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Abstract

Because anti-Marxist's criticism against Fundamental Marxian Theorem(FMT) is based on an assertion that this proof can be understood as a 'sun-power exploitation,' if we assume, for example, 'Sun-power Theory of Value', we should prove not only exploitation but also Labor Theory of Value itself. Therefore, this paper aims to prove Labor Theory of Value mathematically by focusing on the historically conditional proportionality between labor input and amount of products which is assumed in Labor Theory of Value. By this proof of the conditional proportionality, we show that marginalist principle does not disturb Labor Theory of Value in capitalism at all. Furthermore, marginalist principle is important to show the labor process as a subjective optimization process which is not by the sun but only by human beings. In this way, we use anti-Marxist's marginal principle to object anti-Marxist criticism against Labor Theory of Value.

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# A Proof of Labor Theory of Value based on Marginalist Principle

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**Abstract** *Because anti-Marxist's criticism against Fundamental Marxian Theorem(FMT) is based on an assertion that this proof can be understood as a 'sun-power exploitation,' if we assume, for example, 'Sun-power Theory of Value', we should prove not only exploitation but also Labor Theory of Value itself. Therefore, this paper aims to prove Labor Theory of Value mathematically by focusing on the historically conditional proportionality between labor input and amount of products which is assumed in Labor Theory of Value. By this proof of the conditional proportionality, we show that marginalist principle does not disturb Labor Theory of Value in capitalism at all. Furthermore, marginalist principle is important to show the labor process as a subjective optimization process which is not by the sun but only by human beings. In this way, we use anti-Marxist's marginal principle to object anti-Marxist criticism against Labor Theory of Value.*

**Key Words; Fundamental Marxian Theorem, labor theory of value, marginalist principle, constant return to labor, machine-based production system**

## I. Defense of FMT and Marginalist Principle

Fundamental Marxian Theorem (FMT) is the most important achievement in mathematical Marxian economics. Because this proof by prof. Okishio is robust and strong in mathematics, many mainstream economists have taken an attitude to refuse it in a non-mathematical way. Their argument is that it is mathematically true not only in the case of labor theory of value but also in any type of value theory. That is, if we accept sun power theory of value, for example, this theorem has just proved 'sun power exploitation.' In this sense, if Marxists want to prove labor exploitation, Marxists have to prove labor theory of value first of all.

Although, of course, many Marxist economists fought against this contradiction, the most important points should be the difference between the human being and the sun. The human being has a will to make life better, while the sun does not have any will. The human being expenses labor power for this purpose, while the sun expenses sun power without any purpose. Therefore, this must be understood as the problem whether there is a subjective object to be realized by such a subjective action. In my opinion, existence of this subjective object in labor theory of value should be shown clearly in the form of a subjective optimization problem. In other words, labor theory of value should be modelled as a model where the agents maximize a certain target.

However, if we want to formulate a subjective optimization problem, it gives rise to an another question whether or not we should accept the marginalist principle. It is because linear functions do not have any non-corner solutions of the optimization problem and it is not realistic. In this sense, some researches tried to link with the labor theory of value with the marginalist principles.

For example, Johansen(1963) linked linear production functions to the principle that marginal utility of each goods is proportional to each price. By this way, he proved that the influence of marginal utilities on prices is limited. Furthermore, a similar research has been provided by a Chinese mathematical Marxists Bai Baoli and his daughter in Bai & Bai (2014b)[1]. A prominent superiority of their research is that a linear production function is led as a long-run horizontal supply curve in perfectly competitive market based on a short-run upward sloping marginal cost curve in the second subsection of chapter three of Bai & Bai (2014b). This theoretical framework is basically same with the long-run horizontal supply curve in the textbooks of mainstream microeconomics, but a very important difference between them is that Bai & Bai's marginal cost is the marginal labor expense. Therefore, it is obvious that their intention is to support the labor theory of value.[2] In this way, the proportionality between the marginal utility and the labor expense could be explained.

