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Abstract

We calculate explicit representations of locally risk-minimizing of call and put options for the Barndorff-Nielsen and Shephard models.

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Local risk-minimization for Barndorff-Nielsen and Shephard models

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Abstract

We aim to obtain explicit representations of locally risk-minimizing of call and put options for the Barndorff-Nielsen and Shephard models, which are Ornstein-Uhlenbeck type stochastic volatility models. Arai and Suzuki [1] obtained a formula of locally risk-minimizing for Lévy markets under many additional conditions by using Malliavin calculus for Lévy processes. In this paper, supposing mild conditions, we make sure that the Barndorff-Nielsen and Shephard models satisfy all the conditions imposed in [1]. Among others, we investigate the Malliavin differentiability of the density of the minimal martingale measure.

Keywords: Local risk-minimization, Barndorff-Nielsen and Shephard models, Stochastic volatility models, Malliavin calculus, Lévy processes.

1 Introduction

The aim of this paper is to obtain explicit representations of locally risk-minimizing (LRM, for short) of call and put options for the Barndorff-Nielsen and Shephard models (BNS model, for short). Here the BNS models are Ornstein-Uhlenbeck (OU, for short) type stochastic volatility models undertaken by Barndorff-Nielsen and Shephard [2], [3]. On the other hand, LRM is a very well-known quadratic hedging method of contingent claims for incomplete financial markets. Although its theoretical aspects have been developed well, little is known about its explicit representations. Accordingly, Arai and Suzuki [1] have developed this problem for Lévy markets by using Malliavin calculus for Lévy processes. They gave in Theorem 3.7 of their paper an explicit formula for LRM including some Malliavin derivatives. Now, Lévy markets mean models whose asset price process is described by a solution to the

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following stochastic differential equation (SDE, for short):

$$dS_t = S_{t-}\left[\alpha_t dt + \beta_t dW_t + \int_{\mathbb{R}\setminus\{0\}} \gamma_{t,z}\widetilde{N}(dt,dz)\right], \quad S_0 > 0, \tag{1.1}$$

where W is a 1-dimensional Brownian motion, $\widetilde{N}$ is a compensated Poisson random measure; and α, β and γ are predictable processes. If α, β and γ are deterministic, a representation of LRM is given simply under some mild conditions. Indeed, [1] calculated explicitly LRM of call options, Asian options and lookback options for the deterministic coefficients case. On the other hand, according to Theorem 3.7 in [1], we need to impose many additional conditions on models with random coefficients. Thus, they postponed concrete calculations for such models. Since the BNS model is one of typical examples of the random coefficients case, we treat it in this paper.

Now, we introduce the BNS models. Many empirical studies say that the BNS models capture many stylized facts of financial time series. The square volatility process σ^2 of a BNS model is given as an OU process driven by a subordinator, that is, a nondecreasing Lévy process. More precisely, σ^2 is given as a solution to the following SDE:

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dH_{\lambda t}, \quad \sigma_0^2 > 0, \tag{1.2}$$

where $\lambda > 0$, H is a subordinator without drift. Now, the asset price process S of a BNS model is described as

$$S_t = S_0 \exp\left\{\int_0^t \left(\mu - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s + \rho H_{\lambda t}\right\}, \tag{1.3}$$

where $S_0 > 0$, $\rho \leq 0$, $\mu \in \mathbb{R}$. Note that the last term $\rho H_{\lambda t}$ accounts for the leverage effect, which is a stylized fact such that the asset price declines at the moment when the volatility increases. Moreover, defining $J_t := H_{\lambda t}$, we denote by N the Poisson random measure of J. So that, we have $J_t = \int_0^\infty xN([0,t],dx)$. Denoting by ν the Lévy measure of J, we have that $\widetilde{N}(dt,dx) := N(dt,dx) - \nu(dx)dt$ is the compensated Poisson random measure. Then, the asset price process S given in (1.3) is a solution to the following SDE:

$$dS_t = S_{t-}\left\{\alpha dt + \sigma_t dW_t + \int_0^\infty (e^{\rho x} - 1)\widetilde{N}(dt,dx)\right\}, \tag{1.4}$$

where $\alpha := \mu + \int_0^\infty (e^{\rho x} - 1)\nu(dx)$. Thus, the BNS models are corresponding to the case where β in (1.1) is random.

In this paper, we shall derive LRM for BNS models by using Theorem 3.7 of [1]. Thus, the primal part of our discussion lies in confirmation of all the conditions imposed on Theorem 3.7 of [1]. In particular, we need to investigate the Malliavin differentiability of the density of the minimal martingale measure (MMM, for short), which is an indispensable equivalent martingale measure to discuss LRM. To the best of our knowledge, there is no literature on LRM for BNS models. On the other hand, there are some preceding research on the mean-variance hedging, which is an alternative quadratic hedging method, for BNS models. Cont, Tankov and Voltchkova [6], and Kallsen and Pauwels [10] studied this problem under the assumption that S is a martingale. Kallsen and Vierthauer [11] treated the case where $\rho = 0$. Recently, Benth and Detering [4] dealt with the BNS model framework to represent a future price process on electricity under the assumption that S is a martingale and $\rho = 0$.

Outline of this paper is as follows. After giving preliminaries in Section 2, we address the main results in Section 3. Theorem 3.1 gives an explicit representation of LRM for put options. LRM for call options is provided as its corollary. A proof of Theorem 3.1 is discussed in Section 4. Section 5 is devoted to the Malliavin differentiability of the density of the MMM. Conclusions are given in Section 6. Some additional calculations are provided in Appendix.

2 Preliminaries

We consider a financial market model in which only one risky asset and one riskless asset are tradable. For simplicity, we assume that the interest rate is given by 0. Let $T > 0$ be the finite time horizon. The fluctuation of the risky asset is described as a process S given by (1.3). We adapt the same mathematical framework as [1]. The structure of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will be discussed in Subsection 2.3 below. Remark that the Poisson random measure N and the Lévy measure ν of J are defined on $[0, T] \times (0, \infty)$ and $(0, \infty)$, respectively. Remark that

$$\int_0^\infty (x \wedge 1)\nu(dx) < \infty$$

by Proposition 3.10 of Cont and Tankov [5]. Letting ν^H be the Lévy measure of H, we have $\nu(dx) = \lambda\nu^H(dx)$. Denoting $A_t := \int_0^t S_{s-}\alpha ds$ and $M_t := S_t - S_0 - A_t$, we have $S_t = S_0 + M_t + A_t$, which is the canonical decomposition of S. Further, we denote $L_t := \log(S_t/S_0)$ for $t \in [0, T]$, that is,

$$L_t = \int_0^t \left(\mu - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s + \rho J_t. \tag{2.1}$$

Remark 2.1 *Noting that $\sigma_{t-} = \sigma_t$ a.s. for any $t \in [0, T]$, we can regard σ_t and σ_t^2 as predictable processes. For example, we may identify $\sigma_t dW_t$ in (1.4) with $\sigma_{t-}dW_t$, if need be.*

Next, we state our standing assumptions as follows:

Assumption 2.2
1. *$\int_1^\infty \exp\{2(\mathcal{B}(T) \vee |\rho|)x\}\nu(dx) < \infty$, where $\mathcal{B}(t) := \frac{1-e^{-\lambda t}}{\lambda}$ for $t \in [0, T]$.*
2. *$\frac{\alpha}{e^{-\lambda T}\sigma_0^2 + C_\rho} > -1$, where $C_\rho := \int_0^\infty (e^{\rho x} - 1)^2\nu(dx)$.*

Remark 2.3 *1. Item 1 in Assumption 2.2 ensures $\int_0^\infty x^2\nu(dx) < \infty$, which means $\mathbb{E}[J_T^2] < \infty$. In addition, we have $\int_0^\infty (e^{\rho x}-1)^2\nu(dx) \leq \int_0^\infty \rho^2 x^2 \nu(dx) < \infty$, since $0 \leq 1-e^{\rho x} \leq -\rho x$.*

2. *As seen in Subsection 2.3 of [1], the so-called (SC) condition is satisfied under Assumption 2.2. For more details on the (SC) condition, see Schweizer [15], [16]. Moreover, Lemma 2.11 of [1] implies that $\mathbb{E}\left[\sup_{t\in[0,T]}|S_t|^2\right] < \infty$.*

3. *By (A.2) in Appendix, item 2 ensures that $\frac{\alpha}{\sigma_t^2+C_\rho} > -1$ for any $t \in [0,T]$.*

Remark 2.4 *As introduced in Nicolato and Venardos [12], there are two representative examples of σ^2. The first is the case where ν^H is given as*

$$\nu^H(dx) = \frac{a}{2\sqrt{2\pi}} x^{-\frac{3}{2}}(1+b^2x)e^{-\frac{1}{2}b^2x}\mathbf{1}_{(0,\infty)}(x)dx$$

where $a > 0$ and $b > 0$. In this case, the invariant distribution of the squared volatility process σ^2 follows an inverse-Gaussian distribution with parameters $a > 0$ and $b > 0$. σ^2 is called an IG-OU process. If $\frac{b^2}{2} > 2(\mathcal{B}(T) \vee |\rho|)$, then item 1 of Assumption 2.2 is satisfied. The second example is what we call Gamma-OU process, that is, the case where the invariant distribution of σ^2 is given by a Gamma distribution with parameters $a > 0$ and $b > 0$. In this case, ν^H is described as

$$\nu^H(dx) = abe^{-bx}\mathbf{1}_{(0,\infty)}(x)dx.$$

As well as the IG-OU case, item 1 of Assumption 2.2 is satisfied if $b > 2(\mathcal{B}(T) \vee |\rho|)$. For more details on this topic, see also Schoutens [14].

2.1 Locally risk-minimizing

In this subsection, we give a definition of LRM based on Theorem 1.6 of [16].

Definition 2.5 *1. Θ_S denotes the space of all $\mathbb{R}$-valued predictable processes ξ satisfying $\mathbb{E}\left[\int_0^T \xi_t^2 d\langle M\rangle_t + (\int_0^T |\xi_t dA_t|)^2\right] < \infty$.*

2. *An L^2-strategy is given by a pair $\varphi = (\xi,\eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi S + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0,T]$. Note that ξ_t (resp. η_t) represents the amount of units of the risky asset (resp. the risk-free asset) an investor holds at time t.*

3. *For claim $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by $C_t^F(\varphi) := F1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$ is called the cost process of $\varphi = (\xi,\eta)$ for F.*

4. *An L^2-strategy φ is said locally risk-minimizing for claim F if $V_T(\varphi) = 0$ and $C^F(\varphi)$ is a martingale orthogonal to M, that is, $[C^F(\varphi), M]$ is a uniformly integrable martingale.*

5. *An $F \in L^2(\mathbb{P})$ admits a Föllmer-Schweizer decomposition (FS decomposition, for short) if it can be described by*

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F, \tag{2.2}$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

For more details on LRM, see [15], [16]. Now, we introduce Proposition 5.2 of [16].

Proposition 2.6 (Proposition 5.2 of [16]) *Under Assumption 2.2, an LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition; and its relationship is given by*

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F1_{\{t=T\}} - \xi_t^F S_t.$$

Thus, it suffices to get a representation of ξ^F in (2.2) in order to obtain LRM for claim F. Henceforth, we identify ξ^F with LRM for F.

