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Tetsutaro Hatakeyama, Onur Kesten, Morimitsu Kurino

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Institute for Economic Studies, Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan
ies-office@adst.keio.ac.jp
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Tetsutaro Hatakeyama
Graduate School of Economics, Keio University
t.hatakeyama@keio.jp

Onur Kesten
School of Economics, University of Sydney
onur.kesten@sydney.edu.au

Morimitsu Kurino
Faculty of Economics, Keio University
kurino@keio.jp

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Tetsutaro Hatakeyama† Onur Kesten‡ Morimitsu Kurino§

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†Keio University, Graduate School of Economics, 2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan; email: t.hatakeyama@keio.jp

‡University of Sydney, School of Economics, Australia; email: onur.kesten@sydney.edu.au

§Keio University, Faculty of Economics, 2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan; email: kurino@keio.jp

1 Introduction

In numerous real-world allocation scenarios, priorities serve as the cornerstone for distributing sought-after goods and services. This is evident across various sectors: healthcare professionals must arrange patient surgeries, government authorities must determine immigration priorities, educational institutions must rank applicants, and during crises like pandemics, critical resources such as vaccines and ventilators must be allocated according to predetermined criteria.

The vast literature on assignment problems often take priorities as a primitive given exogenously at the outset and focus on the design of institutions for equitable and efficient allocations in light of given priorities (see Sönmez and Ünver (2024a,b) for excellent surveys). Nevertheless, it is inevitable that carefully designed institutions on the premise of equity can only be fair inasmuch the underlying priorities are the result of a transparent, objective and consistent protocol.

In this paper, we are interested in the problem of *endogenously* designing priorities[1] to align with the primary objectives of a planner. Unlike situations where agents share a common attribute, such as test scores for university admissions or arrival times for service queues, prioritization becomes notably challenging when agents possess diverse attributes or belong to different identity groups. For instance, a major dilemma faced by policymakers during the COVID-19 pandemic pertained to determining the vaccination order among various groups of individuals in society including healthcare personnel, essential workers, the elderly individuals etc. The absence of consensus led to the emergence of varied rationing protocols in various countries and regions, highlighting the complexity of prioritization in heterogeneous populations (Pathak et al., 2024).

Similar examples are abundant. When scheduling surgeries at a hospital, prioritizing patients becomes imperative. Within each surgical category, scoring-based systems are often utilized to rank patients based on the urgency and severity of their conditions, facilitating the allocation of limited surgical resources effectively. In allocating research funds across disciplines, government bodies determine which applications to award grants. Each application undergoes review and ranking within its respective discipline, with a fixed total number of grants available. The challenge lies in deciding which researchers and disciplines should receive funding, taking into account the unique needs and priorities of each field. In allocating visas or green cards, applications from various job categories or nationalities are considered. Within each job category, candidates are evaluated based on merit, and a fixed number of visas or green cards are allocated. Allocating sabbatical positions within a university faculty presents similar complexities, as applicants come from diverse departments seeking leaves at the faculty level. Decisions regarding sabbatical awards are made centrally, with applicants within each department ranked against their peers. Here, the task is to prioritize individuals for sabbaticals, considering the academic contributions and needs of each department while maintaining fairness. Furthermore, affirmative action policies in school admissions

[1] Using the more technical terminology, we are pursuing the design of *choice functions* that meet the inherent goals of the mechanism designer.

highlight the importance of prioritizing students from various backgrounds, such as ethnicity and socioeconomic status. When faced with binary choices, schools aim to admit students from multiple applicant types to promote diversity and equity. However, in scenarios involving more than two groups, schools must also determine the composition of each class, balancing factors like academic merit and inclusivity to create a diverse student body that reflects the institution's values. These scenarios illustrate the multifaceted nature of prioritization in allocation problems, ranging from binary choices to complex, multi-group considerations. Throughout these allocation processes, the challenge lies in devising prioritization mechanisms that account for the heterogeneity of agents while ensuring fairness and efficiency.

In this paper, we aim to elucidate the principles behind endogenously designing priorities in allocation problems, offering insights into assignment problems at the intersection of economics, management policy, and social justice. Our formal model builds on a problem faced by universities in Japan to assign students from various faculties to exchange programs. Keio University in Tokyo specifically consulted the third author to develop a protocol to solve this problem. This application constitutes the basis for our canonical model and the terminology we adopt in the rest of the paper.

Universities all over the world have exchange programs in which millions of students with various majors study abroad at partner universities.[2] The common allocation method for assigning students to exchange programs is a choice based on priority, also commonly referred as a *serial dictatorship* mechanism in the assignment literature.[3] Before applying this mechanism, a complete priority order must be constructed. While GPA is a natural basis for determining priorities, the process is further complicated by the fact that students actually come from different faculties with different class sizes and different grading metrics. Universities often address this complication by resorting to ad-hoc internal decision processes. For example, universities such as Keio usually hold face-to-face interviews with each student to prioritize all exchange program applicants. Nonetheless, the sheer administrative cost of conducting such interviews led to the reform in which the centralized prioritization process proposed in this paper was adopted.

We formulate a *prioritization problem* as the problem of assigning a set of agents to an exogenously given K priority groups of arbitrary sizes. Each agent prefers to be in a higher (lower-indexed) group. Each agent inherently belongs to a category, e.g., the faculty a student is affiliated with.[4] Existing priorities are only partial in that only agents from the same category can be ranked, i.e., any two agents across different categories are incomparable. A systematic solution to this problem is a prioritization *protocol* that returns a complete and transitive (not necessarily linear) priority over agents, or equivalently, an assignment of agents to K priority groups (see Figure 1 for an illustration). When agents are otherwise indistinguishable, to achieve fairness in an ex ante sense we also allow for prioritization protocols that allow for lotteries. Our goal is to identify

[2]For example, see the webpage `https://www.statista.com/chart/3624/the-countries-with-the-most-students-studying-abroad/` for details (accessed October 23, 2024).

[3]See, for example, Svensson (1994; 1999) for a first axiomatic account on this method.

[4]We relax this assumption later in the paper and allow for multi-category membership.

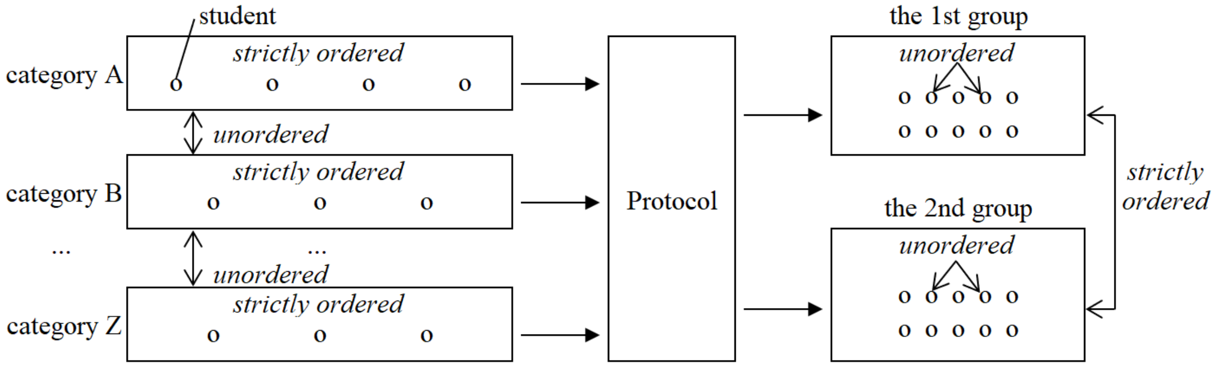

Figure 1: A prioritization problem

Note: Each "o" represents a student. At the outset (on the left), students are ordered only within the same category. After applying a protocol (on the right), all students are weakly ordered. Whereas students within the same group have the same priority, students in the 1st group have strictly higher priority than those in the 2nd group.

the types of prioritization protocols that emerge if one insists on minimal forms of efficiency and fairness. Importantly, when prioritization becomes a central design issue, it expands the concept of *choice rule* that have attracted much attention in recent matching theory and market design (e.g., see Alva and Dogan (2023) for a survey).[5]

As a minimal form of efficiency, we consider *non-wasteful* assignments where no student is assigned to a group while a higher-priority group has a vacancy. An assignment *respects priorities* if no student is assigned to a higher priority group than a higher priority student from the same category. Our first result is a complete characterization of the set of non-wasteful deterministic assignments that respect priorities (Proposition 1).

We consider two plausible metrics for gauging the fairness of assignments depending on how one evaluates two students from different categories. A natural starting point is an *absolute* priority metric, which considers the rank of the student within her category, regardless of the category's size. In many applications, population size is a major determinant for the resource distribution. For example, in public health funding, resources like vaccines, hospital beds, medical staff, and funding are distributed proportionally to the size of different areas or groups. Government funding for public schools is often allocated based on the number of students. Public services such as police, fire departments, and public transportation allocate resources based on the population size of different districts or neighborhoods. Seats in legislative bodies (e.g., House of Representatives in the US) are often allocated based on population size.[6] Motivated by the abundance of applications

[5] A choice rule is a systematic way of selecting subsets from a given set of agents. A choice rule essentially solves a special case of a prioritization problem with $K = 2$. Whereas existing choice rules are deterministic, our analysis also allows for stochastic choices. See the related literature section.

