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Shigeru Makioka*

December 26, 2014

Abstract

In a basic model of search and match, thanks to the assumption that producer-sellers and consumer-buyers pay constant search costs per one unit of a single type of goods, it suffices to consider the *retail transactions* between producer-sellers and consumer-buyers. We extend this model to allow for the possibilities of *economies and diseconomies of scopes in search activities* over two types goods. We show that producer-sellers make *wholesale transactions* with one another when the benefit of economies of scope is strong enough. But when the benefit of economies of scope in search activities for buyers compensates the loss of diseconomies of scope in search activities for sellers, there are multiple equilibria: Matched pairs of producer-sellers always make wholesale transactions in one equilibrium. But they never make those in another, so that there only are retail transactions.

Keywords: search and match; economies of scope; wholesale transaction
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1 Introduction

In a market with search frictions between producers and consumers, it is natural that there is a role for wholesalers to reduce these frictions. However in a basic model of search and match, no wholesalers play such a role. In our point of view, this is due to the implicit assumption that there are no economies of scale and/or scope in search activity. The purpose of our paper is to explore how wholesalers play their role once we no longer impose this assumption.

In basic models of search and match, each transaction between producer-sellers and consumer-buyers involves a single unit of goods and hence it never involves wholesale transactions among sellers which enable them to keep inventory of goods and to sell multiple units of goods to consumer-buyers. This setup is justified under the assumption of constant search cost per unit of goods. Under this assumption, the search values for each unit of goods becomes the same for sellers with different size of inventories. This implies that there is no need for a seller to keep a larger inventory by buying goods from other sellers, i.e., there is no role played by wholesale transactions.[1] However, this is no longer the case if search cost per unit of goods is variable both for sellers and buyers, i.e., if there are (dis)economies of scope or scale in search activities. In such a case, the search values per unit of goods for sellers depend on the amount and variety of goods they are endowed with. Thus, to optimize its inventory size, each seller may buy or sell goods upon meeting other sellers. This wholesale transaction should be explicitly taken into account once the model is extended to allow for the cases where there are economies or diseconomies of scope (or scale) in search activities for both sellers and buyers.

As a first step to these extensions, we consider a model of search and match in which there two kinds of goods, which makes us possible to define the concept of *economies and diseconomies of scope* in search activities for both sellers and buyers in section 2.[2]

[1] Basic models of search and match typically take up a following setup: being endowed with a single unit of goods, each seller enters the market, sells it to a matched buyer who wants to consume just one unit of goods and then exits. This standard setup of search and match is employed, for example, by Gale (1987) and Mortensen and Wright (2002). This setup is justifiable when both sellers and buyers are assumed to incur constant search costs per unit of goods at each moment of time while they search for their trading partners. As a result, any seller who is endowed with multiple units of goods can be considered as a collection of sellers each being endowed with a single unit of goods. This implies that inventory size has no influence on the resource allocation and hence it does not matter whether wholesale transactions between sellers are discarded or not.

[2] Burdett and Mauleg (1981), Carlson and McAfee (1984) and Zhou (2012) also introduce economies of scope in search activities but only those for buyers. Furthermore they consider an extreme case of degree of economies of scope in which no more cost is required to search for wider varieties of goods.

The main question we ask in the analysis of the model is the following one: How do economies and diseconomies of scope in search actitivies affect the equilibrium probability of wholesale transaction? In section 3, we show the equilibrium probability of wholesale transaction as a mapping from combinations of economies and diseconomies of scope in search activities for sellers and buyers. This wholesale transaction, in our model, takes place only between a matched pair of sellers, each of whom has one unit of different goods.[3] Evidently, each seller has a higher (lower) incentive to make a wholesale transaction if doing so reduces (raises) its own search cost per unit of goods, i.e., if there are economies (diseconomies) of scope in search activities for sellers. We find that a higher degree of economies of scope in search activities for sellers leads to a higher equilibrium probability of wholesale transaction whenever there are diseconomies of scope in search activities for buyers. But if there are economies of scope in search for buyers, a lower degree of diseconomies of scope for sellers may not lead to a higher equilibrium probability of wholesale transaction. In such a case, there exist multiple equilibria; the equilibrium probability of wholesale transaction can be both zero and one. These results are summarized in Theorem and Corollary.

Lastly, in section 4, we discuss the implicit assumptions we made on search costs and then we conclude by giving some implications of our theorem to the literature of middlemen. In the literature, the gain from making wholesale transaction is due to either a reduction in search cost for sellers (which we call wholesale gain) or expected reduction in search cost for buyers (which we call retail gain). As a result of this setup, there always are wholesale transactions in equilibrium and that such an equilibrium is unique. This fact is not well recognized in the literature and that the case of multiple equilibria, in other words, the case where there are a wholesale loss and a retail gain, has been ignored.

2 Model

To investigate how the equilibrium probability of wholesale transaction depends on economies and diseconomies of scope in search activities, we extend a standard model of search and match in which there is only one kind of goods to the one in which there are two kinds of goods called x and y.[4]

[3] Matched sellers are indifferent between buying the other variety of goods and selling their own variety of goods if they have incentive to make a wholesale transaction. What matters here is whether a wholesale transaction is made or not by a pair of matched sellers.

[4] We consider a model with two kinds of goods to introduce the concept of economies and diseconomies of scope in search activities. But the widening of goods' variety is not the crucial point. Allowing for wholesale transaction

Each potential seller is endowed with one unit of goods x or y. Upon entering the market, a potential seller becomes either state x or y seller holding one unit of goods x or y as an inventory. Each incurs a search cost $c_{s(x)}$ or $c_{s(y)}$ at each moment of continuous time while searching for its trading partners. Upon being matched with a trading partner, a state x or y seller can sell the goods and exit the market.

The point of departure from the standard setup is that, in addition to this sell-and-exit option, we also allow both state x and y sellers to buy the other kind of good from matched sellers and stay in the market. I.e., there is an option of making *wholesale transaction*. By making wholesale transaction, a seller becomes a state xy seller who holds a pair of goods x and y as an inventory, and incurs a search cost $c_{s(xy)}$ at each moment of time.

Buyers enter the market and become state xy buyers who want to consume both goods x and y. Upon being matched with a state xy seller, a state xy buyer exits the market at once, by consuming both goods and enjoying utility $2v$. But upon being matched with a state x or y seller, a state xy buyer buys and consumes only one kind of goods, and stays in the market as a state y or x buyer. At this moment this buyer enjoys utility v. A state x or y buyer exits the market and enjoys utility v by buying and consuming the goods upon being matched with a seller who has a wanted good.

Each of all these state xy, x and y buyers respectively incurs a cost $c_{b(xy)}$, $c_{b(x)}$, and $c_{b(y)}$ at each moment of time while searching for a seller.

Given these search cost parameters for sellers and buyers, we define the index of *economies of scope in search activities for sellers* as $ESS \equiv c_{s(x)} + c_{s(y)} - c_{s(xy)}$ and that of *economies of scope in search activities for buyers* as $ESB \equiv c_{b(x)} + c_{b(y)} - c_{b(xy)}$. If these are positive, we say that there are economies of scope. If negative, there are diseconomies of scope. Economies of scope in search activity is limited to the extent that it costs more if inventory is larger, i.e., $c_{s(xy)} > c_{s(i)}$ and $c_{b(xy)} > c_{b(i)}$ for $i \in \{x, y\}$. The indices of economies of scope for sellers and buyers, ESS and ESB, are bounded above by $c_s(x)$ and $c_b(x)$, if the search costs are symmetric so that $c_{s(x)} = c_{s(y)}$ and $c_{b(x)} = c_{b(y)}$. We assume this symmetry for the rest of this paper.

The states of sellers and buyers other than those described above are not allowed in our model by implicitly assuming that they incur prohibitively high search costs. Moreover, we assume symmetry of agents so that our focus can be set on the search activities of representative sellers and buyers in state x, y, and xy, which we denote by $s(i)$ and $b(i)$ where $i \in \{x, y, xy\}$. The set of these representative agents is defined as $A \equiv \{s(x), s(y), s(xy), b(x), b(y), b(xy)\}$.

over different kinds of goods between sellers is the most crucial point in our extension of the standard model.

The search value for an agent $a \in A$ is determined as follows. While searching for a trading partner, the agent has possibilities of meeting another agent $a' \in A$ according to a Poisson process whose arrival rate is identical to its population denoted by $\pi_{a'} \geq 0$. Upon meeting a trading partner, the agent a chooses a probability to agree on the terms of trade determined as a result of bargaining, which we denote by $\sigma_a(a') \in [0, 1]$. The vector of these probabilities $\sigma_a \equiv (\sigma_a(b(xy)),$ $\sigma_a(b(x)),$ $\sigma_a(b(y)),$ $\sigma_a(s(x)),$ $\sigma_a(s(y)),$ $\sigma_a(s(xy)))$ is called a *stationary strategy* for agent $a \in A$. Once they reach an agreement in bargaining, there are changes in search values for both agents from their state changes. And if one of them consumes goods, he enjoys utility. The sum of these changes in search values and the utility is defined as a surplus, $S_{aa'}$ for a matched pair of agents a and a'. The surplus is equally divided between the two agents. Engaging in the above described search activities and given a profile of stationary strategies for all agents $\sigma \equiv \{\sigma_a\}_{a \in A}$, an agent $a \in A$ has the search value $V_a(\sigma)$, which is defined as

$$rV_a(\sigma) = -c_a + \sum_{a' \in A} \pi_{a'} \sigma_a(a') \sigma_{a'}(a) \cdot \frac{1}{2} S_{aa'}(\sigma), \tag{1}$$

where r is the discount rate common to all agents.

With these search values, we can specify surpluses for all matched pair of agents. The surplus for a matched pair of state x and y sellers is just the sum of changes in search values from their state changes since consumption never takes place with wholesale transaction,

$$S_{s(x)s(y)}(\sigma) = S_{s(y)s(x)}(\sigma) = V_{s(xy)}(\sigma) - V_{s(x)}(\sigma) - V_{s(y)}(\sigma). \tag{2}$$

It is clear, from the definition of search values (1), that this surplus is positive (negative) if the degree of economies (diseconomies) of scope in search activities for *sellers* is large enough. But if the degree is not so large, it is not clear whether the sign of this surplus is positive or negative. The purpose of our investigation is to show how the sign of this surplus depends also on the degree of economies and diseconomies of scope in search activities for *buyers* in equilibrium.