2.2 Minimal martingale measure

We need to study upon the MMM in order to discuss FS decomposition. A probability measure $\mathbb{P}^* \sim \mathbb{P}$ is called the MMM, if S is a $\mathbb{P}^*$-martingale; and any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale under $\mathbb{P}^*$. Now, we consider the following SDE:

$$dZ_t = -Z_{t-}\Lambda_t dM_t, \quad Z_0 = 1, \tag{2.3}$$

where $\Lambda_t := \frac{1}{S_{t-}} \frac{\alpha}{\sigma_t^2 + C_\rho}$. The solution to (2.3) is a stochastic exponential of $-\int_0^\cdot \Lambda_t dM_t$. More precisely, denoting

$$u_s := \Lambda_s S_{s-} \sigma_s = \frac{\alpha \sigma_s}{\sigma_s^2 + C_\rho} \quad \text{and} \quad \theta_{s,x} := \Lambda_s S_{s-}(e^{\rho x} - 1) = \frac{\alpha(e^{\rho x} - 1)}{\sigma_s^2 + C_\rho}$$

for $s \in [0, T]$ and $x \in (0, \infty)$, we have $\Lambda_t dM_t = u_t dW_t + \int_0^\infty \theta_{t,z} \widetilde{N}(dt, dz)$; and

$$Z_t = \exp\bigg\{ -\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds + \int_0^t \int_0^\infty \log(1 - \theta_{s,x}) \widetilde{N}(ds, dx) + \int_0^t \int_0^\infty (\log(1 - \theta_{s,x}) + \theta_{s,x}) \nu(dx) ds \bigg\}. \tag{2.4}$$

Now, remark that

$$\int_0^T \int_0^\infty \left\{ |\log(1 - \theta_{s,x})|^2 + \theta_{s,x}^2 \right\} \nu(dx) ds \leq 2TC_\theta^2 \rho^2 \int_0^\infty x^2 \nu(dx) < \infty$$

by Lemma A.6. Noting that the boundedness of u_s by Lemma A.6, and

$$(1-\theta_{s,x})\log(1-\theta_{s,x})+\theta_{s,x} \leq (1-\theta_{s,x})(-\theta_{s,x})+\theta_{s,x} = \theta_{s,x}^2,$$

we have the martingale property of Z by Theorem 1.4 of Ishikawa [9]. Now, we see the following proposition:

Proposition 2.7 *1. $Z_T \in L^2(\mathbb{P})$.*

2. A probability measure $\mathbb{P}^*$ defined as $\frac{d\mathbb{P}^*}{d\mathbb{P}} = Z_T$ is the MMM.

Proof. We can see Item 2 immediately from item 1 and the martingale property of Z. Then, we show item 1. Here (2.4) and Lemma A.6 imply that

$$\begin{aligned}
Z_T^2 &= \exp\left\{ -\int_0^T 2u_s dW_s - \frac{1}{2}\int_0^T 4u_s^2 ds + \int_0^T\int_0^\infty \log(1-\delta_{s,x})\widetilde{N}(ds,dx) \right. \\
&\quad \left. + \int_0^T\int_0^\infty \left[\log(1-\delta_{s,x})+\delta_{s,x}+\theta_{s,x}^2\right]\nu(dx)ds + \int_0^T u_s^2 ds \right\} \\
&\leq \exp\left\{ -\int_0^T 2u_s dW_s - \frac{1}{2}\int_0^T 4u_s^2 ds + \int_0^T\int_0^\infty \log(1-\delta_{s,x})\widetilde{N}(ds,dx) \right. \\
&\quad \left. + \int_0^T\int_0^\infty [\log(1-\delta_{s,x})+\delta_{s,x}]\nu(dx)ds + T(C_\theta^2 C_\rho + C_u^2) \right\}
\end{aligned}$$

where $\delta_{s,x} := 2\theta_{s,x} - \theta_{s,x}^2$. That is, denoting

$$\begin{aligned}
Y_t &:= \exp\left\{ -\int_0^T 2u_s dW_s - \frac{1}{2}\int_0^T 4u_s^2 ds + \int_0^T\int_0^\infty \log(1-\delta_{s,x})\widetilde{N}(ds,dx) \right. \\
&\quad \left. + \int_0^T\int_0^\infty [\log(1-\delta_{s,x})+\delta_{s,x}]\nu(dx)ds \right\} \qquad (2.5)
\end{aligned}$$

for $t \in [0,T]$, we have

$$Z_T^2 \leq Y_T \exp\{T(C_\theta^2 C_\rho + C_u^2)\}. \qquad (2.6)$$

Thus, we need only to show the process Y is a martingale. Lemma A.6 again yields that

$$\int_0^T\int_0^\infty |\log(1-\delta_{s,x})|^2\nu(dx)ds \leq \int_0^T\int_0^\infty 4C_\theta^2\rho^2x^2\nu(dx)ds < \infty;$$

and $\delta_{s,x}^2 = \theta_{s,x}^2(2-\theta_{s,x})^2 \leq C_\theta^2\rho^2x^2(2+C_\theta)^2$, that is, $\int_0^T\int_0^\infty \delta_{s,x}^2\nu(dx)ds < \infty$. In addition, we have

$$\int_0^T\int_0^\infty [(1-\delta_{s,x})\log(1-\delta_{s,x})+\delta_{s,x}]\nu(dx)ds \leq \int_0^T\int_0^\infty \delta_{s,x}^2\nu(dx)ds < \infty.$$

Hence, all the conditions in Theorem 1.4 of [9] are satisfied, from which Y is a martingale. This completes the proof of Proposition 2.7. □

2.3 Malliavin calculus

In this subsection, we prepare Malliavin calculus based on the canonical Lévy space framework undertaken by Solé, Utzet and Vives [19]. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be given by $(\Omega_W \times \Omega_J, \mathcal{F}_W \times \mathcal{F}_J, \mathbb{P}_W \times \mathbb{P}_J)$, where $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ is a 1-dimensional Wiener space on $[0, T]$ with coordinate mapping process W; and $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ is the canonical Lévy space for J, that is, $\Omega_J = \cup_{n=0}^{\infty}([0, T] \times (0, \infty))^n$; and $J_t(\omega_J) = \sum_{i=1}^{n} z_i \mathbf{1}_{\{t_i \leq t\}}$ for $t \in [0, T]$ and $\omega_J = ((t_1, z_1), \ldots, (t_n, z_n)) \in ([0, T] \times (0, \infty))^n$. Note that $([0, T] \times (0, \infty))^0$ represents an empty sequence. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ be the canonical filtration completed for $\mathbb{P}$. For more details, see Delong and Imkeller [7], and [19].

First of all, we define measures q and Q on $[0, T] \times [0, \infty)$ as

$$q(E) := \int_E \delta_0(dz)dt + \int_E z^2 \nu(dz)dt,$$

and

$$Q(E) := \int_E \delta_0(dz)dW_t + \int_E z\widetilde{N}(dt, dz),$$

where $E \in \mathcal{B}([0, T] \times [0, \infty))$ and δ_0 is the Dirac measure at 0. For $n \in \mathbb{N}$, we denote by $L^2_{T,q,n}$ the set of product measurable, deterministic functions $h : ([0, T] \times [0, \infty))^n \to \mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,q,n}} := \int_{([0,T]\times[0,\infty))^n} |h((t_1, z_1), \cdots, (t_n, z_n))|^2 q(dt_1, dz_1) \cdots q(t_n, z_n) < \infty.$$

For $n \in \mathbb{N}$ and $h \in L^2_{T,q,n}$, we define

$$I_n(h) := \int_{([0,T]\times[0,\infty))^n} h((t_1, z_1), \cdots, (t_n, z_n))Q(dt_1, dz_1) \cdots Q(dt_n, dz_n).$$

Formally, we denote $L^2_{T,q,0} := \mathbb{R}$ and $I_0(h) := h$ for $h \in \mathbb{R}$. Under this setting, any $F \in L^2(\mathbb{P})$ has the unique representation $F = \sum_{n=0}^{\infty} I_n(h_n)$ with functions $h_n \in L^2_{T,q,n}$ that are symmetric in the n pairs $(t_i, z_i), 1 \leq i \leq n$, and we have $\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n!\|h_n\|^2_{L^2_{T,q,n}}$. We define a Malliavin derivative operator.

Definition 2.8 *1. Let $\mathbb{D}^{1,2}$ denote the set of $\mathcal{F}$-measurable random variables $F \in L^2(\mathbb{P})$ with $F = \sum_{n=0}^{\infty} I_n(h_n)$ satisfying $\sum_{n=1}^{\infty} nn!\|h_n\|^2_{L^2_{T,q,n}} < \infty$.*

2. For any $F \in \mathbb{D}^{1,2}$, a Malliavin derivative $DF : [0, T] \times [0, \infty) \times \Omega \to \mathbb{R}$ is defined as

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(h_n((t, z), \cdot))$$

for q-a.e. $(t, z) \in [0, T] \times [0, \infty)$, $\mathbb{P}$-a.s.

3 Main results

In this section, we introduce explicit representations of LRM for call and put options as the main results of this paper by using the framework of Theorem 3.7 of [1]. To this end, denoting by F the underlying contingent claim, we need $Z_T F \in L^2(\mathbb{P})$. When F is a call option, this condition is not necessarily satisfied in our setting. On the other hand, since put options are bounded, we do not need to care about any integrability condition for them. Thus, we treat put options firstly; and derive LRM for call options from put-call parity. Due to this idea, we can do without any additional assumption.

Theorem 3.1 *For $K > 0$, LRM $\xi^{(K-S_T)^+}$ of put option $(K - S_T)^+$ is represented as*

$$\begin{aligned}\xi_t^{(K-S_T)^+} &= \frac{1}{S_{t-}(\sigma_t^2 + C_\rho)} \Big\{ \sigma_t^2 \mathbb{E}_{\mathbb{P}^*}[-\mathbf{1}_{\{S_T<K\}} S_T | \mathcal{F}_{t-}] \\ &\quad + \int_0^\infty \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+ (H_{t,z}^* - 1) + z H_{t,z}^* D_{t,z} (K - S_T)^+ | \mathcal{F}_{t-}](e^{\rho z} - 1)\nu(dz) \Big\},\end{aligned} \tag{3.1}$$

where $D_{t,z}(K - S_T)^+$ is given by Proposition 4.1; and

$$H_{t,z}^* := \exp\{z D_{t,z} \log Z_T - \log(1 - \theta_{t,z})\}$$

for $(t, z) \in [0, T] \times (0, \infty)$. Note that $D_{t,z} \log Z_T$ is provided in Proposition A.10.

Remark 3.2 *In order to obtain a more explicit representation of $\xi_t^{(K-S_T)^+}$, we calculate the conditional expectation in the second term of (3.1) as follows:*

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+ (H_{t,z}^* - 1) + z H_{t,z}^* D_{t,z} (K - S_T)^+ | \mathcal{F}_{t-}] \\ &= \mathbb{E}_{\mathbb{P}^*}[H_{t,z}^* \{(K - S_T)^+ + z D_{t,z} (K - S_T)^+\} - (K - S_T)^+ | \mathcal{F}_{t-}] \\ &= \frac{\mathbb{E}[Z_T H_{t,z}^* \{(K - S_T)^+ + z D_{t,z} (K - S_T)^+\} | \mathcal{F}_{t-}]}{Z_{t-}} - \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+ | \mathcal{F}_{t-}] \\ &= \frac{\mathbb{E}[Z_T H_{t,z}^* (K - S_T \exp\{z D_{t,z} L_T\})^+ | \mathcal{F}_{t-}]}{Z_{t-}} - \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+ | \mathcal{F}_{t-}],\end{aligned}$$

where $D_{t,z} L_T$ is given explicitly by Proposition A.5.

Now, we calculate $\frac{Z_T H_{t,z}^*}{Z_{t-}}$. For $t \in [0, T]$, $z \in (0, \infty)$, $s \in [t, T]$ and $x \in (0, \infty)$, we denote

$$A_{t,z,s}^u := u_s + z D_{t,z} u_s = f_u\left(\sqrt{\sigma^2 + z e^{-\lambda(s-t)}}\right) = \frac{\alpha\sqrt{\sigma_s^2 + z e^{-\lambda(s-t)}}}{\sigma_s^2 + z e^{-\lambda(s-t)} + C_\rho},$$

and

$$A_{t,z,s,x}^\theta := \theta_{s,x} + z D_{t,z} \theta_{s,x} = f_\theta\left(\sqrt{\sigma_s^2 + z e^{-\lambda(s-t)}}\right)(e^{\rho x} - 1) = \frac{\alpha(e^{\rho x} - 1)}{\sigma_s^2 + z e^{-\lambda(s-t)} + C_\rho} \tag{3.2}$$

by Lemmas A.7 and A.8. We obtain then, by (2.4), Lemmas A.7–A.9 and Proposition A.10,

$$
\begin{aligned}
\frac{Z_T H^*_{t,z}}{Z_{t-}} &= \exp\Bigg\{ -\int_t^T (u_s + zD_{t,z}u_s)dW_s - \frac{1}{2}\int_t^T (u_s + zD_{t,z}u_s)^2 ds \\
&\quad + \int_{t-}^T \int_0^\infty [\log(1-\theta_{s,x}) + zD_{t,z}\log(1-\theta_{s,x})]\, \widetilde{N}(ds,dx) \\
&\quad + \int_t^T \int_0^\infty [\log(1-\theta_{s,x}) + zD_{t,z}\log(1-\theta_{s,x}) + \theta_{s,x} + zD_{t,z}\theta_{s,x}]\, \nu(dx)ds \Bigg\} \\
&= \exp\Bigg\{ -\int_t^T A^u_{t,z,s} dW_s - \frac{1}{2}\int_t^T (A^u_{t,z,s})^2 ds \\
&\quad + \int_{t-}^T \int_0^\infty \log(1-A^\theta_{t,z,s,x}) \widetilde{N}(ds,dx) \\
&\quad + \int_t^T \int_0^\infty \Big[\log(1-A^\theta_{t,z,s,x}) + A^\theta_{t,z,s,x}\Big]\, \nu(dx)ds \Bigg\}.
\end{aligned}
$$

Note that $A^u_{t,z,s}$ is bounded. Moreover, (3.2) and (A.8) imply that $\int_0^\infty (A^\theta_{t,z,s,x})^2 \nu(dx) < C^2_\theta C_\rho$ and $A^\theta_{t,z,s,x} \le 1 - e^{\rho x}$. We have then

$$
|\log(1-A^\theta_{t,z,s,x})|^2 \le \begin{cases} \rho^2 x^2, & \text{if } A^\theta_{t,z,s,x} > 0, \\ (A^\theta_{t,z,s,x})^2, & \text{otherwise,} \end{cases}
$$

which implies that $\int_0^\infty |\log(1-A^\theta_{t,z,s,x})|^2 \nu(dx) < \infty$. As a result, we have

$$
\mathbb{E}\left[\frac{Z_T H^*_{t,z}}{Z_{t-}} \middle| \mathcal{F}_{t-} \right] = 1 \tag{3.3}
$$

from the view of Theorem 1.4 in [9].