[6] Other examples include scholarships and quota assignment in educational institutions (based on the population size of different regions or demographic groups), electoral systems (e.g., political party seats allocated based on the proportion of votes they receive), which indirectly relates to the population size supporting each party, business and marketing (e.g., companies allocate marketing budgets based on the size of different consumer groups), social welfare programs (e.g., programs like food assistance, unemployment benefits, and social security is allocated based on the number of eligible individuals in different areas or groups), housing assistance (e.g., housing vouchers and

where population size matters, we define a *relative* priority metric. It refines the absolute priority metric by considering the percentiles agents in which agents lie with respect to their categories, i.e., rank is defined relative to the size of the category.

Our central fairness notion for prioritization protocols is *equal treatment of students with equal priority metrics*: any pair of students who have the same priority metric must receive the same probability of getting assigned to a given priority group.

Our main result shows that two types of prioritization protocols stand out when the planner requires non-wastefulness, respecting priorities, consistency[7] and equal treatment of equals while using randomization only to break ties. These protocols assign probability distributions over deterministic assignments. Each deterministic assignment in the support is obtained by allowing students to successively compete to fill the positions in the priority groups starting with the highest priority group moving down to the lowest. Each student is equipped with a "filling speed." Students fill positions based first on priority and then speed. When two students from two different categories are tied to fill a position in all relevant characteristics, a tie-breaker determines who gets the position.

When fairness is pursued in an absolute sense, the protocol has to be a *uniform prioritization protocol*: each student's speed is identical and a uniformly random tie-breaker is used (Theorem 1). On the other hand, when fairness is pursued in a relative sense, e.g., students in similar percentiles in their respective categories should be treated the same, the protocol has to be a *proportional prioritization protocol*: each student's speed is proportional to the size of her category and a uniformly random tie-breaker is used (Theorem 2).

It is important to emphasize that the protocols we characterize "minimally" rely on randomization. More specifically, the filling algorithms that constitute the backbone of the two protocols do not treat the positions in the priority groups as a continuum mass of students. Instead, protocols allow students to fill positions one by one to generate deterministic assignments and randomization is applied only "locally" to students that are otherwise indistinguishable.

A major challenge in our approach is that the fairness notion requires making comparisons across deterministic assignments, e.g., do similar agents appear in similar groups with same frequency? This makes working with random assignments (i.e., assignment matrices showing the probabilities with which category members end up in which group) rather intractable. To overcome this challenge, one needs to directly construct lotteries over deterministic assignments rather than resorting to standard machinery of random assignment decomposition, e.g., Birkhoff von Neuman type of methods (Birkhoff, 1946; von Neumann, 1953) are inapplicable.[8] In this sense, despite

subsidies are distributed based on the population size within different regions), disaster relief (e.g., resources for disaster response, such as emergency shelters, medical supplies, personnel, and relief funds are allocated based on the affected population size), environmental conservation (e.g., funding for conservation projects can be allocated based on the population size of endangered species or the extent of habitat areas) and pollution control (e.g., resources for pollution control measures may be distributed based on the population density of affected areas).

[7]A protocol is *consistent* if its assignment is robust to the departure of some of the students from the problem.

[8]This can be seen akin to uncovering the subtle relationships between ex post vs. ordinally efficient assignments

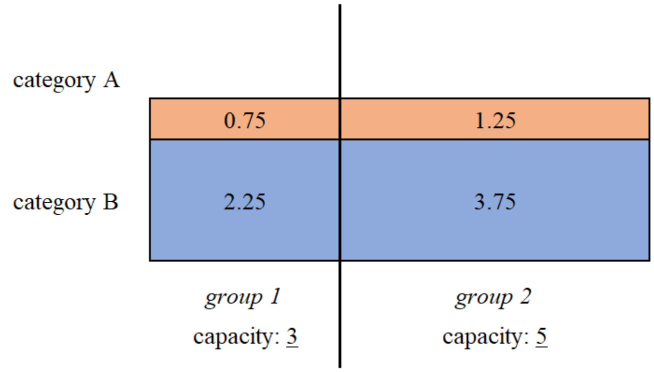

Figure 2: Continuum version of the assignment problem

Note: A set of agents each belonging to one of two categories need to be divided into two groups. Agents (a^1, a^2) are members of category A and agents $(b^1, ..., b^6)$ are members of category B. Group 1 will admit three agents while group 2 five. Group assignments are separated by the thick black boundary. The numbers in each cell represent capacity shares assigned to each category based on category size.

the use of similar terminology,[9] our protocols are markedly different than well-known stochastic rules such as the probabilistic serial (Bogomolnaia and Moulin, 2001) or competitive equilibrium (Hylland and Zeckhauser, 1979) that have attracted much attention in the assignment literature. We next illustrate this difficulty via an example.

Lotteries in lieu of random assignments: An illustration

An alternative and simpler approach to solve a prioritization problem could be based on tackling a continuum version of the problem in two-steps akin to the standard methodology in probabilistic assignment problems: First, assign agents probability shares over priority groups, and then decompose the resulting random assignment matrix into a lottery over deterministic assignments. While this idea sounds simpler and natural, it is incompatible with equal treatment of equals. The following example illustrates this point.

Consider the following simple prioritization problem (Figure 2) where category A consists of two agents (a^1, a^2) and category B of six agents $(b^1, ..., b^6)$, where only agents within each category are ranked (with lower index representing higher priority). The collection of agents are to be assigned into two priority groups where group 1 is of capacity of three, and group 2 is of capacity of five, i.e, $K = 2$. Suppose we convert this problem to a continuum of agents and capacities, where masses correspond to aggregate probability shares in group assignments. Consider an intuitive aggregate random allocation where the capacity of each priority group is allocated to categories

or ex ante vs ex post stable assignments (Abdulkadiroğlu and Sönmez, 2003; Kesten and Ünver, 2015; Aziz et al., 2022). However, the task at hand is more involved since in our context ex ante fairness does not necessarily imply ex post fairness as demonstrated in the upcoming illustration.

[9] Bogomolnaia and Moulin (2001) use the term *eating speed* to represent continuum shares of goods consumed by an agent. In their model, these eaten amounts are probability shares that need to be used to for a final decomposition of the random assignment into a lottery over deterministic assignments.

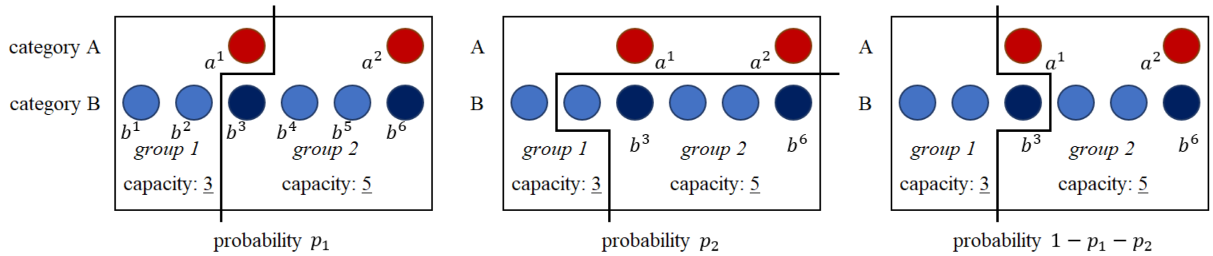

Figure 3: Discrete decompositions of the assignment

The continuum assignment in Fig 2 leads to one of the three assignments depicted in the figure. In both of figures, the circles indicate agents in categories and agents are ordered from left to right in each category. The agents left to the lines are assigned to group 1 in both of the figures. The left assignment is selected with probability p_1, the middle with probability p_2, and the right with probability $1 - p_1 - p_2$. To be a decomposition of the aggregate random assignment in Figure 2, the expected mass of category A students assigned to group 1 should be 0.75, i.e., $p_1 + 2p_2 = 0.75$.

proportionally to the size of each category (see Figure 2). For example, giving the three slots for group 1 is apportioned into a mass of 0.75 for category A with a population size of 2 and a mass of 2.25 for category B with a population size of 6.[10]

Clearly, this is merely an aggregate random allocation that does not yet specify which agents are assigned to which groups. Since the actual implementation of this random assignment requires computation of outcome-equivalent deterministic assignment, one would then resort to a lottery decomposition over assignments that respect priorities where a higher priority student falls into a weakly higher group than a lower priority student from the same category. Figure 3 shows all possible decompositions over such assignment: the left assignment chosen with probability p_1, the middle assignment with p_2, and the right assignment with $1 - p_1 - p_2$, where $p_1 + 2p_2 = 0.75$ (eq. 1) by feasibility. Although the proportional allocation appears to be fair, the lottery it induces *cannot* respect the equal treatment of students with equal relative priority positions.[11] To see this, consider agent a^1 of category A (the red circle in Figure 3) and agent b^3 of category B (the black circle). Since they share the same *relative priority positions* in their respective categories, i.e., they are both in the 50th percentile in their respective populations putting them both on the margin of getting into group 1, both of them should be assigned with equal probability to each group. Since a^1 is assigned to group 1 in the left and the middle assignments whereas b^3 in the right, we must have $p_1 + p_2 = 1 - p_1 - p_2$. Together with (eq. 1), this implies that the unique candidate for a fair decomposition is $(p_1, p_2, 1 - p_1 - p_2) = (0.25, 0.25, 0.5)$. Now consider agents a^2 and b^6, who also share the same relative priority positions (i.e., both in the 0th percentile in their respective populations). Whereas agent a^2 is selected for group 1 with positive probability, b^6 never makes

[10] For example, if we simply assume each student in each category has equal chance of being selected to a group, this mass distribution implies that a randomly selected member of each group is three times more likely to be a member of category B whereas a randomly selected agent from a given category is 1.67 times more likely to fall into the second group.

