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キーワード: Phillips curve, net export, potential output, output gap, exchange rate, productivity, international competitiveness, HP filter

【要旨】

We propose a simple method for estimating multiple natural rates in a system of simultaneous equations. Our estimators of natural rates are closely related to the HP filter and accessible by many practitioners. As an application, Japan's potential output and natural foreign exchange rate are estimated. It is shown that Japan's potential output has been growing, but the natural foreign exchange rate has experienced stepwise downward shifts since the beginning of the 21st century. While Japan suffered the long-lasting stagnation, emerging markets, particularly China, achieved tremendous economic growth. The declines in the natural foreign exchange rate indicate Japan's lost competitiveness in the world economy clearly.

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Koichiro Kamada*

March 18, 2026

Abstract

We propose a simple method for estimating multiple natural rates in a system of simultaneous equations. Our estimators of natural rates are closely related to the HP filter and accessible by many practitioners. As an application, Japan's potential output and natural foreign exchange rate are estimated. It is shown that Japan's potential output has been growing, but the natural foreign exchange rate has experienced stepwise downward shifts since the beginning of the 21st century. While Japan suffered the long-lasting stagnation, emerging markets, particularly China, achieved tremendous economic growth. The declines in the natural foreign exchange rate indicate Japan's lost competitiveness in the world economy clearly.

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JEL classification: *C13, C32, E31, E32, F14, F41, O47*

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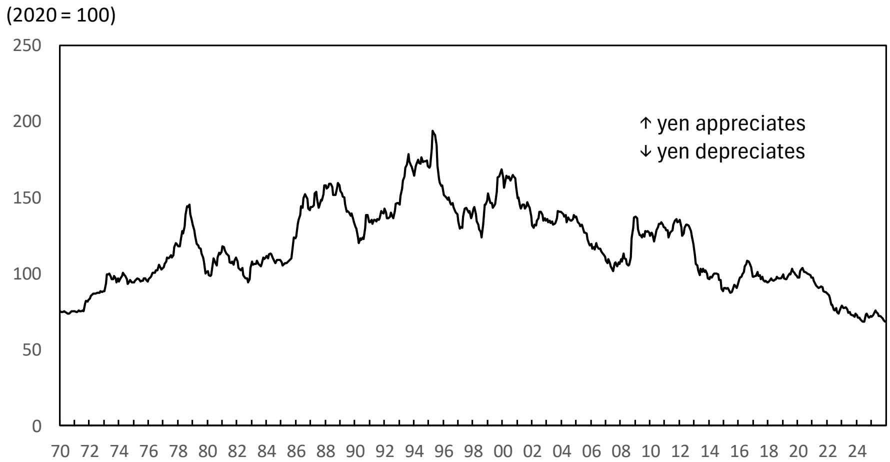
1. Introduction

Natural rates are indispensable variables in economic modeling. The behavior of economic variables is often modeled to be driven by the gap between a certain variable and its natural rate. Back to the 19th century, for instance, a Swedish economist Johan Gustaf Knut Wicksell hypothesized in his classic work, *Geldzins und Güterpreise* (*Interest and Prices*), that the inflation rate is influenced by the gap between a real interest rate and its natural rate (Wicksell, 1898). In the 20th century, Milton Friedman postulated in his natural rate hypothesis of unemployment that the inflation rate is affected by the gap between an unemployment rate and its natural rate (Friedman, 1968). Recently, the New Keynesian Phillips Curve has increasingly accepted among academic economists, saying that the inflation rate is driven by the output gap, defined as a deviation rate of actual output from potential (e.g., Roberts, 1995).

Potential output is one of the key variables in economic modeling and policy making. It is defined as the maximum sustainable output and is thought of as consistent with the stability of inflation rates (e.g., Congressional Budget Office, 2001). The potential rate of growth is one of the critical economic variables referred by the government, for instance, in formulating fiscal policy and reforming a social security system. Potential output, however, is not observable directly. In the context of monetary policy, when the inflation rate rises beyond its target, the central bank has to judge whether it is caused by strong demand or by a decline in potential growth. If demand has strengthened, the central bank raises its policy rate above the natural rate; if potential growth has slowed, monetary policy is not an appropriate policy measure and some structural reforms are necessary.

Recently, real exchange rates have attracted much attention in Japan. As shown in Figure 1, the real effective exchange rate (REER) of the Japanese yen has depreciated largely from 193.95 at the peak in April, 1995 to 68.26 in December, 2025. The REER is a multilateral version of a bilateral real exchange rate. The REER appreciates (rises) when the yen appreciates against foreign currencies and/or when the inflation rate in Japan is higher than those in foreign countries. Two interpretations are possible. First, the actual foreign exchange rate deviates from the natural rate only temporarily; second, the



Sourses: The Bank of Japan.

Figure 1. The real effective exchange rate of the yen

natural foreign exchange rate has depreciated and the actual rate follows it. If the latter is the case, it indicates that Japan has lost competitiveness in the global market and thus government intervention has only temporary effects in the foreign exchange market.

In general, we cannot observe natural rates directly and thus have to estimate it empirically. Many methods have been proposed to estimate natural rates. Since publicized by Hodrick and Prescott (1981), the HP filter has been used as an analytical tool to extract a natural rate from a time series data. The filter decomposes a time series into two: a trend component and a cyclical component. The former is used as a proxy for the natural rate of the original time series. The HP filter is so popular that statistical application software often has a special command for the filter. The HP filter, however, transforms an original time series into a smooth curve only mechanically and has no economic theory in itself. We are not sure in what sense a trend component extracted by the HP filter is natural.

Hirose and Kamada (2003) proposed a method to overcome the above problem of the HP filter. They pick up an economic model that includes a natural rate, solve for the natural rate as a function of observable variables, substitute the data into the function, and apply the HP filter to this numerical series. The trend component thus obtained is a natural rate. Consider the Phillips curve, for instance, where the inflation rate is explained by the output gap, i.e., the deviation rate of actual output from potential.

Given initial values for the parameters of the Phillips curve, potential output is expressed as a function of actual output and other observable variables, which is calculated as a numerical series with the data. They apply the HP filter to this numerical series and obtain the trend component as potential output. Given this potential output, they estimate the parameters of the Phillips curve by OLS. This process is repeated until the parameter values and the potential output converge. In this way, Hirose and Kamada incorporated economic theory into the HP filter.

We extend Hirose and Kamada's (2003) method in two ways. First, we generalize their method, which calculates a single natural rate in a single equation, to the one that calculates multiple natural rates in a system of simultaneous equations. This makes their method applicable to a broader class of economic models. As an example, we estimate potential output and the natural foreign exchange rate simultaneously in this paper. Second, we present the estimators of natural rates in analytical forms unlike Hirose and Kamada, which saves us from tedious convergence calculation they had to implement in estimation.

The remainder of this paper is organized as follows. Section 2 reviews the preceding literature. Section 3 presents the method of estimation. Section 4 applies our method to the estimation of potential output and the natural foreign exchange rate. Section 5 evaluates the uncertainty of the estimates. Section 6 concludes the paper.

2. Literature Review

The starting point of this paper is Hodrick and Prescott (1981). The HP filter decomposes a time series into a trend component and a cyclical component. Denote a trend component of x_t by $\tilde{x}_t$ $(t \in \{1, \cdots, T\})$, which is the minimizer of the following cost function.

$$\mathcal{L} \equiv \Sigma_{t=1}^{T}(x_t - \tilde{x}_t)^2 + \lambda \Sigma_{t=3}^{T}(\Delta^2 \tilde{x}_t)^2, \tag{1}$$

where Δ^2 is a second-difference operator. The first term is the cost of deviations of a trend component from an original time series. The second term is the cost of changes in a direction of the movement of a trend component. In equation (1), λ is called a smoothing parameter and governs the smoothness of the movement of a trend component. As λ increases, a trend component approaches to a straight line. Conversely, as λ decreases, a trend component approaches to the original time series.

To simplify the following argument, we rewrite equation (1) in matrices.

$$\mathcal{L} = (\boldsymbol{x} - \widetilde{\boldsymbol{x}})'(\boldsymbol{x} - \widetilde{\boldsymbol{x}}) + \lambda \widetilde{\boldsymbol{x}}'\boldsymbol{D}'\boldsymbol{D}\widetilde{\boldsymbol{x}}, \tag{2}$$

where $\boldsymbol{x}$ is a T- length column vector with element x_t; $\widetilde{\boldsymbol{x}}$ is a T- length column vector with element $\tilde{x}_t$; $\boldsymbol{D}$ is a $(T-2) \times T$ matrix of a second-difference operator given by

$$\boldsymbol{D} \equiv \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \end{bmatrix}. \tag{3}$$

We differentiate equation (2) with respect to $\widetilde{\boldsymbol{x}}$ and set the result equal to $\boldsymbol{0}$. Then we obtain the HP filter, $\boldsymbol{H}$, as follows.

$$\widetilde{\boldsymbol{x}} = \boldsymbol{H}\boldsymbol{x}; \tag{4}$$

$$\boldsymbol{H} \equiv (\boldsymbol{I} + \lambda \boldsymbol{D}'\boldsymbol{D})^{-1}. \tag{5}$$

Below, we call $\widetilde{\boldsymbol{x}}$ the HP filter trend and $\boldsymbol{x} - \widetilde{\boldsymbol{x}}$ the HP filter cycle.

The HP filter has become a popular tool for extracting a trend component from a time series, but at the same time, criticism has also grown against the usage of the HP filter. There are two criticisms broadly. The first criticism is related to the Yule-Slutzky effect, which says that we can create a cyclical time series with a certain frequency from any time series by taking its moving average, even if that time series is generated randomly. Clearly, the cyclicality thus created is a false one. The HP filter is one of moving average operators and thus not always immune to the Yule-Slutzky effect. The second criticism is that the HP filter is a measurement without theory. Surely as above, no economic theory is included in the generation of the filter.

Hirose and Kamada (2003) responded to the second criticism above and extended the HP filter by incorporating economic theory into it. We explain their methodology by a simple example. Suppose that we have the following economic model.

$$y_t = \beta(x_t - \tilde{x}_t) + \varepsilon_t, \tag{6}$$

where $\tilde{x}_t$ is the natural rate of x_t to be estimated together with parameter β. Hirose and Kamada estimated them in two steps. In the first step, given an initial value of β, equation (6) is transformed as follows.

$$\tilde{x}_t - \varepsilon_t/\beta = x_t - y_t/\beta. \tag{7}$$

Then $\tilde{x}$ is estimated by applying the HP filter to the right-hand side of the above equation, which can be given as a numerical series with data. In the second step, β is estimated in equation (6) by OLS with $\tilde{x}_t$ as given. The above process is repeated until the estimates of $\tilde{x}$ and β converge.