Corollary 3.3 *LRM for call option $(S_T - K)^+$ is given as $\xi^{(S_T-K)^+} = 1 + \xi^{(K-S_T)^+}$.*

Proof. Note that S is a $\mathbb{P}^*$-martingale by Remark 2.3 and Proposition 2.7. We have then

$$
\begin{aligned}
(S_T - K)^+ &= S_T - K + (K - S_T)^+ \\
&= S_0 + \int_0^T dS_t - K + \mathbb{E}_{\mathbb{P}^*}[(K-S_T)^+] + \int_0^T \xi_t^{(K-S_T)^+} dS_t + L_T^{(K-S_T)^+} \\
&= \mathbb{E}_{\mathbb{P}^*}\left[S_T - K + (K-S_T)^+\right] + \int_0^T \left(1 + \xi_t^{(K-S_T)^+}\right) dS_t + L_T^{(K-S_T)^+} \\
&= \mathbb{E}_{\mathbb{P}^*}\left[(S_T - K)^+\right] + \int_0^T \left(1 + \xi_t^{(K-S_T)^+}\right) dS_t + L_T^{(K-S_T)^+},
\end{aligned}
$$

where $L^{(K-S_T)^+}$ is defined in (2.2). This is an FS decomposition of $(S_T - K)^+$ since $1 \in \Theta_S$ by the (SC) condition. □

4 Proof of Theorem 3.1

We begin with the Malliavin derivatives of put options.

Proposition 4.1 *For $K > 0$, we have $(K - S_T)^+ \in \mathbb{D}^{1,2}$ and*

$$D_{t,z}(K - S_T)^+ = -\mathbf{1}_{\{S_T < K\}} S_T D_{t,0} L_T \cdot \mathbf{1}_{\{0\}}(z) + \frac{(K - S_T e^{zD_{t,z}L_T})^+ - (K - S_T)^+}{z} \mathbf{1}_{(0,\infty)}(z).$$

Proof. First of all, note that $S_T = S_0 e^{L_T}$, and $L_T \in \mathbb{D}^{1,2}$ by Proposition A.5. We now denote

$$f_K(r) := \begin{cases} S_0 e^r, & \text{if } r \leq \log(K/S_0), \\ Kr + K(1 - \log(K/S_0)), & \text{if } r > \log(K/S_0). \end{cases}$$

Then, $f_k \in C^1(\mathbb{R})$ and $0 < f_K'(r) \leq K$ for any $r \in \mathbb{R}$. We also note $(K - S_T)^+ = (K - f_K(L_T))^+$. Proposition 2.6 in [18] implies that $f_K(L_T) \in \mathbb{D}^{1,2}$ and

$$D_{t,z} f_K(L_T) = f_K'(L_T) D_{t,0} L_T \cdot \mathbf{1}_{\{0\}}(z) + \frac{f_K(L_T + zD_{t,z}L_T) - f_K(L_T)}{z} \mathbf{1}_{(0,\infty)}(z).$$

The same argument as Theorem 4.1 of [1] implies that, for q-a.e. $(t, z) \in [0, T] \times [0, \infty)$,

$$\begin{aligned} D_{t,z}(K - S_T)^+ &= D_{t,z}(K - f_K(L_T))^+ \\ &= -\mathbf{1}_{\{f_K(L_T) < K\}} D_{t,0} f_K(L_T) \cdot \mathbf{1}_{\{0\}}(z) \\ &\quad + \frac{(K - f_K(L_T) - zD_{t,z} f_K(L_T))^+ - (K - f_K(L_T))^+}{z} \mathbf{1}_{(0,\infty)}(z) \\ &= -\mathbf{1}_{\{S_T < K\}} S_T D_{t,0} L_T \cdot \mathbf{1}_{\{0\}}(z) \\ &\quad + \frac{(K - f_K(L_T + zD_{t,z}L_T))^+ - (K - f_K(L_T))^+}{z} \mathbf{1}_{(0,\infty)}(z) \\ &= -\mathbf{1}_{\{S_T < K\}} S_T D_{t,0} L_T \cdot \mathbf{1}_{\{0\}}(z) \\ &\quad + \frac{(K - S_T e^{zD_{t,z}L_T})^+ - (K - S_T)^+}{z} \mathbf{1}_{(0,\infty)}(z). \end{aligned}$$

□

Now, we show Theorem 3.1 through Theorem 3.7 of [1]. To this end, we need only to make sure of all the conditions imposed there. Since Assumptions 2.1 and 2.6 in [1] are satisfied by Remark 2.3, Proposition 2.7 and the boundedness of $(K - S_T)^+ =: F$, it remains to see Assumption 3.4 and (3.1) in [1]. First of all, we confirm Assumption 3.4 listed as below:

C1 $u, u^2 \in \mathbb{L}_0^{1,2}$; and $2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2 \in L^2(q \times \mathbb{P})$ for a.e. $s \in [0, T]$.

C2 $\theta + \log(1 - \theta) \in \widetilde{\mathbb{L}}_1^{1,2}$, and $\log(1 - \theta) \in \mathbb{L}_1^{1,2}$.

C3 For q-a.e. $(s,x) \in [0,T] \times (0,\infty)$, there is an $\varepsilon_{s,x} \in (0,1)$ such that $\theta_{s,x} < 1 - \varepsilon_{s,x}$.

C4 $Z_T \left\{ D_{t,0} \log Z_T \mathbf{1}_{\{0\}}(z) + \frac{e^{zD_{t,z} \log Z_T} - 1}{z} \mathbf{1}_{(0,\infty)}(z) \right\} \in L^2(q \times \mathbb{P})$.

C5 $F \in \mathbb{D}^{1,2}$; and $Z_T D_{t,z} F + F D_{t,z} Z_T + z D_{t,z} F \cdot D_{t,z} Z_T \in L^2(q \times \mathbb{P})$.

C6 $F H^*_{t,z}$, $H^*_{t,z} D_{t,z} F \in L^1(\mathbb{P}^*)$ for q-a.e. $(t,z) \in [0,T] \times (0,\infty)$.

Here $\mathbb{L}^{1,2}_0$, $\mathbb{L}^{1,2}_1$ and $\widetilde{\mathbb{L}}^{1,2}_1$ are defined as follows:

- $\mathbb{L}^{1,2}_0$ denotes the space of $G : [0,T] \times \Omega \to \mathbb{R}$ satisfying
  (a) $G_s \in \mathbb{D}^{1,2}$ for a.e. $s \in [0,T]$,
  (b) $\mathbb{E}\left[\int_{[0,T]} |G_s|^2 ds\right] < \infty$,
  (c) $\mathbb{E}\left[\int_{[0,T]\times[0,\infty)} \int_0^T |D_{t,z} G_s|^2 ds q(dt,dz)\right] < \infty$.
- $\mathbb{L}^{1,2}_1$ is defined as the space of $G : [0,T] \times (0,\infty) \times \Omega \to \mathbb{R}$ such that
  (d) $G_{s,x} \in \mathbb{D}^{1,2}$ for q-a.e. $(s,x) \in [0,T] \times (0,\infty)$,
  (e) $\mathbb{E}\left[\int_{[0,T]\times(0,\infty)} |G_{s,x}|^2 \nu(dx) ds\right] < \infty$,
  (f) $\mathbb{E}\left[\int_{[0,T]\times[0,\infty)} \int_{[0,T]\times(0,\infty)} |D_{t,z} G_{s,x}|^2 \nu(dx) ds q(dt,dz)\right] < \infty$.
- $\widetilde{\mathbb{L}}^{1,2}_1$ is defined as the space of $G \in \mathbb{L}^{1,2}_1$ such that
  (g) $\mathbb{E}\left[\left(\int_{[0,T]\times(0,\infty)} |G_{s,x}| \nu(dx) ds\right)^2\right] < \infty$,
  (h) $\mathbb{E}\left[\int_{[0,T]\times[0,\infty)} \left(\int_{[0,T]\times(0,\infty)} |D_{t,z} G_{s,x}| \nu(dx) ds\right)^2 q(dt,dz)\right] < \infty$.

Condition C1: First, we see $u \in \mathbb{L}^{1,2}_0$. To this end, we check items (a)-(c) in the definition of $\mathbb{L}^{1,2}_0$. Lemmas A.7 and A.6 ensure items (a) and (b), respectively. To see item (c), Lemma A.7 implies

$$\mathbb{E}\left[\int_{[0,T]\times[0,\infty)} \int_0^T |D_{t,z} u_s|^2 ds q(dt,dz)\right] \leq \int_{[0,T]\times[0,\infty)} (T-t) \frac{C_u^2}{z} z^2 \nu(dz) dt < \infty,$$

from which $u \in \mathbb{L}^{1,2}_0$ follows.

Next, we see $2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2 \in L^2(q \times \mathbb{P})$ as

$$\begin{aligned} &\mathbb{E}\left[\int_{[0,T]\times[0,\infty)} (2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2)^2 q(dt,dz)\right] \\ &\leq 2C_u^4 \int_{[0,T]\times[0,\infty)} \left(\frac{4}{z} + 1\right) z^2 \nu(dz) dt < \infty \end{aligned} \tag{4.1}$$

by Lemmas A.6 and A.7.

Lastly, we see $u^2 \in \mathbb{L}_0^{1,2}$. Item (b) holds by Lemma A.6. Since $u_s \in \mathbb{D}^{1,2}$ and $u_s^2 \in L^2(\mathbb{P})$, Propositions 5.1 and 5.4 of [19], together with (4.1), imply item (a) and $D_{t,z}u_s^2 = 2u_s D_{t,z}u_s + z(D_{t,z}u_s)^2$. Moreover, a similar calculation with (4.1) gives item (c) as follows:

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,T]\times[0,\infty)}\int_0^T (D_{t,z}u_s^2)^2 ds q(dt,dz)\right] \\
&= \mathbb{E}\left[\int_{[0,T]\times[0,\infty)}\int_0^T (2u_s D_{t,z}u_s + z(D_{t,z}u_s)^2)^2 ds q(dt,dz)\right] < \infty.
\end{aligned}$$

□

Condition C2: We see $\log(1-\theta) \in \mathbb{L}_1^{1,2}$ firstly. Items (d) and (e) in the definition of $\mathbb{L}_1^{1,2}$ are given by Lemmas A.9 and A.6, respectively. As for item (f), Lemmas A.8 and A.9 imply

$$|D_{t,z}\log(1-\theta_{s,x})|^2 \leq \frac{(C'_\theta)^2}{z} e^{-2\rho x}(1-e^{\rho x})^2.$$

Since $\int_0^\infty e^{-2\rho x}(1-e^{\rho x})^2 \nu(dx) \leq \int_0^1 e^{-2\rho}\rho^2 x^2 \nu(dx) + \int_1^\infty e^{-2\rho x}\nu(dx) < \infty$ by Assumption 2.2, item (f) follows.