[11] In general one can construct a similar example showing the impossibility equal treatment of equals for *any* given priority metric.

the cut for group 1. This implies that there is no fair decomposition of the random assignment in Figure 2. More generally, transforming a seemingly fair random assignment based on a continuum interpretation may result in an unfair outcome in its lottery-equivalent.

We next illustrate what the uniform and proportional protocols would recommend for this example. The uniform protocol which embodies the principle of absolute equal treatment of equals would insist that agents a^1 vs. b^1 and a^2 vs. b^2 should be treated equally. This entails that the left and middle assignments are selected with equal probability while the right assignment is never selected, i.e., $p_1 = p_2 = 0.5$. The proportional protocol, on the other hand, embodies the principle of relative equal treatment of equals and would insist that agents a^1 vs. b^3 and a^2 vs. b^6 should be treated equally. This entails that the left and right assignments are selected with equal probability while the middle assignment is never selected, i.e., $p_1 = 0.5$ and $p_2 = 0$.

Related Literature

Since the pioneering work of Shapley and Scarf (1974) who first studied indivisible goods allocation problems without monetary transfers, many studies including assignment problems (possibly with prior claims) (Abdulkadiroğlu and Sönmez, 1999) and school choice (Abdulkadiroğlu and Sönmez, 2003), kidney exchange (Roth et al., 2004), reserve design (Dur et al., 2018; Pathak et al., 2024, 2023) predominantly assume exogenously given strict priority orders as primitives of their model.[12]

A series of papers deal with complete but weak priorities in school choice (Erdil and Ergin, 2008; Erdil and Kumano, 2019) and exchange economies (Balbuzanov and Kotowski, 2019).[13] In a contemporaneous to ours, Anno and Takahashi (2023) apply core concepts to priority mechanisms with incomplete priorities while characterizing a deterministic rule similar to our proportional protocol with their fairness notions.[14]

When the number of priority groups is restricted to be only two ($K = 2$), a prioritization problem can be viewed similarly to an apportionment problem (Young, 1995; Evren and Khanna, 2021), e.g., apportionment of representation among political constituencies or the problem of designing choice rules. In the former type of problems, proportional apportionment is accepted as the primary method while the goal is determining the suitable rounding of quotients. In the

[12]There are notable exceptions to this tradition. Álvarez and Medina (2024) consider a school choice problem with students' transferable characteristics, the source of higher priority, and introduce an algorithm which determines both priority and student-school matching simultaneously. Recently, starting from a baseline priority order called the order-of-merit list, Greenberg et al. (2024) axiomatically characterize the cumulative offer mechanism together with a particular choice rule in the context of cadet-branch matching. In reassignment situation of teachers to schools, Combe et al. (2022) design priorities (or choice rules) over experienced and non-experienced teachers, as well as the allocation mechanism, to achieve certain distributional objectives.

[13]In particular, Balbuzanov and Kotowski (2019) study exchange economies in which priority may be partial, but their focus is much different from ours; they define new concepts of cores based on property rights and characterize them by a generalized top trading cycles algorithm.

[14]The third author of this paper was consulted to design the problem faced by the Keio University. He subsequently shared it with Hidekazu Anno. Since then each group worked separately while occasionally exchanging ideas. Anno and Takahashi (2023)'s model focuses on unit capacities of priority groups.

latter type of problem, a choice rule specifies which subset of agents should be chosen from a given set of agents. A number of papers have dealt with axiomatic characterization of appealing choice rules for various applications (Echenique and Yenmez, 2015; Imamura, 2020; Sönmez and Yenmez, 2022; Pathak et al., 2024). [15] In addition to dealing only with the case of $K = 2$, as far as we are aware, all choice rules in this literature are deterministic whereas we also allow for randomization. In recent work, Imamura and Tomoeda (2023) show that for deterministic choice rules equal treatment of equals is effectively incompatible with commonly-used axioms in that literature such as substitutability and size monotonicity.[16] This result provides further justification to our use of randomization only when absolutely necessary. More generally, the methodological tools we develop here can be useful for future exploration of stochastic choice rules.

The structure of this paper is as follows. Section 2 introduces the model and three equity axioms. Section 3 characterizes the prioritization protocol. Section 4 endogenizes the number of priority groups. Section 5 discusses the modeling and an application. Section 6 concludes the paper. All the proofs can be found in the appendix.

2 Formal Analysis

2.1 Model

We consider the problem of dividing a set of students into ordered groups. The groups are interpreted as priority tiers or objects (e.g., time slots for vaccination) over which all agents have the same preferences. Each agent initially belongs to one category (race, occupation, or department) and will be assigned to exactly one group. The set of **(priority) groups** is $K = \{1, \dots, \bar{k}\}$. Group k has **capacity** $q^k > 0$, which is the maximum number of agents assigned to group k. Let $q = (q^k)_{k \in K} \in \mathbb{N}^K$ be the list of capacities. To allow arbitrary numbers of categories and agents, we first define the sets of categories and agents that will potentially be involved. Let $\mathcal{C}$ be a set of **categories,** c or d being generic elements of $\mathcal{C}$.[17] Let $\mathcal{S}_c$ be the set of agents, or **students**, belonging to category $c \in \mathcal{C}$. For simplicity, let $\mathcal{C}$ and $\mathcal{S}_c$ be countable. Each student belongs to only one category, that is, for each pair $c, c' \in \mathcal{C}$ with $c \neq c'$, $\mathcal{S}_c \cap \mathcal{S}_{c'} = \emptyset$. Within each category $c \in \mathcal{C}$, there is a linear order $\succeq_c$ on $\mathcal{S}_c$, which means that $s \succeq_c s'$ and $s' \succeq_c s$ imply $s = s'$. We call $\succeq_c$ a **priority**. Across distinct categories students are not ordered.

Let $C \subsetneq \mathcal{C}$ and $S_c \subsetneq \mathcal{S}_c$ be finite sets of categories and students of category c. A (prioritization) **problem** is a list $P = (C, \{S_c\}_{c \in C}, K, q)$. For each problem, $|C| \geq 2$, $|K| = \bar{k} \geq 2$, and $\sum_{k \in K} q^k \geq$

[15] Similar to some of these papers, we also consider the axiom of consistency for protocols, which is similar to consistency for choice rules, when $K = 2$ (Aygün and Sönmez, 2013; Chambers and Yenmez, 2017; Imamura and Tomoeda, 2023). Also see by Moulin (1985) and Alva and Dogan (2023) for two excellent surveys on choice rules.

[16] More precisely, Imamura and Tomoeda (2023) show that any individual for whom equal treatment applies must either be always selected or never selected, which would violate either feasibility or non-wastefulness in our context.

[17] In the case of universities, a category is a faculty to which a student belongs.

$\sum_{c\in C}|S_c|$. The last inequality is assumed without loss of generality.[18] Students have a common and strict preferences over groups: group 1 is the most preferred, group 2 the second most preferred, and so on. Being in a given group is interpreted as having a more favorable treatment than any member in a higher-indexed group, e.g., receiving a service or privilege before members of later groups.

An (priority) **assignment** for problem P is a function $\mu : \bigcup_{c\in C} S_c \to K$ such that for each $k \in K$, $|\mu^{-1}(k)| \leq q^k$. Here a student $s \in \bigcup_{c\in C} S_c$ is assigned a group $\mu(s) \in K$; and $\mu^{-1}(k)$ is the set of students assigned to group k. For simplicity, we denote $S^k = \mu^{-1}(k)$. Note that for a collection $\{S^k\}_{k\in K}$ of subsets of students, we have $\bigcup_{k\in K} S^k = \bigcup_{c\in C} S_c$ and for each pair $k, k' \in K$ with $k \neq k'$, $S^k \cap S^{k'} = \emptyset$. That is, an assignment partitions the student set $\bigcup_{c\in C} S_c$ whose components are indexed by $k \in K$. We simplify the notation $\{S^k\}_{k\in K}$ to $\{S^k\}$. Note that $\mu(s) = k$ if and only if $s \in S^k$. Note that when each group has a unit capacity, an assignment induces a strict priority over all students.

Following the literature on market design, we focus on assignments satisfying two standard requirements. The first requirement is a weak notion of efficiency. An assignment μ is **non-wasteful** at a problem P if each group's capacity is unfilled only if no student is assigned to a worse group. That is, for each $k \in K$, $|\mu^{-1}(k)| < q^k$ implies that for each $k' > k$, $\mu^{-1}(k') = \emptyset$. The second requirement is a weak notion of fairness within each category. An assignment μ **respects priorities** at a problem P if for each category $c \in C$, when a student $s \in S_c$ has higher priority than a student $s' \in S_c$, then the group to which s is assigned is not worse than the group to which s' is assigned. That is, for each $c \in C$ and each pair $s, s' \in S_c$ with $s \succeq_c s'$, we have $\mu(s) \leq \mu(s')$.