However, what we should note is our purpose is to identify the value determination as a subjective optimization problem, and for this purpose, we should compare the utility of the product not with the labor expense itself but with the disutility of the labor expense. Therefore, we will define an optimization problem to maximize net utility by equalizing the marginal utility of the produced goods and the marginal disutility of the input labor to produce this goods.

## II. The Simplest Version of the Value-determination Model

For this modelling, what should be done first is to identify the cost and benefit in

[1] They also provided another similar research in Bai & Bai (2014a), but this research has strong assumptions such as a linear and homogeneous utility function and constant ratios of consumed goods. Therefore, this paper focuses on only their research Bai & Bai (2014b).

[2] In their understanding, marginal productivity theory is a theory which explains the 'contribution' of capital on the production process and justifies capitalists' profit taking. In this sense, they criticized marginal productivity theory, but accepted short-run upward sloping marginal cost curve.

this subjective action: labor process. In other words, subjective purpose is to maximize benefit minus cost. If so, what is the cost and what is the benefit in this subjective action? Because this subjective action has been existing not only in the capitalist era but also pre-capitalist era, the benefit should be the utility which is called labor expense. Therefore, at the point to maximize this benefit minus cost, the marginal utility of the products produced in this labor process should be equal to the marginal disutility of the labor expense. That is,

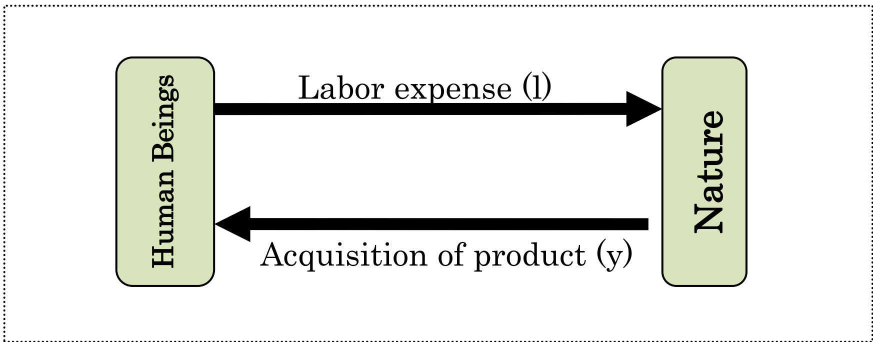
$$\frac{\mathrm{dD}}{\mathrm{dl}} = \frac{dU}{dy} \cdot \frac{dy}{dl}$$

Or in short,

$$\mathrm{dD(l)=dU(y(l))},$$

where l, y, D and U express the amount of labor expense and produced goods, the disutility of labor expense and the utility by consuming y produced by labor. Humans make decision how much labor should be expensed to maximize net utility (=utility minus disutility) as the metabolism between human beings and nature (See Figure-1). This is the determination of labor input, and the determination of value. In this case, value of one unit of product can be expressed by ($l^*/y^*$), that is the ratio of optimized input labor($l^*$) to produced amount of goods($y^*$).

**Figure 1　Labor Expense and Acquisition of Product as the Metabolism between Human Beings and Nature**




Then, let me introduce much concrete and mathematical formulation of the value-determination by unifying the utility of y and disutility of l by the following utility function;

$$\mathrm{U=U(y,l)} \quad \frac{\partial \mathrm{U}}{\partial \mathrm{y}} > 0,\ \frac{\partial \mathrm{U}}{\partial \mathrm{l}} < 0$$

where U is the increasing function of y and the decreasing function of l. However, in order to translate this function into the most popular Cobb-Douglas type of form, we replaced labor expense with free time as an explaining variable of this non-linear utility function as follows;

$$U=Cy^{\alpha}(H-l)^{\beta} \quad \frac{\partial U}{\partial y}>0, \frac{\partial U}{\partial (H-l)}>0$$

where H means the total time that is held by the human beings (eg. 24 hours in one day) and therefore (H-l) is the free time. Furthermore, we assume positive α and β in order to set this function as an increasing function of consumption (y) and a decreasing function of labor time (l). Furthermore, in order to be consistent to the above explanation, we also assume $0<\alpha,\beta<1$ which means diminishing marginal utility to both factors.