Next, we see $\theta + \log(1-\theta) \in \widetilde{\mathbb{L}}_1^{1,2}$. Note that we can see $\theta \in \mathbb{L}_1^{1,2}$ by the same manner as the proof of condition C1. Thus, we see items (g) and (h) in the definition of $\widetilde{\mathbb{L}}_1^{1,2}$. Since $|\theta_{s,x} + \log(1-\theta_{s,x})| \leq 2C_\theta|\rho|x$, item (g) follows. Next, Lemmas A.9 and A.8, and Assumption 2.2 imply

$$\begin{aligned}
&\int_{[0,T]\times(0,\infty)} |D_{t,z}(\theta_{s,x} + \log(1-\theta_{s,x}))|\nu(dx)ds \\
&\leq \int_{[0,T]\times(0,\infty)} |D_{t,z}\theta_{s,x}|(1+e^{-\rho x})\nu(dx)ds \\
&\leq \int_{[0,T]\times(0,\infty)} \frac{C'_\theta}{\sqrt{z}}(1-e^{\rho x})(1+e^{-\rho x})\nu(dx)ds \leq \frac{CT}{\sqrt{z}}
\end{aligned}$$

for some $C > 0$, from which item (h) follows. □

Condition C3: This is given by Lemma A.6. □

Condition C4: Proposition A.10 implies that $\log Z_T \in \mathbb{D}^{1,2}$, and $D_{t,0}\log Z_T = u_t$, from which $\mathbb{E}\left[\int_0^T (Z_T D_{t,0}\log Z_T)^2 dt\right] < \infty$ follows by Lemma A.6 and Proposition 2.7. Next, let $\Psi_{t,z}$ be the increment quoting operator defined in [19]. Since $Z_T \in \mathbb{D}^{1,2}$ by Section 5, Proposition 5.4 of [19] yields that, for $z > 0$,

$$\begin{aligned}
D_{t,z}Z_T &= \Psi_{t,z}Z_T = \Psi_{t,z}\exp\{\log Z_T\} \\
&= \frac{\exp\{\log Z_T(\omega_W, \omega_J^{t,z})\} - \exp\{\log Z_T(\omega_W, \omega_J)\}}{z} \\
&= \frac{\exp\{\log Z_T + z\frac{\log Z_T(\omega_W,\omega_J^{t,z}) - \log Z_T(\omega_W,\omega_J)}{z}\} - \exp\{\log Z_T\}}{z}
\end{aligned}$$

$$= \frac{\exp\{\log Z_T + z\Psi_{t,z} \log Z_T\} - \exp\{\log Z_T\}}{z}$$
$$= \frac{\exp\{\log Z_T + zD_{t,z} \log Z_T\} - \exp\{\log Z_T\}}{z}$$
$$= Z_T \frac{\exp(zD_{t,z} \log Z_T) - 1}{z}, \tag{4.2}$$

where $\omega_W \in \Omega_W$ and $\omega_J \in \Omega_J$. Here, when $\omega_J = ((t_1, z_1), \ldots, (t_n, z_n))$, we denote $\omega_J^{t,z} := ((t, z), (t_1, z_1), \ldots, (t_n, z_n))$. As a result, condition C4 follows. □

Condition C5: Noting that $|F + zD_{t,z}F| \leq K$ by Theorem 4.1, we have $FD_{t,z}Z_T + zD_{t,z}F \cdot D_{t,z}Z_T \in L^2(q \times \mathbb{P})$, since $Z_T \in \mathbb{D}^{1,2}$. Thus, it suffices to see $Z_T D_{t,z}F \in L^2(q \times \mathbb{P})$. To this end, we prove that $\mathbb{E}\left[\int_0^T (Z_T D_{t,0}F)^2 dt\right] < \infty$ firstly. Since $D_{t,0}F = -\mathbf{1}_{\{S_T < K\}} S_T D_{t,0} L_T = -\mathbf{1}_{\{S_T < K\}} S_T \sigma_t$ by Propositions 4.1 and A.5, we have $\mathbb{E}\left[\int_0^T (Z_T D_{t,0}F)^2 dt\right] \leq \mathbb{E}\left[Z_T^2 K^2 \int_0^T \sigma_t^2 dt\right]$. Thus, we have only to show $\mathbb{E}[Z_T^2 J_T] < \infty$ from the view of (A.3). Now, as seen in the proof of Proposition 2.7, Y defined in (2.5) is a positive martingale. Thus, we can define a probability measure $\mathbb{P}_Y$ as $d\mathbb{P}_Y = Y_T d\mathbb{P}$; and we have

$$\mathbb{E}[Y_T J_T] = \mathbb{E}_{\mathbb{P}_Y}[J_T] = \mathbb{E}_{\mathbb{P}_Y}\left[\int_0^T \int_0^\infty (1 - \delta_{s,x}) x \nu(dx) ds\right] < \infty,$$

since $(1 - \delta_{s,x})x = (1 - \theta_{s,x})^2 x \leq (1 + C_\theta)^2 x$. Hence, (2.6) implies that $\mathbb{E}[Z_T^2 J_T] < \infty$.

Next, we show $\mathbb{E}\left[\int_0^T \int_0^\infty (Z_T D_{t,z}F)^2 z^2 \nu(dz) dt\right] < \infty$. Note that

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_1^\infty (Z_T D_{t,z}F)^2 z^2 \nu(dz) dt\right] &\leq \mathbb{E}\left[\int_0^T \int_1^\infty \left(Z_T \frac{K}{z}\right)^2 z^2 \nu(dz) dt\right] \\ &\leq K^2 \mathbb{E}\left[Z_T^2 \int_0^T \int_1^\infty \nu(dz) dt\right] < \infty.\end{aligned}$$

Hence, we have only to show $\mathbb{E}\left[\int_0^T \int_0^1 Z_T^2 |D_{t,z}F|^2 z^2 \nu(dz) dt\right] < \infty$. If we have

$$|D_{t,z}F| \leq K|D_{t,z}L_T|, \tag{4.3}$$

$$\text{there is } C > 0 \text{ such that } \mathbb{E}\left[Z_T^2 |D_{t,z}L_T|^2\right] < \frac{C}{z} \text{ for any } z \in (0,1), \tag{4.4}$$

then we obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_0^1 Z_T^2 |D_{t,z}F|^2 z^2 \nu(dz) dt\right] &\leq K^2 \int_0^T \int_0^1 \mathbb{E}\left[Z_T^2 |D_{t,z}L_T|^2\right] z^2 \nu(dz) dt \\ &\leq K^2 C \int_0^T \int_0^1 z \nu(dz) dt < \infty.\end{aligned}$$

It remains to show (4.3) and (4.4). (4.3) is shown as

$$
\begin{aligned}
|D_{t,z}F| &= \frac{|(K - f_K(L_T + zD_{t,z}L_T))^+ - (K - f_K(L_T))^+|}{|z|} \\
&\le \frac{|f_K(L_T + zD_{t,z}L_T)) - f_K(L_T)|}{|z|} \le \frac{K|zD_{t,z}L_T|}{|z|} = K|D_{t,z}L_T|.
\end{aligned}
$$

Next, in order to see (4.4), it suffices to show that $\mathbb{E}_{\mathbb{P}^Y}[|D_{t,z}L_T|^2] < Cz^{-1}$ for some $C > 0$. The process W^Y defined as $dW_s^Y := dW_s + 2u_s ds$ is a Brownian motion under $\mathbb{P}^Y$. Noting that $\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}} - \sigma_s \le \sqrt{z}$ for $s \in [t, T]$, we have

$$
|D_{t,z}L_T| \le C_1 + \left| \int_t^T \frac{\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}} - \sigma_s}{z} dW_s^Y \right| + \frac{2C_u(T-t)}{\sqrt{z}}
$$

for some $C_1 > 0$ by Proposition A.5. Hence, we have

$$
\mathbb{E}_{\mathbb{P}^Y}[|D_{t,z}L_T|^2] \le 3C_1^2 + 3\mathbb{E}_{\mathbb{P}^Y}\left[\int_t^T \frac{1}{z} ds\right] + \frac{12C_u^2(T-t)^2}{z} \le \frac{C}{z}
$$

for some $C > 0$, since $0 < z < 1$. $\square$

Condition C6: In order to see $FH_{t,z}^* \in L^1(\mathbb{P}^*)$ for q-a.e. $(t,z) \in [0,T] \times (0,\infty)$, it suffices to show $\mathbb{E}[Z_T H_{t,z}^*] < \infty$, since F is bounded. Now, we have

$$
Z_T H_{t,z}^* = Z_T \frac{e^{zD_{t,z}\log Z_T}}{1 - \theta_{t,z}} = \frac{zD_{t,z}Z_T + Z_T}{1 - \theta_{t,z}} \le \hat{C}_\theta\{zD_{t,z}Z_T + Z_T\}
$$

by (4.2) and item 5 of Lemma A.6. Since $Z_T \in \mathbb{D}^{1,2}$ by Section 5, we have $D_{t,z}Z_T \in L^1(\mathbb{P})$ for q-a.e. $(t,z) \in [0,T] \times (0,\infty)$. Hence, $\mathbb{E}[Z_T H_{t,z}^*] < \infty$. Besides, since $D_{t,z}F \le \frac{K}{z}$, we have $H_{t,z}^* D_{t,z}F \in L^1(\mathbb{P}^*)$ for q-a.e. (t,z). $\square$

Condition (3.1) in [1]: As the last part of the proof of Theorem 3.1, we make sure of (3.1) in [1], which is given as follows:

$$
\mathbb{E}\left[\int_0^T \left\{ (h_t^0)^2 + \int_0^\infty (h_{t,z}^1)^2 \nu(dz) \right\} dt\right] < \infty, \tag{4.5}
$$

where $h_{t,z}^1 := \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F | \mathcal{F}_{t-}]$, and

$$
\begin{aligned}
h_t^0 &:= \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - F\left[\int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} + \int_0^T \int_0^\infty \frac{D_{t,0}\theta_{s,x}}{1 - \theta_{s,x}} \widetilde{N}^{\mathbb{P}^*}(ds, dx)\right] \middle| \mathcal{F}_{t-}\right] \\
&= -\mathbb{E}_{\mathbb{P}^*}\left[\mathbf{1}_{\{S_T < K\}} S_T \sigma_t \middle| \mathcal{F}_{t-} \right].
\end{aligned}
$$

Here $dW_t^{\mathbb{P}^*} := dW_t + u_t dt$ and $\widetilde{N}^{\mathbb{P}^*}(dt,dz) := \widetilde{N}(dt,dz) + \theta_{t,z}\nu(dz)dt$ are a Brownian motion and the compensated Poisson random measure of N under $\mathbb{P}^*$, respectively.

First of all, we have $\mathbb{E}\left[\int_0^T (h_t^0)^2 dt\right] \le K^2\mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] < \infty$ by (A.3). Next, we show $\mathbb{E}\left[\int_0^T \int_0^\infty (h_{t,z}^1)^2 \nu(dz)dt\right] < \infty$. Noting that $h_{t,z}^1 = \mathbb{E}_{\mathbb{P}^*}[(F + zD_{t,z}F)H_{t,z}^* - F|\mathcal{F}_{t-}]$, we have

$$h_{t,z}^1 \le \mathbb{E}_{\mathbb{P}^*}[(F + zD_{t,z}F)H_{t,z}^*|\mathcal{F}_{t-}] \le K\mathbb{E}_{\mathbb{P}^*}[H_{t,z}^*|\mathcal{F}_{t-}] = K,$$

since F and $H_{t,z}^*$ are nonnegative, $0 \le F + zD_{t,z}F \le K$ by Proposition 4.1, and $\mathbb{E}_{\mathbb{P}^*}[H_{t,z}^*|\mathcal{F}_{t-}] = 1$ by (3.3). In addition, the following holds:

$$h_{t,z}^1 \ge -\mathbb{E}_{\mathbb{P}^*}[F|\mathcal{F}_{t-}] \ge -K.$$

As a result, $h_{t,z}^1$ is bounded. Hence, we obtain $\mathbb{E}\left[\int_0^T \int_1^\infty (h_{t,z}^1)^2 \nu(dz)dt\right] < \infty$.