The purpose of this paper is to extend Hirose and Kamada's (2003) method in two directions. First, we present the estimators of β and $\tilde{x}_t$ in analytical forms. As shown above, Hirose and Kamada estimated β and $\tilde{x}_t$ by convergence calculation, which is so tedious in practice that users fail to take advantage of the simplicity of the HP filter. Furthermore, no consensus exists regarding the optimal number of repetition and the definitive criterion of convergence. Ball and Mankiw (2002) estimated time-varying NAIRU, which is the natural rate of unemployment with which the inflation rate is not accelerated, but implemented the above estimation process only once, not until convergence was attained. As a result, their NAIRU and other parameters in the model were not completely consistent with each other. In contrast, we give the estimators of β and $\tilde{x}_t$ in analytical forms. We have no need for tedious repetition any longer to obtain consistent estimates of β and $\tilde{x}_t$.

Second, this paper enables us to estimate multiple natural rates in a system of simultaneous equations. Hirose and Kamada (2003) focused on the estimation of a single natural rate in a single equation. However, inflation rates are affected not only by the domestic economy but also by the global economy typically through foreign exchange rates. We incorporate these effects into the Phillips curve as the output gap, the deviation rate of actual output from potential, and as the exchange rate gap, the deviation of the actual exchange rate from the natural rate. As explained later, our method needs two equations for two natural rates. We introduce a net export function as an additional equation, which has the output gap and the foreign exchange rate gap as explanatory variables. The idea of using multiple equations to estimate a natural rate was also pursued by Laxton and Tetlow (1992) in the context of improving the estimate of potential output by incorporating economic theory into the HP filter. Their focus, however, was on improved potential output, not on multiple natural rates. They took as given all the other natural rates. As a result, their model was reduced to a single equation model with a single natural rate.

The method of this paper is thought of as an alternative to the Kalman filter, a classical method of estimating unobservable variables as latent variables. Kuttner (1994) estimated potential output as a latent variable in the Phillips curve. In the current context, we can use the Kalman filter to estimate potential output and the natural foreign exchange rate as latent variables. The Kalman filter is not easy to handle, however, particularly for econometric beginners and does not always work well. Laubach and Williams (2003) used the Kalman filter to estimate the potential output and the natural interest rate. The structure of their model was quite simple, but the model was extremely tough to estimate. They had to use advanced and complex techniques to reach a final answer. In contrast, our method needs no complicated econometric techniques. We transform time series by the HP filter and substitute the results into our formula to obtain natural rates.

3. The Method of Estimation

3.1. Derivation of Estimators

The deviation of a certain variable from its natural rate is called a gap variable. Consider a system of N simultaneous equations including N gap variables. The ℓ-th equation ($\ell \in \{1, \cdots, N\}$) of the system is written as

$$y_{\ell t} = \Sigma_{m=1}^{M_\ell} \alpha_{\ell m} w_{\ell m t} + \Sigma_{n=1}^{N} \beta_{\ell n}(x_{nt} - \tilde{x}_{nt}) + \varepsilon_{\ell t}, \quad (8)$$

where $y_{\ell t}$ is a dependent variable; $w_{\ell m t}$ is one of M_ℓ explanatory variables other than gap variables; $(x_{nt} - \tilde{x}_{nt})$ is one of N gap variables; $\varepsilon_{\ell t}$ is an error term.

The purpose of this paper is to obtain natural rates $\tilde{x}_{nt}$ ($n \in \{1, \cdots, N\}$) together with parameters $\alpha_{\ell m}$ and $\beta_{\ell n}$. For this purpose, we define loss function L_ℓ as follows and define its minimizers as the estimates of $\alpha_{\ell m}$, $\beta_{\ell n}$, and $\tilde{x}_{nt}$.

$$L_\ell \equiv \Sigma_{t=1}^{T}\left\{y_{\ell t} - \Sigma_{m=1}^{M_\ell}\alpha_{\ell m}w_{\ell m t} - \Sigma_{n=1}^{N}\beta_{\ell n}(x_{nt} - \tilde{x}_{nt})\right\}^2 \\ +\lambda\Sigma_{t=3}^{T}\{\Delta^2(\Sigma_{n=1}^{N}\beta_{\ell n}\tilde{x}_{nt})\}^2. \tag{9}$$

Note the similarity of equation (9) to equation (1) in the previous section. The first term is the sum of squared estimation errors; the second term is the sum of squared acceleration rates of natural rates $\tilde{x}_{nt}$, more precisely, a linear combination of them. We need N equations for estimating N natural rates.

To simplify the following argument, we rewrite L_ℓ in matrices as follows.

$$L_\ell = (\boldsymbol{y}_\ell - \boldsymbol{W}_\ell\boldsymbol{\alpha}_\ell - \boldsymbol{Z}\boldsymbol{\beta}_\ell)'(\boldsymbol{y}_\ell - \boldsymbol{W}_\ell\boldsymbol{\alpha}_\ell - \boldsymbol{Z}\boldsymbol{\beta}_\ell) + \lambda\boldsymbol{\beta}_\ell'\widetilde{\boldsymbol{X}}'\boldsymbol{D}'\boldsymbol{D}\widetilde{\boldsymbol{X}}\boldsymbol{\beta}_\ell, \tag{10}$$

where $\boldsymbol{y}_\ell$ is a T- length column vector with element $y_{\ell t}$; $\boldsymbol{W}_\ell$ is a $T \times M_\ell$ matrix with element $w_{\ell m t}$; $\boldsymbol{X}$, $\widetilde{\boldsymbol{X}}$, and $\boldsymbol{Z}(\equiv \boldsymbol{X} - \widetilde{\boldsymbol{X}})$ are $T \times N$ matrices with element x_{nt}, $\tilde{x}_{nt}$, and $(x_{nt} - \tilde{x}_{nt})$, respectively; $\boldsymbol{D}$ is a $(T-2) \times T$ matrix of a second-difference operator; $\boldsymbol{\alpha}_\ell$ is an M_ℓ-length column vector with element $\alpha_{\ell m}$; $\boldsymbol{\beta}_\ell$ is an N-length column vector with element $\beta_{\ell n}$.

Theorem 1. *The minimizers of* L_ℓ *are given by* $\widehat{\boldsymbol{\alpha}}_\ell$, $\widehat{\boldsymbol{\beta}}_\ell$, *and* $\widehat{\widetilde{\boldsymbol{X}}}$ *as follows.*

$$\widehat{\boldsymbol{\gamma}}_\ell = (\boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}\boldsymbol{V}_\ell)^{-1}\boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}\boldsymbol{y}_\ell; \tag{11}$$

$$\widehat{\widetilde{\boldsymbol{X}}} = -\boldsymbol{H}[\boldsymbol{y}_1 - \boldsymbol{V}_1\widehat{\boldsymbol{\gamma}}_1, \cdots, \boldsymbol{y}_N - \boldsymbol{V}_N\widehat{\boldsymbol{\gamma}}_N]\widehat{\boldsymbol{\beta}}^{-\mathbf{1}}, \tag{12}$$

where $\boldsymbol{V}_\ell = [\boldsymbol{W}_\ell, \boldsymbol{X}]$, $\widehat{\boldsymbol{\gamma}}_\ell = \begin{bmatrix}\widehat{\boldsymbol{\alpha}}_\ell \\ \widehat{\boldsymbol{\beta}}_\ell\end{bmatrix}$, *and* $\widehat{\boldsymbol{\beta}} \equiv [\widehat{\boldsymbol{\beta}}_1, \cdots, \widehat{\boldsymbol{\beta}}_N]$.

Proof. Let $\widetilde{\boldsymbol{x}}_n$ be one of N natural rates and a column vector of $\widetilde{\boldsymbol{X}}$. Differentiate L_ℓ with respect to $\boldsymbol{\alpha}_\ell$, $\boldsymbol{\beta}_\ell$, and $\widetilde{\boldsymbol{x}}_n$ and set the results equal to $\mathbf{0}$. Then we have

$$\frac{\partial L_\ell}{\partial \boldsymbol{\alpha}_\ell} = \mathbf{0} \;\Rightarrow\; \boldsymbol{W}_\ell'\left(\boldsymbol{y}_\ell - \boldsymbol{W}_\ell\widehat{\boldsymbol{\alpha}}_\ell - \widehat{\boldsymbol{Z}}\widehat{\boldsymbol{\beta}}_\ell\right) = \mathbf{0}; \tag{13}$$

$$\frac{\partial L_\ell}{\partial \boldsymbol{\beta}_\ell} = \mathbf{0} \;\Rightarrow\; \widehat{\boldsymbol{Z}}'\left(\boldsymbol{y}_\ell - \boldsymbol{W}_\ell\widehat{\boldsymbol{\alpha}}_\ell - \widehat{\boldsymbol{Z}}\widehat{\boldsymbol{\beta}}_\ell\right) = \lambda'\widehat{\widetilde{\boldsymbol{X}}}'\boldsymbol{D}'\boldsymbol{D}\widehat{\widetilde{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}_\ell; \tag{14}$$

$$\frac{\partial L_\ell}{\partial \widetilde{\boldsymbol{x}}_n} = \mathbf{0} \;\Rightarrow\; \boldsymbol{y}_\ell - \boldsymbol{W}_\ell\widehat{\boldsymbol{\alpha}}_\ell - \widehat{\boldsymbol{Z}}\widehat{\boldsymbol{\beta}}_\ell = -\lambda\boldsymbol{D}'\boldsymbol{D}\widehat{\widetilde{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}_\ell, \tag{15}$$

where $\widehat{\boldsymbol{Z}} \equiv \boldsymbol{X} - \widehat{\widetilde{\boldsymbol{X}}}$. Note that the last equation is the same for all $n \in \{1, \cdots N\}$. Substitute equation (15) into equations (13) and (14). Then we have

$$\boldsymbol{W}_\ell'\boldsymbol{D}'\boldsymbol{D}\widehat{\widetilde{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}_\ell = \mathbf{0}; \tag{16}$$

$$\boldsymbol{X}'\boldsymbol{D}'\boldsymbol{D}\widehat{\widetilde{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}_\ell = \mathbf{0}. \tag{17}$$

