $\mathcal{N}$.

Given all $S \in 2^N$, we denote by $v_P : \mathcal{P} \to \mathbb{R}^{2^N}$ a mapping that associates with each allocation problem (N, p) in the domain the unique $2^{|N|}$-dimensional vector whose S-component is the amount coalition S can obtain on its own. By convention, $v_P(N,p)(\emptyset) = 0$. The mapping v_P is the **coalitional game for** $\boldsymbol{N}$ derived from the allocation problem (N, p).

Let $\boldsymbol{\mathcal{V}_P}$ be the class of all coalitional games derived from allocation problems $\mathcal{P}$. Given $(N,p) \in \mathcal{P}$, an **allocation** for (N,p) is a vector $x \in \mathbb{R}^N$ such that $\sum_N x_i = v_P(N,p)(N)$. Let $\mathbf{X}\,(\boldsymbol{N}, \boldsymbol{p})$ be the set of allocations for (N,p). A **solution for coalitional games** is a mapping $\phi : \mathcal{V}_P \to \mathbb{R}^N$ that associates with each coalitional game $v_P(N,p)$ on the domain a non-empty set of allocations in $X(N,p)$. We refer to the composite mapping $\varphi \equiv \phi \circ v_P$ as an **allocation rule,** or simply **a rule,** for allocation problems on the domain of $\mathcal{P}$. For instance, we refer to the composite mapping $\varphi \equiv Sh \circ v_P$ as the **Shapley rule**, and to the composite mapping $\varphi \equiv Nu \circ v_P$ as the **nucleolus rule**.

Given a rule φ on $\mathcal{P}$, **the dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^d}$, is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$\varphi^d(N,p) \equiv \phi[(v_P^d)(N,p)].$$

A rule φ on $\mathcal{P}$ is **self-dual** if for all $(N,p) \in \mathcal{P}$, $\varphi(N,p) = \varphi^d(N,p)$. **Two axioms are dual to each other** if whenever a rule satisfies one of them, the dual of this rule satisfies the other. That is, two axioms A and A' are dual to each other if for all rules that satisfy A, it holds that their duals satisfy A', and conversely, for all rules that satisfy A', it holds that their duals satisfy A. **An axiom is self-dual** if it is its own dual.

Given a rule φ on $\mathcal{P}$, **the anti-dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^{ad}}$, is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$\varphi^{ad}(N,p) \equiv -\phi[(v_P^{ad})(N,p)].$$

A rule φ on $\mathcal{P}$ is **self-anti-dual** if for all $(N,p) \in \mathcal{P}$, $\varphi(N,p) = \varphi^{ad}(N,p)$. **Two axioms are anti-dual** if whenever a rule satisfies one of them, the anti-dual of this rule satisfies the other. That is, two axioms A and A' are anti-dual to each other if for all rules that satisfy A, it holds that their anti-duals satisfy A', and conversely, for all rules that satisfy A', it holds that their anti-duals satisfy A. **An axiom is self-anti-dual** if it is its own anti-dual.

3.2 Basic properties of (anti-)dual axioms for rules

3.2.1 Propositional functions, axioms, and axiomatizations

We show basic properties of duality and anti-duality for rules. In order to show these properties, we introduce the mathematical structure which explicitly deals with (anti-)dual axioms of rules for allocation problems. The basic idea of this mathematical structure follows from Funaki (1998). Funaki (1998) applies the propositional function approach to dealing with axioms of solutions for TU games. We apply the propositional function approach to uncovering properties of (anti-)dual axioms of rules for allocation problems.

Given a class $\mathcal{P}$, a class $\mathcal{V}_P$ and a solution ϕ on $\mathcal{V}_P$, a **propositional function $\boldsymbol{F}$** is generically defined as follows:

$$F : \{((N,p), \varphi(N,p)) : (N,p) \in \mathcal{P},\ \varphi(N,p) \in X(N,p)\} \to \{0,1\},$$

where $\varphi(N,p) \equiv \phi \circ v_P(N,p)$ for all $(N,p) \in \mathcal{P}$ and all $v_P \in \mathcal{V}_P$. We say that the propositional function F with respect to (N,p) and φ is **true** (resp. **false**) if $F((N,p), \varphi(N,p)) = 1$ (resp. $F(\cdot,\cdot) = 0$).

Let $\mathcal{F}$ be the set of all propositional functions. Given a class $\mathcal{P}$, a class $\mathcal{V}_P$ and a solution ϕ on $\mathcal{V}_P$, an **equivalence relation** $\sim_\varphi$ on $\mathcal{F}$ is defined as follows:

$$F \sim_\varphi \bar{F} \quad \Leftrightarrow \quad F((N,p),\varphi(N,p)) = \bar{F}((N,p),\varphi(N,p)) \text{ for all } (N,p) \in \mathcal{P}.$$

Given a propositional function $\bar{F} \in \mathcal{F}$ on $\mathcal{P}$, an **axiom for a rule φ with respect to $\bar{F}$** is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$E_{\bar{F}}(\mathcal{P},\varphi) \equiv \{F : F \sim_\varphi \bar{F}\}.$$

A **rule φ satisfies the axiom $E_{\bar{F}}(\mathcal{P},\varphi)$ with respect to $\bar{F}$** if for all $(N,p) \in \mathcal{P}$, and all $F \in E_{\bar{F}}(\mathcal{P},\varphi)$, $F((N,p),\varphi(N,p)) = 1$.

On $\mathcal{P}$, a **rule φ is axiomatized by the set of axioms** if the rule φ satisfies a set of axioms with respect to some propositional functions and any other rules do not satisfy it.

3.2.2 Dual axioms, and dual axiomatizations

Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and a propositional function $F \in \mathcal{F}$, the **dual of** F, denoted F^d, is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$F^d((N,p),\varphi(N,p)) \equiv F((N,p),\varphi^d(N,p)),$$

where $\varphi^d(N,p) = \phi \circ v_P^d(N,p)$ for all $(N,p) \in \mathcal{P}$ and all $v_P \in \mathcal{V}_P$.

Given a propositional function $\bar{F} \in \mathcal{F}$, a **dual axiom of a rule φ with respect to $\bar{F}$**, denoted $E^d_{\bar{F}}(\mathcal{P},\varphi)$, is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$E^d_{\bar{F}}(\mathcal{P},\varphi) \equiv E_{\bar{F}^d}(\mathcal{P},\varphi).$$

The following theorem shows that one can derive an axiom for the dual of a rule φ, namely φ^d, from taking the dual of an axiom for the original rule φ.