Next, we shall see $\mathbb{E}\left[\int_0^T \int_0^1 (h_{t,z}^1)^2 \nu(dz)dt\right] < \infty$. To this end, we rewrite $h_{t,z}^1$ as

$$h_{t,z}^1 = \mathbb{E}_{\mathbb{P}^*}[(F + zD_{t,z}F)(H_{t,z}^* - 1) + zD_{t,z}F|\mathcal{F}_{t-}].$$

Since $|zD_{t,z}F| \le K$, we have $(\mathbb{E}_{\mathbb{P}^*}[zD_{t,z}F|\mathcal{F}_{t-}])^2 \le K^2$. Thus, it suffices to see

$$\mathbb{E}\left[\int_0^T \int_0^1 \{\mathbb{E}_{\mathbb{P}^*}[(F + zD_{t,z}F)(H_{t,z}^* - 1)|\mathcal{F}_{t-}]\}^2 \nu(dz)dt\right] < \infty. \tag{4.6}$$

(3.3) implies

$$\begin{aligned}
&\left\{\mathbb{E}_{\mathbb{P}^*}\left[(F + zD_{t,z}F)(H_{t,z}^* - 1)|\mathcal{F}_{t-}\right]\right\}^2 \\
&\le K^2\mathbb{E}_{\mathbb{P}^*}\left[(H_{t,z}^* - 1)^2|\mathcal{F}_{t-}\right] \\
&\le K^2\left\{\mathbb{E}_{\mathbb{P}^*}\left[(H_{t,z}^*)^2|\mathcal{F}_{t-}\right] - 2\mathbb{E}_{\mathbb{P}^*}[H_{t,z}^*|\mathcal{F}_{t-}] + 1\right\} \\
&= K^2\left\{\mathbb{E}_{\mathbb{P}^*}\left[(H_{t,z}^*)^2|\mathcal{F}_{t-}\right] - 1\right\}.
\end{aligned} \tag{4.7}$$

Now, we calculate $(H_{t,z}^*)^2$. By the definition of $H_{t,z}^*$ in Theorem 3.1, and Proposition A.10, we have

$$\begin{aligned}
(H_{t,z}^*)^2 = \exp\Bigg\{ &-2z\int_0^T D_{t,z}u_s dW_s - 2z\int_0^T u_s D_{t,z}u_s ds - z^2\int_0^T (D_{t,z}u_s)^2 ds \\
&+ 2z\int_0^T\int_0^\infty D_{t,z}\log(1-\theta_{s,x})\widetilde{N}(ds,dx) \\
&+ 2z\int_0^T\int_0^\infty [D_{t,z}\log(1-\theta_{s,x}) + D_{t,z}\theta_{s,x}]\nu(dx)ds\Bigg\} \\
= \exp\Bigg\{ &-2z\int_0^T D_{t,z}u_s dW_s - 2z\int_0^T u_s D_{t,z}u_s ds - \frac{1}{2}\int_0^T (2zD_{t,z}u_s)^2 ds \\
&+ \int_0^T (zD_{t,z}u_s)^2 ds + \int_0^T\int_0^\infty \log(1-\gamma_{t,z,s,x})\widetilde{N}(ds,dx)
\end{aligned}$$

$$+\int_0^T\int_0^\infty[\log(1-\gamma_{t,z,s,x})+\gamma_{t,z,s,x}]\nu(dx)ds$$
$$-\int_0^T\int_0^\infty\gamma_{t,z,s,x}\theta_{s,x}\nu(dx)ds+\int_0^T\int_0^\infty\frac{(zD_{t,z}\theta_{s,x})^2}{1-\theta_{s,x}}\nu(dx)ds\Bigg\}, \quad (4.8)$$

where $\gamma_{t,z,s,x} := 2\frac{zD_{t,z}\theta_{s,x}}{1-\theta_{s,x}} - \left(\frac{zD_{t,z}\theta_{s,x}}{1-\theta_{s,x}}\right)^2$. Remark that Lemma A.9 implies that

$$\begin{aligned} zD_{t,z}\log(1-\theta_{s,x}) &= \log(1-\theta_{s,x}-zD_{t,z}\theta_{s,x})-\log(1-\theta_{s,x}) \\ &= \log\left(1-\frac{zD_{t,z}\theta_{s,x}}{1-\theta_{s,x}}\right), \end{aligned}$$

that is, $2zD_{t,z}\log(1-\theta_{s,x}) = \log(1-\gamma_{t,z,s,x})$. Now, we have that $(zD_{t,z}u_s)^2 \leq zC_u^2$ by Lemma A.7; and

$$\int_0^\infty\frac{(zD_{t,z}\theta_{s,x})^2}{1-\theta_{s,x}}\nu(dx) \leq z(C'_\theta)^2\hat{C}_\theta C_\rho$$

by Lemmas A.6 and A.8. Thus, we have

R.H.S. of (4.8)

$$\begin{aligned} &\leq \exp\Bigg\{-2z\int_0^T D_{t,z}u_s dW_s - 2z\int_0^T u_s D_{t,z}u_s ds - \frac{1}{2}\int_0^T(2zD_{t,z}u_s)^2 ds \\ &+\int_0^T\int_0^\infty\log(1-\gamma_{t,z,s,x})\widetilde{N}(ds,dx)+\int_0^T\int_0^\infty[\log(1-\gamma_{t,z,s,x})+\gamma_{t,z,s,x}]\nu(dx)ds \\ &-\int_0^T\int_0^\infty\gamma_{t,z,s,x}\theta_{s,x}\nu(dx)ds + Cz\Bigg\} \end{aligned} \quad (4.9)$$

for some $C > 0$. Thus, Lemma 4.2 implies that

$$\mathbb{E}_{\mathbb{P}^*}\left[(H^*_{t,z})^2|\mathcal{F}_{t-}\right] \leq \mathbb{E}_{\mathbb{P}^*}\left[X_T^{t,z}|\mathcal{F}_{t-}\right]e^{Cz} = X_{t-}^{t,z}e^{Cz} = e^{Cz}.$$

Consequently, we have

$$\text{R.H.S. of (4.7)} \leq K^2\left(e^{Cz}-1\right) \leq K^2 z\left(e^C-1\right)$$

for any $z \in (0,1)$. Hence, (4.6) follows, from which we obtain (4.5). This completes the proof of Theorem 3.1. □

Lemma 4.2 *Given $(t,z) \in [0,T]\times(0,\infty)$, we consider the following SDE:*

$$\begin{aligned} dX_s^{t,z} = -X_{s-}^{t,z}\Bigg\{&2zD_{t,z}u_s dW_s + 2zu_s D_{t,z}u_s ds + \int_0^\infty\gamma_{t,z,s,x}\widetilde{N}(ds,dx) \\ &+\int_0^\infty\gamma_{t,z,s,x}\theta_{s,x}\nu(dx)ds\Bigg\}. \end{aligned} \quad (4.10)$$

Then, the solution $X^{t,z}$ is a martingale under $\mathbb{P}^*$ with $X_s^{t,z} = 1$ for any $s \in [0,t)$. In particular, the right hand side of (4.9) is equal to $X_T^{t,z}e^{Cz}$.

Proof. First of all, remark that $zD_{t,z}u_s$ and $zu_sD_{t,z}u_s$ are bounded. In addition, we have

$$\left|\frac{zD_{t,z}\theta_{s,x}}{1-\theta_{s,x}}\right| < 2C_\theta\hat{C}_\theta(1-e^{\rho x}) < 2C_\theta\hat{C}_\theta \tag{4.11}$$

by Lemmas A.6 and A.8. Thus, Lemma A.6 yields

$$\begin{aligned}\int_0^\infty |\gamma_{t,z,s,x}\theta_{s,x}|\nu(dx) &= \int_0^\infty \left|\frac{zD_{t,z}\theta_{s,x}}{1-\theta_{s,x}}\left(2-\frac{zD_{t,z}\theta_{s,x}}{1-\theta_{s,x}}\right)\theta_{s,x}\right|\nu(dx)\\ &\le 2C_\theta\hat{C}_\theta(2+2C_\theta\hat{C}_\theta)\cdot C_\theta|\rho|\int_0^\infty x\nu(dx) < \infty.\end{aligned}$$

Moreover, (4.11) again implies

$$\begin{aligned}\int_0^\infty \gamma^2_{t,z,s,x}\nu(dx) &= \int_0^\infty \frac{(zD_{t,z}\theta_{s,x})^2}{(1-\theta_{s,x})^2}\left(2-\frac{zD_{t,z}\theta_{s,x}}{1-\theta_{s,x}}\right)^2\nu(dx)\\ &\le 4C_\theta^2\hat{C}_\theta^2C_\rho(2+2C_\theta\hat{C}_\theta)^2.\end{aligned}$$

As a result, we can apply Theorem 117 of Situ [17] to (4.10); and then we conclude that (4.10) has a solution $X^{t,z}$ satisfying $\mathbb{E}\left[\sup_{t\le s\le T}|X_s^{t,z}|^2\right] < \infty$, which implies $\mathbb{E}_{\mathbb{P}^*}[\sup_{t\le s\le T}|X_s^{t,z}|] < \infty$ by the $L^2(\mathbb{P})$-property of Z_T. Now, $X^{t,z}$ is a local martingale under $\mathbb{P}^*$, since we can rewrite (4.10) as

$$dX_s^{t,z} = -X_{s-}^{t,z}\left\{2zD_{t,z}u_sdW_s^{\mathbb{P}^*} + \int_0^\infty \gamma_{t,z,s,x}\widetilde{N}^{\mathbb{P}^*}(ds,dx)\right\}.$$

Consequently, Theorem I.51 of Protter [13] implies that $X^{t,z}$ is a $\mathbb{P}^*$-martingale satisfying $X_s^{t,z} = 1$ for any $s \in [0,t)$. Moreover, by Example 9.6 of Di Nunno et al. [8], the right hand side of (4.9) is expressed by $X_T^{t,z}e^{Cz}$. $\square$

5 Malliavin differentiability of Z

This section is devoted to show $Z_t \in \mathbb{D}^{1,2}$ for any $t \in [0,T]$. To this end, for $t \in [0,T]$, we define $Z_t^{(0)} := 1$ and

$$Z_t^{(n+1)} := 1 - \int_0^t Z_{s-}^{(n)}u_sdW_s - \int_0^t\int_0^\infty Z_{s-}^{(n)}\theta_{s,x}\widetilde{N}(ds,dx)$$

for $n \ge 0$. Besides, we denote, for $n \ge 0$,

$$\phi_n(t) := \mathbb{E}\left[\int_{[0,t]\times[0,\infty)}\left(D_{r,z}Z_t^{(n)}\right)^2 q(dr,dz)\right].$$

Note that $\phi_0(t) \equiv 0$.

Lemma 5.1 *We have $Z_t^{(n)} \in \mathbb{D}^{1,2}$ for every $n \geq 0$ and any $t \in [0,T]$. Moreover, there exist constants $k_1 > 0$ and $k_2 > 0$ such that*

$$\phi_{n+1}(t) \leq k_1 + k_2 \int_0^t \phi_n(s)ds$$

for every $n \geq 0$ and any $t \in [0,T]$.

Under Lemma 5.1, we have

$$\begin{aligned}\phi_{n+1}(t) &\leq k_1 + k_2 \int_0^t \phi_n(s)ds \leq k_1 + k_2 \int_0^t \left(k_1 + k_2 \int_0^s \phi_{n-1}(s_1)ds_1\right) ds \\ &\leq \cdots \leq k_1 \sum_{j=0}^n \frac{k_2^j t^j}{j!} < k_1 e^{k_2 t}.\end{aligned}$$

for any $t \in [0,T]$. Thus, $\sup_{n\geq 1} \phi_n(t) < \infty$ holds. Since we have $Z_t^{(n)} \to Z_t$ in $L^2(\mathbb{P})$, Lemma 17.1 of [8] implies that $Z_t \in \mathbb{D}^{1,2}$ for $t \in [0,T]$. Remark that the Malliavin derivative in [8] is defined in a different way from ours. Denoting by $\hat{D}$ the Malliavin derivative operator in [8], we have $\hat{D}_{t,z}F = zD_{t,z}F$ for $z \neq 0$ and $F \in \mathbb{D}^{1,2}$.