On the other hand, because the amount of product of the consumption goods is also a function of labor expense, we also specify the following non-linear production function;

$$y=Al^{\gamma}$$

where $A$ and γ express different kind of labor productivity under the assumption $0<\gamma<1$ which is consistent to the above explanation. In this case, above-mentioned maximization problem becomes the problem to choose the optimal labor input ($l$) to maximize $U=C(Al^{\gamma})^{\alpha}(H-l)^{\beta}$.

Therefore, partially differentiated U with respect to $l$ must be zero. That is,

$$\frac{\partial U}{\partial l}=CA^{\alpha}\alpha l^{\gamma\alpha-1}(H-l)^{\beta}-CA^{\alpha}\beta l^{\gamma\alpha}(H-l)^{\beta-1}=0$$

$$\Leftrightarrow \quad CA^{\alpha}l^{\gamma\alpha-1}(H-l)^{\beta-1}\{\gamma\alpha(H-l)-\beta l\}=0.$$

This can be simplified as $\gamma\alpha H=(\gamma\alpha+\beta)l$, and therefore the optimal labor input becomes

$$l^{*}=\frac{\gamma\alpha}{\gamma\alpha+\beta}H.$$

Furthermore, we can calculate the value for one unit product as follows

$$\frac{l^{*}}{y^{*}}=\frac{\gamma\alpha H}{\gamma\alpha+\beta}\Bigg/A\left(\frac{\gamma\alpha H}{\gamma\alpha+\beta}\right)^{\gamma}=\frac{1}{A}\left(\frac{\gamma\alpha H}{\gamma\alpha+\beta}\right)^{1-\gamma}$$

which indicates that this value is affected by A、γ、H and $\frac{\gamma\alpha}{\gamma\alpha+\beta}\left(=1\Big/\left(1+\frac{\beta}{\gamma\alpha}\right)\right)$. Concretely speaking, ①increase in labor productivity (A) leads decrease in value for one unit product. ②Although γ reflects another kind of labor productivity, γ's effects are not clear.[3] ③Although H is fixed in nature, if necessary domestic labor can be cut by domestic electrification, actual free time can be lengthened and it is same with increase in H. In this case, disutility of labor becomes small and labor supply becomes larger, and finally value for one unit product becomes larger. ④Because increase of $\frac{\gamma\alpha}{\gamma\alpha+\beta}$ makes leisure time less important, disutility of labor expense becomes smaller and value for one unit product becomes larger. It is same with the case of H.[4]

These results show that the value is determined by the technical condition (parameter of production function) such as ① ② and the various conditions on the preference such as ③ ④ (the parameters of the utility function). And since Marx did not conduct a detailed analysis on the disutility of labor, he did not explain the effects of ③ and ④, but only the situation of ① ② was discussed.

However, when arguing a little more, it turns out that the case of $\gamma$ = 1 is the most Marxistic situation in the following sense. In this case, the production function becomes y = Al, which is a typical 'LTV situation' since the input labor per unit product (l / y) is constant as 1 / A. Furthermore, it is important that this situation can be held as a solution of the optimization problem when γ = 1 independently from the various properties of utility function. The reason why Marx ignored the utility side such as ③ ④ can be explained by this way.

Additionally speaking, because the condition $\gamma$ =1 expresses constant productivity, it means that we do not need "marginal productivity principle" to explain how input labor (= value) is determined. Only the marginal utility principle is enough to explain, and this was the case of Johansen(1963) and Bai & Bai(2014a). But they just assumed such a condition, while we have proved.