Theorem 1 *(Existence theorem of dual axioms of rules) Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and a propositional function $F \in \mathcal{F}$, a rule φ satisfies an axiom $E_F(\mathcal{P},\varphi)$ if and only if the dual rule φ^d satisfies the dual axiom $E^d_F(\mathcal{P},\varphi^d)$.*

Proof. Let φ^d be a rule satisfying an axiom $E^d_F(\mathcal{P},\varphi^d)$, that is, for all $(N,p) \in \mathcal{P}$ and all $G \in E^d_F(\mathcal{P},\varphi^d)$,

$$G((N,p),\varphi^d(N,p)) = 1.$$

By the duality of propositional functions,

$$
\begin{aligned}
& E^d_F(\mathcal{P},\varphi^d) \\
= \;& \{G : G((N,p),\varphi^d(N,p)) = F^d((N,p),\varphi^d(N,p)) \text{ for all } (N,p)\in\mathcal{P}\} \\
= \;& \{G : G((N,p),\phi\circ v^d(N,p)) = F^d((N,p),\phi\circ v^d(N,p)) \text{ for all } (N,p)\in\mathcal{P}\} \\
= \;& \{G : G^d((N,p),\varphi(N,p)) = F((N,p),\varphi(N,p)) \text{ for all } (N,p)\in\mathcal{P}\} \\
= \;& \{G : G^d \in E_F(\mathcal{P},\varphi)\}.
\end{aligned}
$$

Since $G((N,p),\varphi^d(N,p)) = 1$, $G^d((N,p),\varphi(N,p)) = 1$. Then, for all $(N,p)\in\mathcal{P}$ and all $G^d \in E_F(\mathcal{P},\varphi)$, $G^d((N,p),\varphi(N,p)) = 1$, which implies that φ satisfies an axiom $E_F(\mathcal{P},\varphi)$. By the same argument, we obtain the opposite implication. ■

Next, the following theorem shows that one can axiomatize the *dual* of a rule φ, namely φ^d, by taking the *dual* of the axioms involved in an axiomatization of the original rule φ.

Theorem 2 *(Axiomatization theorem of dual rules) Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and propositional functions $F_l\in\mathcal{F}$ $(l=1,2,\cdots,k)$, if a rule φ on $\mathcal{P}$ is axiomatized by axioms $E_{F_l}(\mathcal{P},\varphi)$ $(l=1,2,\cdots,k)$, then the dual rule φ^d is axiomatized by the dual axioms $E^d_{F_l}(\mathcal{P},\varphi^d)$ $(l=1,2,\cdots,k)$.*

Proof. Let φ be a rule on $\mathcal{P}$, satisfying $E_{F_l}(\mathcal{P},\varphi)$ $(l=1,2,\cdots,k)$. By Theorem 1, φ^d satisfies $E^d_{F_l}(\mathcal{P},\varphi^d)$ $(l=1,2,\cdots,k)$. Suppose that φ is the unique rule on $\mathcal{P}$, satisfying $E_{F_l}(\mathcal{P},\varphi)$ $(l=1,2,\cdots,k)$, and $\tilde{\varphi}$ is any rule on $\mathcal{P}$, satisfying $E^d_{F_l}(\mathcal{P},\varphi^d)$ $(l=1,2,\cdots,k)$. Since $\tilde{\varphi}=(\tilde{\varphi}^d)^d$, $(\tilde{\varphi}^d)^d$ satisfies $E^d_{F_l}(\mathcal{P},\varphi^d)$ $(l=1,2,\cdots,k)$. Again, by Theorem 1, $\tilde{\varphi}^d$ satisfies $E_{F_l}(\mathcal{P},\varphi)$ $(l=1,2,\cdots,k)$. Hence, $\tilde{\varphi}^d=\varphi$, or equivalently, $\tilde{\varphi}=\varphi^d$, which implies that $\tilde{\varphi}$ is unique. ■

An economic application of Theorem 2 is as follows: Suppose that we have an axiomatization of a rule φ for allocation problems and its dual is φ^d for distinct allocation problems. Furthermore, suppose that in the existing literature no axiomatization of the rule φ^d is investigated. Then just by identifying the dual of each axiom involved in an axiomatization of the rule φ, we obtain an axiomatization of the rule φ^d.

3.3 Anti-dual axioms, and anti-dual axiomatizations

By the same manner as in the case of dual axioms of rules, we introduce the mathematical structure of anti-dual axioms for rules.

Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and a propositional function $F \in \mathcal{F}$, the **anti-dual of $\boldsymbol{F}$**, denoted F^{ad}, is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$F^{ad}((N,p), \varphi(N,p)) \equiv F((N,p), \varphi^{ad}(N,p)),$$

where $\varphi^{ad}(N,p) = -\phi \circ v_P^{ad}(N,p)$ for all $(N,p) \in \mathcal{P}$ and all $v_P \in \mathcal{V}_P$.

Given a propositional function $\bar{F} \in \mathcal{F}$, an **anti-dual axiom for a rule $\boldsymbol{\varphi}$ with respect to $\bar{F}$**, denoted $E_{\bar{F}}^{ad}(\mathcal{P}, \varphi)$, is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$E_{\bar{F}}^{ad}(\mathcal{P}, \varphi) \equiv E_{\bar{F}^{ad}}(\mathcal{P}, \varphi).$$

The following theorem is the anti-dual version of Theorem 1. The proof is the same as that of Theorem 1. We omit it.

Theorem 3 *(Existence theorem of anti-dual axioms of rules) Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and a propositional function $F \in \mathcal{F}$, a rule φ satisfies an axiom $E_F(\mathcal{P}, \varphi)$ if and only if the anti-dual rule φ^{ad} satisfies the anti-dual axiom $E_F^{ad}(\mathcal{P}, \varphi^{ad})$.*

Next, the following theorem is the anti-dual version of Theorem 2. The proof is the same as that of Theorem 2. We omit it.

Theorem 4 *(Axiomatization theorem of anti-dual rules) Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and propositional functions $F_l \in \mathcal{F}$ $(l = 1, 2, \cdots, k)$, if a rule φ on $\mathcal{P}$ is axiomatized by axioms $E_{F_l}(\mathcal{P}, \varphi)$ $(l = 1, 2, \cdots, k)$, then the anti-dual rule φ^{ad} is axiomatized by the anti-dual axioms $E_{F_l}^{ad}(\mathcal{P}, \varphi^{ad})$ $(l = 1, 2, \cdots, k)$.*

An economic application of Theorem 4 is the same as in the case of Theorem 2. In the following sections, we apply the duality and ant-duality approach mentioned to economic problems.

4 Illustration of (anti-)dual axiomatizations of rules

In this section, by using the *duality and anti-duality approach*, we derive new axiomatizations of rules for bidding ring problems.

4.1 Airport problems, and bidding ring problems

There is a set of airlines for whom an airstrip they will jointly use is to be built. Each airline owns one type of aircraft. Airlines have different needs for airstrips, since they own different types of aircraft. An airstrip needed to accommodate the largest aircraft is to be built. The problem is to determine how to share the cost of the airstrip between the airlines (Littlechild and Owen 1973).

An **airport problem** is a pair (N, c), where $N \in \mathcal{N}$ is the set of airlines and $c = (c_i)_{i \in N}$ is the profile of cost parameters, namely c_i is the construction cost of the airstrip for airline i. We assume that the cost is increasing in the length of the airstrip.

For all $N \in \mathcal{N}$ such that $|N| = n$, let $\sigma : N \to \{1, 2, \cdots, n\}$ be a bijection such that $c_{\sigma^{-1}(n)} \geq c_{\sigma^{-1}(n-1)} \geq \cdots \geq c_{\sigma^{-1}(1)} > 0$. These airlines are ordered in terms of their costs. Let $\mathcal{C}$ be the class of all airport problems on $\mathcal{N}$.