Since the surpluses for pairs of a seller and a buyer are the sum of utility from consumption and the gains from their state changes, they are specified as,

$$
\begin{aligned}
S_{s(xy)b(xy)}(\sigma) = \quad S_{b(xy)s(xy)}(\sigma) \quad &= v + v - V_{s(xy)}(\sigma) - V_{b(xy)}(\sigma), && (3)\\
S_{s(x)b(xy)}(\sigma) = \quad S_{b(xy)s(x)}(\sigma) \quad &= v + V_{b(y)}(\sigma) - V_{s(x)}(\sigma) - V_{b(xy)}(\sigma), && (4)\\
S_{s(y)b(xy)}(\sigma) = \quad S_{b(xy)s(y)}(\sigma) \quad &= v + V_{b(x)}(\sigma) - V_{s(y)}(\sigma) - V_{b(xy)}(\sigma), && (5)\\
S_{s(xy)b(x)}(\sigma) = \quad S_{b(x)s(xy)}(\sigma) \quad &= v + V_{s(y)}(\sigma) - V_{s(xy)}(\sigma) - V_{b(x)}(\sigma), && (6)\\
S_{s(xy)b(y)}(\sigma) = \quad S_{b(y)s(xy)}(\sigma) \quad &= v + V_{s(x)}(\sigma) - V_{s(xy)}(\sigma) - V_{b(y)}(\sigma), && (7)\\
S_{s(x)b(x)}(\sigma) = \quad S_{b(x)s(x)}(\sigma) \quad &= v - V_{s(x)}(\sigma) - V_{b(x)}(\sigma), \quad \text{and} && (8)\\
S_{s(y)b(y)}(\sigma) = \quad S_{b(y)s(y)}(\sigma) \quad &= v - V_{s(y)}(\sigma) - V_{b(y)}(\sigma). && (9)
\end{aligned}
$$

All other surpluses for matched pairs of agents are negative since we implicitly assume that their search costs are prohibitively high.

In defining the search values above, we presumed a stationary state. In order to maintain a stationary state, the inflows into and the outflows from the population of each agents must balance. For state x sellers, there are two inflows of agents. The one is an inflow of type x producers who enter the market. They enter at rate μ_x, which is determined endogenously by the free entry condition,

$$V_{s(x)}(\sigma) \quad = \quad 0. \tag{10}$$

The other is an inflow of state xy sellers who sold goods y. The sum of these inflows of agents must balance with the outflux of state x sellers,

$$
\begin{aligned}
\mu_x + \pi_{s(xy)}\pi_{b(y)} \cdot \sigma_{s(xy)}(b(y)) \cdot \sigma_{b(y)}(s(xy)) \quad = \quad & \pi_{s(x)}\pi_{s(y)} \cdot \sigma_{s(x)}(s(y)) \cdot \sigma_{s(y)}(s(x)) \\
+ \quad & \pi_{s(x)}\pi_{b(xy)} \cdot \sigma_{s(x)}(b(xy)) \cdot \sigma_{b(xy)}(s(x)) \\
+ \quad & \pi_{s(x)}\pi_{b(x)} \cdot \sigma_{s(x)}(b(x)) \cdot \sigma_{b(x)}(s(x)). \qquad (11)
\end{aligned}
$$

The first term of the right hand side is the outflux of state x sellers who either bought a unit of goods y or sold goods x to state y sellers. The second and the third terms are those of state x sellers who sold goods x to state xy and x buyers respectively.

Similarly for the population of state y sellers, we must have the free entry condition for type y producers,

$$V_{s(y)}(\sigma) \quad = \quad 0, \tag{12}$$

which determines their rate of entry μ_y, and a stationary state condition,

$$\begin{aligned}
\mu_y + \pi_{s(xy)}\pi_{b(x)} \cdot \sigma_{s(xy)}(b(x)) \cdot \sigma_{b(x)}(s(xy)) &= \pi_{s(y)}\pi_{s(x)} \cdot \sigma_{s(y)}(s(x)) \cdot \sigma_{s(x)}(s(y)) \\
&+ \pi_{s(y)}\pi_{b(xy)} \cdot \sigma_{s(y)}(b(xy)) \cdot \sigma_{b(xy)}(s(y)) \\
&+ \pi_{s(y)}\pi_{b(y)} \cdot \sigma_{s(y)}(b(y)) \cdot \sigma_{b(y)}(s(y)). \qquad (13)
\end{aligned}$$

For state xy sellers, state x and y sellers flows in upon making wholesale transactions. This must balance with an outflux of state xy sellers who sold both or one of their goods;

$$\begin{aligned}
\pi_{s(x)}\pi_{s(y)} \cdot \sigma_{s(x)}(s(y)) \cdot \sigma_{s(y)}(s(x)) &= \pi_{s(xy)}\pi_{b(xy)} \cdot \sigma_{s(xy)}(b(xy)) \cdot \sigma_{b(xy)}(s(xy)) \\
&+ \pi_{s(xy)}\pi_{b(x)} \cdot \sigma_{s(xy)}(b(x)) \cdot \sigma_{b(x)}(s(xy)) \\
&+ \pi_{s(xy)}\pi_{b(y)} \cdot \sigma_{s(xy)}(b(y)) \cdot \sigma_{b(y)}(s(xy)). \qquad (14)
\end{aligned}$$

Unlike producers who enter the market at the endogenously determined rates, consumers enter the market at an exogenous rate μ. In order for this to be consistent with consumers' incentives to enter the market and become state xy buyers, the following participation constraint for state xy buyers must be satisfied:

$$V_{b(xy)}(\sigma) \geq 0. \qquad (15)$$

The inflow of these entrants must balance with an outflow of state xy buyers who bought either one or both goods from sellers,

$$\begin{aligned}
\mu &= \pi_{b(xy)}\pi_{s(xy)} \cdot \sigma_{b(xy)}(s(xy)) \cdot \sigma_{s(xy)}(b(xy)) \\
&+ \pi_{b(xy)}\pi_{s(x)} \cdot \sigma_{b(xy)}(s(x)) \cdot \sigma_{s(x)}(b(xy)) \\
&+ \pi_{b(xy)}\pi_{s(y)} \cdot \sigma_{b(xy)}(s(y)) \cdot \sigma_{s(y)}(b(xy)). \qquad (16)
\end{aligned}$$

For the population of state x buyers, the inflow of the state xy buyers who bought goods y must balance with an outflux of state x buyers who bought goods x from state xy or state x sellers,

$$\begin{aligned}
\pi_{b(xy)}\pi_{s(y)} \cdot \sigma_{b(xy)}(s(y)) \cdot \sigma_{s(y)}(b(xy)) &= \pi_{b(x)}\pi_{s(xy)} \cdot \sigma_{b(x)}(s(xy)) \cdot \sigma_{s(xy)}(b(x)) \\
&+ \pi_{b(x)}\pi_{s(x)} \cdot \sigma_{b(x)}(s(x)) \cdot \sigma_{s(x)}(b(x)). \qquad (17)
\end{aligned}$$

Similarly and lastly, for the population of state y buyers, we must have

$$\begin{aligned}
\pi_{b(xy)}\pi_{s(x)} \cdot \sigma_{b(xy)}(s(x)) \cdot \sigma_{s(x)}(b(xy)) &= \pi_{b(y)}\pi_{s(xy)} \cdot \sigma_{b(y)}(s(xy)) \cdot \sigma_{s(xy)}(b(y)) \\
&+ \pi_{b(y)}\pi_{s(y)} \cdot \sigma_{b(y)}(s(y)) \cdot \sigma_{s(y)}(b(y)), \qquad (18)
\end{aligned}$$

which completes the description of our model.

In order for the stationary state conditions (11), (13), (14), (16), (17), and (18) to be consistent, we must have a condition that both consumers and producers enter the market at the same rate,

$$\mu_x = \mu_y = \mu. \tag{19}$$

Under this symmetric entry rates condition (19), two equations, for example the equations (11) and (13), among the stationary state conditions become redundant. Using this representation of stationary state conditions, a state equilibrium strategy of our model is defined as follows:

Definition. *A stationary strategy profile* $\sigma^* = \{\sigma_a^*\}_{a\in A} \in [0,1]^{6\times 6}$ *is called a stationary equilibrium strategy if it is a fixed point of the best response mapping* $\sigma^{BR}(\cdot) = \{\sigma_a^{BR}(\cdot)\}_{a\in A}$ *from* $[0,1]^{6\times 6}$ *to* $[0,1]^{6\times 6}$*, in which* $\sigma_a^{BR}(\cdot)$ *for all* $a \in A$ *is defined as*

$$\begin{aligned} \sigma_a^{BR}(\sigma) \quad \equiv \quad & \underset{\hat{\sigma}_a \in [0,1]^6}{\arg\max} \sum_{a' \in A} \pi_{a'} \hat{\sigma}_a(a') \sigma_{a'}(a) \cdot \frac{1}{2} S_{aa'}(\sigma), \quad \textit{for all } \sigma \in [0,1]^{6\times 6} \\ & \textit{subject to the participation constraint for buyers } (15), \end{aligned} \tag{20}$$

in which the stationary populations π_a *for all* $a \in A$ *and entry rates* μ_x *and* μ_y *are determined by the free entry conditions for producers and the stationary state conditions,* (10)*,* (12)*,* (14)*,* (16)*,* (17)*, and* (18)*, where the search values and the surpluses are defined by equations* (1) *and* (2) - (9)*, given the symmetric entry rates conditions* (19)*.*

The objective of our paper is to characterize $\sigma_{s(x)}^*(s(y)) \cdot \sigma_{s(y)}^*(s(x))$, which we call *the equilibrium probability of wholesale transaction.*

3 Characterization of Equilibrium Probability of Wholesale Transaction

To answer our main question of the paper, we characterize the equilibrium in terms of the equilibrium probability of wholesale transaction under possible combinations of ESS and ESB. To do so, we set our focus to the case where retail transactions are always made. This is assured if v is large enough. In this case, together with the symmetry assumptions, it turns out that in classifying the equilibrium probability of wholesale transactions we only need to pay attention to ESS and ESB among various parameters of the model.

Theorem. *Suppose* $c_{s(xy)} > c_{s(x)} = c_{s(y)}$ *and* $c_{b(xy)} > c_{b(x)} = c_{b(y)}$. *If* v *is sufficiently large, then the equilibrium probability of wholesale transaction under a stationary equilibrium strategy* σ^* *is given by*

$$\sigma^*_{s(x)}(s(y)) \cdot \sigma^*_{s(y)}(s(x)) = \begin{cases} 1 & \textit{if } ESS \geq \max\{-\hat{k}ESB, -\check{k}ESB\}, \\ \{0, \rho, 1\} & \textit{if } -\hat{k}ESB < ESS < -\check{k}ESB < 0, \\ \rho & \textit{if } -\hat{k}ESB > ESS > -\check{k}ESB > 0, \\ 0 & \textit{if } ESS \leq \min\{-\hat{k}ESB, -\check{k}ESB\}, \quad \textit{or} \\ [0,1] & \textit{if } ESS = ESB = 0, \end{cases} \tag{21}$$

where $\hat{k}$ *and* $\check{k} > 0$ *are fixed numbers such that* $\hat{k} \geq \check{k} > 0$ *and* ρ *is a differentiable function* $\rho(ESS, ESB) : R^2 \to (0,1)$.