Stacking the above equations, we have

$$\boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\widehat{\widehat{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}_\ell = \boldsymbol{0}. \tag{18}$$

Substitute $\widehat{\boldsymbol{Z}} \equiv \boldsymbol{X} - \widehat{\widehat{\boldsymbol{X}}}$ into equation (15). Then we have

$$\widehat{\widehat{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}_\ell = -\boldsymbol{H}(\boldsymbol{y}_\ell - \boldsymbol{V}_\ell\widehat{\boldsymbol{\gamma}}_\ell). \tag{19}$$

Substituting equation (19) into equation (18), we have equation (11). As seen in equation (19), we can obtain only one series $\widehat{\widehat{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}_\ell$, a linear combination of N natural rates, from one equation. We stack N series of $\widehat{\widehat{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}_\ell$ $(\ell \in \{1, \cdots N\})$ to obtain $T \times N$ matrix $\widehat{\widehat{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}}$ as follows.

$$\widehat{\widehat{\boldsymbol{X}}}\widehat{\boldsymbol{\beta}} = [-\boldsymbol{H}(\boldsymbol{y}_1 - \boldsymbol{V}_1\widehat{\boldsymbol{\gamma}}_1), \cdots, -\boldsymbol{H}(\boldsymbol{y}_N - \boldsymbol{V}_N\widehat{\boldsymbol{\gamma}}_N)]. \tag{20}$$

When $N \times N$ matrix $\widehat{\boldsymbol{\beta}}$ has full rank, we obtain equation (12) by multiplying the both sides of equation (20) by $\widehat{\boldsymbol{\beta}}^{-1}$ from right. (Q.E.D.)

Although equations (11) and (12) in Theorem 1 look complicated, we can reformulate them into a simpler system. From equation (5), we have $(\boldsymbol{I} + \lambda\boldsymbol{D}'\boldsymbol{D})\boldsymbol{H} = \boldsymbol{I}$. Thus, we have the following equation.

$$\lambda\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H} = \boldsymbol{I} - \boldsymbol{H}. \tag{21}$$

This means that $\lambda\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}$ is the filter that extracts the HP filter cycle. Denote the HP filter trend of $\boldsymbol{y}_\ell$ by $\boldsymbol{y}_\ell^*$. Denote the HP filter trends and cycles of $\boldsymbol{V}_\ell$ by $\boldsymbol{V}_\ell^*$ and $\boldsymbol{V}_\ell^\#$, respectively. Using equation (21), we have $\lambda\boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H} = \boldsymbol{V}_\ell'(\boldsymbol{I} - \boldsymbol{H}) = \{(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{V}_\ell\}' = \boldsymbol{V}_\ell^{\#\prime}$. Substituting this into equations (11) and (12), we have

$$\widehat{\boldsymbol{\gamma}}_\ell = \left(\boldsymbol{V}_\ell^{\#\prime}\boldsymbol{V}_\ell\right)^{-1}\boldsymbol{V}_\ell^{\#\prime}\boldsymbol{y}_\ell; \tag{22}$$

$$\widehat{\widehat{\boldsymbol{X}}} = -[\boldsymbol{y}_1^* - \boldsymbol{V}_1^*\widehat{\boldsymbol{\gamma}}_1, \cdots, \boldsymbol{y}_N^* - \boldsymbol{V}_N^*\widehat{\boldsymbol{\gamma}}_N]\widehat{\boldsymbol{\beta}}^{-1}. \tag{23}$$

This shows that Theorem 1 is closely related to the HP filter. Applying the HP filter to explanatory and dependent variables, we obtain the estimates by implementing easy matrix algebra.

To understand the meanings of Theorem 1 intuitively, consider a single equation case with a single natural rate and no other explanatory variable. The estimators in Theorem 1 are given by

$$\hat{\beta} = \frac{\boldsymbol{x}^{\#\prime}\boldsymbol{y}}{\boldsymbol{x}^{\#\prime}\boldsymbol{x}}; \tag{24}$$

$$\hat{\hat{\boldsymbol{x}}} = (\boldsymbol{x}^* - \boldsymbol{y}^*/\hat{\beta}). \tag{25}$$

We apply the HP filter to $\boldsymbol{x}$ and $\boldsymbol{y}$ to obtain their trend and cycle components and calculate the inner product of $\boldsymbol{x}^{\#}$ and $\boldsymbol{x}$ and the inner product of $\boldsymbol{x}^{\#}$ and $\boldsymbol{y}$. We obtain $\hat{\beta}$ as their ratio. Given $\hat{\beta}$, we obtain $\hat{\hat{\boldsymbol{x}}}$ from $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$. Note its similarity to the Hirose-Kamada method in estimating $\hat{\hat{\boldsymbol{x}}}$ and a difference in estimating $\hat{\beta}$, which is given analytically in our method, while obtained by convergence calculation in their method. Note also that as shown in equation (25), $\hat{\hat{\boldsymbol{x}}}$ is not computable if $\hat{\beta}$ is zero; $\hat{\hat{\boldsymbol{x}}}$ takes an extreme value if $\hat{\beta}$ is close to zero. In the case of multiple natural rates, $\hat{\hat{\boldsymbol{X}}}$ is not computable if $\hat{\boldsymbol{\beta}}$ is singular; $\hat{\hat{\boldsymbol{X}}}$ takes extreme values if $\hat{\boldsymbol{\beta}}$ is near singular.

3.2. Cautions in Model Specification

Here we give some cautions for the use of our estimators.

Lemma 1. *If* $\boldsymbol{a}$ *is a linear trend, we have the following relationships.*

$$\boldsymbol{D}\boldsymbol{a} = \boldsymbol{0}; \tag{26}$$

$$\boldsymbol{H}\boldsymbol{a} = \boldsymbol{a}; \tag{27}$$

$$(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{a} = \boldsymbol{0}. \tag{28}$$

Proof. Equation (26) is trivial, since $\boldsymbol{D}$ is a second-difference operator. Using equation (26), we have $(\boldsymbol{I} + \boldsymbol{\lambda}\boldsymbol{D}'\boldsymbol{D})\boldsymbol{a} = \boldsymbol{a}$. Multiplying the both sides by $\boldsymbol{H} \equiv (\boldsymbol{I} + \boldsymbol{\lambda}\boldsymbol{D}'\boldsymbol{D})^{-1}$, we obtain equation (27). Equation (28) is obtained directly from equation (27). (Q.E.D.)

Corollary 1. $\hat{\boldsymbol{\gamma}}_\ell$ *is unchanged by detrending explanatory and dependent variables linearly. Detrended factors of dependent variables are added on* $\hat{\hat{\boldsymbol{X}}}$ *and those of explanatory variables are subtracted from* $\hat{\hat{\boldsymbol{X}}}$.

Proof. Denote a column vector of $\boldsymbol{V}_\ell$ by $\boldsymbol{v}_{\ell m}$ and its linear trend by $\boldsymbol{a}_{\ell m}$ ($\ell \in \{1, \cdots N\}$, $m \in \{1, \cdots, M_\ell + N\}$). Define a matrix of the trends by $\boldsymbol{A}_\ell \equiv (\boldsymbol{a}_{\ell 1}, \cdots, \boldsymbol{a}_{\ell M_\ell + N})$. Then a matrix of detrended explanatory variables is given by $\boldsymbol{V}_\ell - \boldsymbol{A}_\ell$. Similarly, by detrending a dependent variable by linear trend $\boldsymbol{b}_\ell$ ($\ell \in \{1, \cdots N\}$), we have detrended dependent variable $\boldsymbol{y}_\ell - \boldsymbol{b}_\ell$. Let $\hat{\hat{\boldsymbol{\gamma}}}_\ell$ be the estimator after detrending. Substituting $\boldsymbol{V}_\ell - \boldsymbol{A}_\ell$ and

$\boldsymbol{y}_\ell - \boldsymbol{b}_\ell$ into $\boldsymbol{V}_\ell$ and $\boldsymbol{y}_\ell$ in equation (11), we have

$$\hat{\hat{\boldsymbol{\gamma}}}_\ell = \{(\boldsymbol{V}_\ell' - \boldsymbol{A}_\ell')\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}(\boldsymbol{V}_\ell - \boldsymbol{A}_\ell)\}^{-1}(\boldsymbol{V}_\ell' - \boldsymbol{A}_\ell')\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}(\boldsymbol{y}_\ell - \boldsymbol{b}_\ell). \quad (29)$$

Applying Lemma 1 to the parts of the right-hand side, we have

$$(\boldsymbol{V}_\ell' - \boldsymbol{A}_\ell')\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}(\boldsymbol{V}_\ell - \boldsymbol{A}_\ell) = \boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}\boldsymbol{V}_\ell; \quad (30)$$

$$(\boldsymbol{V}_\ell' - \boldsymbol{A}_\ell')\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}(\boldsymbol{y}_\ell - \boldsymbol{b}_\ell) = \boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}\boldsymbol{y}_\ell. \quad (31)$$

Therefore, we have $\hat{\hat{\boldsymbol{\gamma}}}_\ell = \hat{\boldsymbol{\gamma}}_\ell$. Denote $\hat{\hat{\boldsymbol{X}}}\hat{\hat{\boldsymbol{\beta}}}_\ell$ after detrending by $\hat{\hat{\hat{\boldsymbol{X}}}}\hat{\hat{\hat{\boldsymbol{\beta}}}}_\ell$. Substituting $\boldsymbol{V}_\ell - \boldsymbol{A}_\ell$ and $\boldsymbol{y}_\ell - \boldsymbol{b}_\ell$ into $\boldsymbol{V}_\ell$ and $\boldsymbol{y}_\ell$ in equation (19), we have

$$\hat{\hat{\hat{\boldsymbol{X}}}}\hat{\hat{\hat{\boldsymbol{\beta}}}}_\ell = -\boldsymbol{H}\{(\boldsymbol{y}_\ell - \boldsymbol{b}_\ell) - (\boldsymbol{V}_\ell - \boldsymbol{A}_\ell)\hat{\boldsymbol{\gamma}}_\ell\} = \hat{\hat{\boldsymbol{X}}}\hat{\hat{\boldsymbol{\beta}}}_\ell + (\boldsymbol{b}_\ell - \boldsymbol{A}_\ell\hat{\boldsymbol{\gamma}}_\ell). \quad (32)$$

As shown in equation (32), the detrended portion of the dependent variable is added on the estimated natural rate; the detrended portions of the explanatory variables are subtracted from the estimated natural rates. (Q.E.D.)