2.2 Random assignments

We consider a **lottery** $\lambda = (\lambda_\mu)$ that is a probability distribution over (deterministic) assignments. We denote by σ_μ the probability that the lottery places on the assignment μ. The resulting probability distribution for a student s over groups, $\sigma_s = (\sigma_{s,k})$, is called a **random allocation.** $\sigma_{s,k} \in [0,1]$, is the probability that student s is assigned group k. A **random assignment** is a matrix $\sigma = (\sigma_{s,k})_{s\in\bigcup_{c\in C} S_c, k\in K}$, a collection of random allocations for all students, such that for each $s \in \bigcup_{c\in C} S_c$, $\sum_{k\in K}\sigma_{s,k} = 1$; and for each $k \in K$, $\sum_{s\in\bigcup_{c\in C} S_c}\sigma_{s,k} \leq q^k$. An assignment can be expressed as a random assignment with each cell being 0 or 1. Clearly, each lottery induces a random assignment. That is, given a lottery $\lambda = (\lambda_\mu)$, its induced random assignment is calculated as $\sigma = \sum_\mu \lambda_\mu \sigma(\mu)$ where $\sigma(\mu)$ is a random assignment representing an assignment μ. Conversely, a random assignment is induced by some lottery (Birkhoff, 1946; von Neumann, 1953; Budish et al., 2013).

[18]Because any problem $(C, \{S_c\}_{c\in C}, K, q)$ with $\sum_{k\in K} q^k < \sum_{c\in C}|S_c|$ is equivalent to another problem $(C, \{S_c\}_{c\in C}, \bar{K}, \bar{q})$ with $\sum_{k\in\bar{K}} \bar{q}^k \geq \sum_{c\in C}|S_c|$ where $\bar{K} = K + 1$ and if $k \leq K$, $\bar{q}^k = q^k$; if $k = K+1$, $\bar{q}^k > \sum_{c\in C}|S_c| - \sum_{k\in K} q^k$.

Finally, we formulate protocols for prioritization. A **protocol** is a function φ that associates a lottery $\varphi(P)$ to each problem P. When a student is assigned a group k with probability 1, we say that her assigned group is **deterministic**. In this case, we simply write $\varphi_s(P) = k$. We say that a protocol satisfies non-wastefulness or respect of priorities when it can be represented as a probability distribution over assignments that are non-wasteful or respect priorities.

In practice, random assignments may not be desirable.[19] Motivated by this observation, we introduce a simple measure over protocols: A protocol φ is **less randomized than** another protocol φ' if, for *each* problem P, the number of students whose assigned groups are non-deterministic is smaller in $\varphi(P)$ than in $\varphi'(P)$, except when $\varphi'(P) = \varphi(P)$. Similarly, among a class of protocols, a protocol is the **least randomized** if it is less randomized than any other protocols within the class.

2.3 Non-wasteful and fair deterministic assignments

We introduce a family of intuitive algorithms to find assignments that are non-wasteful and respect priorities: Let $P = (C, \{S_c\}_{c\in C}, K, q)$ be given. The algorithm is parameterized by two variables (i) a **filling speed** $\omega = (\omega_s)_{s\in\bigcup_{c\in C} S_c}$ where for each student $s \in \bigcup_{c\in C} S_c$, we have $\omega_s \in \mathbb{R}_{++}$, and (ii) a **tie-breaker** $\triangleright$, a partial binary relation defined on $\bigcup_{c\in C} S_c$: We imagine the process taking over time. In step 1, at time 0 the highest-priority student s_c in each category $c \in C$ starts to fill a position with speed ω_{s_c}. Among them, the fastest student who finishes filling positions, i.e., the smallest number in $\{1/\omega_{s_c}\}_{c\in C}$, is assigned the most preferred group. Then, we move to step 2 by reducing the capacities of assigned groups by the number of assigned students, and by removing the assigned students and groups with the capacity becoming zero. Then, in step 2, the highest-priority student s_c, among those who remain in each category $c \in C$, starts to fill a position with speed ω_{s_c}. Among them, the fastest student who finishes filling positions with the total amount of one is assigned the most preferred group.

In this family of algorithms, there might be multiple students who finish filling positions at the same time. For this reason, the assignment of groups to these students uses a tie-breaker $\triangleright$. For each pair $s, s' \in \bigcup_{c\in C} S_c$ with the same total time spent up to $s \in S_c$ and $s' \in S_{c'}$ in order of priority, they are ordered without ties. That is, when $\sum_{s'':s''\succeq_c s} 1/\omega_{s''} = \sum_{s'':s''\succeq_{c'} s'} 1/\omega_{s''}$, either $s \triangleright s'$ or $s' \triangleright s$.

Algorithm $A(\omega, \triangleright)$**.** Given $P = (C, \{S_c\}_{c\in C}, K, q)$, each student is progressively assigned to a group in continuous time starting at time 0.

Step 1. The highest-priority student in each category c, say s_c, fills 1 unit of position at his own speed ω_{s_c}. The set of these students is $S(1)$. We set the filled amount of student s_c at the beginning of the step to be $e_{s_c}(1) = 0$. The first students in $S(1)$ who finish filling exactly 1 units, say $\{s'\}$,

[19] For instance at Keio University in the story of subsection 4.2, we first recommended the well-known random priority protocol but professors from other fields refused our recommendation.

are those with the fastest speeds $\{\omega_{s'}\}$. Denote the set of such students by $S'(1)$.[20] Following the order over these students $\triangleright$, a student is successively assigned the most preferred group one by one.[21] This occurs at time $t'(1) = \min_{s' \in S(1)} \frac{1}{\omega_{s'}}$. We remove the students in $S'(1)$ from the problem, and reduce the capacity of group k to $q^k(2)$ by the number of filled positions.[22] If group k's capacity is $q^k(2) = 0$, remove such a group. The filled amount of positions for student $s \in S(1) \backslash S'(1)$ is updated to $e_s(2) = \omega_s/\omega_{s'}$.

Steps $j \geq 2$. The highest-priority student among those remaining in each category c, say s_c, fills $(1 - e_{s_c}(j))$ unit of positions at his own speed ω_{s_c}. The set of these students is $S(j)$.[23] Each student s_c in $S(j)$ would finish filling positions at time $\left(t'(j-1) + \frac{1-e_s(j)}{\omega_s}\right)$ with the total filled amount being $e_s(j) + (1 - e_s(j)) = 1$. Take students at the fastest time in finishing filling positions among those in $S(j)$. Denote the set of such students by $S'(j)$. Following the order over these students $\triangleright$, a student is successively assigned the most preferred group among those remaining one by one.[24] This occurs at time $t'(j) = \min_{s' \in S(j)} \left(t'(j-1) + \frac{1-e_{s'}(j)}{\omega_{s'}}\right)$. We remove the students in $S'(j)$ from the problem, and reduce the capacity of groups k to $q^k(j+1)$ by the number of filled positions.[25] If a group k's capacity is $q_k(j+1) = 0$, remove such a group. The filled amount of positions for student $s \in S(j) \backslash S'(j)$ is updated to $e_s(j+1) = e_s(j) + \frac{1-e_{s'}(j)}{\omega_{s'}} \omega_s$.

Example 1 (Execution of the algorithm)**.** There are two categories A and B so that $C = \{A, B\}$. The set of students of category A is $S_A = \{a^1, a^2, a^3, a^4\}$, and that of category B is $S_B = \{b^1, b^2\}$. Category A's priority is $\succeq_A$: a^1, a^2, a^3, a^4, while category B's priority is $\succeq_B$: b^1, b^2. There are two priority groups, 1 and 2, whose capacities are $(q^1, q^2) = (3, 3)$.

Consider an algorithm $A(\omega, \triangleright)$ where the filling speed is $\omega = (\omega_{a^1}, \omega_{a^2}, \omega_{a^3}, \omega_{a^4}, \omega_{b^1}, \omega_{b^2}) = (2, 2, 1, 1, 1, 2)$ and a tie-breaker is $a^2 \triangleright b^1$.

Step 1. Students $S(1) = \{a^1, b^1\}$ fill 1 units of positions. Since a^1 is faster ($\omega_{a^1} = 2 > 1 = \omega_{b^1}$), we assign a^1 to group 1 and reduce the capacity of group 1 by 1 ($q^1(2) = 2$). Set $e_{b^1}(2) = \omega_{b^1}/\omega_{a^1} = 0.5$.

Step 2. Students $S(2) = \{a^2, b^1\}$ fill positions. Since they finish at the same time ($\frac{1-e_{b^1}(2)}{\omega_{b^1}} = 0.5 = \frac{1-e_{a^2}(2)}{\omega_{a^2}}$), they are assigned to groups as follows: Since $a^2 \triangleright b^1$, we first assign a^2 to group 1 and then assign b^1 to group 1. We remove group 1.

Steps 3, 4, and 5. Remaining students are assigned to group 2.

[20] $S'(1) = \{s \in S(1) : \frac{1}{\omega_s} = \min_{s' \in S(1)} \frac{1}{\omega_{s'}}\}$

[21] Assign a student s' in $S'(1)$ the priority group $k(s') = k$ where k solves the following: $s' \in \{\text{top } \sum_{k' \leq k} q^{k'}(1) \text{ in } \triangleright|_{S'(1)}\}$ and $s' \notin \{\text{top } \sum_{k' < k} q^{k'}(1) \text{ in } \triangleright|_{S'(1)}\}$.

[22] $q(2) = (q^k(2))_{k \in K} = (q^k(1) - |\{s \in S'(1) : k(s) = k\}|)_{k \in K}$

[23] $S(j) = (S(j-1) \backslash S'(j-1)) \cup \{s \in S \backslash \bigcup_{j' < j} S(j') : \exists s' \in S'(j) \text{ with } c(s') = c(s) \text{ and } \nexists s'' \in S \backslash \bigcup_{j' < j} S(j') \text{ with } s'' \succeq_{c(s)} s\}$

[24] Assign a student s' in $S'(j)$ the priority group $k(s') = k$ where k solves the following: $s \in \{\text{top } \sum_{k' \leq k} q^{k'}(j) \text{ in } \triangleright|_{S'(j)}\}$ and $s \notin \{\text{top } \sum_{k' < k} q^{k'}(j) \text{ in } \triangleright|_{S'(j)}\}$.