Proof. See Appendix A.

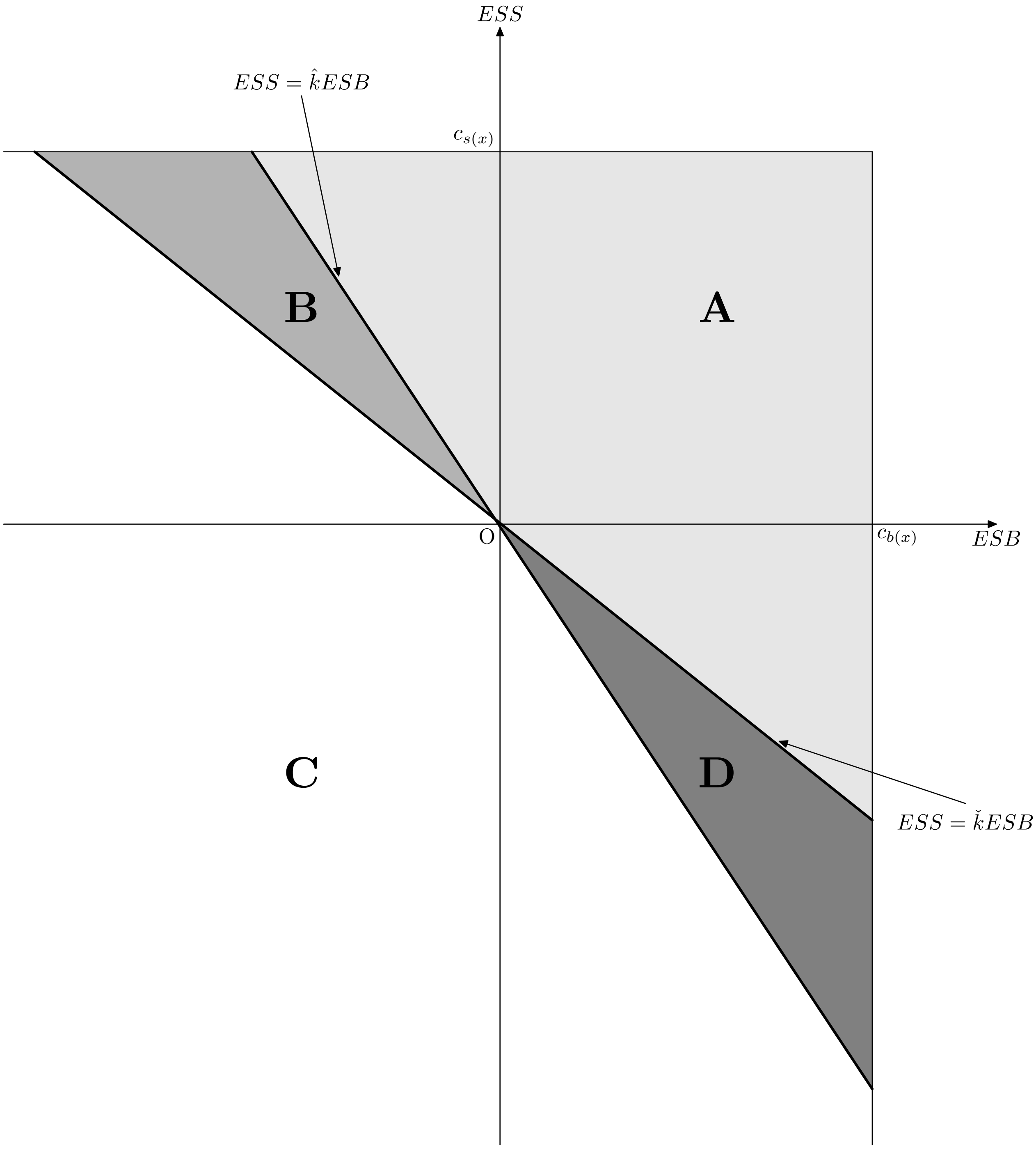

Figure: Classification of ESS and ESB by equilibrium probabilities of wholesale transaction

The straightforward point of the theorem is that matched sellers always make the wholesale transaction when overall economies of scope in search activities for sellers and buyers together is large enough and vice versa. And between these two straightforward cases there always is a mixed strategy equilibrium in which matched sellers are indifferent between trading and not trading. The main point of the theorem is that this mixed strategy equilibrium is a unique equilibrium when $ESB < 0$ and that it is one of multiple equilibria when $ESB > 0$. It is unique when economies of scope in search activities for sellers are cancelled by diseconomies of scope in search activities for buyers. But in addition to this mixed strategy equilibrium there are two other pure strategy equilibria in which a matched pair of sellers always trade or never trade, when diseconomies of scope in search activities for sellers are cancelled by economies of scope in search activities for buyers.

This result can be intuitively understood by considering the stability of the mixed strategy equilibrium: In case of $ESB < 0$, the mixed strategy equilibrium is stable in a sense that the equilibrium probability of wholesale transaction is maintained even when a measurable number of sellers simultaneously deviate from the stationary equilibrium strategy. If they deviate from this mixed strategy, other sellers would take counter actions. As a result, their initial effect on the equilibrium probability of wholesale transaction should be offset by the counter actions of other sellers. Other sellers should take these counter actions when a seller's gain from the wholesale transaction becomes smaller with an increase in the number of the wholesale transaction in the market, i.e. when these mixed actions in equilibrium are strategic substitutes. On the other hand, in case of $ESB > 0$, the mixed strategy equilibrium is unstable in a sense that the equilibrium probability of wholesale transaction cannot be maintained because deviations from the equilibrium strategy will be followed by all sellers. All sellers should follow these deviations when a seller's gain from the wholesale transaction becomes larger with an increase in the number of the wholesale transaction in the market, i.e. if these mixed actions at the equilibrium point are strategic complements.

To see why the mixed actions at the equilibrium are strategic complement when $ESB > 0$, let us examine how it ensures that an increase in the number of the wholesale transaction in the market raises a seller's marginal gain from raising the the probability of the wholesale transaction. A seller's marginal gain from raising the probability of the wholesale transaction is a sum of the *wholesale gain* which is a reduction in search costs for these sellers and the *retail gain* which is an expected reduction in a search cost for a buyer who buys both goods from one of these sellers.[5] This seller's marginal gain from raising the probability of the wholesale transaction can only be

[5]The retail gain is a difference of search costs for state xy buyers between two cases in which state x and y sellers trade or not. If they do not trade and remain to be state x and y sellers, each of them reduces search costs of a state xy buyer by the amounts $c_{b(xy)} - c_{b(x)}$ and $c_{b(xy)} - c_{b(y)}$ respectively. If they trade, one exits and the other stays as a state xy seller who reduces a search cost of a state xy buyer by the amount $c_{b(xy)}$. Hence the difference in the reductions of buyers' search cost between these two cases is $c_{b(x)} - c_{b(y)} - c_{b(xy)}$, which is ESB.

increased with an increase in the population of state xy buyers when $ESB > 0$. And this increase in the population of state xy buyers is induced by the increase in the number of the wholesale transaction in the market.

These stability features are reflected in the comparative statics results on the equilibrium probability of wholesale transaction with respect to economies of scope in search activities for sellers.

Corollary. *The equilibrium probability of wholesale transaction under mixed strategy, ρ, defined in the Theorem has a following property:*

$$\frac{\partial \rho}{\partial ESS} < 0 \quad \textit{iff} \quad -\hat{k}ESB < ESS < -\check{k}ESB < 0, \qquad \textit{and} \tag{22}$$

$$\frac{\partial \rho}{\partial ESS} > 0 \quad \textit{iff} \quad -\hat{k}ESB > ESS > -\check{k}ESB > 0. \tag{23}$$

Proof. See Appendix B.

4 Discussions

From the view point of our model, the standard model of search and match between buyers and sellers can be considered as a special case in which $ESS = 0$ and $ESB = 0$ or in other words both wholesale gain and retail gain are zero. We characterized equilibrium of our model in the simplest possible case in which various values of ESS and ESB are allowed.

One of the most simplifying assumption of our model is that the search costs are prohibitively high for sellers who try to sell more than a unit of the same goods. Because of this assumption, we are able to characterize the equilibrium outcomes just by classifying the equilibrium probabilities of the wholesale transaction in terms of the indices of economies and diseconomies of *scope*. But if search costs are reasonably low for sellers who try to sell more than a unit of the same goods, then we also need to take into account economies and diseconomies of *scale* in search activities as well. The relaxation of our implicit assumption opens up the possibility of explaining various inventory sizes of sellers, which is beyond the scope of this paper.

In our model, sellers who buy goods in trades between matched sellers can be interpreted as a version of middlemen. While our model takes into account both wholesale and retail gains (and losses) from such a trade, the models in the literature of middlemen assumes the cases in which only a wholesale gain or a retail gain exists. For example, the search cost assumption taken up in Rubinstein and Wolinsky (1987) can be considered as a case in which only a wholesale gain exists.

In their model, middlemen have only a wholesale gain in a sense that they can find buyers faster than sellers. On the other hand, in the model of Johri and Leach (2002) only a retail gain exists. In their model, middlemen have a retail gain in a sense that it is easier for a buyer to buy his favorite goods from middlemen than to do so from sellers because middlemen can hold wider varieties of heterogeneous goods as an inventory.

Appendix

A Proof of Theorem

To show that equation (21) is the equilibrium probability of wholesale transaction, we need to find the necessary and sufficient conditions for the parameters given each of possible equilibrium stationary strategy profiles. The possible equilibrium stationary strategy profiles reduce to just three cases (matched pairs of sellers trade, not trade or take mixed actions) under the assumption of v being large, $c_{s(x)} = c_{s(y)}$ and $c_{b(x)} = c_{b(y)}$.