In empirical study, we sometimes detrend our data before analysis. For instance, in time series analysis, we often demean each data to normalize it. This is a kind of linear detrending. Corollary 1 says that demeaning does not affect $\hat{\boldsymbol{\gamma}}_\ell$, while it affects the levels of estimated natural rates. Suppose that we demean the data on the exchange ate. According to the corollary, demeaning does not matter if we are interested in the exchange rate gap, i.e., the deviation rate of the actual exchange rate from the natural rate, but it does matter if we are interested in the levels of the natural exchange rate. Normalization of data is often thought of as harmless, but we have to be careful whether we should implement it or not.

Corollary 2. $\hat{\boldsymbol{\gamma}}_\ell$ *is not obtained if a linear trend is included as an explanatory variable.*

Proof. Add linear trend $\boldsymbol{a}_\ell$ ($\ell \in \{1, \cdots, N\}$) on $\boldsymbol{V}_\ell$ to construct a new matrix of explanatory variables, i.e., $[\boldsymbol{a}_\ell, \boldsymbol{V}_\ell]$. Replace $\boldsymbol{V}_\ell$ in equation (11) with it. By Lemma 1, the inside of the inverse matrix of equation (11) is calculated as follows.

$$\begin{bmatrix} \boldsymbol{a}' \\ \boldsymbol{V}_\ell' \end{bmatrix} \boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}[\boldsymbol{a}, \boldsymbol{V}_\ell] = \begin{bmatrix} \boldsymbol{a}'\boldsymbol{D}' \\ \boldsymbol{V}_\ell'\boldsymbol{D}' \end{bmatrix} [\boldsymbol{D}\boldsymbol{H}\boldsymbol{a}, \boldsymbol{D}\boldsymbol{H}\boldsymbol{V}_\ell] = \begin{bmatrix} 0 & \boldsymbol{0}' \\ \boldsymbol{0} & \boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{H}\boldsymbol{V}_\ell \end{bmatrix}. \quad (33)$$

Obviously, this matrix does not have full rank. Thus, we cannot calculate $\hat{\boldsymbol{\gamma}}_\ell$. (Q.E.D.)

If a dependent variable has a trend, they may affect the estimates of natural rates. You may want to avoid the effects by including a linear trend as an additional explanatory variable. However, Corollary 2 says that you make the system unable to be estimated by doing so. It is recommendable that you detrend the dependent variable before analysis if necessary.

3.3. Selection of the Smoothing Parameter

The HP filter trend depends on the value of smoothing parameter λ (e.g., Hodrick and Prescott, 1981). In general, as λ goes to infinity, the HP filter trend converges to a linear trend. Conversely, as λ goes to zero, the HP filter trend converges to an original series. This general tendency has to be modified in our model.

Theorem 2. *(i) As* λ *converges to zero,* $\hat{\boldsymbol{\gamma}}_\ell$ *converges to the coefficients obtained when* $\boldsymbol{D}\boldsymbol{y}_\ell$ *is regressed on* $\boldsymbol{D}\boldsymbol{V}_\ell$ *by OLS.* $\hat{\tilde{\boldsymbol{X}}}\hat{\boldsymbol{\beta}}_\ell$ *is given by the portion of* $\boldsymbol{y}_\ell$ *that is not explained by* $\boldsymbol{V}_\ell\hat{\boldsymbol{\gamma}}_\ell$*. (ii) As* λ *goes to infinity,* $\hat{\boldsymbol{\gamma}}_\ell$ *converges to the coefficients obtained when* $\boldsymbol{y}_\ell$ *is regressed on* $\boldsymbol{V}_\ell$ *and a linear trend by OLS.* $\hat{\tilde{\boldsymbol{X}}}\hat{\boldsymbol{\beta}}_\ell$ *is given by the estimated linear trend.*

Proof. (i) As $\lambda \to 0$, $\boldsymbol{H} \to \boldsymbol{I}$. Thus, $\hat{\boldsymbol{\gamma}}_\ell \to (\boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{V}_\ell)^{-1}\boldsymbol{V}_\ell'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{y}_\ell$ in equation (11); $\hat{\tilde{\boldsymbol{X}}}\hat{\boldsymbol{\beta}}_\ell \to -(\boldsymbol{y}_\ell - \boldsymbol{V}_\ell\hat{\boldsymbol{\gamma}}_\ell)$ in equation (19). (ii) As $\lambda \to \infty$, the second term of equation (10) grow infinitely. Thus, for the minimization of equation (10), we minimize its second term first. We can minimize it if $\tilde{\boldsymbol{X}}\boldsymbol{\beta}_\ell$ is a linear trend. If this is the case, the second term vanishes. Then we minimize the first term. We can achieve it by regressing $\boldsymbol{y}_\ell$ on a linear trend as well as on $\boldsymbol{V}_\ell$. Once the convergence limits of $\hat{\boldsymbol{\gamma}}_\ell$ and $\hat{\tilde{\boldsymbol{X}}}\hat{\boldsymbol{\beta}}_\ell$ are found for all ℓ, we obtain the convergence limit of $\hat{\tilde{\boldsymbol{X}}}$ by multiplying the convergence limit of $\hat{\tilde{\boldsymbol{X}}}\hat{\boldsymbol{\beta}}$ by that of $\hat{\boldsymbol{\beta}}^{-1}$. (Q.E.D.)

Hodrick and Prescott (1981) recommended $\lambda = 14{,}400$ for a monthly data, $\lambda = 1{,}600$ for a quarterly data, and $\lambda = 100$ for a yearly data.[1] If we have some

[1] Hodrick and Prescott (1981) set the value of λ for quarterly data, based on the following observation about real GNP in the United States: The standard deviation of its cycle component was 5 percent and the standard deviation of the second-order difference of its trend component was 1/8 percent. To balance the two standard deviations, they chose $\lambda = (5/(1/8))) = 1600$.

presumption about natural rates, we can follow it for the value of λ. For instance, if natural rates are known to follow linear trends, we should choose a large value for λ. It is rare, however, that we have such presumption. Moreover, such a presumption is often arbitrary. To avoid arbitrariness, some authors assume a certain probability model and obtain an appropriate value of λ (e.g., Schlicht, 2005; Dermoune, et al., 2008). However, arbitrariness often comes into a model unintendedly. When an author chooses a value for λ different from Hodrick and Prescott's recommendation, we should make sure what is his presumption about natural rates.

4. A Numerical Example

4.1. Data

We use the following data in the empirical analysis below. The data on the consumer price index is issued by the Ministry of Internal Affairs and Communication as a year-on-year change in the *consumer price index, excluding fresh foods* (quarterly average). The data on output is issued by the Cabinet Office as *gross domestic product* (billion yen, real, chained, prices in 2015, seasonally adjusted) in the *System of National Accounts*. The data on the exchange rate is issued by the Bank of Japan as the *real effective exchange rate* (quarterly average). The data on net export is issued by the Cabinet Office as *net export* in the *System of National Accounts*. The data on the world demand is issued by the Organization for Economic Co-operation and Development (OECD) as *Composite Leading Indicators, G20* (quarterly average, deviations from the sample average). Our sample is quarterly and covers the period of 1985Q2 to 2025Q1.

4.2. Model Specifications

Our economic model is a system of two equations: the Phillips curve and the net export function. The Phillips curve is given by

$$\Delta\pi_t = \alpha_{11}\Delta\pi_{t-1} + \beta_{11}(g_t - \tilde{g}_t) + \beta_{12}(f_t - \tilde{f}_t) + \varepsilon_{1t}, \tag{34}$$

where π is the inflation rate and $\Delta\pi$ is the acceleration rate; g is actual real output and $\tilde{g}$ is potential output; f is the actual real foreign exchange rate and $\tilde{f}$ is the natural rate; ε_1 is an error term. It is assumed that as actual output exceeds potential, the inflation rate accelerates. This means $\beta_{11} > 0$ in equation (34). If actual output coincides with potential, there are no pressures on inflation rates from output. Hirose and Kamada (2003) called this type of potential output the NAILO (Non-Accelerating Inflation Level of Output). We assume that as the actual exchange rate appreciates beyond the natural rate, the inflation rate decelerates. In this paper, a rise in f indicates an appreciation of the yen against other currencies in a real term. This means $\beta_{12} < 0$ in equation (34). The foreign exchange rate is often ignored in the estimation of potential output. Here we analyze whether this treatment can be justified or not.

The net export function is given by

$$\Delta n_t = \alpha_{21}\Delta n_{t-1} + \alpha_{22}d_t + \beta_{21}(g_t - \tilde{g}_t) + \beta_{22}(f_t - \tilde{f}_t) + \varepsilon_{2t}, \tag{35}$$

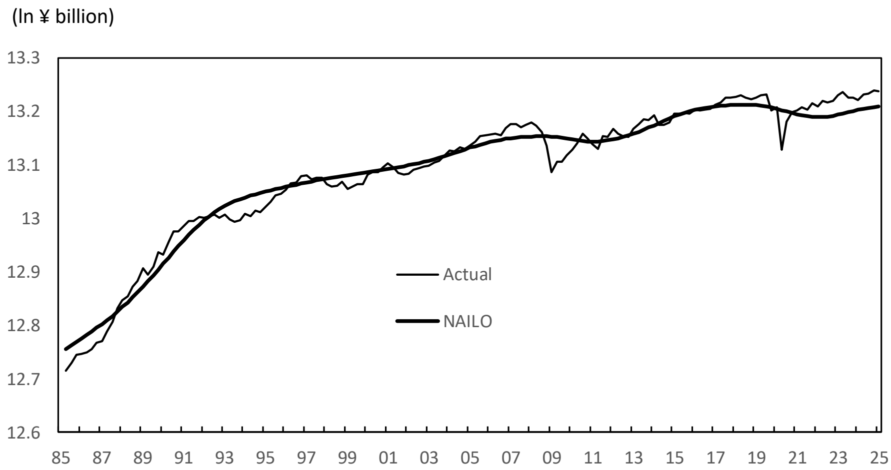
where n is the ratio of net export to output and Δn is its increase; d is a measure of world business conditions; ε_2 is an error term. d is included in equation (35) as a proxy for world demand and we expect $\alpha_{22} > 0$. It is assumed that as actual output exceeds potential, net export decreases, which means $\beta_{21} < 0$. We assume that as the actual exchange rate appreciates beyond the natural rate, net export decreases, which means $\beta_{22} < 0$. If the actual exchange rate is equal to the natural rate, there are no pressures on net export from the exchange rate. The natural foreign exchange rate is called the NAFEX below.