Given $(N, c) \in \mathcal{C}$, the **airport game** is defined by setting, for all $S \subseteq N$,

$$c_A(N, c)(S) \equiv \max_{i \in S} c_i.$$

For all $S \in 2^N$, $c_A(N, c)(S)$ represents the cost of the airstrip needed to accommodate the members of coalition S. It is equal to the cost of the airstrip needed to accommodate the member of the coalition whose cost parameter is the largest.

Let $\mathcal{C}_A$ be the class of all airport games. Given $(N, c) \in \mathcal{C}$, an allocation for (N, c) is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \max_N c_i$ (which is equal to $c_{\sigma^{-1}(n)}$). Let $X(N, c)$ be the set of allocations for (N, c). A **solution for airport games** is a mapping $\phi_A : \mathcal{C}_A \to \mathbb{R}^N$ that associates with each airport game $c_A(N, c)$ in the domain an allocation in $X(N, c)$. We refer to the composite mapping $\varphi_A \equiv \phi_A \circ c_A$ as a **rule for airport problems**. The **Shapley rule for airport problems** is defined by $\varphi_A^{Sh} \equiv Sh \circ c_A$. The **nucleolus rule for airport problems** is defined by $\varphi_A^{Nu} \equiv Nu \circ -c_A$, since airport problems are cost problems and the nucleolus is defined under the situation of profit games.[3]

An **English auction** is an oral auction in which an auctioneer initially sets a bid at a seller's reservation price and then gradually increases the price until only one bidder remains active. There is a set of buyers in a *single-object English auction*. There is no asymmetry of information between the buyers; that is, each buyer has information on the valuations of all buyers for the

[3] The nucleolus can be defined under the situation of cost games. Given $N \in \mathcal{N}$ and $v \in \mathcal{V}^N$, let $I(v)$ be the set of vectors $x \in \mathbb{R}^N$ such that for all $i \in N$, $x_i \leq v(\{i\})$, and $\sum_N x_i = v(N)$. Let $\mathcal{V}^N$ be a class of (cost) games such that for all $v \in \mathcal{V}^N$, $I(v) \neq \emptyset$. For all $x \in I(v)$, let $e(v, x) \in \mathbb{R}^{2^N}$ be defined by setting, for all $S \subseteq N$, $e_S(v, x) \equiv \sum_S x_i - v(S)$. For $N \in \mathcal{N}$ and $v \in \mathcal{V}^N$, the nucleolus is defined as the set of $x \in I(v)$ such that for all $y \in I(v) \backslash \{x\}$ $e(v, x)$ is lexicographically smaller than $e(v, y)$.

object. The valuation of each buyer is positive. The reservation price is zero. A bidding ring is formed by all buyers. The bidding ring wins the auction by making the buyer whose valuation is the largest the sole bidder. The benefit of the ring members' strategy is equal to the valuation of this buyer. The problem for the members in the ring is to determine how to share the benefit of their strategy (Graham et al. 1990).

A **bidding ring problem** is a pair (N, c), where $N \in \mathcal{N}$ is the set of buyers and $c = (c_i)_{i \in N}$ is the profile of valuations for a single object, c_i being the valuation of buyer i. For all $N \in \mathcal{N}$ such that $|N| = n$, let $\sigma : N \to \{1, 2, \cdots, n\}$ be a bijection such that $c_{\sigma^{-1}(n)} \geq c_{\sigma^{-1}(n-1)} \geq \cdots \geq c_{\sigma^{-1}(1)} > 0$. These buyers are ordered in terms of their values. Let $\mathcal{C}$ be the class of all bidding ring problems on $\mathcal{N}$.

Given $(N, c) \in \mathcal{C}$, the **bidding ring game** is defined by setting, for all $S \subseteq N$,

$$v_B(N, c)(S) = \max \left\{ \max_{i \in S} c_i - \max_{j \in N \setminus S} c_j, 0 \right\}.$$

where $\max_{j \notin N} c_j \equiv 0$. The intuition is as follows: First, under the English auction rule, it is a dominant strategy for each bidder to remain active until bidding reaches his valuation. Second, any coalition including buyer $\sigma^{-1}(n)$ with the largest valuation can win the auction, and achieve the net benefit $\max_{i \in S} c_i - \max_{j \notin S} c_j$ by making buyer $\sigma^{-1}(n)$ the sole bidder in the coalition and his bidding $c_{\sigma^{-1}(n)}$. Finally, no coalition that does not include buyer $\sigma^{-1}(n)$ wins the auction, and hence its net benefit is 0.

Let $\mathcal{V}_B$ be the class of all bidding ring games. Given $(N, c) \in \mathcal{C}$, an allocation for (N, c) is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = c_{\sigma^{-1}(n)}$. Let $X(N, c)$ be the set of allocations for (N, c) . A **solution for bidding ring games** is a mapping $\phi_B : \mathcal{V}_B \to \mathbb{R}^N$ that associates with each bidding ring game $v_B(N, c)$ in the domain an allocation in $X(N, c)$. We refer to the composite mapping $\varphi_B \equiv \phi_B \circ v_B$ as a **rule for bidding ring problems**. The **Shapley rule for bidding ring problems** is defined by $\varphi_B^{Sh} \equiv Sh \circ v_B$. The **nucleolus rule for bidding ring problems** is defined by $\varphi_B^{Nu} \equiv Nu \circ v_B$.

Remark 1 *Given an arbitrary pair $(N, c) \in \mathcal{C}$, let c_A be the airport game derived from (N, c), and let v_B be the bidding ring game derived from (N, c).*

(i) c_A and v_B are dual to each other.
(ii) $-c_A$ and v_B are anti-dual to each other.
(ii) The Shapley value of c_A coincides with that of v_B.
(iv) The nucleolus of $-c_A$ coincides with that of v_B multiplied by -1.

The proof of Remark 1 is immediate from Oishi and Nakayama (2009).

4.2 Illustration of dual axiomatizations of the Shapley rule

In the existing literature, the Shapley rule for bidding ring problems has not been axiomatized. Just by identifying the *dual* of each axiom involved in an axiomatization of φ_A^{Sh}, we obtain an axiomatization of φ_B^{Sh}. Let us consider the *dual* of each axiom involved in an axiomatization of the Shapley rule for airport problems (Chun et al. 2012).

First, we consider the following property. Each airline i has the right to use at least the airstrip to accommodate the airline i. It says that each airline i should pay at least an equal share of c_i.

Equal share lower bound for airport problems: For all $(N, c) \in \mathcal{C}$ and all $i \in N$,

$$\varphi_{A[i]}(N, c) \geq \frac{c_i}{n}.$$ [4]

The following property says that each buyer $i \in N$ should gain at least an equal share of his valuation.

Equal share lower bound for bidding ring problems: For all $(N, c) \in \mathcal{C}$ and all $i \in N$,

$$\varphi_{B[i]}(N, c) \geq \frac{c_i}{n}.$$

Next, we consider the following property for airport problems. It requires that if the cost of an airline increases, then all the other airlines should pay at most as much as they did initially.