[25] $q(j+1) = (q^k(j+1))_{k \in K} = (q^k(j) - |\{s \in S'(j) : k(s) = k\}|)_{k \in K}$

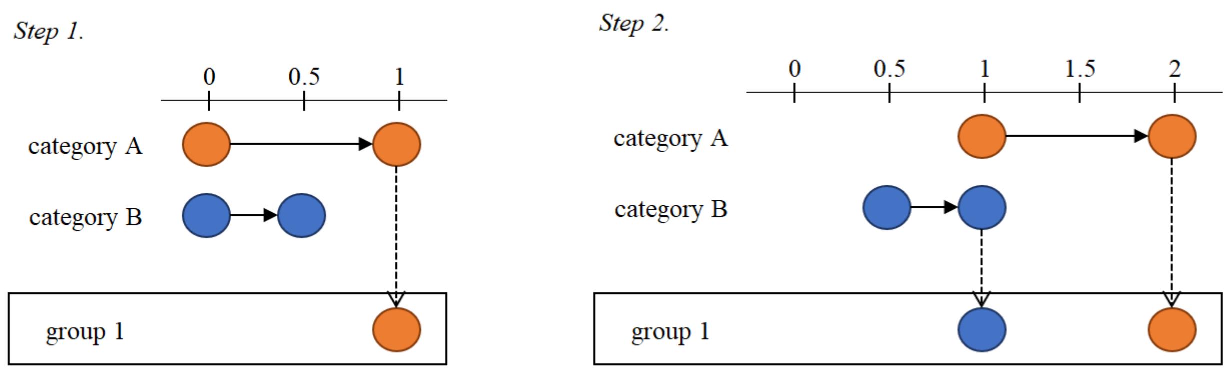

Figure 4: Execution of the algorithm in Example 3

The resulting assignment μ is

$$\mu = \begin{pmatrix} 1 & 2 \\ \{a^1, a^2, b^1\} & \{a^3, a^4, b^2\} \end{pmatrix}$$

We see that this assignment μ is non-wasteful and respects priorities. $\Diamond$

The family of proposed algorithms fully characterize the assignments that are non-wasteful and respect priorities as follows.

Proposition 1. *For each problem P and each assignment μ that is non-wasteful and respects priorities at P, there is a pair $(\omega, \triangleright)$ such that algorithm $A(\omega, \triangleright)$ finds μ. Conversely, for each problem P and each pair $(\omega, \triangleright)$, the assignment that algorithm $A(\omega, \triangleright)$ finds is non-wasteful and respects priorities at P.*

In other words, the collection of assignments obtained by a family of our algorithms coincide with that of assignments that are non-wasteful and respect priorities. Since the algorithm selects an assignment at any problem, Proposition 1 also guarantees the existence of such assignments.

We focus on random assignments placing positive probabilities over assignments that are non-wasteful and respect priorities. By Proposition 1, these random assignments can be obtained by a lottery over assignments achieved by the algorithm under some parameters $(\omega, \triangleright)$. Hence, we sometimes represent a lottery as a distribution over parameters, that is, $\sigma(P) = (\sigma_{(\omega^\ell, \triangleright^\ell)}(P))_{\ell=1}^{L}$.

2.4 Axioms: Consistency and fairness

To obtain a systematic way of assigning priorities to students, we require the protocol to be *consistent*. We need to allow *probabilistic* protocols in the requirement. Let a problem $(C, \{S_c\}_{c \in C}, K, q)$, a group $k \in K$, an assignment μ, and a subset of categories $D \subset C$ be given. We call $q^k_{D,\mu} = q^k - |\bigcup_{c \in C \setminus D} \mu^{-1}(k) \cap S_c|$ a **reduced capacity**, and $(D, \{S_c\}_{c \in D}, K, (q^k_{D,\mu}))$ a **reduced problem** of $(C, \{S_c\}_{c \in C}, K, q)$ with respect to μ. We require a consistent protocol to satisfy the following:

for each reduced problem, each student is assigned to the same group as the one to which she was assigned in the original problem, as long as her assigned group is deterministic. The formal definition is as follows.[26]

Consistency: For each problem $(C, \{S_c\}_{c\in C}, K, q)$, each assignment μ on which φ places a positive probability, each subset of categories $D \subset C$ with $|D| \geq 2$, we have, for each student $s \in \bigcup_{c\in D} S_c$ whose assigned group is deterministic, $\varphi_s(D, \{S_c\}_{c\in D}, K, (q^k_{D,\mu})) = \varphi_s(C, \{S_c\}_{c\in C}, K, q)$.

The above consistency requires that, when students of some categories leave with their assigned groups, it does not affect the remaining students' assigned groups. We only consider reduced problems where all students of a category simultaneously leave because, when an arbitrary subset of students leaves, requiring no effect on the remaining students seems too restrictive. For instance, imagine that a category's students except its bottom one ("he") leave. It is not obvious that we should keep him in the worst group, i.e., allowing him to be reassigned a better group may be reasonable.

In this section, we first formalize fairness ideas across student categories. We introduce the concepts by applying the notion of equal treatment of equals to our problem. We consider two students satisfying specific criteria to be equals and require such students to be assigned to the same group. We can see this is not possible in deterministic assignments by considering the following case: When there are two "equal" students but each group's capacity is one, assigning them to the same group is not possible. Therefore, we consider random assignments (or in ex-ante sense). We use the following two intuitive measurements to determine which students are "equals." Given a problem, we define the **absolute position** of student $s \in S_c$ as $a(s|S_c) = |\{s' \in S_c \mid s' \succeq_c s\}|$. This represents the rank of student s within S_c starting from the *top*. We may omit "S_c" and denote the absolute position of s by $a(s)$. For example, when $S_c = \{s, s', s''\}$ and $s \succeq_c s' \succeq_c s''$, we have $(a(s), a(s'), a(s'')) = (1, 2, 3)$. Thus $(a(s))_{s\in S_c} = (1, 2, ..., |S_c|)$. On the other hand, we define the **relative position** of student $s \in S_c$ as $r(s|S_c) = a(s|S_c)/|S_c|$. We may omit "$S_c$" and denote the relative position of s by $r(s)$. By definition, $(r(s))_{s\in S_c} = (1/|S_c|, 2/|S_c|, ..., 1)$. In the following definitions, we require random assignments to assign the same probability share over groups for students in the same absolute/relative position.

Definition 1.

1. A lottery λ satisfies the **equal treatment of equal absolute positions** if for each pair $s, s' \in \bigcup_{c\in C} S_c$ of students with the same absolute positions $a(s) = a(s')$, their random allocations induced by lottery λ are the same, i.e., $\sigma_s(\lambda) = \sigma_{s'}(\lambda)$.
2. A lottery λ satisfies the **equal treatment of equal relative positions** if for each pair $s, s' \in \bigcup_{c\in C} S_c$ of students with the same relative positions $r(s) = r(s')$, their random allocations induced by lottery λ are the same, i.e., $\sigma_s(\lambda) = \sigma_{s'}(\lambda)$.

[26] Similar requirements are studied well for choice rules, for example, the weak axiom of revealed preference (WARP), the independence of irrelevant alternatives (IIA), and the irrelevance of rejected contracts (IRC). When $K = 2$ and an assignment is chosen for sure, our consistency is quite similar to these ones.

The next protocols are central in this paper. While our protocol picks a probability distribution over assignments achieved through our algorithms, restricting the distributions over parameters enables us to define these protocols. In the definitions, we say that $(\triangleright^\ell)_{\ell=1}^L$ are **perfectly random** if (i) each $\triangleright^\ell$ is selected with equal probability of $\frac{1}{L}$ and (ii) for each pair $s, s' \in \bigcup_{c\in C} S_c$ to whom $\triangleright$ is applied, $|\{\triangleright^\ell : s \triangleright^\ell s'\}| = |\{\triangleright^\ell : s' \triangleright^\ell s\}| = \frac{L}{2}$.

Definition 2.

1. For each problem P, the **uniform protocol** $\mathcal{U}$ associates a lottery $\mathcal{U}(P) = \left(\sigma_{(\omega^\ell,\triangleright^\ell)}(P)\right)_{\ell=1}^L$ where $(\triangleright^\ell)_{\ell=1}^L$ are perfectly random and all students' speeds are the same, that is, for each $\ell \in \{1,\ldots,L\}$ and each pair $s, s' \in \bigcup_{c\in C} S_c$, $\omega_s^\ell = \omega_{s'}^\ell$.
2. For each problem P, the **proportional protocol** $\mathcal{P}$ associates a lottery $\mathcal{P}(P) = \left(\sigma_{(\omega^\ell,\triangleright^\ell)}\right)_{\ell=1}^L$ where $(\triangleright^\ell)_{\ell=1}^L$ are perfectly random and the speed ω_s^ℓ of student s is proportional to the population $|S_c|$ of her category c, that is, for each $\ell \in \{1,\ldots,L\}$ and each pair $s, s' \in \bigcup_{c\in C} S_c$ with $s \in S_c$ and $s' \in S_{c'}$,
$$\frac{\omega_s^\ell}{|S_c|} = \frac{\omega_{s'}^\ell}{|S_{c'}|}.$$

We emphasize that the above two protocols are "almost" deterministic in the sense that we only use a randomly selected tie-breaker when multiple agents with the same favorite group are present in the algorithm.