Note that a natural rate may have multiple meanings in a system of multiple equations. If β_{11} and β_{21} are both significant in equations (34) and (35), respectively, then potential output is the output that is neutral not only to inflation rates but also to net export. Similarly, if β_{12} and β_{22} are both significant in respective equations, then the natural exchange rate is the exchange rate that is neutral not only to net export but also to inflation rates. These possibilities are investigated only by empirical analysis.

Table 1

Estimated parameters of the Phillips curve and net export function

	λ	0	10^{-8}	1.6	160	**1,600**	16,000	1,600,000	10^{8}	$+\infty$
$\Delta\pi$	$\hat{\beta}_{11}$	3.345	3.278	5.589	6.469	**4.452**	2.547	1.085	0.949	0.946
	$\hat{\beta}_{12}$	-0.017	-0.017	-0.015	-0.005	**-0.001**	-0.002	-0.003	-0.004	-0.004
	$\hat{\alpha}_{11}$	-0.667	-0.667	-0.267	0.135	**0.224**	0.271	0.295	0.298	0.298
Δn	$\hat{\beta}_{21}$	23.094	22.457	15.539	-1.341	**-2.352**	-0.019	2.239	2.377	2.380
	$\hat{\beta}_{22}$	-0.016	-0.016	-0.013	-0.014	**-0.015**	-0.012	-0.009	-0.008	-0.008
	$\hat{\alpha}_{21}$	-0.576	-0.576	-0.493	-0.329	**-0.291**	-0.275	-0.265	-0.265	-0.265
	$\hat{\alpha}_{22}$	0.503	0.510	0.396	0.245	**0.206**	0.173	0.153	0.152	0.152



Sourses: The Cabinet office, the Bank of Japan.

Figure 2. The NAILO-based potential output

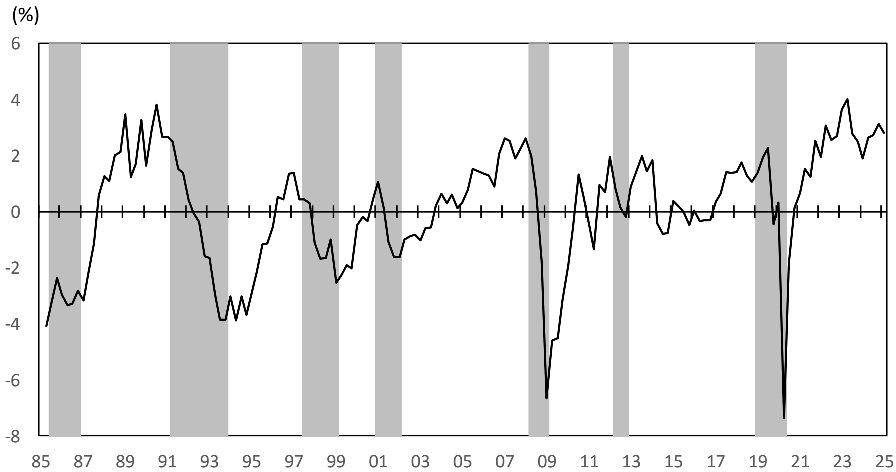
4.3. Estimation Results

We apply our estimation method to the quarterly data with $\lambda = 1600$ and obtain the estimates of α's and β's as presented in the center column in Table 1. All the estimated coefficients in the column have the signs expected theoretically. The estimated potential output, i.e., the NAILO, is displayed in Figure 2. As shown in Figure 3, the NAILO-based potential growth decelerated in the first half of the 1990s and fluctuates around one percent in the recent years. The NAILO-based output gap is displayed in Figure 4, where the shaded areas indicate contraction phases defined by the *Reference Dates of Business Cycle* issued by the Cabinet Office. The graph shows that the output gap is consistent with the official definition of peaks and troughs of business cycle, which gives a guarantee to our estimation.

We are interested in the consistency of the NAFEX with two theoretical

Sourses: The Cabinet office, the Bank of Japan.

Figure 3. The NAILO-based potential growth

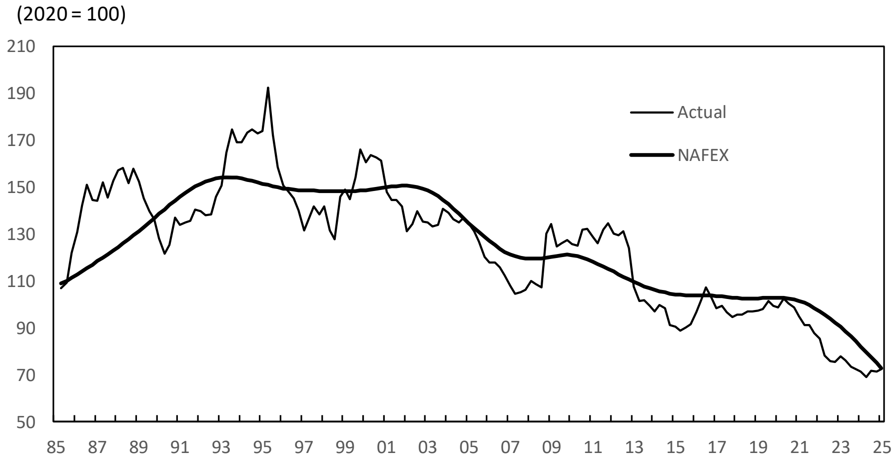


Note: The shaded areas indicate contraction phases by the Reference Dates of Business Cycle.

Sourses: The Cabinet office, the Bank of Japan.

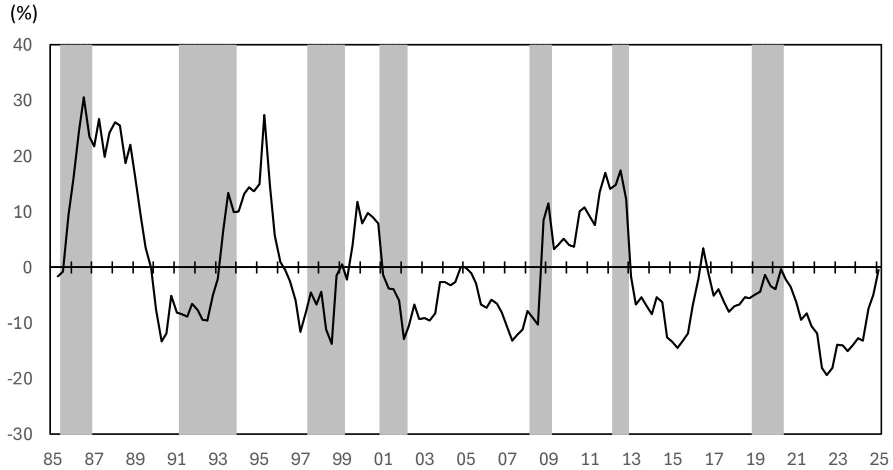
Figure 4. The NAILO-based output gap

hypotheses of the relative purchasing power parity (PPP) and the Balassa-Harrod-Samuelson (BHP) theorem. Many of academic economists believe that the relative PPP holds in the long run. One of the implications of the relative PPP is that the real exchange rate is constant over time. Therefore, if the NAFEX moves horizontally, it proves the validity of the relative PPP hypothesis. On the other hand, the BHP theorem predicts that the real exchange rate varies in the long run. Reflecting differentials in technological progress among countries, inflation rates are higher in fast-growing countries than in slow-growing countries. Thus, if the NAFEX rises, it indicates that Japan grows potentially faster than the rest of the world.



Sourses: The Cabinet office, the Bank of Japan.

Figure 5. The NAFEX-based natural foreign exchange rate



Note: The shaded areas indicate contraction phases by the Reference Dates of Business Cycle.

Sourses: The Cabinet office, Bank of Japan.

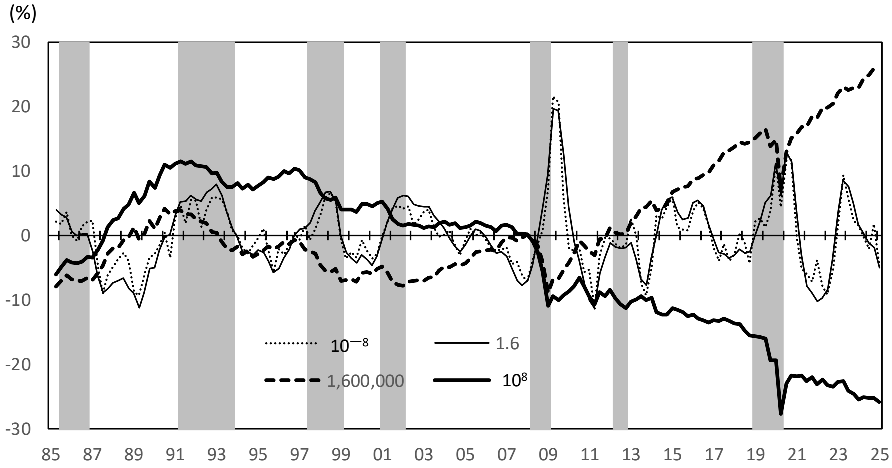
Figure 6. The NAFEX-based exchange rate gap

The estimated natural foreign exchange rate, i.e., the NAFEX, is displayed in Figure 5. Note the stepwise movement of the NAFEX. This is thought of as created by the combination of the relative PPP hypothesis and the BHP theorem. The NAFEX moved flat in the 1990s, the late 2000s, and the 2010s, as predicted by the relative PPP hypothesis. The NAFEX also shifted upward or downward according to the BHS theorem. After its upward shift in the second half of the 1980s, the NAFEX experienced downward shifts three times: first, in the early 2000s after the financial system unease as a legacy of Japan’s asset bubble around 1990; second, around 2010 due to the economic turmoil caused by the global financial crisis (GFC) in the late 2000s; third, in recent years due to the COVID-19 pandemic in the early 2020s. While Japan suffered more than 30 years of stagnation, the emerging markets achieved tremendous development. The development of China is particularly remarkable. The downward shifts of the NAFEX are the evidence that Japan has lost its international competitiveness in the world economy. The NAFEX-based exchange rate gap is presented in Figure 6. Though not always, a rise in the exchange rate gap led to a recession phase, which implies that an exchange rate gap sometimes caused business cycle.