Due to $c_{s(x)} = c_{s(y)}$ and $c_{b(x)} = c_{b(y)}$, we focus on the symmetric equilibrium in which

$$\begin{aligned} \pi_{s(x)} &= \pi_{s(y)}, \\ \pi_{b(x)} &= \pi_{b(y)}, \\ V_{s(x)}(\sigma^*) &= V_{s(y)}(\sigma^*), \\ V_{b(x)}(\sigma^*) &= V_{b(y)}(\sigma^*). \end{aligned} \tag{A.0}$$

The assumption of large enough v makes matched pairs of sellers and buyers trade so that we set $\sigma^*_{b(xy)}(s(x)) = \sigma^*_{b(xy)}(s(y)) = \sigma^*_{b(xy)}(s(xy)) = \sigma^*_{b(x)}(s(x)) = \sigma^*_{b(x)}(s(xy)) = \sigma^*_{b(y)}(s(y)) = \sigma^*_{b(y)}(s(xy)) = \sigma^*_{s(x)}(b(xy)) = \sigma^*_{s(x)}(b(x)) = \sigma^*_{s(y)}(b(xy)) = \sigma^*_{s(y)}(b(y)) = \sigma^*_{s(xy)}(b(xy)) = 1$. As a result, the system of equations (1) - (9), (10), (12), (14), (16), (17), (18) and (A.0), determines the search values and populations for a given stationary strategy profile.

Furthermore, the best response requirement, equation (20), put conditions on the system of equations. If their equilibrium actions are to trade, the conditions are $\sigma^*_{s(x)}(s(y)) = \sigma^*_{s(x)}(s(y)) = 1$ and the surplus for a matched pair of state x and y sellers is non-negative. If their equilibrium actions are not to trade, the conditions are $\sigma^*_{s(x)}(s(y)) \cdot \sigma^*_{s(x)}(s(y)) = 0$ and the surplus for a matched pair of state x and y sellers to be non-positive. And if their equilibrium actions are the mixed ones, the conditions are $0 < \sigma^*_{s(x)}(s(y)) \cdot \sigma^*_{s(x)}(s(y)) < 1$ and the surplus for a matched pair of state x and y sellers to be zero.

A.1 Derivation of conditions that the equilibrium actions are to trade for all matched pairs of state x and y sellers

Supposing that all matched pairs of state x and y sellers trade, we prove that their surpluses are non-negative if and only if $ESS \geq -\hat{k}ESB$, where $\hat{k}$ is a fixed number defined below.

Let $\hat{\sigma}$ be a candidate equilibrium strategy in which all matched pairs of state x and y sellers trade. Then, it should satisfy the followings:

$$\hat{\sigma}^*_{s(x)}(s(y)) = \hat{\sigma}^*_{s(y)}(s(x)) = 1,$$
$$\hat{\sigma}^*_{s(x)}(b(xy)) \cdot \hat{\sigma}^*_{b(xy)}(s(x)) = 1,$$
$$\hat{\sigma}^*_{s(x)}(b(x)) \cdot \hat{\sigma}^*_{b(x)}(s(x)) = 1,$$
$$\hat{\sigma}^*_{s(xy)}(b(x)) \cdot \hat{\sigma}^*_{s(xy)}(b(x)) = 1,$$
$$\hat{\sigma}^*_{s(xy)}(b(xy)) \cdot \hat{\sigma}^*_{s(xy)}(b(xy)) = 1.$$

Substituting these into the definition of the search values, the stationary state conditions and the free entry condition, we can define the candidate equilibrium search values and populations, which we denote as $\hat{V}_a$ and $\hat{\pi}_a$ for all $a \in A$. They should satisfy the followings:

$$\begin{aligned} r\hat{V}_{s(x)} =& - c_{s(x)} + \hat{\pi}_{s(x)} \cdot 0.5(\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}) \\ &+ \hat{\pi}_{b(xy)} \cdot 0.5(v + \hat{V}_{b(x)} - \hat{V}_{b(xy)} - \hat{V}_{s(x)}) + \hat{\pi}_{b(x)} \cdot 0.5(v - \hat{V}_{b(x)} - \hat{V}_{s(x)}), \end{aligned} \tag{A.1.1}$$

$$\begin{aligned} r\hat{V}_{s(xy)} =& - c_{s(xy)} + \hat{\pi}_{b(xy)} \cdot 0.5(2v - \hat{V}_{b(xy)} - \hat{V}_{s(xy)}) \\ &+ 2 \cdot \hat{\pi}_{b(x)} \cdot 0.5(v + \hat{V}_{s(x)} - \hat{V}_{b(x)} - \hat{V}_{s(xy)}), \end{aligned} \tag{A.1.2}$$

$$r\hat{V}_{b(xy)} = - c_{b(xy)} + \hat{\pi}_{s(xy)} \cdot 0.5(2v - \hat{V}_{b(xy)} - \hat{V}_{s(xy)}) + 2 \cdot \hat{\pi}_{s(x)} \cdot 0.5(v + \hat{V}_{b(x)} - \hat{V}_{b(xy)} - \hat{V}_{s(x)}), \tag{A.1.3}$$

$$r\hat{V}_{b(x)} = - c_{b(x)} + \hat{\pi}_{s(xy)} \cdot 0.5(v + \hat{V}_{s(x)} - \hat{V}_{b(x)} - \hat{V}_{s(xy)}) + \hat{\pi}_{s(x)} \cdot 0.5(v - \hat{V}_{b(x)} - \hat{V}_{s(x)}), \tag{A.1.4}$$

$$\hat{\pi}_{s(x)} \cdot \hat{\pi}_{s(x)} = \hat{\pi}_{s(xy)} \cdot \hat{\pi}_{b(xy)} + 2 \cdot \hat{\pi}_{s(xy)} \cdot \hat{\pi}_{b(x)}. \tag{A.1.5}$$

$$\mu = \hat{\pi}_{b(xy)} \cdot \hat{\pi}_{s(xy)} + 2 \cdot \hat{\pi}_{b(xy)} \cdot \hat{\pi}_{s(x)}, \tag{A.1.6}$$

$$\hat{\pi}_{b(xy)} \cdot \hat{\pi}_{s(x)} = \hat{\pi}_{b(x)} \cdot \hat{\pi}_{s(xy)} + \hat{\pi}_{b(x)} \cdot \hat{\pi}_{s(x)}, \tag{A.1.7}$$

and

$$\hat{V}_{s(x)} = 0. \tag{A.1.8}$$

In the rest of this subsection, we first prove that the system of equations (A.1.1) - (A.1.8) has a solution in which the population of each state of sellers and buyers is positive (Claim A.1.1). Given this solution, we define $\hat{k}$ and prove that the surplus for a matched pair of state x and y sellers is zero if and only if $ESS = -\hat{k}ESB$ (Claim A.1.2), and that the surplus for a matched pair of state x and y sellers is positive if and only if $ESS > -\hat{k}ESB$ (Claim A.1.3).

Claim A.1.1. *Under $c_{b(xy)} > c_{b(x)}$, the system of equations (A.1.1) - (A.1.8) has a solution in which $\hat{\pi}_{b(xy)}$, $\hat{\pi}_{b(x)}$, $\hat{\pi}_{s(xy)}$, $\hat{\pi}_{s(x)} > 0$.*

We will prove this in the following manner: Given a population of state xy buyers, the populations of all other agents are given as a solution to the system of equations (A.1.5) - (A.1.7). Lemma A.1.1 will show that these populations are all positive if $\hat{\pi}_{b(xy)} > 0$. With this result, we only need to show that there exists a $\hat{\pi}_{b(xy)} > 0$ which satisfies the free entry condition (A.1.8), where all other search values are determined by the system of equations (A.1.1) to (A.1.4). The left hand side of equation (A.1.8), $\hat{V}_{s(x)}$, can be expressed in terms of the populations, the surplus for a matched pair of state x and y sellers in addition to the parameters of the model by arranging the equations (A.1.1) to (A.1.4). We view this $\hat{V}_{s(x)}$ as a function of $\hat{\pi}_{b(xy)}$ and apply the intermediate value theorem to show that there is a $\hat{\pi}_{b(xy)} > 0$ which satisfies the condition $\hat{V}_{s(x)} = 0$ by using Lemmas A.1.2 and A.1.3.

Lemma A.1.1 *Given $\hat{\pi}_{b(xy)} > 0$, there are positive values $\hat{\pi}_{s(x)}$, $\hat{\pi}_{s(xy)}$, and $\hat{\pi}_{b(x)}$ which satisfy the system of equations (A.1.5) - (A.1.7).*

Proof. From equations (A.1.6) and (A.1.7), we obtain

$$\hat{\pi}_{s(x)} = \frac{\mu - \hat{\pi}_{b(xy)}\hat{\pi}_{s(xy)}}{2\hat{\pi}_{b(xy)}}, \tag{A.1.6'}$$

$$\hat{\pi}_{b(x)} = \frac{\hat{\pi}_{s(x)}}{\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)}}\hat{\pi}_{b(xy)}. \tag{A.1.7'}$$

By substituting these equations into equation (A.1.5), we obtain

$$\left(\frac{\mu - \hat{\pi}_{b(xy)}\hat{\pi}_{s(xy)}}{2\hat{\pi}_{b(xy)}}\right)^2 = \hat{\pi}_{s(xy)}\left(\hat{\pi}_{b(xy)} + \frac{2\frac{\mu - \hat{\pi}_{b(xy)}\hat{\pi}_{s(xy)}}{2\hat{\pi}_{b(xy)}}}{\frac{\mu - \hat{\pi}_{b(xy)}\hat{\pi}_{s(xy)}}{2\hat{\pi}_{b(xy)}} + \hat{\pi}_{s(xy)}}\hat{\pi}_{b(xy)}\right).$$

The left hand side continuously changes from a positive value to zero as $\hat{\pi}_{s(xy)}$ changes from 0 to $\mu/\hat{\pi}_{b(xy)}$. The right hand side continuously changes from zero to a positive value as $\hat{\pi}_{s(xy)}$ changes from 0 to $\mu/\hat{\pi}_{b(xy)}$. By the intermediate value theorem, there exists a value of $\hat{\pi}_{s(xy)} \in (0, \mu/\hat{\pi}_{b(xy)})$ which satisfies this equation. This result together with equations (A.1.6') and (A.1.7') gives us the desired result. ■

Lemma A.1.2. *If $\hat{\pi}_{b(xy)} \to \infty$, then we have*

$$\begin{array}{llll} (i) & \hat{\pi}_{s(x)} & \to & 0, \\ (ii) & \hat{\pi}_{s(xy)} & \to & 0, \\ (iii) & \hat{V}_{s(xy)} - 2\hat{V}_{s(x)} & \to & \dfrac{\hat{\pi}_{b(xy)}}{2\hat{\pi}_{b(x)} + \hat{\pi}_{b(xy)}}\dfrac{2c_{b(x)} - c_{b(xy)}}{r}. \end{array}$$

Proof. It is clear that we need (i) and (ii) since we have (A.1.6) with positive values of $\hat{\pi}_{s(x)}$ and $\hat{\pi}_{s(xy)}$.