Individual monotonicity for airport problems: Fix an arbitrary $N \in \mathcal{N}$. For all $(N, c) \in \mathcal{C}$, all $(N, c') \in \mathcal{C}$, and all $i \in N$, if $c'_i > c_i$, and for all $j \in N \backslash \{i\}$, $c'_j = c_j$, then for all $j \in N \backslash \{i\}$,

$$\varphi_{A[j]}(N, c') \leq \varphi_{A[j]}(N, c).$$

The following property says that if the valuation of a buyer increases, then all the other buyers should share at most as much as they did initially.

Individual monotonicity for bidding ring problems: Fix an arbitrary $N \in \mathcal{N}$. For all $(N, c) \in \mathcal{C}$, all $(N, c') \in \mathcal{C}$, and all $i \in N$, if $c'_i > c_i$, and for all $j \in N \backslash \{i\}$, $c'_j = c_j$, then for all $j \in N \backslash \{i\}$,

$$\varphi_{B[j]}(N, c') \leq \varphi_{B[j]}(N, c).$$

[4] $\varphi_{A[i]}(N, c)$ is the i-th coordinate of $\varphi_A(N, c) \in \mathbb{R}^n$. Similarly, we define $\varphi_{B[i]}(N, c)$ as the i-th coordinate of $\varphi_B(N, c) \in \mathbb{R}^n$.

Next, the following property says that each airline should contribute a non-negative amount, but no more than his individual cost.

Reasonableness for airport problems: For all $(N, c) \in \mathcal{C}$, all $i \in N$ and $x = \varphi_A(N, c)$, $0 \leq x_i \leq c_i$.

The following property says that each buyer should gain a non-negative amount, but no more than his contribution to N.

Reasonableness for bidding problems: For all $(N, c) \in \mathcal{C}$, all $i \in N$ and $x = \varphi_B(N, c)$, $0 \leq x_i \leq c_i$.

Let i^* be an airline with the smallest cost. Imagine that airline i^* pays its contribution $\varphi_{i^*}(N, c)$ and leaves. Let $x \equiv \varphi(N, c)$ and $N' \equiv N \backslash \{i^*\}$. Imagine that the contribution x_{i^*} is intended to cover the construction cost of the part of the airstrip that airline i^* uses. Since the part of the airstrip that airline i^* uses is also used by the remaining airlines, the cost of each remaining airline $j \in N'$ is revised down by the amount x_{i^*}. As a result, the revised cost of each remaining airline $j \in N'$ is $c_j - x_{i^*}$. The following property says that the outcome chosen by a rule should be invariant under the departure of airline i^*.

First-agent airport consistency[5]: For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, all $i \in N$, $N' = N \backslash \{i^*\}$, where $i^* = \arg\min_{i \in N} c_i$, and $x = \varphi_A(N, c)$,

$$(N', \bar{c}^x_{N'}) \in \mathcal{C} \text{ and } x_{N'} = \varphi_A(N', \bar{c}^x_{N'}),$$

where for each $j \in N'$, $(\bar{c}^x_{N'})_{[j]} = c_j - x_{i^*}$.[6]

The following property, referred to as "first-agent transfer agreement equivalence", says that for reduced bidding ring problems where a buyer i^* with the smallest valuation leaves the outcome chosen by a modified rule φ^* should coincide with that chosen by the original rule φ. Given φ, its modified rule φ^* is derived from "transfer agreement" between i^* and any bidding ring in $N \backslash \{i^*\}$.

An explanation of **transfer agreement** is as follows: Let i^* be a buyer with the smallest valuation. Let $N' \equiv N \backslash \{i^*\}$. Imagine that the members of $S \subseteq N'$ with $S \neq \emptyset$ announce that they will cooperate with anybody if they gain $v_B(S)$. Then the remaining buyers in $N \backslash S$ will play a game $v^S_B(T)$, where for all $T \subseteq N \backslash S$ $v^S_B(T) \equiv v_B(S \cup T) - v_B(S)$. As a result of the announcement, each buyer $k \in N \backslash S$ gets $\hat{\varphi}_{B[k]}(N \backslash S, c_{N \backslash S}) \equiv (\phi \circ v^S_B(N \backslash S, c_{N \backslash S}))_{[k]}$.[7] If the members of S do not make this announcement, each

[5]In Chun et al. (2012), this property is called "smallest cost consistency". Smallest cost consistency is originally introduced by Potters and Sudhölter (1999).

[6]For $z \in \mathbb{R}^n$ and $N' \subset N$, $z_{N'}$ is the projection of z onto N', that is, $z_{N''} \equiv (z_k)_{k \in N'}$.

[7]$(\phi \circ v^S_B(N \backslash S, c_{N \backslash S}))_{[k]}$ is the k-th coordinate of $\phi \circ v^S_B(N \backslash S, c_{N \backslash S}) \in \mathbb{R}^{N \backslash S}$.

buyer $k \in N\backslash S$ gets $\varphi_{B_{[k]}}(N,c)$. Therefore buyer i^* gets $\hat{\varphi}_{i^*}(N\backslash S, c_{N\backslash S})$ if the announcement is made, and $\varphi_{B_{[i^*]}}(N,c)$ otherwise. Imagine that the members of S and buyer i^* agree upon that the difference $\hat{\varphi}_{i^*}(N\backslash S, c_{N\backslash S}) - \varphi_{B_{[i^*]}}(N,c)$ is transferred from buyer i^* to the members of S. This agreement is referred to as **transfer agreement** between i^* and $S \subseteq N'$. First-agent transfer agreement equivalence requires that the outcome chosen by a rule φ should be invariant even if transfer agreement between i^* and any bidding ring in $N\backslash\{i^*\}$ is made.

First-agent transfer agreement equivalence: For all $(N,c) \in \mathcal{C}$ with $n \geq 2$, $N' \equiv N\backslash\{i^*\}$, where $i^* = \arg\min_{i\in N} c_i$, and $x = \varphi_B(N,c)$,

$$(N', c_{N'}) \in \mathcal{C} \text{ and } x_{N'} = \varphi^*(N', c_{N'}),$$

where
(i) $\varphi^*(N', c_{N'}) \equiv \phi \circ \hat{w}(N', c_{N'})$, and
(ii) for each $S \subseteq N'$

$$\begin{aligned} &\hat{w}(N', c_{N'})(S) \\ = &\begin{cases} \max\left\{\max\limits_{i\in S} c_i - \max\limits_{j\in N\backslash S} c_j, 0\right\} + \hat{\varphi}_{i^*}(N\backslash S, c_{N\backslash S}) - x_{i^*} & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset, \end{cases}\end{aligned}$$

where $\hat{\varphi}(N\backslash S, c_{N\backslash S}) \equiv \phi \circ v_B^S(N\backslash S, c_{N\backslash S})$ and v_B^S is the game for $N\backslash S$ defined by setting for all $T \subseteq N\backslash S$, $v_B^S(T) \equiv v_B(S \cup T) - v_B(S)$.

Theorem A *(Chun et al. 2012) For airport problems, the Shapley rule is the only rule satisfying reasonableness, the equal share lower bound, individual monotonicity, and first-agent airport consistency.*

Using the duality approach, we obtain the following axiomatization of φ_B^{Sh} that is dual of the axiomatization appearing in Theorem A.