4.3. Choice of the Smoothing Parameter

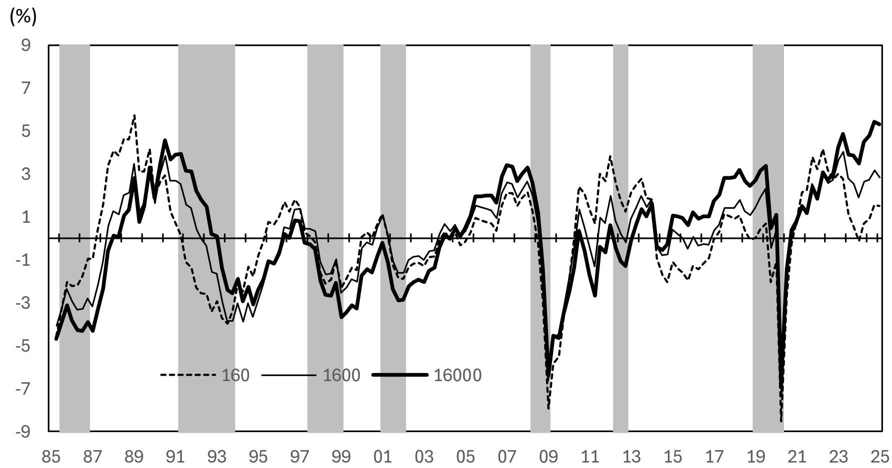
As discussed in Section 3, the estimates of natural rates are affected by the choice of smoothing parameter λ. Table 1 also presents the estimates of α's and β's corresponding to various values of λ. We also calculate the convergence limits of the coefficients by Theorem 2 and present them in the first and last columns in the table. Observe that the estimates converge to those in the first column as $\lambda \to 0$ and to those in the last column as $\lambda \to +\infty$. $\hat{\beta}_{11}$ is the coefficient of the output gap in the Phillips curve and is expected to be positive. The table shows that $\hat{\beta}_{11}$ is positive regardless of the choice of λ. $\hat{\beta}_{12}$ is the coefficient of the exchange rate gap and is expected to be negative. The table shows that $\hat{\beta}_{12}$ is negative regardless of the choice of λ. Thus, the theoretical consistency is kept regardless of the choice of λ in the Phillips curve of equation (34).

We do the same exercise as above for the net export function. $\hat{\beta}_{22}$ is the coefficient of the foreign exchange rate gap in the net export function, which is expected to be negative. The table shows that $\hat{\beta}_{22}$ is always negative regardless of the choice of λ. $\hat{\beta}_{21}$ is the coefficient of the output gap, which is expected to be negative. The table shows that $\hat{\beta}_{21}$ is negative only if $\lambda =$ 160, 1600, and 16000, while it is positive otherwise. The theoretical consistency of the net export function in equation (35) is not robust against the choice of extreme values for λ. However, as long as λ is not far from 1600, the net export function satisfies theoretical sign requirements.



Sourses: The Cabinet office, the Bank of Japan.

Figure 7. The NAILO-based output gaps with extreme smoothing parameter values

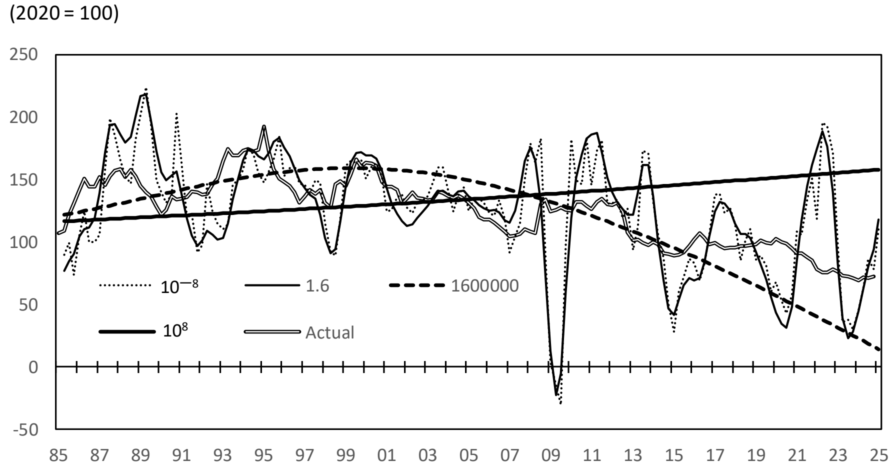


Sourses: The Cabinet office, the Bank of Japan.

Figure 8. The NAILO-based output gaps with moderate smoothing parameter values

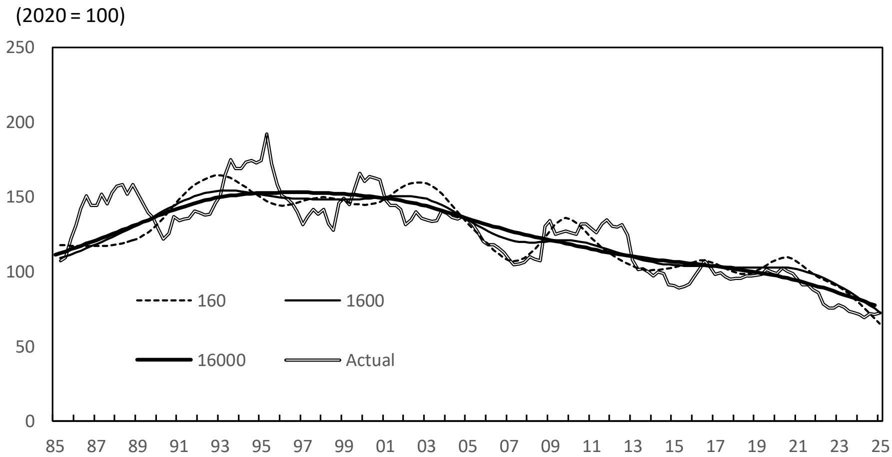
We can show that the estimated natural rates depict counterintuitive trajectories if λ takes an extreme value. Figure 7 displays the NAILO-based output gap that are obtained with $\lambda = 10^{-8}$, 1.6, 1600000, and 10^{8}. With extremely small values for λ, the output gap loses the consistency with the official reference dates of business cycle. With extremely large values for λ, the output gap at the end of the sample becomes unrealistically large. In contrast, as in Figure 8, when we use moderate values not far from 1600, the output gap keeps its consistency with the official reference dates of business cycle.

Figure 9 shows the NAFEX with $\lambda = 10^{-8}$, 1.6, 1600000, and 10^{8}. With extremely



Sourses: The Cabinet office, the Bank of Japan.

Figure 9. The NAFEX-based natural exchange rate with extreme smoothing parameter values



Sourses: The Cabinet office, the Bank of Japan.

Figure 10. The NAFEX-based natural exchange rate with moderate smoothing parameter values

small values for λ, the NAFEX becomes more volatile than the actual rate and takes negative values in 2009. These results are against our intuition that natural rates are stabler than actual rates. With extremely large values for λ, the movement of the natural exchange rate is no more stepwise and thus loses the consistency with the relative PPP and the BHS theorem. In contrast, when we use moderate values not far from 1600, the stepwise movements are still observed in the natural exchange rate, as seen in Figure 10.

5. Uncertainty of the Estimates

5.1. The General Procedure for the Bootstrap Method

In this paper, we use the bootstrap method to see statistical properties on $\hat{\boldsymbol{\gamma}}_\ell$ and $\hat{\hat{\boldsymbol{X}}}$. To begin with, we give a brief explanation of the bootstrap method generally, following Efron and Tibshirani (1993, ch.8) and Hirose and Kamada (2003).

Step 1: We estimate $\hat{\boldsymbol{\gamma}}_\ell$ and $\hat{\hat{\boldsymbol{X}}}$ from $\boldsymbol{y}_\ell$ and $\boldsymbol{V}_\ell$ by equations (11) and (12). Given $\hat{\boldsymbol{\gamma}}_\ell$ and $\hat{\hat{\boldsymbol{X}}}$, we calculate *approximate disturbances* $\hat{\boldsymbol{\varepsilon}}_\ell = (\hat{\varepsilon}_{\ell 1}, \cdots, \hat{\varepsilon}_{\ell T})$ as follows.

$$\hat{\boldsymbol{\varepsilon}}_\ell = \boldsymbol{y}_\ell - \left(\boldsymbol{V}_\ell \hat{\boldsymbol{\gamma}}_\ell - \hat{\hat{\boldsymbol{X}}}\hat{\boldsymbol{\beta}}_\ell\right). \tag{36}$$

Step 2: Design an *empirical distribution* from which we draw *bootstrap disturbances* $\boldsymbol{\varepsilon}_\ell^b = (\varepsilon_{\ell 1}^b, \cdots, \varepsilon_{\ell T}^b)$ by random sampling with replacement. A naïve one is that $\varepsilon_{\ell t}^b$ $(t \in \{1, \cdots, T\})$ is drawn from $\hat{\varepsilon}_{\ell 1}, \cdots, \hat{\varepsilon}_{\ell T}$ each with probability $1/T$ independently with replacement.

Step 3: Calculate *bootstrap sample* $\boldsymbol{y}_\ell^b$ and $\boldsymbol{V}_\ell^b$ as follows.

$$\boldsymbol{y}_\ell^b = \boldsymbol{V}_\ell^b \hat{\boldsymbol{\gamma}}_\ell - \hat{\hat{\boldsymbol{X}}}\hat{\boldsymbol{\beta}}_\ell + \hat{\boldsymbol{\varepsilon}}_\ell^b. \tag{37}$$

When a model includes the lags of dependent variables as explanatory variables, not only $\boldsymbol{y}_\ell$ is replaced by $\boldsymbol{y}_\ell^b$, but also $\boldsymbol{V}_\ell$ is by $\boldsymbol{V}_\ell^b$.

Step 4: We estimate *bootstrap replications* $\hat{\boldsymbol{\gamma}}_\ell^b$ and $\hat{\hat{\boldsymbol{X}}}^b$ from $\boldsymbol{y}_\ell^b$ and $\boldsymbol{V}_\ell^b$ by equations (11) and (12).