Proof of Lemma 5.1. We take an integer $n \geq 0$ arbitrarily. Suppose that $Z_t^{(n)} \in \mathbb{D}^{1,2}$ and $\int_0^t \phi_n(s)ds < \infty$ for any $t \in [0,T]$. Lemma 5.2 below and Lemma 3.3 of [7] imply that $Z_t^{(n+1)} \in \mathbb{D}^{1,2}$ for any $t \in [0,T]$; and, for any $t \in [r,T]$ and any $z \in (0,\infty)$,

$$\begin{aligned}D_{r,0}Z_t^{(n+1)} &= -D_{r,0} \int_{[0,T]\times[0,\infty)} Z_{s-}^{(n)} \left\{u_s \mathbf{1}_{\{0\}}(x) + \frac{\theta_{s,x}}{x}\mathbf{1}_{(0,\infty)}(x)\right\} \mathbf{1}_{[0,t]}(s)Q(ds,dx) \\ &= -Z_{r-}^{(n)}u_r - \int_r^t D_{r,0}(Z_{s-}^{(n)}u_s)dW_s - \int_r^t \int_0^\infty D_{r,0}\left(Z_{s-}^{(n)}\frac{\theta_{s,x}}{x}\right) x\widetilde{N}(ds,dx) \\ &= -Z_{r-}^{(n)}u_r - \int_r^t u_s D_{r,0}Z_{s-}^{(n)}dW_s - \int_r^t \int_0^\infty \theta_{s,x} D_{r,0}Z_{s-}^{(n)}\widetilde{N}(ds,dx)\end{aligned} \tag{5.1}$$

and

$$D_{r,z}Z_t^{(n+1)} = -Z_{r-}^{(n)}\frac{\theta_{r,z}}{z} - \int_r^t D_{r,z}\left(Z_{s-}^{(n)}u_s\right)dW_s - \int_r^t \int_0^\infty D_{r,z}\left(Z_{s-}^{(n)}\frac{\theta_{s,x}}{x}\right) x\widetilde{N}(ds,dx). \tag{5.2}$$

Now, we fix $t \in [0,T]$ arbitrarily. We have then

$$\phi_{n+1}(t) = \mathbb{E}\left[\int_0^t \left(D_{r,0}Z_t^{(n+1)}\right)^2 dr\right] + \mathbb{E}\left[\int_0^t \int_0^\infty \left(D_{r,z}Z_t^{(n+1)}\right)^2 z^2\nu(dz)dr\right]. \tag{5.3}$$

(5.1) implies

The first term of (5.3)

$$\leq 3\mathbb{E}\left[\int_0^t \left(Z_{r-}^{(n)} u_r\right)^2 dr\right] + 3\mathbb{E}\left[\int_0^t \left(\int_r^t u_s D_{r,0} Z_{s-}^{(n)} dW_s\right)^2 dr\right]$$
$$+ 3\mathbb{E}\left[\int_0^t \left(\int_r^t \int_0^\infty \theta_{s,x} D_{r,0} Z_{s-}^{(n)} \widetilde{N}(ds, dx)\right)^2 dr\right]. \tag{5.4}$$

We evaluate each term in the right hand side of (5.4). Lemma A.6 implies

$$\mathbb{E}\left[\int_0^t \left(Z_{r-}^{(n)} u_r\right)^2 dr\right] \leq C_u^2 \mathbb{E}\left[\int_0^t \left(Z_{r-}^{(n)}\right)^2 dr\right] \leq C_u^2 T \mathbb{E}\left[\sup_{0\leq s\leq T} \left(Z_s^{(n)}\right)^2\right]$$

and

$$\mathbb{E}\left[\int_0^t \left(\int_r^t u_s D_{r,0} Z_{s-}^{(n)} dW_s\right)^2 dr\right] \leq C_u^2 \int_0^t \mathbb{E}\left[\int_r^t \left(D_{r,0} Z_{s-}^{(n)}\right)^2 ds\right] dr.$$

The same argument implies that

$$\mathbb{E}\left[\int_0^t \left(\int_r^t \int_0^\infty \theta_{s,x} D_{r,0} Z_{s-}^{(n)} \widetilde{N}(ds, dx)\right)^2 dr\right]$$
$$= \int_0^t \mathbb{E}\left[\int_r^t \int_0^\infty \left(\theta_{s,x} D_{r,0} Z_{s-}^{(n)}\right)^2 \nu(dx) ds\right] dr \leq C_\theta^2 C_\rho \int_0^t \mathbb{E}\left[\int_r^t \left(D_{r,0} Z_{s-}^{(n)}\right)^2 ds\right] dr.$$

As a result, we obtain

The first term of (5.3)

$$\leq 3C_u^2 T \mathbb{E}\left[\sup_{0\leq s\leq T} \left(Z_s^{(n)}\right)^2\right] + 3(C_u^2 + C_\theta^2 C_\rho) \int_0^t \mathbb{E}\left[\int_r^t \left(D_{r,0} Z_{s-}^{(n)}\right)^2 ds\right] dr. \tag{5.5}$$

Next, (5.2) yields

The second term of (5.3)

$$\leq 3\mathbb{E}\left[\int_0^t \int_0^\infty \left(Z_{r-}^{(n)} \frac{\theta_{r,z}}{z}\right)^2 z^2 \nu(dz) dr\right]$$
$$+ 3\mathbb{E}\left[\int_0^t \int_0^\infty \left(\int_r^t D_{r,z}\left(Z_{s-}^{(n)} u_s\right) dW_s\right)^2 z^2 \nu(dz) dr\right]$$
$$+ 3\mathbb{E}\left[\int_0^t \int_0^\infty \left(\int_r^t \int_0^\infty D_{r,z}\left(Z_{s-}^{(n)} \frac{\theta_{s,x}}{x}\right) x\widetilde{N}(ds, dx)\right)^2 z^2 \nu(dz) dr\right]. \tag{5.6}$$

Now, we calculate each term of the right hand side of (5.6).

The first term of (5.6) $\leq 3C_\theta^2 C_\rho \mathbb{E}\left[\int_0^t \left(Z_{r-}^{(n)}\right)^2 dr\right] \leq 3C_\theta^2 C_\rho T \mathbb{E}\left[\sup_{0\leq s\leq T} \left(Z_s^{(n)}\right)^2\right]. \qquad (5.7)$

Next, Lemma A.7 implies

$$
\begin{aligned}
&\text{The second term of (5.6)}\\
&= 3\int_0^t \int_0^\infty \mathbb{E}\left[\int_r^t \left(D_{r,z}\left(Z_{s-}^{(n)} u_s\right)\right)^2 ds\right] z^2 \nu(dz) dr\\
&= 3\int_0^t \int_0^\infty \mathbb{E}\left[\int_r^t \left(u_s D_{r,z} Z_{s-}^{(n)} + Z_{s-}^{(n)} D_{r,z} u_s + z D_{r,z} Z_{s-}^{(n)} \cdot D_{r,z} u_s\right)^2 ds\right] z^2 \nu(dz) dr\\
&\le 9\int_0^t \int_0^\infty \left\{ C_u^2 \mathbb{E}\left[\int_r^t \left(D_{r,z} Z_{s-}^{(n)}\right)^2 ds\right] + \frac{C_u^2}{z}\mathbb{E}\left[\int_r^t \left(Z_{s-}^{(n)}\right)^2 ds\right]\right.\\
&\quad \left. + (C_u')^2 \mathbb{E}\left[\int_r^t \left(D_{r,z} Z_{s-}^{(n)}\right)^2 ds\right]\right\} z^2 \nu(dz) dr\\
&\le 9 C_u^2 \int_0^\infty z\nu(dz) T^2 \mathbb{E}\left[\sup_{0\le s\le T}\left(Z_s^{(n)}\right)^2\right]\\
&\quad + 9\left(C_u^2 + (C_u')^2\right)\int_0^t \int_0^\infty \mathbb{E}\left[\int_r^t \left(D_{r,z} Z_{s-}^{(n)}\right)^2 ds\right] z^2 \nu(dz) dr. \qquad (5.8)
\end{aligned}
$$

Moreover, we evaluate the third term of (5.6). By Lemma A.8, we obtain

$$
\begin{aligned}
&\text{The third term of (5.6)}\\
&= 3\int_0^t \int_0^\infty \mathbb{E}\Bigg[\int_r^t \int_0^\infty \left\{ D_{r,z} Z_{s-}^{(n)} \cdot \frac{\theta_{s,x}}{x} + Z_{s-}^{(n)} D_{r,z}\frac{\theta_{s,x}}{x}\right.\\
&\quad \left. + z D_{r,z} Z_{s-}^{(n)} \cdot D_{r,z}\frac{\theta_{s,x}}{x}\right\}^2 x^2 \nu(dx) ds\Bigg] z^2 \nu(dz) dr\\
&\le 9\int_0^t \int_0^\infty \left\{ C_\theta^2 C_\rho \mathbb{E}\left[\int_r^t \left(D_{r,z} Z_{s-}^{(n)}\right)^2 ds\right] + \frac{(C_\theta')^2 C_\rho}{z}\mathbb{E}\left[\int_r^t \left(Z_{s-}^{(n)}\right)^2 ds\right]\right.\\
&\quad \left. + z^2 \frac{4 C_\theta^2 C_\rho}{z^2}\mathbb{E}\left[\int_r^t \left(D_{r,z} Z_{s-}^{(n)}\right)^2 ds\right]\right\} z^2 \nu(dz) dr\\
&\le 9 (C_\theta')^2 C_\rho \int_0^\infty z\nu(dz) T^2 \mathbb{E}\left[\sup_{0\le s\le T}\left(Z_s^{(n)}\right)^2\right]\\
&\quad + 45 C_\theta^2 C_\rho \int_0^t \int_0^\infty \mathbb{E}\left[\int_r^t \left(D_{r,z} Z_{s-}^{(n)}\right)^2 ds\right] z^2 \nu(dz) dr. \qquad (5.9)
\end{aligned}
$$

Consequently, by (5.3), (5.5)–(5.9) and Lemma 5.3 below, there are constants $k_1 > 0$ and $k_2 > 0$ such that

$$
\begin{aligned}
\phi_{n+1}(t) &\le k_1 \mathbb{E}\left[\sup_{0\le s\le T}\left(Z_s^{(n)}\right)^2\right] + k_2 \int_{[0,t]\times[0,\infty)} \mathbb{E}\left[\int_r^t \left(D_{r,z} Z_{s-}^{(n)}\right)^2 ds\right] q(dr,dz)\\
&\le k_1 \sup_{n\ge 1}\mathbb{E}\left[\sup_{0\le s\le T}\left(Z_s^{(n)}\right)^2\right] + k_2 \int_0^t \mathbb{E}\left[\int_{[0,s]\times[0,\infty)} \left(D_{r,z} Z_{s-}^{(n)}\right)^2 q(dr,dz)\right] ds
\end{aligned}
$$

$$= k_1 \sup_{n\geq 1} \mathbb{E}\left[\sup_{0\leq s\leq T}\left(Z_s^{(n)}\right)^2\right] + k_2\int_0^t \mathbb{E}\left[\int_{[0,s]\times[0,\infty)}\left(D_{r,z}Z_s^{(n)}\right)^2 q(dr,dz)\right]ds$$
$$\leq k_1 + k_2\int_0^t \phi_n(s)ds,$$

where k_1 and k_2 may vary from line to line. □

Lemma 5.2 *Fix $n \geq 0$ arbitrarily. Assume that $Z_t^{(n)} \in \mathbb{D}^{1,2}$ and $\int_0^t \phi_n(s)ds < \infty$ for any $t \in [0,T]$. We have $Z_-^{(n)}u \in \mathbb{L}_0^{1,2}$ and $Z_-^{(n)}\theta \in \mathbb{L}_1^{1,2}$.*

Proof. We show $Z_-^{(n)}u \in \mathbb{L}_0^{1,2}$. Since we can see that $Z_{s-}^{(n)}D_{t,z}u_s + u_sD_{t,z}Z_{s-}^{(n)} + zD_{t,z}Z_{s-}^{(n)}\cdot D_{t,z}u_s \in L^2(q\times\mathbb{P})$ for any $s \in [0,T]$ by $Z_t^{(n)} \in \mathbb{D}^{1,2}$, and Lemmas A.6 and A.7. Thus, item (a) in the definition of $\mathbb{L}_0^{1,2}$ is given by Propositions 5.1 and 5.4 of [19]. Next, item (b) is satisfied by Lemma A.6. As for item (c), there exist two constants $C_1 > 0$ and $C_2 > 0$ such that $(D_{t,z}(Z_{s-}^{(n)}u_s))^2 \leq \frac{C_1}{z}(Z_{s-}^{(n)})^2 + C_2(D_{t,z}Z_{s-}^{(n)})^2$. In addition, we have

$$\mathbb{E}\left[\int_{[0,T]\times[0,\infty)}\int_0^T\left(D_{t,z}Z_{s-}^{(n)}\right)^2 dsq(dt,dz)\right]$$
$$= \int_0^T \mathbb{E}\left[\int_{[0,T]\times[0,\infty)}\left(D_{t,z}Z_{s-}^{(n)}\right)^2 q(dt,dz)\right]ds = \int_0^T \phi_n(s)ds < \infty.$$