Theorem 5 *(Dual of Theorem A) For bidding ring problems, the Shapley rule is the only rule satisfying reasonableness, the equal share lower bound, individual monotonicity, and first-agent transfer agreement equivalence.*

Proof. We compute the dual of each axiom involved in an axiomatization of the Shapley rule appearing in Theorem A. We have two steps.

Step 1: *Reasonableness* for bidding ring problems is self-dual to that for airport problems. Using the propositional function form, we restate the following axioms.

Reasonableness for airport problems (propositional function form): For all $(N, c) \in \mathcal{C}$, $F((N, c), \phi \circ c_A(N, c)) = 1$ iff for all $i \in N$ and $x = \phi \circ c_A(N, c)$, $0 \le x_i \le c_i$.

Reasonableness for bidding problems (propositional function form): For all $(N, c) \in \mathcal{C}$, $F((N, c), \phi \circ v_B(N, c)) = 1$ iff for all $i \in N$ and $x = \phi \circ v_B(N, c)$, $0 \le x_i \le c_i$.

Let us take the dual of reasonableness for airport problems. Since $c_A = (v_B)^d$, the following formula holds.

(1) For all $(N, c) \in \mathcal{C}$, $F((N, c), \phi \circ (v_B)^d(N, c)) = 1$ iff for all $i \in N$ and $x = \phi \circ (v_B)^d(N, c)$, $0 \le x_i \le c_i$.

By Theorem 1 and the fact that $((v_B)^d)^d = v_B$, the dual of the formula (1) is the following formula.

(2) For all $(N, c) \in \mathcal{C}$, $F^d((N, c), \phi \circ (v_B)(N, c)) = 1$ iff for all $i \in N$ and $x = \phi \circ v_B(N, c)$, $0 \le x_i \le c_i$.

Therefore, the dual of reasonableness of airport problems is reasonableness for bidding ring problems, that is, reasonableness is self-dual. By the same manner as in the argument mentioned, *the equal share lower bound* and *individual monotonicity* are self-duals, respectively.

Step 2: We show that *first-agent transfer agreement equivalence* is dual to *first-agent airport consistency.*

First, we write the propositional function of first-agent airport consistency.

(3) For all $(N, c) \in \mathcal{C}$ with $n \ge 2$, $F((N, c), \phi \circ c_A(N, c)) = 1$ iff for all $i \in N$, $N' = N \backslash \{i^*\}$, where $i^* = \arg\min_{i \in N} c_i$, and $x = \phi \circ c_A(N, c)$, $(N', \bar{c}^x_{N'}) \in \mathcal{C}$ and

$$x_{N'} = \phi \circ c_A(N', \bar{c}^x_{N'}),$$

where (i) for all $j \in N'$, $(\bar{c}^x_{N'})_{[j]} = c_j - x_{i^*}$, and (ii) for all $S \subseteq N'$ $c_A(N', \bar{c}^x_{N'})(S) = \max_{j \in S} (\bar{c}^x_{N'})_{[j]}$.

Let $x = \varphi(N, c)$ and $N' = N \backslash \{i^*\}$, where $i^* = \arg\min_{i \in N} c_i$. First, we claim that the Davis-Maschler reduced game[8] on N' with respect to c_A and x

[8]For a game $v \in \mathcal{V}$, a vector $x \in \mathbb{R}^N$ and non-empty subset N' of N, the *Davis-Maschler reduced game* (Davis and Maschler 1965) on N' with respect to v and x is the

is the Hart-Mas-Colell reduced game[9] on N' with respect to c_A and x. Given an airport game c_A and $N' \subset N$ with $N' \neq \emptyset$, the subgame of N', denoted $c_A|_{N'}$, is defined by setting for all $S \in 2^{N'}$, $c_A|_{N'}(S) = c_A(S)$. Since for each $S \subset N'$ with $S \neq \emptyset$ the subgame $c_A\big|_{S\cup(N\backslash N')} = c_A(S \cup \{i^*\})$ and c_A $c_A(S \cup \{i^*\}) = c_A(S)$,

$$\begin{aligned} c_A(S \cup (N\backslash N')) - \phi_{i^*}(c_A\big|_{S\cup(N\backslash N')}) &= c_A(S \cup \{i^*\}) - \phi_{i^*}(c_A) \\ &= \min\{c_A(S \cup \{i^*\}) - \phi_{i^*}(c_A), c_A(S)\}, \end{aligned}$$

the desired claim.

Since the Davis-Maschler reduced game on N' with respect to c_A and x is the Hart-Mas-Colell reduced game on N' with respect to c_A and x, we obtain the following proposition function of first-agent airport consistency.

(4) For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $F((N, c), \phi \circ c_A(N, c)) = 1$ iff for $N' = N\backslash\{i^*\}$, where $i^* = \arg\min_{i\in N} c_i$, and $x = \phi \circ c_A(N, c)$, $x_{N'} = \phi \circ w(N', c_{N'})$ and for each $S \subseteq N'$

$$w(S) = \begin{cases} c_A\big(S \cup (N\backslash N')\big) - \phi_{i^*}(c_A\big|_{S\cup(N\backslash N')}) & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Since $(v_B)^d = c_A$, the formula (4) is rewritten as follows.

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game $(N', r^x_{N'}(v)) \in \mathcal{V}$ defined by setting for all $S \subseteq N'$,

$$r^x_{N'}(v)(S) = \begin{cases} v(N) - \sum_{i\in N\backslash N'} x_i & \text{if } S = N', \\ \max_{T\subseteq N\backslash N'} \big[v(S \cup T) - \sum_{i\in T} x_i\big] & \text{if } S \neq N', \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

A solution f on a subclass $\mathcal{V}'$ of $\mathcal{V}$ satisfies the *Davis-Maschler consistency* (Davis and Maschler 1965) if for all $(N, v) \in \mathcal{V}'$ and every non-empty $N' \subsetneq N$ it holds that for all $i \in N$ $f_i(N, v) = f_i(N', r^x_{N'}(v))$, where $x \in f(N, v)$.

[9] Let f be a single-valued solution on $\mathcal{V}$. For a game $v \in \mathcal{V}$, a vector $x \in \mathbb{R}^N$ and non-empty subset N' of N, the *Hart-Mas-Colell reduced game* (Hart and Mas-Collel 1989) on N' with respect to v and x is the game $(N', r^x_{N'}(v)) \in \mathcal{V}$ defined by setting for all $S \subseteq N'$,

$$r^x_{N'}(v)(S) = \begin{cases} v(S \cup (N\backslash N')) - \sum_{i\in N\backslash N'} f_i(v\big|_{S\cup(N\backslash N')}) & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

A single-valued solution f on a subclass $\mathcal{V}'$ of $\mathcal{V}$ satisfies the *Hart-Mas-Colell consistency* (Hart and Mas-Colell 1989) if for all $(N, v) \in \mathcal{V}'$ and every non-empty $N' \subsetneq N$ it holds that for all $i \in N$ $f_i(N, v) = f_i(N', r^x_{N'}(v))$, where $x = f(N, v)$.