We return to Step 2 after Step 4 and repeat the process K times to accumulate $\hat{\boldsymbol{\gamma}}_\ell^b$ and $\hat{\hat{\boldsymbol{X}}}^b$ for $b \in \{1, \cdots, K\}$, from which we can obtain various statistics on $\hat{\boldsymbol{\gamma}}_\ell$ and $\hat{\hat{\boldsymbol{X}}}$. The possibility of $\hat{\boldsymbol{\beta}}^b \equiv \left[\hat{\boldsymbol{\beta}}_1^b, \cdots, \hat{\boldsymbol{\beta}}_N^b\right]$ being a singular matrix is small, if an empirical model is designed appropriately and a sufficiently large sample exists. We cannot completely avoid the possibility, however. If $\hat{\boldsymbol{\beta}}^b$ is singular, we skip that round of estimation and start another.

5.2. Evaluating the Estimates Stochastically

Following the procedure shown above, we inspect the estimates of α's, β's, $\tilde{g}$, and $\tilde{f}$ obtained in the previous section from the stochastic point of view. In Step 1, given the estimates, $\hat{\alpha}$'s, $\hat{\beta}$'s, $\hat{\hat{g}}$, and $\hat{\hat{f}}$, we estimate approximate disturbances $\hat{\varepsilon}_{1t}$ and $\hat{\varepsilon}_{2t}$ ($t \in \{1, \cdots, T\}$). In Step 2, with the empirical distributions of $\hat{\varepsilon}_{1t}$ and $\hat{\varepsilon}_{2t}$ in hand, we construct bootstrap disturbances ε_{1t}^b and ε_{2t}^b ($t \in \{1, \cdots, T\}$) by random sampling. In Step 3, substituting ε_{1t}^b and ε_{2t}^b in equations (34) and (35), we obtain bootstrap sample $\Delta\pi_t^b$ and Δn_t^b ($t \in \{1, \cdots, T\}$). In doing so, we should note that equations (34) and (35) are autoregressive. We obtain bootstrap samples quarter by quarter. For instance, $\Delta\pi_t^b$ ($t \in \{1, \cdots, T\}$) is obtained as follows.

$$
\begin{aligned}
\Delta\pi_1^b &= \hat{\alpha}_{11}\Delta\pi_0 + \hat{\beta}_{11}\left(g_1 - \hat{\hat{g}}_1\right) + \hat{\beta}_{12}\left(f_1 - \hat{\hat{f}}_1\right) + \varepsilon_{11}^b; \\
\Delta\pi_2^b &= \hat{\alpha}_{11}\Delta\pi_1^b + \hat{\beta}_{11}\left(g_2 - \hat{\hat{g}}_2\right) + \hat{\beta}_{12}\left(f_2 - \hat{\hat{f}}_2\right) + \varepsilon_{12}^b; \\
&\vdots \\
\Delta\pi_T^b &= \hat{\alpha}_{11}\Delta\pi_{T-1}^b + \hat{\beta}_{11}\left(g_T - \hat{\hat{g}}_T\right) + \hat{\beta}_{12}\left(f_T - \hat{\hat{f}}_T\right) + \varepsilon_{1T}^b.
\end{aligned}
\tag{38}
$$

Given $\Delta\pi_0$, we calculate $\Delta\pi_1^b$ in the first equation. Then, $\Delta\pi_2^b$ is calculated with $\Delta\pi_1^b$ substituted as one of the explanatory variables in the second equation. We repeat this procedure until $\Delta\pi_T^b$ is obtained. We can make bootstrap sample Δn_t^b ($t \in \{1, \cdots, T\}$) by equation (35) in a similar fashion.[2] In Step 4, based on the bootstrap sample, we estimate the model to obtain bootstrap replications $\hat{\alpha}^b$'s, $\hat{\beta}^b$'s, $\hat{\hat{g}}^b$, and $\hat{\hat{f}}^b$. Repeating this process many times, say 10,000 times, we have distributions of $\hat{\alpha}^b$'s, $\hat{\beta}^b$'s, $\hat{\hat{g}}^b$, and $\hat{\hat{f}}^b$, which allow us to make various statistical inference.

The standard deviations and *t*-statistics of $\hat{\alpha}$'s and $\hat{\beta}$'s are presented in the second and third columns of Table 2. We are particularly interested in the significance of $\hat{\beta}$'s. $\hat{\beta}_{11}$ and $\hat{\beta}_{22}$ are both significant and have the signs expected theoretically. Therefore, it is statistically correct to call $\hat{\hat{g}}$ the NAILO and $\hat{\hat{f}}$ the NAFEX. In contrast, neither $\hat{\beta}_{21}$ nor $\hat{\beta}_{12}$ are significant, although they have theoretically correct signs. This implies that the output gap does not play an important role in the net export function, and that the exchange rate gap does no plays an effective role in the Phillips curve. We

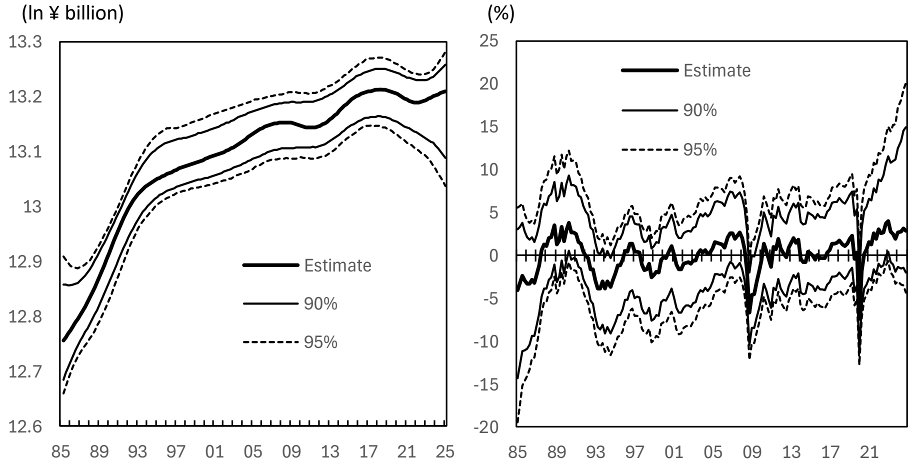
[2] In this exercise, it is assumed that both potential output and the real effective exchange rate are unaffected by changes in disturbances.

Table 2

Significance of the estimates of the Phillips curve and net export function

			Naïve		Elaborated	
		Estimate	S.d.	t-value	S.d.	t-value
$\Delta\pi$	$\hat{\beta}_{11}$	4.452	1.599	2.784	1.477	3.014
	$\hat{\beta}_{12}$	-0.001	0.003	-0.429	0.002	-0.532
	$\hat{\alpha}_{11}$	0.224	0.076	2.944	0.086	2.618
Δn	$\hat{\beta}_{21}$	-2.352	2.472	-0.952	2.030	-1.159
	$\hat{\beta}_{22}$	-0.015	0.004	-3.730	0.003	-5.040
	$\hat{\alpha}_{21}$	-0.291	0.070	-4.144	0.055	-5.329
	$\hat{\alpha}_{22}$	0.206	0.041	5.078	0.032	6.426

Notes: S.d. indicates the standard deviation of the estimate.



Sourses: The Cabinet office, the Bank of Japan.

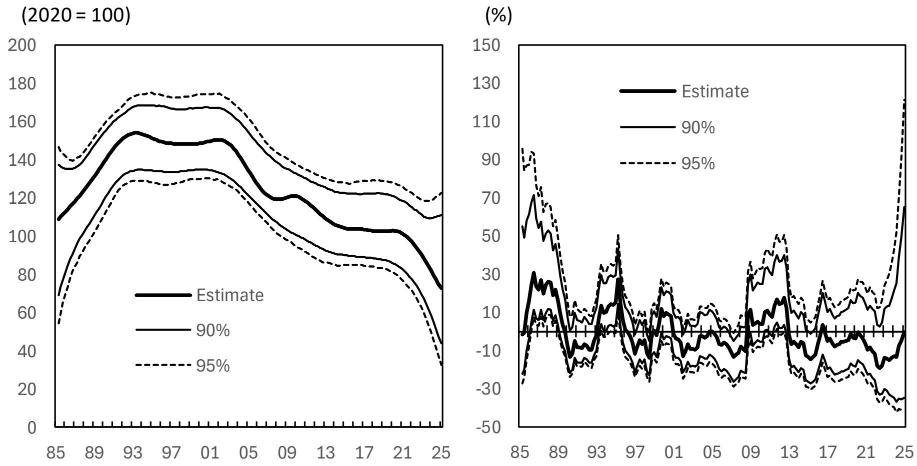
Figure 11. Naïve confidence intervals of the NAILO and output gap

see the robustness of these results below.

We can also construct confidence intervals (CIs) of the NAILO, $\hat{\hat{g}}$, and the NAFEX, $\hat{\hat{f}}$. To construct the 95% CI, for instance, we pick up the 2.5 percentile point and the 97.5 percentile point each quarter. Figure 11 presents the CIs of the NAILO and the associated output gap; Figure 12 shows the CIs of the NAFEX and the associated exchange rate gap. They are hopelessly wide, particularly at the end of the sample. We should be cautious when referring to the levels of the NAILO and the NAFEX, particularly in discussing their recent developments. We see how robust this result is below.

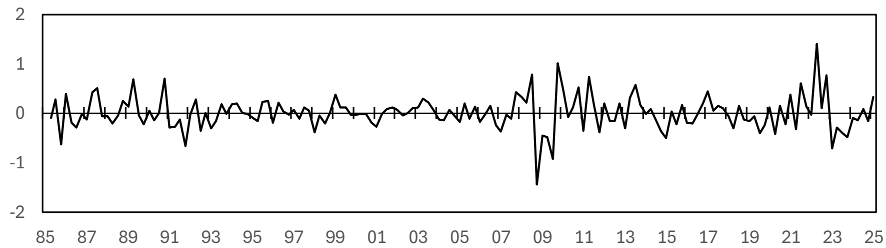
5.3. Elaborating the Bootstrap Distribution

One of the causes of wide confidence intervals is the naïve assumption we made about



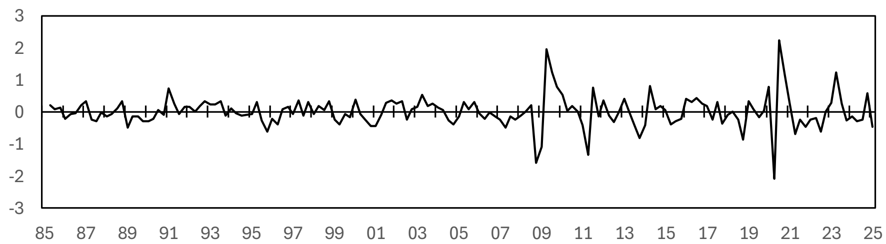
Sourses: The Cabinet office, the Bank of Japan.