To see (iii), we express $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}$ in terms of the parameters and the populations by rearranging equations (A.1.1) - (A.1.4),

$$\left[r + \hat{\pi}_{s(x)} + 2\hat{\pi}_{b(x)} \cdot 0.5 + \hat{\pi}_{b(xy)} \cdot 0.5\frac{r + \hat{\pi}_{s(x)}}{r + 0.5(2\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})}\right](\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}) = 2c_{s(x)} - c_{s(xy)}$$
$$+\hat{\pi}_{b(xy)}0.5\frac{2c_{b(x)} - c_{b(xy)}}{r + 0.5(2\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})}.$$

Applying (i) and (ii) to this equation, we obtain the desired result. ■

Lemma A.1.3. *If $\hat{\pi}_{b(xy)} \downarrow 0$, then we have*

$$\begin{array}{llll} (i) & \hat{\pi}_{b(x)} & \to & 0, \\ (ii) & \hat{V}_{s(xy)} - 2\hat{V}_{s(x)} & \to & \dfrac{2c_{s(x)} - c_{s(xy)}}{r + \hat{\pi}_{s(x)}}. \end{array}$$

Proof. The part (i) is clear since we have (A.1.7'). The part (ii) can be obtained in a similar manner to the proof of part (iii) of Lemma A.1.2. ■

To complete the proof of Claim A.1.1., we first derive an expression for $\hat{V}_{s(x)}$, which can be seen as a continuous function of $\hat{\pi}_{b(xy)}$. Then we apply the intermediate value theorem to it in order to show that there is $\hat{\pi}_{b(xy)} > 0$ which satisfies $\hat{V}_{s(x)} = 0$.

By substituting equations (A.1.3) and (A.1.4) into equation (A.1.1) and rearranging it, $\hat{V}_{s(x)}$ can be expressed as,

$$\begin{aligned} \hat{V}_{s(x)} = \Bigg[& - (2c_{s(x)} - c_{s(xy)}) - (c_{s(xy)} - c_{s(x)}) \\ & + \hat{\pi}_{s(x)} \cdot 0.5 \cdot (\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}) \\ & - \hat{\pi}_{b(xy)} \cdot 0.5 \frac{(2c_{b(x)} - c_{b(xy)}) + \hat{\pi}_{s(xy)} \cdot 0.5(\hat{V}_{s(xy)} - 2\hat{V}_{s(x)})}{r + \theta(2\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})} \\ & + (\hat{\pi}_{b(x)} + \hat{\pi}_{b(xy)})(1-\theta) \frac{rv + c_{b(x)} + \hat{\pi}_{s(xy)}\theta(\hat{V}_{s(xy)} - 2\hat{V}_{s(x)})}{r + (\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})\theta} \Bigg] \\ & \cdot \left[r + (\hat{\pi}_{b(x)} + \hat{\pi}_{b(xy)})(1-\theta) \frac{r + \hat{\pi}_{s(x)}\theta}{r + (\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})\theta} \right]^{-1}. \end{aligned} \tag{A.1.9}$$

By Lemma A.1.1, we can view the right hand side of this equation as a continuous function of $\hat{\pi}_{b(xy)}$. If it becomes positive as $\hat{\pi}_{b(xy)} \to \infty$ and if it becomes negative as $\hat{\pi}_{b(xy)} \to 0$, then the intermediate value theorem tells us that there exists a $\hat{\pi}_{b(xy)} > 0$ which makes the right hand side of this equation zero.

By using Lemma A.1.2, it is clear that as $\hat{\pi}_{b(xy)}$ goes to ∞, the right hand side of equation (A.1.9) goes to

$$\left[\hat{\pi}_{b(xy)} \frac{c_{b(xy)} - c_{b(x)}}{r} + \hat{\pi}_{b(xy)} v + \hat{\pi}_{b(x)} v + \hat{\pi}_{b(x)} \frac{c_{b(x)}}{r} \right] (\hat{\pi}_{b(x)} + \hat{\pi}_{b(xy)})^{-1},$$

which is positive by Assumption $c_{b(xy)} > c_{b(x)}$.

Moreover, by using Lemma A.1.4, it is clear that as $\hat{\pi}_{b(xy)}$ goes to 0, the right hand side of equation (A.1.9) goes to

$$\left[- \left(1 - \frac{\pi_{s(x)}}{r + \hat{\pi}_{s(x)}} \right) c_{s(x)} - \hat{\pi}_{s(x)} \cdot 0.5 \cdot \frac{c_{s(xy)}}{r + \hat{\pi}_{s(x)}} \right] / r,$$

which is negative. This completes the proof of Claim A.1.1.

Claim A.1.2. *Suppose that assumption $c_{s(xy)} > c_{s(x)}$ holds. Then, $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)} = 0$ if and only if $(2c_{s(x)} - c_{s(xy)}) = -\hat{k}(2c_{b(x)} - c_{b(xy)})$, where $\hat{k}$ is defined as follows:*

$$\hat{k} \equiv 0.5\hat{\pi}_{b(xy)}/(r + 0.5\mu/\hat{\pi}_{b(xy)}),$$

where $(\hat{\pi}_{b(xy)}, \hat{\pi}_{b(x)}, \hat{\pi}_{s(xy)}, \hat{\pi}_{s(x)})$ is the solution to the system of equations,

$$\begin{aligned}
c_{s(xy)} - c_{s(x)} &= 0.5(\hat{\pi}_{b(xy)} + \hat{\pi}_{b(x)})\frac{rv + c_{b(x)}}{r + 0.5\mu/(\hat{\pi}_{b(xy)} + \hat{\pi}_{b(x)})}, \\
\mu &= \hat{\pi}_{b(xy)}(\hat{\pi}_{s(xy)} + 2\hat{\pi}_{s(x)}), \\
\hat{\pi}_{b(xy)}\hat{\pi}_{s(x)} &= \hat{\pi}_{b(x)}(\hat{\pi}_{s(xy)} + \hat{\pi}_{s(x)}), \\
\hat{\pi}_{s(x)}\hat{\pi}_{s(x)} &= \hat{\pi}_{s(xy)}(\hat{\pi}_{b(xy)} + 2\hat{\pi}_{b(x)}).
\end{aligned}$$

Proof. From equations (A.1.1) - (A.1.4), the surplus for a matched pair of state x and y sellers, $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}$, can be expressed in terms of the parameters and the populations,

$$\begin{aligned}
&\left[r + \hat{\pi}_{s(x)} + 2\cdot 0.5\hat{\pi}_{b(x)} + 0.5\hat{\pi}_{b(xy)}\frac{r + 2\cdot 0.5\hat{\pi}_{s(x)}}{r + 0.5(2\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})}\right](\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}) \\
&\qquad = (2c_{s(x)} - c_{s(xy)}) + \frac{0.5\hat{\pi}_{b(xy)}}{r + 0.5(2\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})}(2c_{b(x)} - c_{b(xy)}). \qquad \text{(A.1.10)}
\end{aligned}$$

Thus,

$$\text{sign of } \left(\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}\right) = \text{sign of } \left((2c_{s(x)} - c_{s(xy)}) + 0.5\hat{\pi}_{b(xy)}\frac{(2c_{b(x)} - c_{b(xy)})}{r + 0.5(\hat{\pi}_{s(xy)} + 2\hat{\pi}_{s(x)})}\right). \qquad \text{(A.1.10')}$$

Since $\hat{\pi}_{s(x)}$, $\hat{\pi}_{s(xy)}$ and $\hat{\pi}_{b(xy)}$ are determined by the system of equations (A.1.1) - (A.1.8), we have $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)} = 0$ if and only if

$$\begin{aligned}
0 &= (2c_{s(x)} - c_{s(xy)}) + \frac{0.5\hat{\pi}_{b(xy)}}{r + 0.5(\hat{\pi}_{s(xy)} + 2\hat{\pi}_{s(x)})}(2c_{b(x)} - c_{b(xy)}), && \text{(A.1.11)} \\
c_{s(xy)} - c_{s(x)} &= 0.5(\hat{\pi}_{b(xy)} + \hat{\pi}_{b(x)})\frac{rv + c_{b(x)}}{r + 0.5(\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})}, && \text{(A.1.12)} \\
\mu &= \hat{\pi}_{b(xy)}(\hat{\pi}_{s(xy)} + 2\hat{\pi}_{s(x)}), && \text{(A.1.13)} \\
\hat{\pi}_{b(xy)}\hat{\pi}_{s(x)} &= \hat{\pi}_{b(x)}(\hat{\pi}_{s(xy)} + \hat{\pi}_{s(x)}), && \text{(A.1.14)} \\
\hat{\pi}_{s(x)}\hat{\pi}_{s(x)} &= \hat{\pi}_{s(xy)}(\hat{\pi}_{b(xy)} + 2\hat{\pi}_{b(x)}). && \text{(A.1.15)}
\end{aligned}$$

Equation (A.1.11) is a requirement from the sign condition (A.1.10') when $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)} = 0$. It is clear from this equation that $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)} = 0$ if only if $(2c_{s(x)} - c_{s(xy)}) = -\hat{k}(2c_{b(x)} - c_{b(xy)})$ where $\hat{k} \equiv 0.5\hat{\pi}_{b(xy)}/[r + 0.5\mu/\hat{\pi}_{b(xy)}]$. We can easily see that this $\hat{k}$ equals to the coefficient of the second term of the right hand side of equation (A.1.10') by substituting (A.1.12) into it. Moreover, equation (A.1.12) is the free entry condition which is an (A.1.9) set to zero. Notice that assumption $c_{s(xy)} > c_{s(x)}$ is necessary for the equation (A.1.12) to hold. ■

Claim A.1.3. *Suppose that assumption $c_{s(xy)} > c_{s(x)}$ holds. Then, for any $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$,*

$$\hat{V}_{s(xy)} - 2\hat{V}_{s(x)} > 0 \textit{ if and only if } (2c_{s(x)} - c_{s(xy)}) > -\hat{k}(2c_{b(x)} - c_{b(xy)}).$$

Proof. In the case where the equilibrium actions for all matched pairs of state x and y sellers are to trade, the surplus $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}$ and the equilibrium populations are determined by the system of equations (A.1.10), (A.1.13), (A.1.14), (A.1.15) and an entry condition (A.1.8) which can be rewritten as a condition that the right hand side of equation (A.1.9) equals to zero,

$$\begin{aligned} 0 = \Bigg[& -(2c_{s(x)} - c_{s(xy)}) - (c_{s(xy)} - c_{s(x)}) \\ & + 0.5\hat{\pi}_{s(x)}(\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}) \\ & - 0.5\hat{\pi}_{b(xy)} \frac{(2c_{b(x)} - c_{b(xy)}) + 0.5\hat{\pi}_{s(xy)}(\hat{V}_{s(xy)} - 2\hat{V}_{s(x)})}{r + 0.5(2\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})} \\ & + 0.5(\hat{\pi}_{b(x)} + \hat{\pi}_{b(xy)}) \frac{rv + c_{b(x)} + 0.5\hat{\pi}_{s(xy)}(\hat{V}_{s(xy)} - 2\hat{V}_{s(x)})}{r + 0.5(\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})} \Bigg] \\ & \cdot \left[r + 0.5(\hat{\pi}_{b(x)} + \hat{\pi}_{b(xy)}) \frac{r + 0.5\hat{\pi}_{s(x)}}{r + 0.5(\hat{\pi}_{s(x)} + \hat{\pi}_{s(xy)})} \right]^{-1}. \end{aligned} \tag{A.1.9'}$$

The system of equations above can be reduced to the system of two equations (A.1.10) and (A.1.9'), which determine the surplus and $\pi_{b(xy)}$, because all other populations are determined by equations (A.1.13), (A.1.14), and (A.1.15).