(5) For all $(N,c) \in \mathcal{C}$ with $n \geq 2$, $F((N,c), \phi \circ (v_B)^d(N,c)) = 1$ iff for $N' = N \backslash \{i^*\}$, where $i^* = \arg\min_{i \in N} c_i$, and $x = \phi \circ (v_B)^d(N,c)$, $x_{N'} = \phi \circ w(N', c_{N'})$, where for each $S \subseteq N'$

$$w(S) = \begin{cases} (v_B)^d\big(S \cup (N \backslash N')\big) - \phi_{i^*}((v_B)^d \big|_{S \cup (N \backslash N')}) & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

For each $T \subseteq N \backslash S$,

$$\begin{aligned} (c_A \big|_{N \backslash S})^d(T) &= (c_A \big|_{N \backslash S})(N \backslash S) - (c_A \big|_{N \backslash S})((N \backslash S) \backslash T) \\ &= ((v_B)^d \big|_{N \backslash S})(N \backslash S) - ((v_B)^d \big|_{N \backslash S})((N \backslash S) \backslash T) \\ &= v_B(N) - v_B(N \backslash (N \backslash S)) - v_B(N) + v_B(N \backslash ((N \backslash S) \backslash T)) \\ &= v_B(S \cup T) - v_B(S). \end{aligned}$$

By this observation together with the fact that $w^d(\emptyset) = 0$ and for each $\emptyset \neq S \subseteq N'$,

$$\begin{aligned} w^d(S) &= w(N') - w(N' \backslash S) \\ &= c_A(N) - \phi_{i^*}(c_A) - c_A(N \backslash S) + \phi_{i^*}(c_A \big|_{N \backslash S}) \\ &= c_A(N) - c_A(N \backslash S) - \phi_{i^*}(c_A) + \phi_{i^*}(c_A \big|_{N \backslash S}) \\ &= v_B(S) - \phi_{i^*}((v_B)^d) + \phi_{i^*}((v_B^S)^d), \end{aligned}$$

where each $T \subseteq N \backslash S$, $v_B^S(T) \equiv v_B(S \cup T) - v_B(S)$.

By Theorem 1 and the fact that $((v_B)^d)^d = v_B$, the dual of the formula (5) is the following formula.

(6) For all $(N,c) \in \mathcal{C}$ with $n \geq 2$, $F^d((N,c), \phi \circ v_B(N,c)) = 1$ iff for $N' = N \backslash \{i^*\}$ and , where $i^* = \arg\min_{i \in N} c_i$, and $x = \phi \circ v_B(N,c)$, $x_{N'} = \phi \circ \tilde{w}(N', c_{N'})$, and for each $S \subseteq N'$

$$\tilde{w}(S) = \begin{cases} v_B(S) + \tilde{\varphi}_{i^*}(N \backslash S, c_{N \backslash S}) - x_{i^*} & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset, \end{cases}$$

such that $\tilde{\varphi}(N \backslash S, c_{N \backslash S}) \equiv \phi \circ v_B^S(N \backslash S, c_{N \backslash S})$ and v_B^S is the game for $N \backslash S$ defined by setting for each $T \subseteq N \backslash S$, $v_B^S(T) \equiv v_B(S \cup T) - v_B(S)$.

Therefore, the dual of first-agent airport consistency is first-agent transfer agreement equivalence. ■

Thanks to Theorem 2, logical independence of the axioms appearing in Theorem 5 holds. The argument of the formula (6) is inspired from the fact that the dual of the *Hart-Mas-Colell consistency* is *transfer-agreement consistency*

(Oishi et al. 2016).[10] Since transfer-agreement consistency is a property that should be satisfied by a single-valued solution on the domain of all TU games, it is not directly applicable to the argument mentioned above. As shown in the proof of Theorem 5, calculation of a series of propositional functions based on Theorem 1 is necessary for our argument.

4.3 Illustration of anti-dual axiomatizations of the nucleolus rule

In the existing literature, the nucleolus rule for bidding ring problems has not been axiomatized. Just by identifying the *anti-dual* of each axiom involved in an axiomatization of φ_A^{Nu}, we obtain an axiomatization of φ_B^{Nu}. Let us consider the *anti-dual* of each axiom involved in an axiomatization of the nucleolus rule for airport problems (Yeh 2004, Hwang and Yeh 2012).

First, we consider the following property. It says that airlines with equal costs should contribute equal amounts.

Equal treatment of equals for airport problems: For each $(N,c) \in \mathcal{C}$ and each pair $\{i,j\} \subseteq N$, if $c_i = c_j$, then $\varphi_{A[i]}(N,c) = \varphi_{A[j]}(N,c)$.

The following property says that buyers with equal valuations should gain equal amounts.

Equal treatment of equals for bidding ring problems: For each $(N,c) \in \mathcal{C}$ and each pair $\{i,j\} \subseteq N$, if $c_i = c_j$, then $\varphi_{B[i]}(N,c) = \varphi_{B[j]}(N,c)$.

Next, we consider the following property. It says that if the cost of an airline with the largest cost increases by δ, then all other airlines should contribute the same amounts as they did initially.

Last-agent additivity for airport problems: For each pair $\{(N,c),(N,c')\}$ of elements of $\mathcal{C}$, each $\delta \in \mathbb{R}_+$, and $i^* = \arg\max_{i\in N} c_i$, if $c'_{i^*} = c_{i^*} + \delta$ and for

[10] Let f be a single-valued solution on $\mathcal{V}$. For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $N' \subset N$ with $N' \neq \emptyset$, and all $w \in \mathbb{R}^{2^{N'}}$, if for all $S \subseteq N'$,

$$w(S) = \begin{cases} v(N) - \sum_{i\in N\setminus N'} f_i(v) & \text{if } S = N', \\ v(S) + \sum_{i\in N\setminus N'} f_i(v^S) - \sum_{i\in N\setminus N'} f_i(v) & \text{if } S \neq N', \emptyset, \\ 0 & \text{if } S = \emptyset, \end{cases}$$

where v^S is the game for $N\setminus S$ defined by setting for all $T \subseteq N\setminus S$, $v^S(T) \equiv v(S\cup T) - v(S)$, then $w \in \mathcal{V}^{N'}$ and for all $i \in N'$, $f_i(w) = f_i(v)$.

each $j \in N\backslash\{i^*\}$ $c'_j = c_j$, then $\varphi_{A[i^*]}(N, c') = \varphi_{A[i^*]}(N, c) + \delta$ and for each $j \in N\backslash\{i^*\}$, $\varphi_{A[j]}(N, c') = \varphi_{A[j]}(N, c)$.

The following property says that if the valuation of a buyer with the largest valuation increases by δ, then all other buyers should gain the same amounts as they did initially.

Last-agent additivity for bidding ring problems: For each pair $\{(N, c), (N, c')\}$ of elements of $\mathcal{C}$, each $\delta \in \mathbb{R}_+$, and and $i^* = \arg\max_{i \in N} c_i$, if $c'_{i^*} = c_{i^*} + \delta$ and for each $j \in N\backslash\{i^*\}$ $c'_j = c_j$, then $\varphi_{B[i^*]}(N, c') = \varphi_{B[i^*]}(N, c) + \delta$ and for each $j \in N\backslash\{i^*\}$, $\varphi_{B[j]}(N, c') = \varphi_{B[j]}(N, c)$.