Example 2. Consider the same problem as in Example 1.

1. Uniform protocol. A tie-breaker is applied to pairs $\{a^1, b^1\}$ and $\{a^2, b^2\}$. $(\triangleright^\ell)_{\ell=1}^4$ comprises the following:
   - $\triangleright^1 :\ a^1 \triangleright^1 b^1$ and $a^2 \triangleright^1 b^2$,
   - $\triangleright^2 :\ a^1 \triangleright^2 b^1$ and $b^2 \triangleright^2 a^2$,
   - $\triangleright^3 :\ b^1 \triangleright^3 a^1$ and $a^2 \triangleright^3 b^2$,
   - $\triangleright^4 :\ b^1 \triangleright^4 a^1$ and $b^2 \triangleright^4 a^2$.

   The algorithms $A(\omega, \triangleright^1)$ and $A(\omega, \triangleright^3)$ select assignment μ, while $A(\omega, \triangleright^2)$ and $A(\omega, \triangleright^4)$ select assignment μ' where
   $$\mu = \begin{pmatrix} 1 & 2 \\ \{a^1, a^2, b^1\} & \{a^3, a^4, b^2\} \end{pmatrix} \quad \text{and} \quad \mu' = \begin{pmatrix} 1 & 2 \\ \{a^1, b^1, b^2\} & \{a^2, a^3, a^4\} \end{pmatrix}.$$
   The uniform protocol selects μ and μ' with equal probabilities.
2. Proportional protocol. A tie-breaker is applied to pairs $\{a^2, b^1\}$ and $\{a^4, b^2\}$. $(\triangleright^\ell)_{\ell=5}^8$ comprises the following:

- $\rhd^5$: $a^2 \rhd^5 b^1$ and $a^4 \rhd^5 b^2$,
- $\rhd^6$: $a^2 \rhd^6 b^1$ and $b^2 \rhd^6 a^4$,
- $\rhd^7$: $b^1 \rhd^7 a^2$ and $a^4 \rhd^7 b^2$,
- $\rhd^8$: $b^1 \rhd^8 a^2$ and $b^2 \rhd^8 a^4$.

The algorithms $A(\omega, \rhd^5)$, $A(\omega, \rhd^6)$, $A(\omega, \rhd^7)$, and $A(\omega, \rhd^8)$ select assignment μ. The proportional protocol select μ with probability 1. $\Diamond$

The next two theorems are the main results of the paper. The first is the characterization of the uniform protocol.

Theorem 1. *The uniform protocol satisfies (i) non-wastefulness, (ii) respect of priorities, (iii) equal treatment of equal absolute positions, and (iv) consistency. Among the protocols satisfying (i)-(iii), the uniform protocol is the least randomized.*

The next is the characterization of the proportional protocol.

Theorem 2. *The proportional protocol satisfies (i) non-wastefulness, (ii) respect of priorities, (iii) equal treatment of equal relative positions, and (iv) consistency. Among the protocols satisfying (i)-(iv), the proportional protocol is the least randomized.*

Note that one central protocol is characterized in each result by four axioms plus least randomization, where three standard axioms are used in both results. Therefore, the difference in protocols purely comes from difference in the notion of fairness across student categories.

The logical independence of axioms is shown in the following example.

Example 3. Let P be $C = \{c, c'\}$, $(S_c, S_{c'}) = (\{s_1, s_2\}, \{s'_1, s'_2\})$ where $s_1 \succeq_c s_2$ and $s'_1 \succeq_{c'} s'_2$, $K = 2$, and $q = (2, 4)$. $\mathcal{U}(P) = \mathcal{P}(P) = \mu^*$ (see below). Since P has no reduced problems, the following protocols are consistent. For a protocol violating consistency, see the proof of Lemma 2 in the Appendix. Protocols φ^1, φ^2, and φ^3, choosing deterministic assignments for P, are the least-randomized.

$$\mu^* = \begin{pmatrix} 1 & 2 \\ \{s_1, s'_1\} & \{s_2, s'_2\} \end{pmatrix} \qquad \mu^1 = \begin{pmatrix} 1 & 2 \\ \emptyset & \{s_1, s_2, s'_1, s'_2\} \end{pmatrix} \qquad \mu^2 = \begin{pmatrix} 1 & 2 \\ \{s_2, s'_2\} & \{s_1, s'_1\} \end{pmatrix}$$

$$\mu^3 = \begin{pmatrix} 1 & 2 \\ \{s_1, s_2\} & \{s'_1, s'_2\} \end{pmatrix} \qquad \mu^4 = \begin{pmatrix} 1 & 2 \\ \{s'_1, s'_2\} & \{s_1, s_2\} \end{pmatrix}$$

1. φ^1 selects μ^1 for P. This assignment is wasteful but it respects priorities and satisfies equal treatment of equal absolute/relative positions.

2. φ^2 selects μ^2 for P. This assignment does not respect priorities but it is non-wasteful and satisfies equal treatment of equal absolute/relative positions.

3. φ^3 selects μ^3 for P. This assignment violates equal treatment of equal absolute/relative positions but it is non-wasteful and respects priorities.

4. φ^4 selects μ^3 and μ^4 with equal probabilities for P. Both $\mathcal{U}$ and $\mathcal{P}$ are less randomized than this protocol. However, this assignment is non-wasteful, respects priorities, and satisfies equal treatment of equal absolute/relative positions. $\Diamond$

3 Several Extensions

3.1 Giving more advantages to specific categories

In reality, to achieve fairness, we sometimes apply differentiated treatment for categories, e.g., medical rationing and affirmative action. To allow this, we introduce (favoring) **weight** for each category, denoted by p_c, for category c. The higher the weight, the more favorable its category in the prioritization process. Denote this vector by $p = (p_c)_{c \in C}$.

We extend the definition of a problem to $P = (C, p, \{S_c\}_{c \in C}, K, q)$ where $p = (p_c)_{c \in C}$. Reduced problems are defined accordingly. We modify our fairness notions as follows.

Definition 3. 1. A lottery λ satisfies the **p-equal treatment of equal absolute positions** if for each pair $s \in S_c$, $s' \in S_{c'}$ of students with $c \neq c'$ and $\frac{a(s)}{p_c} = \frac{a(s')}{p_{c'}}$, their random allocations induced by lottery λ are the same, i.e., $\sigma_s(\lambda) = \sigma_{s'}(\lambda)$.

2. A lottery λ satisfies the **p-equal treatment of equal relative positions** if for each pair $s \in S_c$, $s' \in S_{c'}$ of students with $c \neq c'$ and $\frac{r(s)}{p_c} = \frac{r(s')}{p_{c'}}$, their random allocations induced by lottery λ are the same, i.e., $\sigma_s(\lambda) = \sigma_{s'}(\lambda)$.

For the p-equal treatment of equal absolute positions, category c with a larger p_c is treated as a higher absolute position, and we apply the equal treatment of equals with this adjusted absolute position. Similarly, for p-equal treatment of equal relative positions, category c with a larger p_c is treated as a higher relative position, and we apply the equal treatment of equals with this adjusted relative position.

We embed the favoring weights to our central protocols of the uniform and the proportional protocol by adjusting the speeds in the algorithm.

Definition 4. 1. For each problem P, the **p-uniform protocol** $\mathcal{U}^p$ associates a lottery $\mathcal{U}^p(P) = \left(\sigma_{(\omega^\ell, \rhd^\ell)}(P)\right)_{\ell=1}^{L}$ where $(\rhd^\ell)_{\ell=1}^{L}$ are perfectly random and satisfies the following: for each pair $s, s' \in \bigcup_{c \in C} S_c$ with $s \in S_c$ and $s' \in S_{c'}$,

$$\frac{\omega_{s'}}{\omega_s} = \frac{p_{c'}}{p_c}$$

2. For each problem P, the p-**proportional protocol** $\mathcal{P}^p$ associates a lottery $\mathcal{P}^p(P) = \left(\sigma_{(\omega^\ell, \rhd^\ell)}(P)\right)_{\ell=1}^L$ where $(\rhd^\ell)_{\ell=1}^L$ are perfectly random and satisfies the following: for each pair $s, s' \in \bigcup_{c \in C} S_c$ with $s \in S_c$ and $s' \in S_{c'}$,
$$\frac{\omega_{s'}}{\omega_s} = \frac{p_{c'}}{p_c} \times \frac{|S_{c'}|}{|S_c|}$$

The difference from our protocols is the speeds: For the p-uniform protocol, the ratio of speed ω_s to $\omega_{s'}$ is equal to the ratio of favoring weight p_c to $p_{c'}$. Therefore, the larger the weight of the student, the faster the student's speed. On the other hand, for the p-proportional protocol, the ratio of speed ω_s to $\omega_{s'}$ is equal to the proportional ratio of the weight p_c to $p_{c'}$ regarding the population.

Notably, when the weights are equal across categories (i.e., $p_c = p_{c'}$), the p-uniform protocol coincides with the uniform protocol and p-proportional protocol coincides with the proportional protocol.

Example 4. Consider the same problem as in Examples 1 and 2. Let the weights be $p = (p_A, p_B) = (1, 2)$ and $p' = (p'_A, p'_B) = (1, 4)$. Assignments μ and μ' are shown in Example 2 where μ' favors category B more than μ.

- $\mathcal{U}$ chooses μ and μ' with equal probabilities where $\mathcal{P}$ choose μ for sure.
- $\mathcal{U}^p$ chooses μ' for sure where $\mathcal{P}^p$ choose μ for sure.
- $\mathcal{U}^{p'}$ and $\mathcal{P}^{p'}$ choose μ' for sure.