## III. Means of Labor and "LTV Situation"

Based on the above findings, what we must pay attention to in the next step is much more technological reality after the industrial revolution. It is because the industrial

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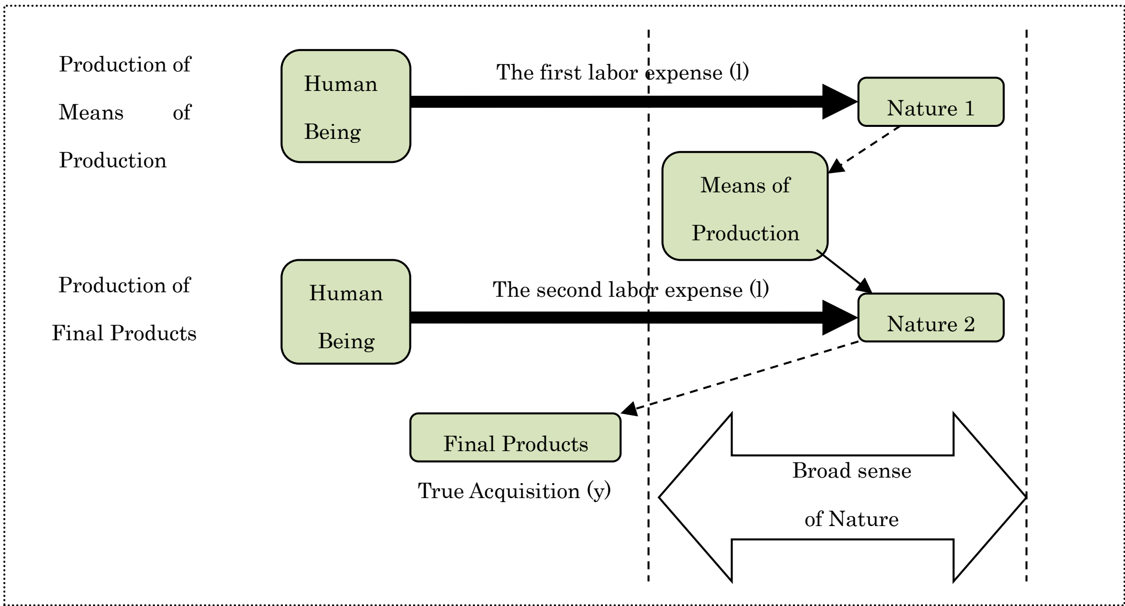
[3] Differentiating $\left(\frac{\gamma\alpha H}{\gamma\alpha+\beta}\right)^{1-\gamma}$ with respect to γ becomes $\left(\frac{\gamma\alpha H}{\gamma\alpha+\beta}\right)^{1-\gamma}\left[-log\frac{\gamma\alpha H}{\gamma\alpha+\beta}+(1-\gamma)\frac{\beta}{\gamma\alpha+\beta}\right]$. It shows, by a complex calculations, that larger γ increases l*/y* if γ is close to zero, and decreases l*/y* if γ is close to one in the range 0<γ≦1.

[4] Strictly speaking, all these explanations are just the determination of the 'individual value'. However, we can assume that all the agents in the society behave in same way. Therefore, even if there are differences among 'individual values', social value can be calculated just by averaging them.

revolution has changed the form of production functions. Before the industrial revolution, means of labor was not so important, and the productive activities were covered mainly by human powers. However, after the industrial revolution, the machines had become the most decisive factor of production.

However, what we must know here is that this transformation of production system made labor expend twice; first to produce means of production, and secondly to produce final goods as shown in **Figure 2**, and this duplicity can provide a crucially important and preferable characteristic for labor theory of value. It is because labor theory of value needs a proportionality of labor input or 'value[5]' and physical amount of products, and this proportionality can be led only by the above duplicity of labor input which makes technology constant return to labor. In other words, although the original single production system had a diminishing return to labor as shown as the production function $y=Al^{\gamma}$ $(0<\gamma<1)$ in the previous section, the above duplicity of labor input changes the ultimate form of production function. Now we will introduce this ultimate form of new production function.