Figure 12. Naïve confidence intervals of the NAFEX and exchange rate gap



Sourses: The Cabinet office, the Bank of Japan.

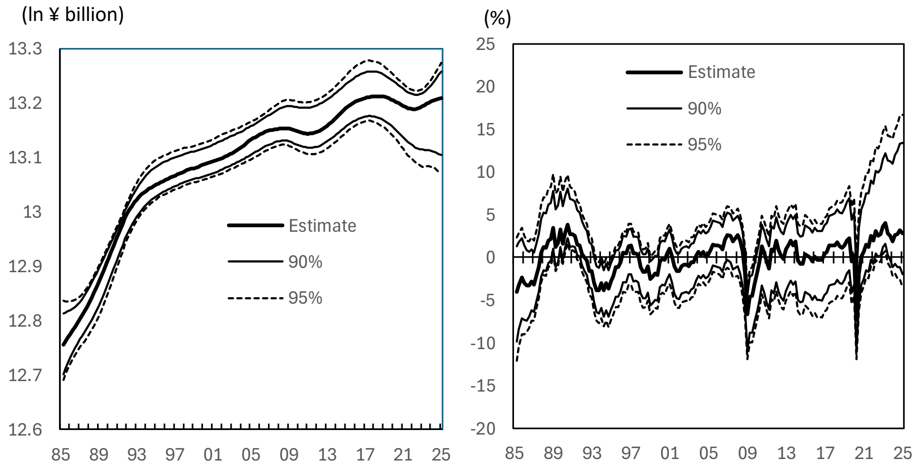
Figure 13. Approximate disturbances in the Phillips curve



Sourses: The Cabinet office, the Bank of Japan.

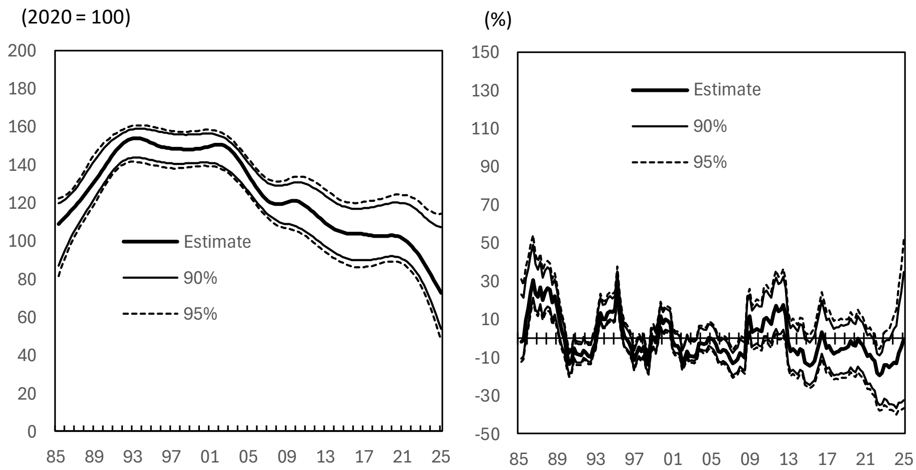
Figure 14. Approximate disturbances in the net export function

the empirical distribution from which we draw bootstrap disturbances ε_{1t}^{b} and ε_{2t}^{b} ($t \in \{1, \cdots, T\}$) in Step 2 of the general bootstrap procedure. Figures 13 and 14 display the developments of $\hat{\varepsilon}_{1t}$ and $\hat{\varepsilon}_{2t}$ over the sample period, respectively. Two points are notable. First, the shocks after the GFC are larger than those before on average. Second, there are extremely large shocks observed around the collapse of the Lehman Brothers and around the pandemic of COVID-19.



Sourses: The Cabinet office, the Bank of Japan.

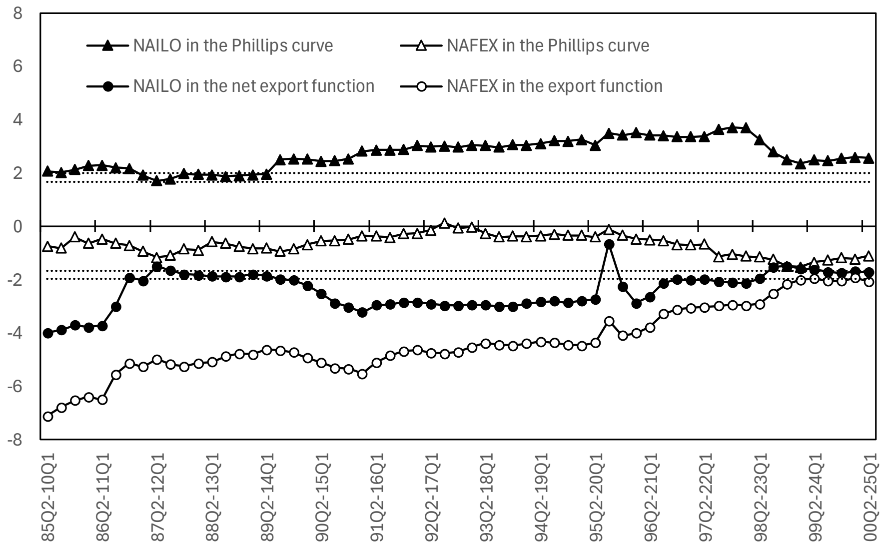
Figure 15. Elaborated confidence intervals of the NAILO and output gap



Sourses: The Cabinet office, the Bank of Japan.

Figure 16. Elaborated confidence intervals of the NAFEX and exchange rate gap

Taking the above points into consideration, we elaborate an empirical distribution from which we draw bootstrap disturbances: (i) the disturbances are divided before and after the GFC; (ii) the shocks that are three times larger than the standard deviation are treated as deterministic. Treatment (ii) prevents rare shocks from being drawn many times. The results are shown in Figures 15 and 16. The CIs are narrowed, but still too wide for practical use. The associated standard deviations and *t*-statistics of $\hat{\alpha}$'s and $\hat{\beta}$'s are presented in the fourth and fifth columns of Table 2, respectively. Significance of $\hat{\alpha}$'s and $\hat{\beta}$'s are slightly improved, but almost the same as obtained under the naïve assumption.



Note: The dotted lines indicate the leves of 1.960, 1.645, -1.645, and -1.960.

Sourses: The Cabinet office, the Bank of Japan.

Figure 17. t-values of the coefficients of the Phillips curve and net export function

5.4. Changing the Sample Period for Estimation

We use a rolling sample window to see the effects of the sample period on the estimates. The window size is 25 years (100 quarters). An initial window covers the period of 1985Q2 to 2010Q1; a final one covers the period of 2000Q2 to 2025Q1. We have 61 windows in total. We calculate *t*-values of $\hat{\beta}$'s for each window. We make the same elaboration on the empirical distribution as above.

The results are displayed in Figure 17. For reference, we also indicates the significance levels of two-sided 5% and 10% tests in the figure. All of the $\hat{\beta}$'s have theoretically correct signs for all of the sub-samples, that is, $\hat{\beta}_{11} > 0$ and $\hat{\beta}_{12}$, $\hat{\beta}_{21}$, $\hat{\beta}_{22} < 0$. As for the Phillips curve, despite sample change, the significance of the NAILO and NAFEX does not change, that is, $\hat{\beta}_{11}$ is significant, but $\hat{\beta}_{12}$ is insignificant. As for the net export function, the significance of the NAILO and NAFEX is changed, that is, $\hat{\beta}_{22}$ and $\hat{\beta}_{21}$ are both significant, despite sample change. The NAILO has double meanings, the output level neutral to net export as well as to inflation rates. Note, however, that their significance decreases from the pandemic of COVID-19. This exercise shows clearly that the COVID-19 pandemic has changed the relationship between the Japanese and global economy.

6. Conclusion

We generalize Hirose and Kamada's (2003) method of estimating potential output and propose a method of estimating multiple natural rates in a system of simultaneous equations. Unlike Hirose and Kamada, we provide the estimators of natural rates in analytical forms, which saves us from implementing tedious convergence calculation. We find that the estimators are closely related to the popular filter proposed by Hodrick and Prescott (1981). Our method is intuitive and accessible by many practitioners who are not familiar with advanced statistical methods like the Kalman filter.

As a numerical application, we apply our method to calculating potential output and the natural foreign exchange rate in Japan simultaneously. The former is called the non-accelerating inflation level of output, the NAILO, by Hirose and Kamada (2003). The estimation of the latter was not tried by them and is attempted in this paper for the first time. We call it the NAFEX in this paper. Both are important in evaluating a country's economic capability: The NAILO indicates its domestic productivity, while the NAFEX indicates its international competitiveness.

The result of estimation shows that the NAILO has been growing, but the NAFEX has experienced downward shift three times, due to Japan's financial system unease in the late 1990s, the global financial crisis in the late 2000s, and the COVID-19 pandemic in the early 2020s. While Japan suffered the long-lasting stagnation, emerging markets, particularly China, achieved tremendous development. Japan is growing potentially, but its growth is slower than emerging markets obviously. The decline in the NAFEX shows Japan's lost competitiveness in the global economy clearly.

We examine the uncertainty of our estimates by the bootstrap method formulated by Efron and Tibshirani (1993, ch.8). The result shows that the NAILO does not play a significant role in determining Japan's net export and that the NAFEX has no significant effects on Japan's inflation rates. It is also shown that their confidence intervals are too wide to use in practice. We find that the performance of the net export function has deteriorated since the pandemic of COVID-19. The precision of the NAFEX and thereby of the NAILO depends on that function. Thus, a remaining issue we should work on is to find an appropriate specification of the net export function that reflects changes in the structure of the global economy and in Japan's role in the world economy.

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