As a result, item (c) follows. This completes the proof of $Z_-^{(n)}u \in \mathbb{L}_0^{1,2}$. $Z_-^{(n)}\theta \in \mathbb{L}_1^{1,2}$ is also shown similarly. □

Lemma 5.3 $\sup_{n\geq 1}\mathbb{E}\left[\sup_{0\leq s\leq T}\left(Z_s^{(n)}\right)^2\right] < \infty.$

Proof. First of all, we can see inductively that $Z^{(n)}$ is a martingale with $Z_T^{(n)} \in L^2(\mathbb{P})$. Denoting $\zeta_n(t) := \mathbb{E}\left[\sup_{0\leq s\leq t}\left(Z_s^{(n)}\right)^2\right]$ for $t \in [0,T]$ and $n \geq 1$, we have

$$\zeta_n(T) \leq 4\mathbb{E}\left[\left\{1 - \int_0^T Z_{s-}^{(n-1)}u_s dW_s - \int_0^T\int_0^\infty Z_{s-}^{(n-1)}\theta_{s,x}\widetilde{N}(ds,dx)\right\}^2\right]$$
$$\leq 4\left\{1 + \mathbb{E}\left[\int_0^T\left(Z_{s-}^{(n-1)}\right)^2\left\{u_s^2 + \int_0^\infty \theta_{s,x}^2\nu(dx)\right\}ds\right]\right\}$$
$$\leq 4 + 4(C_u^2 + C_\theta^2 C_\rho)\int_0^T \zeta_{n-1}(s)ds \leq 4\exp\{4(C_u^2 + C_\theta^2 C_\rho)T\}$$

by Doob's inequality and Lemma A.6. □

6 Conclusions

We obtain explicit representations of LRM of call and put options for the BNS models given by (1.2) and (1.3). We impose only Assumption 2.2 as the standing assumptions. Recall that Assumption 2.2 does not exclude the two important examples, IG-OU and Gamma-OU, although parameters are restricted. Our discussion is based on the framework of [1]. We spend many pages to make sure of the additional conditions imposed in [1]. Above all, we need some integrability conditions on the underlying contingent claim F. For example, we need $Z_T F \in L^2(\mathbb{P})$, which is almost equivalent to $Z_T S_T \in L^2(\mathbb{P})$ if F is a call option. However, $Z_T S_T$ is not in $L^2(\mathbb{P})$ in our setting, which means that an additional condition is needed in order to treat call options directly in the framework of [1]. Thus, we consider put options first in this paper, since they are bounded. LRM for call options are given as a corollary. By this small idea, we do not need to impose any additional condition.

Moreover, in order to see condition C4, we need to investigate the Malliavin differentiability of the process Z. Note that Z is a solution to the SDE (2.3). [8] showed the Malliavin differentiability of solutions to Markovian type SDEs with the Lipschitz condition. However, the SDE (2.3) is not the case, since u_s and $\theta_{s,x}$ are random. In Section 5, as an extension of Section 17 in [8], we show that $Z_t \in \mathbb{D}^{1,2}$. This result should be a valuable mathematical contribution in its own right. Recall that u_s and $\theta_{s,x}$ are bounded by Lemma A.6; and the Malliavin derivatives of u_s and $\theta_{s,x}$ are equivalent to $O(1/z)$ and $O(1/\sqrt{z})$ simultaneously by Lemmas A.7 and A.8. These facts play a vital role to see the Malliavin differentiability of Z.

We consider, throughout the paper, the BNS models whose asset price process is given by (1.3). Actually, the general form of the BNS models is as follows:

$$S_t = S_0 \exp\left\{ \int_0^t (\mu + \beta\sigma_s^2) ds + \int_0^t \sigma_s dW_s + \rho J_t \right\},$$

where the parameter $\beta \in \mathbb{R}$ is called the volatility risk premium. In other words, we restrict β to $-1/2$. When $\beta \neq -1/2$, the boundedness of u_s and $\theta_{s,x}$ no longer hold, from which it is not easy to show that $Z_T \in \mathbb{D}^{1,2}$. Thus, we need some new ideas to generalize our results. It remains to future research. Moreover, we put off comparison with delta hedge, and development of numerical scheme for future work.

A Appendix

A.1 Properties of σ_t, and related Malliavin derivatives

The squared volatility process σ_t^2, given as a solution to the SDE (1.2), is represented as

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dJ_s. \tag{A.1}$$

Remark that we have

$$\sigma_t^2 \geq e^{-\lambda t}\sigma_0^2 \geq e^{-\lambda T}\sigma_0^2, \tag{A.2}$$

and

$$\int_0^t \sigma_s^2 ds = \frac{1}{\lambda}(J_t - \sigma_t^2 + \sigma_0^2) \leq \frac{1}{\lambda}(J_t + \sigma_0^2). \tag{A.3}$$

Next, we calculate some related Malliavin derivatives.

Lemma A.1 *For any $s \in [0,T]$, we have $\sigma_s^2 \in \mathbb{D}^{1,2}$; and*

$$D_{t,z}\sigma_s^2 = e^{-\lambda(s-t)}\mathbf{1}_{[0,s]\times(0,\infty)}(t,z) \tag{A.4}$$

for $(t,z) \in [0,T]\times[0,\infty)$.

Proof. We can rewrite (A.1) as

$$\begin{aligned}\sigma_s^2 = e^{-\lambda s}\sigma_0^2 &+ \int_0^s\int_0^\infty e^{-\lambda(s-u)}x\nu(dx)du \\ &+ \int_{[0,T]\times[0,\infty)} e^{-\lambda(s-u)}\mathbf{1}_{[0,s]\times(0,\infty)}(u,x)Q(du,dx).\end{aligned}$$

Moreover, we have $\int_{[0,T]\times[0,\infty)} e^{-2\lambda(s-u)}\mathbf{1}_{[0,s]\times(0,\infty)}(u,x)q(du,dx) < \infty$. By Definition 2.8, the lemma follows. □

Lemma A.2 *For any $s \in [0,T]$, we have $\sigma_s \in \mathbb{D}^{1,2}$; and*

$$D_{t,z}\sigma_s = \frac{\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}} - \sigma_s}{z}\mathbf{1}_{[0,s]\times(0,\infty)}(t,z)$$

for $(t,z) \in [0,T]\times[0,\infty)$. Moreover, we have $0 \leq D_{t,z}\sigma_s \leq \frac{1}{\sqrt{z}}\mathbf{1}_{[0,s]}(t)$ for $z > 0$.

Proof. Taking a C^1-function f such that f' is bounded; and $f(r) = \sqrt{r}$ for $r \geq e^{-\lambda T}\sigma_0^2$, we have $\sigma_s = f(\sigma_s^2)$ by (A.2). Proposition 2.6 in [18] implies $\sigma_s \in \mathbb{D}^{1,2}$, $D_{t,0}\sigma_s = f'(\sigma_s^2)D_{t,0}\sigma_s^2 = 0$; and

$$D_{t,z}\sigma_s = \frac{f(\sigma_s^2 + zD_{t,z}\sigma_s^2) - f(\sigma_s^2)}{z} = \frac{\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}} - \sigma_s}{z}\mathbf{1}_{[0,s]}(t)$$

for $z > 0$, since $D_{t,z}\sigma_s^2$ is nonnegative by (A.4). In addition, we have $D_{t,z}\sigma_s \leq \frac{\sqrt{ze^{-\lambda(s-t)}}}{z}\mathbf{1}_{[0,s]}(t) \leq \frac{1}{\sqrt{z}}\mathbf{1}_{[0,s]}(t)$ for $z > 0$. □

Lemma A.3 *We have $\int_0^T \sigma_s^2 ds \in \mathbb{D}^{1,2}$; and*

$$D_{t,z}\int_0^T \sigma_s^2 ds = \mathcal{B}(T-t)\mathbf{1}_{(0,\infty)}(z)$$

for $(t,z) \in [0,T]\times[0,\infty)$, where the function $\mathcal{B}$ is defined in Assumption 2.2.

Proof. First of all, we have

$$\int_0^T \sigma_s^2 ds = \sigma_0^2 \int_0^T e^{-\lambda s} ds + \int_0^T \int_0^s e^{-\lambda(s-u)} dJ_u ds$$
$$= \sigma_0^2 \frac{1-e^{-\lambda T}}{\lambda} + \int_0^T \int_u^T e^{-\lambda(s-u)} ds dJ_u = \sigma_0^2 \mathcal{B}(T) + \int_0^T \mathcal{B}(T-u) dJ_u.$$

From the view of Definition 2.8, we obtain that $\int_0^T \sigma_s^2 ds \in \mathbb{D}^{1,2}$; and $D_{t,z} \int_0^T \sigma_s^2 ds = \mathcal{B}(T-t)\mathbf{1}_{(0,\infty)}(z)$ for $(t,z) \in [0,T] \times (0,\infty)$. □

Lemma A.4 *We have $\int_0^T \sigma_s dW_s \in \mathbb{D}^{1,2}$; and*

$$D_{t,z} \int_0^T \sigma_s dW_s = \sigma_t \mathbf{1}_{\{0\}}(z) + \int_t^T \frac{\sqrt{\sigma_s^2 + z e^{-\lambda(s-t)}} - \sigma_s}{z} dW_s \mathbf{1}_{(0,\infty)}(z).$$

for $(t,z) \in [0,T] \times [0,\infty)$.

Proof. First of all, we show $\sigma \in \mathbb{L}_0^{1,2}$. Lemma A.2 implies $\sigma_s \in \mathbb{D}^{1,2}$ for any $s \in [0,T]$. We have $\mathbb{E}\left[\int_0^T \sigma_s^2 ds\right] < \infty$ by (A.3) and the integrability of J_T. Since $|D_{t,z}\sigma_s|^2 \le \frac{1}{z}$ by Lemma A.2, item (c) of the definition of $\mathbb{L}_0^{1,2}$ is satisfied. Thus, Lemma 3.3 in [7] provides that $\int_0^T \sigma_s dW_s \in \mathbb{D}^{1,2}$; and

$$D_{t,z} \int_0^T \sigma_s dW_s = D_{t,z} \int_{[0,T]\times[0,\infty)} \sigma_s \cdot \mathbf{1}_{\{0\}}(x) Q(ds,dx) = \sigma_t \mathbf{1}_{\{0\}}(z) + \int_0^T D_{t,z}\sigma_s dW_s$$
$$= \sigma_t \mathbf{1}_{\{0\}}(z) + \int_t^T \frac{\sqrt{\sigma_s^2 + z e^{-\lambda(s-t)}} - \sigma_s}{z} dW_s \mathbf{1}_{(0,\infty)}(z)$$

for $(t,z) \in [0,T] \times [0,\infty)$ by Lemma A.2. □

Lastly, we calculate $D_{t,z}L_T$ as follows:

Proposition A.5 *$L_T \in \mathbb{D}^{1,2}$ and, for $(t,z) \in [0,T] \times [0,\infty)$, we have*

$$D_{t,z} L_T = \sigma_t \mathbf{1}_{\{0\}}(z) + \left\{ -\frac{1}{2}\mathcal{B}(T-t) + \int_t^T \frac{\sqrt{\sigma_s^2 + z e^{-\lambda(s-t)}} - \sigma_s}{z} dW_s + \rho \right\} \mathbf{1}_{(0,\infty)}(z).$$

Proof. By (2.1), we have $L_T = \mu T - \frac{1}{2}\int_0^T \sigma_s^2 ds + \int_0^T \sigma_s dW_s + \rho J_T$. Since $J_T \in \mathbb{D}^{1,2}$ and $D_{t,z}J_T = \mathbf{1}_{(0,\infty)}(z)$, we can see this proposition by Lemmas A.3 and A.4. □

A.2 Properties of u_s and $\theta_{s,x}$, and related Malliavin derivatives

We begin with the definition of two constants as follows:

$$C_u := \max\left\{\frac{|\alpha|e^{\frac{\lambda T}{2}}}{\sigma_0}, \frac{|\alpha|}{C_\rho}\right\}; \quad \text{and} \quad C_\theta := \max\left\{\frac{|\alpha|}{C_\rho}, 1\right\}.$$

The next lemma is referred throughout the paper over and over again.