The following property says that the outcome chosen by a rule should be invariant under the departure of an airline with the largest cost. Let i^* be an airline with the largest cost. Imagine that airline i^* pays its contribution $\varphi_{i^*}(N, c)$ and leaves. Let $x \equiv \varphi(N, c)$ and $N' \equiv N\backslash\{i^*\}$. Imagine that for all $j \neq i^*$ the contribution x_{i^*} is intended to cover $c_{i^*} - c_j$. Airline j may or may not benefit from x_{i^*} under a situation that depends on the difference between x_{i^*} and $c_{i^*} - c_j$. Consider two cases. If $x_{i^*} \geq c_{i^*} - c_j$, airline j benefits since the cost of j is revised down by $x_{i^*} - (c_{i^*} - c_j)$. Otherwise, airline j does not benefit since the cost of j is not revised down. As a result, the revised cost of each remaining airline $j \in N'$ is $c_j - \max\{x_{i^*} - (c_{i^*} - c_j), 0\}$, namely $\min\{c_j, c_{i^*} - x_{i^*}\}$. For the details of this property, see Yeh (2004), and Hwang and Yeh (2012).

Last-agent airport consistency[11]: For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, all $i \in N$, $N' = N\backslash\{i^*\}$, where $i^* = \arg\max_{i \in N} c_i$, and $x = \varphi_A(N, c)$,

$$(N', \bar{c}^x_{N'}) \in \mathcal{C} \text{ and } x_{N'} = \varphi_A(N', \bar{c}^x_{N'}),$$

where for each $j \in N'$, $(\bar{c}^x_{N'})_{[j]} = \min\{c_j, c_{i^*} - x_{i^*}\}$.

The following property, referred to as "last-agent secret agreement equivalence", says that for reduced bidding ring problems where a buyer i^* with the largest valuation leaves the outcome chosen by a modified rule φ^* should coincide with that chosen by the original rule φ. Given φ, its modified rule φ^* is derived from "secret agreement" between i^* and any bidding ring in $N\backslash\{i^*\}$.

For the details of this property, we will explain the revised valuation and a game that any bidding ring in $N\backslash\{i^*\}$ play.

[11] In Yeh (2004), this property is called "last-agent consistency". In Hwang and Yeh (2012), they used "consistency" in their axiomatization of the nucleolus rule for airport problems. This property is originally introduced by Potters and Sudhölter (1999). The last-agent consistency is a weaker version of consistency.

First, the scenario of the revised valuation is as follows. Let i^* be a buyer with the largest valuation. Let $N' \equiv N\backslash\{i^*\}$. Notice that the valuation c_i of each buyer $i \in N$ is reinterpreted as his contribution to the bidding ring N. This is because c_i is the difference between the gain of N (i.e. c_{i^*}) and that of $N\backslash\{i\}$ (i.e. $c_{i^*} - c_i$). Thus the profile of valuations $(c_i)_{i\in N}$ is reinterpreted as buyers' contribution to N. Imagine that the bidding ring N' pays x_{i^*} to buyer i^* as a reward for his cooperation, and buyer i^* leaves. Thus the gain of N' is $c_{i^*} - x_{i^*}$. On the other hand, imagine that buyer $j \in N'$ competes with the members of $N'\backslash\{j\}$, and buyer j leaves from N'. Here, two scenarios are possible: (a) buyer i^* behaves cooperatively for $N'\backslash\{j\}$ or (b) buyer i^* behaves non-cooperatively for $N'\backslash\{j\}$. In the case of (a), since buyer i^* obtains x_{i^*} as a reward for his cooperation for N', the coalition $N'\backslash\{j\}$ can avoid competing with buyer i^*. Thus, in the case of (a), the gain of $N'\backslash\{j\}$ is $c_{i^*} - c_j - x_{i^*}$. In the case of (b), since $N'\backslash\{j\}$ competes with buyers i^* and j, the gain of $N'\backslash\{j\}$ is 0. Thus, the possibly highest gain of $N'\backslash\{j\}$ is $\max\{c_{i^*} - c_j - x_{i^*}, 0\}$. Therefore, the contribution of buyer j is revised as $(c_{i^*} - x_{i^*}) - \max\{c_{i^*} - c_j - x_{i^*}, 0\}$, which means "buyer j's contribution to the coalition N'". As a result, the revised valuation of each remaining buyer $j \in N'$, denoted $\bar{c}^x_{N'}$, is $(c_{i^*} - x_{i^*}) - \max\{c_{i^*} - c_j - x_{i^*}, 0\}$, namely, for each $j \in N'$ $(\bar{c}^x_{N'})_{[j]} = \min\{c_j, c_{i^*} - x_{i^*}\}$.

Next, the scenario of a game that the members of any bidding ring in N' play is as follows. Imagine that each agent $j \in N'$ has his revised valuation $\bar{c}^x_{N'[j]}$ mentioned above and that the members of $S \subseteq N'$ with $S \neq \emptyset$ compete with the members of $N'\backslash S$ in the English auction. Furthermore imagine that in this competition the members of S and buyer i^* agree upon that buyer i^*cooperates with S and they pay x_{i^*} to buyer i^* as his reward. Therefore the members of S, where $\emptyset \neq S \subseteq N'$, always win and play a game, denoted $W(S)$, where $W(S) = c_{i^*} - \max_{j\in N'\backslash S}(\bar{c}^x_{N'})_{[j]} - x_{i^*}$. Here, the number $c_{i^*} - \max_{j\in i^{*\prime}\backslash S}(\bar{c}^x_{N'})_{[j]}$ is the gain of S in the English auction, and the number x_{i^*} is the payment of S to buyer i^*. Notice that $W(\emptyset) = 0$.

An explanation of **last-agent secret agreement equivalence** is as follows: Imagine that the members of $S \subseteq N'$ with $S \neq \emptyset$ compete with the members of $N'\backslash S$ in the English auction. Buyer i^* and the members of S agree upon that buyer i^* cooperates with S and the members of S pay x_{i^*} to buyer i^* as his reward. This agreement is referred to as "secret agreement" between i^* and $S \subseteq N'$. Then the remaining buyers in N' will play a game $W(S)$, and thus each buyer $k \in N'$ gets $\varphi^*_{[k]}(N', \bar{c}^x_{N'}) \equiv (\phi \circ W(N', \bar{c}^x_{N'}))_{[k]}$, where for all $S \subseteq N'$ with $S \neq \emptyset$ $W(S) = c_{i^*} - \max_{j\in N'\backslash S}(\bar{c}^x_{N'})_{[j]} - x_{i^*}$ and $W(\emptyset) = 0$.[12] This property requires that the outcome chosen by a rule φ should be invariant even if secret agreement between i^* and any bidding ring $S \subseteq N\backslash\{i^*\}$ is made.

[12] $(\phi \circ W(N', \bar{c}^x_{N'}))_{[k]}$ is the k-th coordinate of $\phi \circ W(N', \bar{c}^x_{N'}) \in \mathbb{R}^{N'}$.