For any given pair of parameters $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$, the surplus $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}$ and the population $\pi_{b(xy)}$ can be viewed as a continuous function of $ESS = (2c_{s(x)} - c_{s(xy)})$ and $ESB = (2c_{b(x)} - c_{b(xy)})$. And note that $\hat{k}$ is fixed for a given pair of parameters $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$. Moreover, since we have Claim A.1.1, we know that there is a solution to the system of equations with positive values of populations. All we need to show is that the surplus $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}$ is positive if and only if $ESS > -\hat{k}ESB$.

We show that $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}$ is positive if $ESS > -\hat{k}ESB$. Consider a point (ESB, ESS) where $ESS > -\hat{k}ESB$. Then the surplus must not be zero under this combination of (ESB, ESS) since we have Claim A.1.2. If the surplus is positive under this (ESB, ESS), then under all other points (ESS', ESB') such that $ESS' > -\hat{k}ESB'$, the surplus must also be positive. If not and the surplus is negative, then there exists a convex combination of these two points (ESS'', ESB'') under which the surplus is zero and $ESS'' > -\hat{k}ESB''$. This contradicts Claim A.1.2. It is clear from equation (A.1.10) that the surplus is positive if both ESS and ESB are positive.

We can also show that $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}$ is negative if $ESS < -\hat{k}ESB$. This can be done by making similar arguments as above. ■

A.2 Derivation of conditions that the equilibrium actions are not to trade for all matched pairs of state x and y sellers

Supposing that all matched pairs of state x and y sellers do not trade, we show that their surplus is non-positive if and only if $ESS \leq -\check{k}ESB$, where $\check{k}$ is a fixed number defined below.

Let $\check{\sigma}$ be a candidate equilibrium strategy in which all matched pairs of state x and y sellers trade. Then, it should satisfy the followings:

$$\begin{aligned}
\check{\sigma}^*_{s(x)}(s(y)) = \check{\sigma}^*_{s(y)}(s(x)) &= 0, \\
\check{\sigma}^*_{s(x)}(b(xy)) \cdot \check{\sigma}^*_{b(xy)}(s(x)) &= 1, \\
\check{\sigma}^*_{s(x)}(b(x)) \cdot \check{\sigma}^*_{b(x)}(s(x)) &= 1, \\
\check{\sigma}^*_{s(xy)}(b(x)) \cdot \check{\sigma}^*_{s(xy)}(b(x)) &= 1, \\
\check{\sigma}^*_{s(xy)}(b(xy)) \cdot \check{\sigma}^*_{s(xy)}(b(xy)) &= 1.
\end{aligned}$$

By substituting these into the definition of search values, the stationary state conditions and the free entry condition, we define the candidate equilibrium search values and populations. They are denoted as $\check{V}_a$ and $\check{\pi}_a$ for all $a \in A$ and satisfy the following:

$$r\check{V}_{s(x)} = -c_{s(x)} + 0.5\check{\pi}_{b(xy)}(v + \check{V}_{b(x)} - \check{V}_{b(xy)} - \check{V}_{s(x)}) + 0.5\check{\pi}_{b(x)}(v - \check{V}_{b(x)} - \check{V}_{s(x)}), \tag{A.2.1}$$

$$r\check{V}_{b(xy)} = -c_{b(xy)} + 2 \cdot 0.5 \cdot \check{\pi}_{s(x)}(v + \check{V}_{b(x)} - \check{V}_{b(xy)} - \check{V}_{s(x)}), \tag{A.2.2}$$

$$r\check{V}_{b(x)} = -c_{b(x)} + 0.5\check{\pi}_{s(x)}(v - \check{V}_{b(x)} - \check{V}_{s(x)}), \tag{A.2.3}$$

$$\begin{aligned} r\check{V}_{s(xy)} = &-c_{s(xy)} + 0.5\check{\pi}_{b(xy)}(2v - \check{V}_{b(xy)} - \check{V}_{s(xy)}) \\ &+ 2 \cdot 0.5 \cdot \check{\pi}_{b(x)}(v + \check{V}_{s(x)} - \check{V}_{b(x)} - \check{V}_{s(xy)}), \end{aligned} \tag{A.2.4}$$

$$0 = \check{\pi}_{s(xy)}(\check{\pi}_{b(xy)} + 2\check{\pi}_{b(x)}) \tag{A.2.5}$$

$$\mu = \check{\pi}_{b(xy)} \cdot \check{\pi}_{s(xy)} + 2 \cdot \check{\pi}_{b(xy)} \cdot \check{\pi}_{s(x)}, \tag{A.2.6}$$

$$\check{\pi}_{b(xy)} \cdot \check{\pi}_{s(x)} = \check{\pi}_{b(x)} \cdot \check{\pi}_{s(xy)} + \check{\pi}_{b(x)} \cdot \check{\pi}_{s(x)}, \tag{A.2.7}$$

and

$$\check{V}_{s(x)} = 0. \tag{A.2.8}$$

In the rest of this subsection, we first prove that the system of equations (A.2.1) - (A.2.8) has a solution in which $\check{\pi}_{b(xy)}$, $\check{\pi}_{b(x)}$, $\check{\pi}_{s(x)} > 0$ and $\check{\pi}_{s(xy)} = 0$ (Claim A.2.1). Given this solution, we define $\check{k}$ and prove that the surplus for a matched pair of state x and y sellers is non-positive if and only if $ESS \leq -\check{k}ESB$ (Claim A.2.2).

Claim A.2.1. *The system of equations (A.2.1) - (A.2.8) has a solution in which* $\check{\pi}_{b(xy)}$, $\check{\pi}_{b(x)}$, $\check{\pi}_{s(x)} > 0$ *and* $\check{\pi}_{s(xy)} = 0$.

Proof. Given a population of state xy buyers, it is clear that the populations of all other agents are determined as a solution to the system of equations (A.2.5) - (A.2.7). We can see that $\check{\pi}_{b(x)}$, $\check{\pi}_{s(x)} > 0$ and $\check{\pi}_{s(xy)} = 0$ if $\hat{\pi}_{b(xy)} > 0$. And given these populations, the search values are determined as a solution to the system of equations (A.2.1) to (A.2.4). We only need to show that this solution satisfies the condition (A.2.8) for a positive value of $\check{\pi}_{b(xy)} > 0$.

By rearranging equation (A.2.1) - (A.2.7), the left hand side of equation (A.2.8) can be expressed as

$$\check{V}_{s(x)} = \left[-c_{s(x)} + 2\cdot 0.5\cdot \check{\pi}_{b(xy)}\frac{rv + c_{b(x)}}{r + 0.5\mu/2\check{\pi}_{b(xy)}} - 0.5\check{\pi}_{b(xy)}\frac{2c_{b(x)} - c_{b(xy)}}{r + 0.5\mu/\check{\pi}_{b(xy)}}\right]\left[r + r\frac{2\cdot 0.5\cdot \check{\pi}_{b(xy)}}{r + 0.5\mu/2\check{\pi}_{b(xy)}}\right]^{-1}. \tag{A.2.9}$$

The right hand side of this equation becomes positive as $\check{\pi}_{b(xy)} \to \infty$ and becomes negative as $\check{\pi}_{b(xy)} \to 0$. Thus, by the intermediate value theorem, there exists $\check{\pi}_{b(xy)} > 0$ such that $\check{V}_{s(x)} = 0$ which can be expressed as

$$0 = -c_{s(x)} + 2\cdot 0.5\cdot \check{\pi}_{b(xy)}\frac{rv + c_{b(x)}}{r + 0.5\mu/2\check{\pi}_{b(xy)}} - 0.5\check{\pi}_{b(xy)}\frac{2c_{b(x)} - c_{b(xy)}}{r + 0.5\mu/\check{\pi}_{b(xy)}}. \tag{A.2.10}$$

■

Claim A.2.2. *Suppose that assumption $c_{s(xy)} > c_{s(x)}$ holds. For any $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$,*

$$V_{s(xy)} - 2V_{s(x)} \leq 0 \textit{ if and only if } (2c_{s(x)} - c_{s(xy)}) \leq -\check{k}(2c_{b(x)} - c_{b(xy)}),$$

where $\check{k}$ is defined as follows:

$$\check{k} \equiv 0.5\check{\pi}_{b(xy)}/(r + 0.5\mu/\check{\pi}_{b(xy)}) > 0,$$

where $\check{\pi}_{b(xy)}$ is the solution to the equation,

$$c_{s(xy)} - c_{s(x)} = 2\cdot 0.5\cdot \check{\pi}_{b(xy)}\frac{rv + c_{b(x)}}{r + 0.5\mu/2\check{\pi}_{b(xy)}}.$$

Proof: We first show that

$$V_{s(xy)} - 2V_{s(x)} = 0 \text{ if and only if } (2c_{s(x)} - c_{s(xy)}) = -\check{k}(2c_{b(x)} - c_{b(xy)}), \tag{A.2.11}$$

which can be straightforwardly seen by rearranging equations (A.2.1) and (A.2.4),

$$[r + 0.5(\check{\pi}_{b(xy)} + 2\check{\pi}_{b(x)})](V_{s(xy)} - 2V_{s(x)}) = (2c_{s(x)} - c_{s(xy)}) + \frac{0.5\check{\pi}_{b(xy)}}{r + 0.5\mu/\check{\pi}_{b(xy)}}(2c_{b(x)} - c_{b(xy)}). \tag{A.2.12}$$

Substituting (A.2.10) into (A.2.11), we obtain the following equation

$$c_{s(xy)} - c_{s(x)} = 2\cdot 0.5\cdot \check{\pi}_{b(xy)}\frac{rv + c_{b(x)}}{r + 0.5\mu/2\check{\pi}_{b(xy)}}, \tag{A.2.13}$$

which determines $\check{\pi}_{b(xy)}$. Note here that all populations and $\check{k}$ are fixed once parameter values $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$ are fixed. Notice that assumption $c_{s(xy)} > c_{s(x)}$ is necessary for this equation to hold.