Lemma A.6 *For any $s \in [0,T]$ and any $x \in (0,\infty)$, the following hold:*

1. $|u_s| \leq C_u$,
2. $|\theta_{s,x}| \leq C_\theta$; *and* $|\theta_{s,x}| \leq C_\theta(1 - e^{\rho x}) \leq C_\theta|\rho|x$,
3. $\theta_{s,x} < 1 - e^{\rho x}$,
4. $|\log(1-\theta_{s,x})| \leq C_\theta|\rho|x$,
5. $\frac{1}{1-\theta_{s,x}} < \hat{C}_\theta$ *for some* $\hat{C}_\theta > 0$.

Proof. 1. We have $|u_s| \leq \frac{|\alpha|}{\sigma_s} \leq \frac{|\alpha|e^{\frac{\lambda T}{2}}}{\sigma_0}$ for any $s \in [0,T]$ by (A.2).
2. $|\theta_{s,x}| \leq \frac{|\alpha|}{C_\rho}(1-e^{\rho x}) \leq C_\theta$; and $1 - e^{\rho x} \leq |\rho|x$ for any $x > 0$.
3. As seen in Remark 2.3, $\frac{\alpha}{\sigma_s^2 + C_\rho} > -1$ for any $s \in [0,T]$. We have then $\theta_{s,x} < 1 - e^{\rho x}$.
4. When $\theta_{s,x} \geq 0$, we have $0 \geq \log(1-\theta_{s,x}) > \log(1-(1-e^{\rho x})) = \rho x \geq C_\theta \rho x$. On the other hand, if $\theta_{s,x} < 0$, then $0 < \log(1-\theta_{s,x}) \leq -\theta_{s,x} \leq C_\theta|\rho|x$.
5. If $\theta_{s,x} \leq 0$, then $\frac{1}{1-\theta_{s,x}} \leq 1$. Else if $\theta_{s,x} > 0$, equivalently $\alpha < 0$, then $1 - \theta_{s,x} = 1 + \frac{\alpha}{\sigma_s^2 + C_\rho}(1-e^{\rho x}) \geq 1 + \frac{\alpha}{\sigma_s^2 + C_\rho} \geq 1 + \frac{\alpha}{e^{-\lambda T}\sigma_0^2 + C_\rho} > 0$ by Assumption 2.2. This completes the proof. □

Next, we calculate some Malliavin derivatives related to u_s and $\theta_{s,x}$.

Lemma A.7 *For any $s \in [0,T]$, we have $u_s \in \mathbb{D}^{1,2}$; and*

$$\begin{aligned} D_{t,z}u_s &= \frac{f_u(\sigma_s + zD_{t,z}\sigma_s) - f_u(\sigma_s)}{z}\mathbf{1}_{[0,s]\times(0,\infty)}(t,z) \\ &= \frac{f_u\left(\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}}\right) - f_u(\sigma_s)}{z}\mathbf{1}_{[0,s]\times(0,\infty)}(t,z) \end{aligned} \tag{A.5}$$

for $(t,z) \in [0,T] \times [0,\infty)$, where $f_u(r) := \frac{\alpha r}{r^2 + C_\rho}$ for $r \in \mathbb{R}$. Moreover, we have

$$|D_{t,z}u_s| \leq \frac{C_u}{\sqrt{z}}\mathbf{1}_{[0,s]}(t) \quad \textit{and} \quad |D_{t,z}u_s| \leq \frac{C'_u}{z}\mathbf{1}_{[0,s]}(t)$$

for some $C'_u > 0$.

Proof. Note that $f_u'(r) = \alpha \frac{C_\rho - r^2}{(r^2+C_\rho)^2}$ and $|f_u'(r)| \le \frac{|\alpha|}{C_\rho} \le C_u$. Since $u_s = f_u(\sigma_s)$ and $\sigma_s \in \mathbb{D}^{1,2}$, Proposition 2.6 in [18], together with Lemma A.2, implies $u_s \in \mathbb{D}^{1,2}$ and (A.5). In particular, we have $D_{t,0}u_s = f_u'(\sigma_s)D_{t,0}\sigma_s = 0$. On the other hand, Lemma A.2 again yields that $|D_{t,z}u_s| \le \frac{1}{z}|zD_{t,z}\sigma_s|C_u \le \frac{1}{\sqrt{z}}\mathbf{1}_{[0,s]}(t)C_u$. Moreover, since $f_u(r)$ is bounded, we can find a $C_u' > 0$ such that $|D_{t,z}u_s| \le \frac{C_u'}{z}$. □

Lemma A.8 *For any $(s,x) \in [0,T] \times (0,\infty)$, we have $\theta_{s,x} \in \mathbb{D}^{1,2}$; and*

$$\begin{aligned} D_{t,z}\theta_{s,x} &= \frac{f_\theta(\sigma_s + zD_{t,z}\sigma_s) - f_\theta(\sigma_s)}{z}(e^{\rho x} - 1)\mathbf{1}_{[0,s]\times(0,\infty)}(t,z) \\ &= \frac{f_\theta\left(\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}}\right) - f_\theta(\sigma_s)}{z}(e^{\rho x} - 1)\mathbf{1}_{[0,s]\times(0,\infty)}(t,z) \qquad \text{(A.6)} \end{aligned}$$

for $(t,z) \in [0,T] \times [0,\infty)$, where $f_\theta(r) := \frac{\alpha}{r^2 + C_\rho}$ for $r \in \mathbb{R}$. Moreover, we have

$$|D_{t,z}\theta_{s,x}| \le \frac{C_\theta'}{\sqrt{z}}(1 - e^{\rho x})\mathbf{1}_{[0,s]}(t) \quad \text{and} \quad |D_{t,z}\theta_{s,x}| \le \frac{2C_\theta}{z}(1 - e^{\rho x})\mathbf{1}_{[0,s]}(t) \qquad \text{(A.7)}$$

for some $C_\theta' > 0$.

Proof. Note that $\theta_{s,x} = f_\theta(\sigma_s)(e^{\rho x} - 1)$; and $f_\theta'(r) = -\frac{2\alpha r}{(r^2+C_\rho)^2}$. So that, $|f_\theta'(r)|$ is bounded. Thus, the same argument as Lemma A.7 implies (A.6). In addition, (A.7) is given by the boundedness of f_θ and f_θ'. □

Lemma A.9 *For any $(s,x) \in [0,T] \times (0,\infty)$, we have $\log(1-\theta_{s,x}) \in \mathbb{D}^{1,2}$; and*

$$D_{t,z}\log(1-\theta_{s,x}) = \frac{\log(1-\theta_{s,x} - zD_{t,z}\theta_{s,x}) - \log(1-\theta_{s,x})}{z}\mathbf{1}_{(0,\infty)}(z)$$

for $(t,z) \in [0,T] \times [0,\infty)$. Moreover, we have $|D_{t,z}\log(1-\theta_{s,x})| \le |D_{t,z}\theta_{s,x}|e^{-\rho x}$.

Proof. For $x > 0$, we denote

$$g_x(r) := \begin{cases} \log(1-r), & r < 1 - e^{\rho x}, \\ -e^{-\rho x}r + e^{-\rho x} - 1 + \rho x, & r \ge 1 - e^{\rho x}. \end{cases}$$

Note that g_x is a C^1-function satisfying $|g_x'(r)| \le e^{-\rho x}$ for all $r \in \mathbb{R}$. Since $\theta_{s,x} \in \mathbb{D}^{1,2}$ and $\log(1-\theta_{s,x}) = g_x(\theta_{s,x})$ by item 3 of Lemma A.6, we have

$$D_{t,z}\log(1-\theta_{s,x}) = \frac{g_x(\theta_{s,x} + zD_{t,z}\theta_{s,x}) - g_x(\theta_{s,x})}{z}\mathbf{1}_{(0,\infty)}(z).$$

Lemma A.8 implies, for $t \in [0,s]$ and $z \in (0,\infty)$,

$$\theta_{s,x} + zD_{t,z}\theta_{s,x} = f_\theta\left(\sqrt{\sigma^2 + ze^{-\lambda(s-t)}}\right)(e^{\rho x} - 1)$$

$$= \frac{\alpha(e^{\rho x}-1)}{\sigma_s^2 + ze^{-\lambda(s-t)} + C_\rho} < 1 - e^{\rho x}. \tag{A.8}$$

We have then $g_x(\theta_{s,x} + zD_{t,z}\theta_{s,x}) = \log(1 - \theta_{s,x} - zD_{t,z}\theta_{s,x})$. $\square$

A.3 On $D_{t,z} \log Z_T$

We show $\log Z_T \in \mathbb{D}^{1,2}$ and calculate $D_{t,z}\log Z_T$. (2.4) implies that

$$\begin{aligned}\log Z_T = &-\int_0^T u_s dW_s - \frac{1}{2}\int_0^T u_s^2 ds + \int_0^T\int_0^\infty \log(1-\theta_{s,x})\widetilde{N}(ds,dx) \\ &+ \int_0^T\int_0^\infty [\log(1-\theta_{s,x}) + \theta_{s,x}]\nu(dx)ds. \end{aligned} \tag{A.9}$$

We discuss each term of (A.9) separately. As seen in Section 4, we have $u \in \mathbb{L}_0^{1,2}$. Thus, Lemma 3.3 of [7] implies that $D_{t,0}\int_0^T u_s dW_s = u_t + \int_0^T D_{t,0}u_s dW_s = u_t$, and $D_{t,z}\int_0^T u_s dW_s = \int_0^T D_{t,z}u_s dW_s$ for $z > 0$. Similarly, we have $D_{t,0}\int_0^T\int_0^\infty \log(1-\theta_{s,x})\widetilde{N}(ds,dx) = 0$, and

$$D_{t,z}\int_0^T\int_0^\infty \log(1-\theta_{s,x})\widetilde{N}(ds,dx) = \frac{\log(1-\theta_{t,z})}{z} + \int_0^T\int_0^\infty D_{t,z}\log(1-\theta_{s,x})\widetilde{N}(ds,dx)$$

for $z > 0$. As for $D_{t,z}\int_0^T u_s^2 ds$, since $u^2 \in \mathbb{L}_0^{1,2}$ by Section 4, Lemma 3.2 of [7] yields

$$D_{t,z}\int_0^T u_s^2 ds = \int_0^T D_{t,z}u_s^2 ds = 2\int_0^T u_s D_{t,z}u_s ds + z\int_0^T (D_{t,z}u_s)^2 ds$$

for $z \geq 0$. In particular, $D_{t,0}\int_0^T u_s^2 ds = 0$. For the fourth term of (A.9), since $\log(1-\theta) + \theta \in \widetilde{\mathbb{L}}_1^{1,2}$, Proposition 3.5 of [18] implies

$$D_{t,z}\int_0^T\int_0^\infty [\log(1-\theta_{s,x}) + \theta_{s,x}]\nu(dx)ds = \int_0^T\int_0^\infty [D_{t,z}\log(1-\theta_{s,x}) + D_{t,z}\theta_{s,x}]\nu(dx)ds$$

for $z \geq 0$. Collectively, we conclude the following:

Proposition A.10 *We have* $\log Z_T \in \mathbb{D}^{1,2}$, $D_{t,0}\log Z_T = u_t$*; and*

$$\begin{aligned} D_{t,z}\log Z_T = &-\int_0^T D_{t,z}u_s dW_s - \int_0^T u_s D_{t,z}u_s ds - \frac{z}{2}\int_0^T (D_{t,z}u_s)^2 ds \\ &+ \int_0^T\int_0^\infty D_{t,z}\log(1-\theta_{s,x})\widetilde{N}(ds,dx) \\ &+ \int_0^T\int_0^\infty [D_{t,z}\log(1-\theta_{s,x}) + D_{t,z}\theta_{s,x}]\nu(dx)ds + \frac{\log(1-\theta_{t,z})}{z} \end{aligned}$$

for $z > 0$.

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