**Figure 2 Production of Means of Labor and Final Products**




In this case, the most normal type of new production function which includes means of production might be

[5] Strictly speaking, because substance of value is input labor in Marxian economics, there is no difference between value and input labor. In this sense, the word 'value' here just means a certain substance imagined as 'value'.

$$y = Al^{\gamma_1}k^{\gamma_2}.$$ [6]

Here, we assume $k$ is a 'flow' variable for simplification, even though actual k is a stock variable. This simplification is not essential on this issue. And we also assume $0<\gamma_1<1$, $0<\gamma_2<1$ to express that there is diminishing return to both factors. Furthermore, we need to introduce one more production function of means of production as follows;

$$k = Bl^{\gamma_3}.$$

In order to simplify again, we neglected the input of means of production for this production. In this case, combining these two production functions leads

$$y = Al^{\gamma_1}(Bl^{\gamma_3})^{\gamma_2} = AB^{\gamma_2}l^{\gamma_1}l^{\gamma_3\gamma_2} = AB^{\gamma_2}l^{\gamma_1+\gamma_3\gamma_2}.$$

Therefore, we can understand that above-mentioned situation is of $\gamma_1 + \gamma_3\gamma_2 = 1$ which can be held under the condition that $\gamma 1$, $\gamma 2$ and $\gamma 3$ are all under 1. It means that this condition does not the diminishing returns to individual factors of production. However, as long as $\gamma 3 <1$, this condition needs that $\gamma 1 + \gamma 2$ must be over 1, that is increasing return to scale. Although such a situation cannot be assumed in macro economies, it is natural to assume on the level of micro economics under capitalism. This is an assumption of scale merit that larger companies run more efficiently. In this case, technology becomes constant return to labor ultimately, and guarantees the proportionality between labor input or 'value' and physical amount of products. I want to call this situation "LTV(Labor Theory of Value) situation" realized by the condition $\gamma_1 + \gamma_3\gamma_2 = 1$.

## Ⅳ. Another Explanation by Minimum Necessary Investment

Such a focus on means of production can provide another type of explanation to lead the situation $\gamma=1$. One of the important characteristics of the machinery system after the industrial revolution is that it has a certain minimum amount to be activated. Companies cannot operate them below this level, but can do only when means of production are accumulated over this level. If we express this level as $k_0$, this relation can be shown as the following production function of the final goods;

$$y = A(k - k_0)^{\gamma_4}.$$

Here, we assume the diminishing return to capital as $0<\gamma_4<1$, but at the same time, we want to introduce the same production function of the means of production with the last section as $k = Bl^{\gamma_3}$, and also assume the diminishing return to labor as $0<\gamma_3<1$. In this

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[6] Differently from Leontief type of production function, Cobb-Douglas type of production function is better to accept substitutability of factors of production as a kind of marginalist analysis. Marginal substitution and marginal productivity of each factor are closely related. Bai & Bai(2014b) pointed out Sraffa's limitation to assume unsubstitutability of factors of production in page 116.

case, above production function of the final goods can be translated into

$$y = A(Bl^{\gamma_3} - Bl_0^{\gamma_3})^{\gamma_4},$$ [7]

and labor productivity[8] becomes

$$\frac{y}{l} = \frac{AB^{\gamma_4}(l^{\gamma_3} - l_0^{\gamma_3})^{\gamma_4}}{l}.$$