Given these fixed values of populations and equation (A.2.12), the value of surplus $V_{s(xy)} - 2V_{s(x)}$ can now be considered as a continuous function of $2c_{s(x)} - c_{s(xy)}$ and $2c_{b(x)} - c_{b(xy)}$. At this point, using property (A.2.11) and equation (A.2.12), we can take a procedure similar to the one we have taken in the proof of Claim A.1.3 to show that

$$V_{s(xy)} - 2V_{s(x)} \leq 0 \Longleftrightarrow 2c_{s(x)} - c_{s(xy)} + \frac{0.5\check{\pi}_{b(xy)}}{r + 0.5\mu/\check{\pi}_{b(xy)}}(2c_{b(x)} - c_{b(xy)}) \leq 0.$$

■

A.3 Derivation of conditions that the equilibrium actions are mixed ones for all matched pairs of state x and y sellers

Supposing that the equilibrium actions are mixed ones for all matched pairs of state x and y sellers, we show that their surpluses is zero if and only if either $-\check{k}(2c_{b(x)} - c_{b(xy)}) > (2c_{s(x)} - c_{s(xy)}) > -\hat{k}(2c_{b(x)} - c_{b(xy)})$ or $-\hat{k}(2c_{b(x)} - c_{b(xy)}) > (2c_{s(x)} - c_{s(xy)}) > -\check{k}(2c_{b(x)} - c_{b(xy)})$.

Let $\tilde{\sigma}$ be a candidate equilibrium strategy in which matched pairs of state x and y sellers trade with probability ρ. Then, it should satisfy the following:

$$\begin{aligned}
\tilde{\sigma}^*_{s(x)}(s(y)) = \tilde{\sigma}^*_{s(y)}(s(x)) &= \sqrt{\rho}, \\
\tilde{\sigma}^*_{s(x)}(b(xy)) \cdot \tilde{\sigma}^*_{b(xy)}(s(x)) &= 1, \\
\tilde{\sigma}^*_{s(x)}(b(x)) \cdot \tilde{\sigma}^*_{b(x)}(s(x)) &= 1, \\
\tilde{\sigma}^*_{s(xy)}(b(x)) \cdot \tilde{\sigma}^*_{s(xy)}(b(x)) &= 1, \\
\tilde{\sigma}^*_{s(xy)}(b(xy)) \cdot \tilde{\sigma}^*_{s(xy)}(b(xy)) &= 1.
\end{aligned}$$

By substituting these into the definition of search values, the stationary state conditions and the free entry condition, we define the candidate equilibrium probability, search values and populations. They are denoted as ρ, $\tilde{V}_a$ and $\tilde{\pi}_a$ for all $a \in A$ and satisfy the followings:

$$\begin{aligned}
r\tilde{V}_{s(x)} =& - c_{s(x)} + 0.5\tilde{\pi}_{s(x)}\rho(\tilde{V}_{s(xy)} - 2\tilde{V}_{s(x)}) \\
& + 0.5\tilde{\pi}_{b(xy)}(v + \tilde{V}_{b(x)} - \tilde{V}_{b(xy)} - \tilde{V}_{s(x)}) + 0.5\tilde{\pi}_{b(x)}(v - \tilde{V}_{b(x)} - \tilde{V}_{s(x)}), && \text{(A.3.1)} \\
r\tilde{V}_{s(xy)} =& - c_{s(xy)} + 0.5\tilde{\pi}_{b(xy)}(2v - \tilde{V}_{b(xy)} - \tilde{V}_{s(xy)}) \\
& + 2 \cdot 0.5 \cdot \tilde{\pi}_{b(x)}(v + \tilde{V}_{s(x)} - \tilde{V}_{b(x)} - \tilde{V}_{s(xy)}), && \text{(A.3.2)} \\
r\tilde{V}_{b(xy)} =& - c_{b(xy)} + 0.5\tilde{\pi}_{s(xy)}(2v - \tilde{V}_{b(xy)} - \tilde{V}_{s(xy)}) + 2 \cdot 0.5 \cdot \tilde{\pi}_{s(x)}(v + \tilde{V}_{b(x)} - \tilde{V}_{b(xy)} - \tilde{V}_{s(x)}), && \text{(A.3.3)} \\
r\tilde{V}_{b(x)} =& - c_{b(x)} + 0.5\tilde{\pi}_{s(xy)}(v + \tilde{V}_{s(x)} - \tilde{V}_{b(x)} - \tilde{V}_{s(xy)}) + 0.5\tilde{\pi}_{s(x)}(v - \tilde{V}_{b(x)} - \tilde{V}_{s(x)}), && \text{(A.3.4)}
\end{aligned}$$

$$\begin{aligned}
\rho \cdot \tilde{\pi}_{s(x)} \cdot \tilde{\pi}_{s(x)} &= \tilde{\pi}_{s(xy)} \cdot \tilde{\pi}_{b(xy)} + 2 \cdot \tilde{\pi}_{s(xy)} \cdot \tilde{\pi}_{b(x)}. && \text{(A.3.5)} \\
\mu &= \tilde{\pi}_{b(xy)} \cdot \tilde{\pi}_{s(xy)} + 2 \cdot \tilde{\pi}_{b(xy)} \cdot \tilde{\pi}_{s(x)}, && \text{(A.3.6)} \\
\tilde{\pi}_{b(xy)} \cdot \tilde{\pi}_{s(x)} &= \tilde{\pi}_{b(x)} \cdot \tilde{\pi}_{s(xy)} + \tilde{\pi}_{b(x)} \cdot \tilde{\pi}_{s(x)}, && \text{(A.3.7)}
\end{aligned}$$

and

$$\tilde{V}_{s(x)} = 0. \qquad \text{(A.3.8)}$$

Following a procedure similar to that we have taken in Claim A.1.1, under assumption $c_{b(xy)} > c_{b(x)}$, we can prove that the system of equations (A.3.1) - (A.3.8) has a solution in which the population of each state of sellers and buyers is positive. Thus, in the rest of this subsection, we prove that there exists a unique $\rho \in (0, 1)$ such that $\tilde{V}_{s(xy)} - 2\tilde{V}_{s(x)} = 0$ if and only if $-\check{k}(2c_{b(x)} - c_{b(xy)}) > (2c_{s(x)} - c_{s(xy)}) > -\hat{k}(2c_{b(x)} - c_{b(xy)})$

or $-\hat{k}(2c_{b(x)} - c_{b(xy)}) > (2c_{s(x)} - c_{s(xy)}) > -\check{k}(2c_{b(x)} - c_{b(xy)})$ (Claim A.3.1). The uniqueness of ρ is shown in Lemma A.3.1.

Lemma A.3.1. *Suppose that assumption $c_{s(xy)} > c_{s(x)}$ holds. For any $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$, the system of equations (A.3.1) - (A.3.8) and $\tilde{V}_{s(xy)} - 2\tilde{V}_{s(x)} = 0$ has a unique solution. And the equilibrium probability of wholesale transaction and the equilibrium populations under this solution $(\rho, \tilde{\pi}_{b(xy)}, \tilde{\pi}_{b(x)}, \tilde{\pi}_{s(xy)}, \tilde{\pi}_{s(x)})$ can be considered as a function of $(2c_{s(x)} - c_{s(xy)}, 2c_{b(x)} - c_{b(xy)})$.*

Proof. By rearranging equations (A.3.1) - (A.3.8) and $\tilde{V}_{s(xy)} - 2\tilde{V}_{s(x)} = 0$ as we did in Claim A.1.2, we see that $(\rho, \tilde{\pi}_{b(xy)}, \tilde{\pi}_{b(x)}, \tilde{\pi}_{s(xy)}, \tilde{\pi}_{s(x)})$ is a solution to the following system of equations:

$$0 = (2c_{s(x)} - c_{s(xy)}) + (2c_{b(x)} - c_{b(xy)}) \cdot 0.5\tilde{\pi}_{b(xy)}/(r + 0.5\mu/\tilde{\pi}_{b(xy)}), \tag{A.3.9}$$

$$\rho\tilde{\pi}_{s(x)}\tilde{\pi}_{s(x)} = \tilde{\pi}_{s(xy)}(\tilde{\pi}_{b(xy)} + 2\tilde{\pi}_{b(x)}), \tag{A.3.10}$$

$$c_{s(xy)} - c_{s(x)} = 0.5(\tilde{\pi}_{b(xy)} + \tilde{\pi}_{b(x)})\frac{rv + c_{b(x)}}{r + 0.5\mu/(\tilde{\pi}_{b(xy)} + \tilde{\pi}_{b(x)})}, \tag{A.3.11}$$

$$\mu = \tilde{\pi}_{b(xy)}(\tilde{\pi}_{s(xy)} + 2\tilde{\pi}_{s(x)}), \tag{A.3.12}$$

$$\tilde{\pi}_{b(xy)}\tilde{\pi}_{s(x)} = \tilde{\pi}_{b(x)}(\tilde{\pi}_{s(xy)} + \tilde{\pi}_{s(x)}). \tag{A.3.13}$$

We show that this system of equations (A.3.9) - (A.3.13) has a unique solution. To see this, we first notice that equation (A.3.9) determines a unique value of $\tilde{\pi}_{b(xy)}$. Then, equation (A.3.11) determines a unique value of $\tilde{\pi}_{b(xy)} + \tilde{\pi}_{b(x)}$ under assumption $c_{s(xy)} > c_{s(x)}$. From this, we obtain a unique value of $\tilde{\pi}_{b(x)}$ because we already obtained the unique value of $\tilde{\pi}_{b(xy)}$. Given these $\tilde{\pi}_{b(xy)}$ and $\tilde{\pi}_{b(x)}$, equations (A.3.12) and (A.3.13) determine unique values of $\tilde{\pi}_{s(xy)}$ and $\tilde{\pi}_{s(x)}$. Since we obtained unique values of $(\tilde{\pi}_{b(xy)}, \tilde{\pi}_{b(x)}, \tilde{\pi}_{s(xy)}, \tilde{\pi}_{s(x)})$, equation (A.3.10) determines a unique value of ρ. Once $(\rho, \tilde{\pi}_{b(xy)}, \tilde{\pi}_{b(x)}, \tilde{\pi}_{s(xy)}, \tilde{\pi}_{s(x)})$ is determined, the search values are uniquely obtained from equations (A.3.1) - (A.3.4). ■