Last-agent secret agreement equivalence: For all $(N,c) \in \mathcal{C}$ with $n \geq 2$, $N' \equiv N \backslash \{i^*\}$, where $i^* = \arg\max_{i \in N} c_i$, and $x = \varphi_B(N,c)$,

$$(N', \bar{c}^x_{N'}) \in \mathcal{C} \text{ and } x_{N'} = \varphi^*(N', \bar{c}^x_{N'}),$$

where

(i) $\varphi^*(N', \bar{c}^x_{N'}) \equiv \phi \circ W(N', \bar{c}^x_{N'})$, where for all $j \in N'$ $(\bar{c}^x_{N'})_{[j]} = \min\{c_j, c_{i^*} - x_{i^*}\}$, and

(ii) for all $S \subseteq N'$

$$W(N', \bar{c}^x_{N'})(S) = \begin{cases} c_{i^*} - \max_{j \in N' \backslash S} (\bar{c}^x_{N'})_{[j]} - x_{i^*} & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}.$$

Theorem B *(Yeh 2004, Hwang and Yeh 2012) For airport problems, the nucleolus rule is the only rule satisfying equal treatment of equals, last-agent additivity, and last-agent airport consistency.*

We obtain the following axiomatization of solution φ^{Nu}_B that is the anti-dual of the axiomatization of φ^{Nu}_A appearing in Theorem B.

Theorem 6 *(Anti-dual of Theorem B) For bidding ring problems, the nucleolus rule is the only rule satisfying equal treatment of equals, last-agent additivity, and last-agent secret agreement equivalence.*

Proof. We compute the anti-dual of each axiom involved in an axiomatization of the nucleolus rule appearing in Theorem B. *Equal treatment of equals,* and *last-agent additivity* are self-anti-duals, respectively. The proof is omitted. We show that *last-agent secret agreement equivalence* is anti-dual to *last-agent airport consistency.* By using the proportional function form, we can express *last-agent airport consistency* as follows.

(7) For all $(N,c) \in \mathcal{C}$ with $n \geq 2$, $F((N,c), -\phi \circ -c_A(N,c)) = 1$ iff for $N' \equiv N \backslash \{i^*\}$, where $i^* = \arg\max_{i \in N} c_i$, and $x = -\phi \circ -c_A(N,c)$, $(N', \bar{c}^x_{N'}) \in \mathcal{C}$ and $x_{N'} = -\phi \circ w(N', \bar{c}^x_{N'})$, where for all $j \in N'$ $(\bar{c}^x_{N'})_{[j]} = \min\{c_j, c_{i^*} - x_{i^*}\}$ and for all $S \subseteq N'$ $w(S) \equiv -\max_{j \in S} (\bar{c}^x_{N'})_{[j]}$.

Since $(v_B)^{ad} = -c_A$, the formula (7) can be rewritten by the following formula.

(8) For all $(N,c) \in \mathcal{C}$ with $n \geq 2$, $F((N,c), -\phi \circ (v_B)^{ad}(N,c)) = 1$ iff for $N' \equiv N \backslash \{i^*\}$, where $i^* = \arg\max_{i \in N} c_i$, and $x = -\phi \circ (v_B)^{ad}(N,c)$, $(N', \bar{c}^x_{N'}) \in \mathcal{C}$ and $x_{N'} = -\phi \circ w(N', \bar{c}^x_{N'})$, where for all $j \in N'$ $(\bar{c}^x_{N'})_{[j]} = \min\{c_j, c_{i^*} - x_{i^*}\}$ for all $S \subseteq N'$ $w(S) \equiv -\max_{j \in S} (\bar{c}^x_{N'})_{[j]}$.

For all $S \subseteq N'$ with $S \neq \emptyset$,

$$\begin{aligned} w^{ad}(S) &= -w(N') + w(N' \backslash S) \\ &= c_{i^*} - x_{i^*} - \max_{j \in N' \backslash S} (\bar{c}^x_{N'})_{[j]}, \end{aligned}$$

and $w^{ad}(\emptyset) = 0$. For all $S \subseteq N'$ let $W(S) \equiv w^{ad}(S)$. The formula (8) can be rewritten by the following formula.

(9) For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $F((N, c), -\phi \circ (v_B)^{ad}(N, c)) = 1$ iff for $N' \equiv N \backslash \{i^*\}$, where $i^* = \arg\max_{i \in N} c_i$, and $x = -\phi \circ (v_B)^{ad}(N, c)$, $(N', \bar{c}^x_{N'}) \in \mathcal{C}$ and $x_{N'} = -\phi \circ W^{ad}(N', \bar{c}^x_{N'})$, where for all $j \in N'$ $(\bar{c}^x_{N'})_{[j]} = \min\{c_j, c_{i^*} - x_{i^*}\}$ for all $S \subseteq N'$ $W^{ad}(S) = -\max_{j \in N' \backslash S} (\bar{c}^x_{N'})_{[j]}$.

By Theorem 3 and the fact that $((v_B)^{ad})^{ad} = v_B$, the anti-dual of the formula (9) is the following formula.

(10) For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $F^{ad}((N, c), \phi \circ v_B(N, c)) = 1$ iff for all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $N' \equiv N \backslash \{i^*\}$, where $i^* = \arg\max_{i \in N} c_i$, $x = \varphi_B(N, c)$, $N' = N \backslash \{i^*\}$, and all $S \subseteq N'$

$$(N', \bar{c}^x_{N'}) \in \mathcal{C} \text{ and } x_{N'} = \varphi^*(N', \bar{c}^x_{N'}),$$

where (i) $\varphi^*(N', \bar{c}^x_{N'}) \equiv \phi \circ W(N', \bar{c}^x_{N'})$, where for all $j \in N'$ $(\bar{c}^x_{N'})_{[j]} = \min\{c_j, c_{i^*} - x_{i^*}\}$, and (ii) for all $S \subseteq N'$

$$W(N', \bar{c}^x_{N'})(S) = \begin{cases} c_{i^*} - \max_{j \in N' \backslash S} (\bar{c}^x_{N'})_{[j]} - x_{i^*} & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Therefore, the anti-dual of last-agent airport consistency is last-agent secret agreement equivalence. ■

Thanks to Theorem 4, logical independence of the axioms appearing in Theorem 6 holds.

5 Concluding remarks

Finally, we discuss a generalization of duality and anti-duality approach. A TU game $v \in \mathcal{V}$ can be considered as an element of $\mathbb{R}^{2^N - 1}$, its dual can be expressed by a linear transformation $w = Mv$, where M is a $(2n-1) \times (2n-1)$ matrix, on the space.[13] If there exists the inverse matrix M^{-1} and $(Mv)(N) =$

[13] For instance, Faigle and Grabisch (2016) investigated a relation between such a linear transformation as the Möbius transformation and solutions for TU games.

$v(N)$, one can define M-transformation of solutions and axioms for TU games. Using these notions, one can generalize duality and anti-duality approach for axiomatic analysis of solutions for TU games. Hokari et al. (2017) show a non-trivial M-transformation for which the Shapley value is invariant, that is for all $v \in 2^{N\setminus\{\emptyset\}}$, $Sh(Mv) = Sh(v)$. Analogously to this approach, one can define M-transformation of rules, and axioms for allocation problems. Whether the M-(anti-) duality approach for axiomatic analysis of rules for allocation problems is useful may deserve investigation, which we leave to the future research.

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