To check the return to labor, differentiating it with l leads

$$\frac{\partial}{\partial l}\left(\frac{y}{l}\right) = \frac{AB^{\gamma_4}\gamma_4\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4 - 1} \cdot \gamma_3 l^{\gamma_3 - 1} \cdot l - AB^{\gamma_4}\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4}}{l^2}$$

$$= \frac{AB^{\gamma_4}\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4 - 1}\left\{\gamma_4\gamma_3 l^{\gamma_3} - \left(l^{\gamma_3} - l_0^{\gamma_3}\right)\right\}}{l^2}$$

$$= \frac{AB^{\gamma_4}\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4 - 1}\left\{l_0^{\gamma_3} - l^{\gamma_3}(1 - \gamma_4\gamma_3)\right\}}{l^2}$$

$$>0 \quad \text{when} \quad l < l_0(1 - \gamma_3\gamma_4)^{-\frac{1}{3}}$$

$$=0 \quad \text{when} \quad l = l_0(1 - \gamma_3\gamma_4)^{-\frac{1}{3}}$$

$$<0 \quad \text{when} \quad l > l_0(1 - \gamma_3\gamma_4)^{-\frac{1}{3}} \ .$$

Here, we assumed $0 < \gamma_3\gamma_4 \leqq 1$, and in this case, we could find an area where is the increasing return and constant return, if we introduce a concept of minimum necessary amount of capital and even if we assume the diminishing return on the production functions. For us, the most important case is the second case where technology becomes constant return to labor as we saw in the previous section. This is the 'Labor Theory of Value (LTV) situation.'

This situation can be realized if human beings choose the maximum efficiency in terms of labor input. It is true that this section does not consider any influence of the utility at all differently from the second section of this paper, and in this sense, there is a possibility that $\tilde{l}$ is not chosen as $l^*$. However, if producers of a certain product compete each other in the market, only the most efficient producers can survive by applying the same technological condition in the long-run, and that condition should be the point shown as $l = \tilde{l}$. In this case, each producer's size of production is fixed, but total demand for this product is covered by a certain number of producers whose number is determined

[7] Here, $l_0$ is the necessary input labor to produce the minimum necessary investment.

[8] Take note that the labor productivity is the inverse of the value for one unit product which we discussed in the second section of this paper..

by $\frac{total\ demand\ determined\ by\ this\ price\ level}{size\ of\ production\ of\ each\ producers}$[9]. This situation makes long-run supply curve horizontal[10], and realize the 'LTV situation' in the sense that y/l ratio becomes constant.

In conclusion, as shown in these ways, the proportionality between its input (embodied) labor or "value" and physical amount of products could be realistic conditionally in capitalism where means of labor have played a more important role in production process. As we know that there had not been a concept of 'commodity value' in the pre-market societies before capitalism, generalization of the market system was the crucial condition for the modern concept of value. But now we also need to know another important condition of the proportionality between input (embodied) labor or "value" and physical products which has been introduced by the machine-based production system after the industrial revolution.[11] In this way, Labor Theory of Value can be proved.

**References**

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Bai, Baoli and Bai, Ruixue(2014b), *General Theory of Value and Price, volume 3, Comparison between Various Value Theories*, pp.155-163.

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9 This explanation is completely same with the ordinary explanation of the long-run horizontal supply curve in the textbooks of mainstream microeconomics. In this case, all the companies, in the long-run, choose the amount of production to minimize the unit cost, and then the number of companies for this product is determined by $\frac{total\ demand\ determined\ by\ this\ price\ level}{amount\ of\ production\ of\ each\ companies}$.

10 In the same way, Bai & Bai(2014b) introduced this horizontal supply curve.

11 In fact, the machine-based production system which was decisive for capitalism provided two other ways to establish the concept of commodity value. It's first way is that it destroyed workers' skill, standardized the human labor into simple labor, and substantiated the concept of 'abstract human labor.' It has been the materialistic base of the concept of commodity value. Secondly, the machine-based production system had realized full market system by its 'scale-merit' technology which accelerated the social division of labor. Because the concept of commodity value needs a situation where all of the wealth can be exchanged and therefore can be imagined to include a certain substance, society should be marketized at all by its 'scale-merit' technology. Consequently, both of characteristics show that the concept of "value" is historical.

Johansen, Leif(1963), “Labor Theory of Value and Marginal Utilities,” *Economics of Planning*, vol.3, no.2.