From above, $\mathcal{U}^p$ weakly favors category B more than $\mathcal{U}$, and $\mathcal{U}^{p'}$ weakly favors more than $\mathcal{U}^p$. $\mathcal{P}^p$ weakly favors category B more than $\mathcal{P}$, and $\mathcal{P}^{p'}$ weakly favors more than $\mathcal{P}^p$. $\lozenge$

We obtain the following characterization results in parallel to Theorems 1 and 2 for the no-weight case. The proof uses the same logic, and so is omitted.

Proposition 2. *1. The p-uniform protocol satisfies (i) non-wastefulness, (ii) respect of priorities, (iii) p-equal treatment of equal absolute positions, and (iv) consistency. Among the protocols satisfying all of (i)-(iv), the p-uniform protocol is the least randomized.*

2. The p-proportional protocol satisfies (i) non-wastefulness, (ii) respect of priorities, (iii) p-equal treatment of equal relative positions, and (iv) consistency. Among the protocols satisfying all of (i)-(iv), the p-proportional protocol is the least randomized.

By adjusting favoring weights, we obtain the following results.

Corollary 1. 1. The proportional protocol is the p-uniform protocol with $p = (|S_c|)_{c \in C}$.

2. The uniform protocol is the p'-proportional protocol with $p' = (1/|S_c|)_{c \in C}$.

3. The p-uniform protocol is the p'-proportional protocol where for each $c \in C$, $p'_c = p_c/|S_c|$.

3.2 Managing overlapping categories

A critical assumption made so far is that each student belongs to a single category. For some applications, categories overlap so that a student belongs to multiple categories. For example, in vaccine allocation, job and region-based categories coexist and overlap. We relax this assumption as follows: (i) students may initially belong to more than one categories (i.e., $c \neq c'$ does not necessarily imply $S_c \cap S_{c'} = \emptyset$), and (ii) a student has an absolute/relative position in every category to which she belongs. We re-denote the absolute position (relative position) of student s in category c as $a(s, c)$ ($r(s, c)$) instead of $a(s)$ ($r(s)$). To accommodate this overlapping case, we adjust our algorithm by letting *all* students start to fill positions from time 0.

Algorithm $\tilde{A}(\omega, \triangleright)$**.** *Step 1. Each* student s fills 1 unit of position at his own speed ω_s. Following $\triangleright$, the first students who finish filling 1 units are assigned to the most preferred groups. Set $(q^k(2))$ and $(e_s(2))$ in the same manner as Section 2.
Steps $j \geq 2$. *Each* student s who has not finished fills $1 - e_s(j)$ unit of position at his own speed ω_s. Take the fastest students and assign them to the most preferred groups. Set $(q^k(j+1))$ and $(e_s(j+1))$ in the same manner as Section 2.

As in Proposition 1, which characterizes the original algorithm by two axioms, we characterize this adjusted algorithm only by non-wastefulness.

Proposition 3. *For each problem P and each non-wasteful assignment μ at P, there is a pair $(\omega, \triangleright)$ such that algorithm $\tilde{A}(\omega, \triangleright)$ finds μ. For each problem P and each pair $(\omega, \triangleright)$, the assignment that algorithm $\tilde{A}(\omega, \triangleright)$ finds is non-wasteful at P.*

Since each student s is endowed with multiple absolute positions $\{a(s, c)\}_{c \in C}$, we introduce a **statistic for absolute positions**, denoted by $\tilde{a}(s)$, that aggregates into one representative absolute position. Similarly, we introduce a **statistic for relative positions**, denoted by $\tilde{r}(s)$, that aggregates her various positions $\{r(s, c)\}_{c \in C}$. We focus on the following statistics.

- Simple average: $\tilde{a}^{\mathrm{AVE}}(s) := \frac{\sum_{c \in C; s \in S_c} a(s,c)}{|\{c \in C; s \in S_c\}|}$ and $\tilde{r}^{\mathrm{AVE}}(s) := \frac{\sum_{c \in C; s \in S_c} r(s,c)}{|\{c \in C; s \in S_c\}|}$.
- Weighted average: $\tilde{a}^{\mathrm{WA}}(s) := \frac{\sum_{c \in C; s \in S_c} w_c a(s,c)}{\sum_{c \in C; s \in S_c} w_c}$ and $\tilde{r}^{\mathrm{WA}}(s) := \frac{\sum_{c \in C; s \in S_c} w_c r(s,c)}{\sum_{c \in C; s \in S_c} w_c}$ with respect to weights $(w_c)_{c \in C} \in \mathbb{R}^{|C|}_{++}$ on categories.

- Maximum: $\tilde{a}^{\mathrm{MAX}}(s) := \max_{c \in C; s \in S_c} \{a(s,c)\}$ and $\tilde{r}^{\mathrm{MAX}}(s) := \max_{c \in C; s \in S_c} \{r(s,c)\}$.
- Minimum: $\tilde{a}^{\mathrm{MIN}}(s) := \min_{c \in C; s \in S_c} \{a(s,c)\}$ and $\tilde{r}^{\mathrm{MIN}}(s) := \min_{c \in C; s \in S_c} \{r(s,c)\}$.
- Median: $\tilde{a}^{\mathrm{MED}}(s) := \operatorname{med}_{c \in C; s \in S_c} \{a(s,c)\}$ and $\tilde{r}^{\mathrm{MED}}(s) := \operatorname{med}_{c \in C; s \in S_c} \{r(s,c)\}$.

According to these statistics, we re-define the notions of respecting priorities and equal treatment of equal absolute/relative positions.

- An assignment μ *respects priorities* at a problem P if for each pair of students s, s' who belongs to the same subset of categories, when student s has higher priority than student s' at each category, the group s is assigned is not worse than the group s' is assigned. That is, for each pair s, s' with $\{c \in C : s \in S_c\} = \{c \in C : s' \in S_c\}$, [for each c with $s, s' \in S_c$, $s \succeq_c s'$] implies $[\mu(s) \leq \mu(s')]$.
- A lottery λ satisfies the $\tilde{a}(\cdot)$-*equal treatment of equal absolute positions* ($\tilde{r}(\cdot)$-*equal treatment of equal relative positions*) if for each pair s, s' with the same value of statics $\tilde{a}(s) = \tilde{a}(s')$ (resp. $\tilde{r}(s) = \tilde{r}(s')$), their random allocations induced by λ are the same, i.e., $\sigma_s(\lambda) = \sigma_{s'}(\lambda)$.

Using the adjusted algorithm and statistics, we re-define the uniform and proportional protocols as follows:

1. For each $x \in \{\mathrm{AVE}, \mathrm{WA}, \mathrm{MAX}, \mathrm{MIN}, \mathrm{MED}\}$, the *$x$-uniform protocol* $\tilde{\mathcal{U}}^x$ associates a lottery for a problem where tie-breakers $(\triangleright^\ell)_{\ell=1}^L$ are perfectly random and speeds $((\omega_s^\ell)_{s \in \bigcup_{c \in C} S_c})_{\ell=1}^L$ are for each $\ell \in \{1, \ldots, L\}$ and each $s \in \bigcup_{c \in C} S_c$, $\omega_s^\ell = 1/\tilde{a}^x(s)$.
2. For each $x \in \{\mathrm{AVE}, \mathrm{WA}, \mathrm{MAX}, \mathrm{MIN}, \mathrm{MED}\}$, the *$x$-proportional protocol* $\tilde{\mathcal{P}}^x$ associates a lottery for a problem where tie-breakers $(\triangleright^\ell)_{\ell=1}^L$ are perfectly random and speeds $((\omega_s^\ell)_{s \in \bigcup_{c \in C} S_c})_{\ell=1}^L$ are for each $\ell \in \{1, \ldots, L\}$ and each $s \in \bigcup_{c \in C} S_c$, $\omega_s^\ell = 1/\tilde{r}^x(s)$.

Proposition 4 shows that the restated versions of the uniform and proportional protocols satisfy the desirable properties.

Proposition 4. *Let the statistic be either simple average, weighted average, maximum, minimum, or median, i.e., $x \in \{AVE, WA, MAX, MIN, MED\}$.*

1. *The x-uniform protocol $\tilde{\mathcal{U}}^x$ is non-wasteful, respects priorities, and satisfies the $\tilde{a}^x(\cdot)$-equal treatment of equal absolute positions.*
2. *The x-proportional protocol $\tilde{\mathcal{P}}^x$ is non-wasteful, respects priorities, and satisfies the $\tilde{r}^x(\cdot)$-equal treatment of equal relative positions.*

The following is an example of the uniform and proportional protocols for the simple average.