Claim A.3.1. *Suppose that assumption $c_{s(xy)} > c_{s(x)}$ holds. For any $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$, there exists a $\rho \in (0, 1)$ such that $\tilde{V}_{s(xy)} - 2\tilde{V}_{s(x)} = 0$ if and only if either $-\check{k}(2c_{b(x)} - c_{b(xy)}) > (2c_{s(x)} - c_{s(xy)}) > -\hat{k}(2c_{b(x)} - c_{b(xy)})$ or $-\hat{k}(2c_{b(x)} - c_{b(xy)}) > (2c_{s(x)} - c_{s(xy)}) > -\check{k}(2c_{b(x)} - c_{b(xy)})$.*

Proof. By rearranging equations (A.3.1) - (A.3.4), the surplus $\tilde{V}_{s(xy)} - 2\tilde{V}_{s(x)}$ can be expressed as,

$$\begin{aligned} &\tilde{V}_{s(xy)} - 2\tilde{V}_{s(x)} \\ &= \left[(2c_{s(x)} - c_{s(xy)}) + \frac{0.5\tilde{\pi}_{b(xy)}}{r + 0.5\mu/\tilde{\pi}_{b(xy)}}(2c_{b(x)} - c_{b(xy)})\right] \\ &\cdot \left[r + \rho\tilde{\pi}_{s(x)} + 2 \cdot 0.5 \cdot \tilde{\pi}_{b(x)} + 0.5\tilde{\pi}_{b(xy)}\frac{r + 2\theta\tilde{\pi}_{s(x)}}{r + 0.5\mu/\tilde{\pi}_{b(xy)}}\right]^{-1}, \end{aligned} \tag{A.3.14}$$

which can be considered as a function of $(2c_{s(x)} - c_{s(xy)}, 2c_{b(x)} - c_{b(xy)})$ for a fixed pair of $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$ since we have Lemma A.3.1. The question we now ask is under what value of $(2c_{s(x)} - c_{s(xy)}, 2c_{b(x)} - c_{b(xy)})$, ρ is in between 0 and 1.

To check this, we divide possible values of $(2c_{s(x)} - c_{s(xy)},\, 2c_{b(x)} - c_{b(xy)})$ into the following four cases since we have fixed values of $\hat{k}$ and $\check{k}$ for a fixed pair of $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$: Case (i) $-\check{k}(2c_{b(x)} - c_{b(xy)}) > (2c_{s(x)} - c_{s(xy)}) > -\hat{k}(2c_{b(x)} - c_{b(xy)})$, Case (ii) $-\hat{k}(2c_{b(x)} - c_{b(xy)}) > (2c_{s(x)} - c_{s(xy)}) > -\check{k}(2c_{b(x)} - c_{b(xy)})$, Case (iii) $(2c_{s(x)} - c_{s(xy)}) \geq \max\{-\hat{k}(2c_{b(x)} - c_{b(xy)}), -\check{k}(2c_{b(x)} - c_{b(xy)})\}$ and Case (iv) $(2c_{s(x)} - c_{s(xy)}) \leq \min\{-\hat{k}(2c_{b(x)} - c_{b(xy)}), -\check{k}(2c_{b(x)} - c_{b(xy)})\}$.

In case (i), we show that there exists a $\rho \in (0, 1)$ such that the right hand side of equation (A.3.14) equals zero. When ρ goes to 1, the system of equations (A.3.1) - (A.3.8) becomes the same as the system of equation (A.1.1) - (A.1.8). Thus, when ρ goes to 1, the right hand side of equation (A.3.14) goes to $\hat{V}_{s(xy)} - 2\hat{V}_{s(x)}$, which is positive if $(2c_{s(x)} - c_{s(xy)}) > -\hat{k}(2c_{b(x)} - c_{b(xy)})$ (Claim A.1.3). Similarly, when ρ goes to 0, the system of equations (A.3.1) - (A.3.8) becomes the same as the system of equation (A.2.1) - (A.2.8) so that the right hand side of equation (A.3.14) goes to $\check{V}_{s(xy)} - 2\check{V}_{s(x)}$ which is negative if $(2c_{s(x)} - c_{s(xy)}) < -\check{k}(2c_{b(x)} - c_{b(xy)})$ (Claim A.2.2). Since the right hand side of equation (A.3.14) is a continuous function of ρ, we can apply the intermediate value theorem to it and obtain the desired result.

Similarly, in case (ii), we can show that there exists a $\rho \in (0, 1)$ such that the right hand side of equation (A.3.14) equals zero. When ρ goes to 1, Claim A.1.3 tells us that the right hand side of equation (A.3.14) is negative since $(2c_{s(x)} - c_{s(xy)}) < -\hat{k}(2c_{b(x)} - c_{b(xy)})$. When ρ goes to 0, Claim A.2.2 tells us the right hand side of equation (A.3.14) is positive since $(2c_{s(x)} - c_{s(xy)}) > -\check{k}(2c_{b(x)} - c_{b(xy)})$. Again by using the intermediate value theorem, we obtain the desired result.

On the other hand, in case (iii), we can show that there exists no $\rho \in (0, 1)$ such that the right hand side of equation (A.3.14) equals zero. When ρ goes to 1, Claims A.1.2 and A.1.3 tell us the right hand side of equation (A.3.14) is non-negative since $(2c_{s(x)} - c_{s(xy)}) \geq -\hat{k}(2c_{b(x)} - c_{b(xy)})$. When ρ goes to 0, Claim A.2.2 tells us it is non-negative since $(2c_{s(x)} - c_{s(xy)}) \geq -\check{k}(2c_{b(x)} - c_{b(xy)})$. These imply what we wanted to show since ρ is unique (Lemma A.3.1).

Similarly, in case (iv), we can show that there exists no $\rho \in (0, 1)$ such that the right hand side of equation (A.3.14) is zero. In this case, when ρ goes to either zero or one, the right hand side of equation (A.3.14) goes to non-positive. ■

A.4 Proof of Theorem

We notice that all Claims in Appendix A hold for any given $c_{s(xy)} - c_{s(x)}$ and $c_{b(x)}$. By combining all Claims, the Theorem is established once we show $\hat{k} \geq \check{k}$. Showing this is equivalent to showing $\hat{\pi}_{b(xy)} \geq \check{\pi}_{b(xy)}$ due to the definitions of $\hat{k}$ and $\check{k}$.

We show $\hat{\pi}_{b(xy)} \geq \check{\pi}_{b(xy)}$ as follows: From equations (A.1.12) and (A.2.13),

$$2 \cdot 0.5 \cdot \check{\pi}_{b(xy)} \frac{rv + c_{b(x)}}{r + 0.5\mu/2\check{\pi}_{b(xy)}} = c_{s(xy)} - c_{s(x)} = 0.5(\hat{\pi}_{b(xy)} + \hat{\pi}_{b(x)}) \frac{rv + c_{b(x)}}{r + 0.5\mu/(\hat{\pi}_{b(xy)} + \hat{\pi}_{b(x)})}.$$

This equation implies $\hat{\pi}_{b(xy)} + \hat{\pi}_{b(x)} = 2\check{\pi}_{b(xy)}$. Also, we have $\hat{\pi}_{b(xy)} \geq \hat{\pi}_{b(x)}$ since $\hat{\pi}_{b(xy)}\hat{\pi}_{s(x)} = \hat{\pi}_{b(x)}(\hat{\pi}_{s(xy)} + \hat{\pi}_{s(x)})$. These results imply $\hat{\pi}_{b(xy)} \geq \check{\pi}_{b(xy)}$.

B Proof of Corollary 1

In Appendix B, under equations (A.3.9) - (A.3.13), we show that $\partial\rho/\partial ESS > 0$ if $ESB < 0$ and that $\partial\rho/\partial ESS < 0$ if $ESB > 0$. We write $\partial\rho/\partial ESS > 0$ as follows:

$$\frac{\partial\rho}{\partial ESS} = \frac{\partial\rho}{\partial\tilde{\pi}_{s(x)}}\frac{\partial\tilde{\pi}_{s(x)}}{\partial\tilde{\pi}_{b(xy)}}\frac{\partial\tilde{\pi}_{b(xy)}}{\partial ESS}.$$

We look at the sign of $\partial\rho/\partial ESS$ by looking at the sign of each term on the right hand side.

First, we look at the sign of $\partial\tilde{\pi}_{b(xy)}/\partial ESS$. From equation (A.3.9), the sign is positive if $ESB < 0$, and negative if $ESB > 0$.

Second, we look at the sign of $\partial\tilde{\pi}_{s(x)}/\partial\tilde{\pi}_{b(xy)}$. To investigate the sign, we note that $\tilde{\pi}_b \equiv \tilde{\pi}_{b(xy)} + \tilde{\pi}_{b(x)}$ is constant by equation (A.3.11). We also note that $\tilde{\pi}_s \equiv \tilde{\pi}_{s(xy)} + \tilde{\pi}_{s(x)}$ is constant because it can be shown $\mu = (\tilde{\pi}_{s(xy)} + \tilde{\pi}_{s(x)})(\tilde{\pi}_{b(xy)} + \tilde{\pi}_{b(x)})$ as we did in Claim A.1.2. With these results, equation (A.3.13) can be rewritten as

$$\tilde{\pi}_{b(x)} = \frac{\tilde{\pi}_b\tilde{\pi}_{s(x)}}{\tilde{\pi}_s + \tilde{\pi}_{s(x)}}\tilde{\pi}_b. \tag{B.1}$$

This implies that $\partial\tilde{\pi}_{s(x)}/\partial\tilde{\pi}_{b(xy)} = \partial\tilde{\pi}_{s(x)}/\partial(\tilde{\pi}_b - \tilde{\pi}_{b(x)}) < 0$.

Finally, we look at the sign of $\partial\rho/\partial\tilde{\pi}_{s(x)}$. By substituting equation (B.1), $\tilde{\pi}_b = \tilde{\pi}_{b(xy)} + \tilde{\pi}_{b(x)}$ and $\tilde{\pi}_s = \tilde{\pi}_{s(xy)} + \tilde{\pi}_{s(x)}$ into equation (A.3.10), we obtain

$$\rho = \frac{\tilde{\pi}_s - \tilde{\pi}_{s(x)}}{\tilde{\pi}_{s(x)}\tilde{\pi}_{s(x)}}\frac{\tilde{\pi}_s + 2\tilde{\pi}_{s(x)}}{\tilde{\pi}_s + \tilde{\pi}_{s(x)}}\tilde{\pi}_b. \tag{B.2}$$

This equation implies $\partial\rho/\partial\tilde{\pi}_{s(x)} < 0$.

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