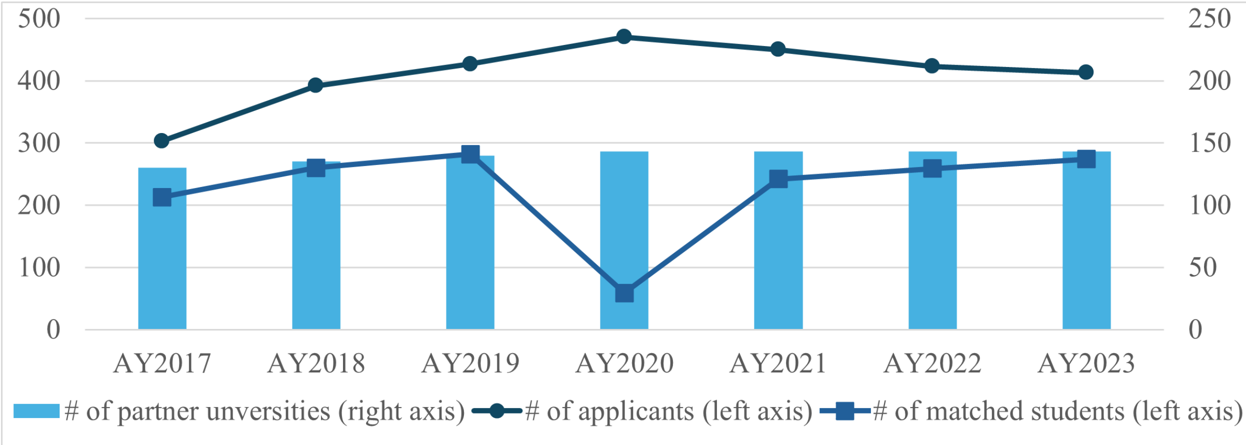

Figure 5: Numbers of applicants and partner universities at Keio University

Example 5. Let P be $C = \{c, c'\}$, $(S_c, S_{c'}) = (\{s_1, s_2, \tilde{s}\}, \{\tilde{s}, s_2', s_3'\})$ where $\succeq_c$: $s_1, s_2, \tilde{s}$ and $\succeq_{c'}$: $\tilde{s}, s_2', s_3'$, $K = 2$, and $q = (2, 3)$. Only $\tilde{s}$ belongs to both categories. Fix x to simple average. Since $|S_c| = |S_{c'}|$, two protocols $\tilde{\mathcal{U}}^x$ and $\tilde{\mathcal{P}}^x$ choose the same lottery for P. In $\tilde{\mathcal{U}}^x$, speeds are $(\omega_{s_1}, \omega_{s_2}, \omega_{\tilde{s}}, \omega_{s_2'}, \omega_{s_3'}) = (1, 1/2, 1/(\frac{3+1}{2}), 1/2, 1/3)$ and it chooses the following assignments with equal probabilities.

$$\mu = \begin{pmatrix} 1 & 2 \\ \{s_1, s_2\} & \{\tilde{s}, s_2', s_3'\} \end{pmatrix}, \quad \mu' = \begin{pmatrix} 1 & 2 \\ \{s_1, s_2'\} & \{s_2, \tilde{s}, s_3'\} \end{pmatrix}, \quad \tilde{\mu} = \begin{pmatrix} 1 & 2 \\ \{s_1, \tilde{s}\} & \{s_2, s_2', s_3'\} \end{pmatrix}.$$

- The protocol is non-wasteful. Group 1 is filled up to its capacity at each assignment.
- The protocol respects priorities. All pairs belonging to the same subset of categories are $(s_1, s_2) \subsetneq S_c$ and $(s_2', s_3') \subsetneq S_{c'}$. s_1 (s_2') is never assigned a worse group than s_2 (s_3').
- The protocol satisfies x-equal treatment of equal absolute (relative) positions. Students $(s_2, s_2', \tilde{s})$ are at the same absolute (relative) positions on average, and are assigned the same random assignments. $\diamondsuit$

3.3 Case study: Prioritization for exchange programs at Keio University

Keio University, one of the largest private universities in Japan, offers exchange program to its undergraduate students. The exchange program has steadily expanded year by year in terms of the numbers of applicants and partner universities though there have been drops due to the pandemic (See Figure 5).

Previously applicants were matched to their desired institutions through one-day interviews and faculty meeting three times a year. On a specific day, approximately 20 faculty members gathered to conduct interviews with applicants. The applicants were divided into approximately 10 groups, with each group interviewed by two faculty members and one staff member. These interviews were conducted simultaneously. After the interviews, all faculty and staff convened in one room to decide on matching applicants to universities.

Following the increasing number of applicants, this type of evaluation and matching through interviews had become unsustainable owing to high administrative and time costs. In 2020, the International Center of Keio University decided to change to document-based evaluation with a centralized matching mechanism. The third author of this paper was invited to design the mechanism.

Regarding the new mechanism, the evaluation was based on a weighted average of the GPA and the scores of the application documents.[27] The mechanism we intended to use was a serial dictatorship in which a complete priority order over students is exogenously given and then following the priority order higher-priority students are assigned their favorite universities. The challenge was that all students could be prioritized completely, but such a priority was determined not to be used owing to its unfairness across the grading policies of different faculties. Therefore, we began our project to develop axiom-based protocols for complete priority ranking based on the partial ordering of each faculty.

The prioritization problem at Keio University is that the categories are ten faculties, applicants in each faculty are priority-ordered, and priority groups are set so that each group should contain at least one applicant from each faculty, considering the balance across faculties. Therefore, the number of priority groups, K, is equal to the number of students of the faculty with the fewest applicants. The average value of K is about 20 in the past few years. Moreover, the capacity q^k is set to be equal among priority groups. The university requested the protocol to satisfy non-wastefulness, respect of priorities, and equal treatment of equal relative positions. In addition, the request for minimal use of randomization prompted us to formalize the concept of least randomization. Thus, based on Theorem 2, we recommended the use of the proportional protocol.

After applying the proportional protocol, it is not sufficient to apply the serial dictatorship mechanism because the priority order is not linear owing to the possibility of multiple students in the same priority groups. Therefore, within each priority group, students are linearly ordered based on the weighted average of their scores. The reason for this indirect method of obtaining linear orderings is that the prioritization induces higher GPA students to be assigned higher priority groups to reflect their learning efforts at Keio, and linearization within each priority group gives applicants incentives to write application documents seriously.

The serial dictatorship mechanism with linear order had a problem in which some students did not satisfy the criteria set by partner universities, and therefore, could not be assigned. To avoid this, we adjusted the priority ordering of each partner university so that such students are not assigned. This makes partner universities' priority orders differ, making the serial dictatorship inapplicable. This drove the adoption of the Gale and Shapley (1962)'s students-proposing deferred

[27] Keio University told authors not to reveal the weights.

acceptance mechanism. The whole assignment procedure was implemented in 2022, and has been in use at Keio University.

4 Conclusion

Our study delves into the intricate issue of designing endogenous priorities for heterogeneous groups of agents, a fundamental challenge in many real-world allocation problems. From scheduling surgeries and allocating visas to assigning students to university exchange programs, the need for a fair and efficient prioritization protocol is evident. Our research emphasizes the importance of creating a transparent, objective, and consistent prioritization system that aligns with the primary objectives of a planner.

We introduced the concept of a prioritization problem, where agents are assigned to priority groups based on their attributes, with the aim of achieving fairness and efficiency. Our analysis identified two types of prioritization protocols—uniform prioritization and proportional prioritization—that stand out when adhering to principles of non-wastefulness, respect for priorities, consistency, and equal treatment of equals. These protocols utilize randomization only as a tie-breaking mechanism, ensuring minimal reliance on randomness while maintaining fairness.

The uniform prioritization protocol applies when fairness is absolute, treating all students equally irrespective of category size. In contrast, the proportional prioritization protocol is suited for scenarios where fairness is relative, considering the size of each category and assigning probabilities proportionally. Our findings highlight the necessity of direct construction of lotteries over deterministic assignments, a more nuanced approach compared to traditional random assignment methods.

Ultimately, our research provides a robust framework for designing prioritization protocols that can be applied across various domains, offering insights into achieving equitable and efficient allocations in complex, multi-attribute environments.

5 Appendix

Proof of Footnote 10. Precisely, we have the following impossibility on the alternative approach described in the Introduction.

Proposition 5. *For each priority metric (p_A, p_B) satisfying $\frac{p_A}{p_B} \in \mathbb{N}$, there is an instance where the alternative approach violates (p_A, p_B)-equal treatment of equal relative positions.*

Proof. Fix $\ell := \frac{p_A}{p_B} \in \mathbb{N}$. Consider an instance where agents are (a^1, a^2) for category A and $(b^1, b^2, ..., b^{6\ell})$ for category B and priority groups are $K = \{1, 2\}$ with capacity 3 for group 1. The alternative approach proceeds as follows.

1. Capacity 3 for group 1 is allocated to categories proportionally to $p_A|S_A|$ and $p_B|S_B|$, i.e.,

Table 1: Assignemnts in Proof of Proposition 5.

	a^1	a^2	b^3	b^6
μ_0	group 2	group 2	group 1	group 2
μ_1	group 1	group 2	group 2	group 2
μ_2	group 1	group 1	group 2	group 2

$3 \times \frac{2\ell}{2\ell+6\ell} = 0.75$ for category A and $3 \times \frac{6\ell}{2\ell+6\ell} = 2.25$ for category B.
2. Since $2 < 2.25 < 3$, possible decompositions are (μ_0, μ_1, μ_2) where for each μ_j, capacity of j is assigned to agents of category A and capacity of $3-j$ is assigned to agents of category B. Let (p_0, p_1, p_2) be probabilities for these decompositions. Clearly, $p_0 + p_1 + p_2 = 1$. Feasibility implies $3p_0 + 2p_1 + p_2 = 2.25$ (eq.1*). Since RHS> 2, $p_1 < 1$, or equivalently $p_0 + p_2 > 0$.
3. Since $\frac{r(a^1)}{p_A} = \frac{1/2}{\ell p_B} = \frac{3/6\ell}{p_B} = \frac{r(b^3)}{p_B}$ and $\frac{r(a^2)}{p_A} = \frac{2/2}{\ell p_B} = \frac{6/6\ell}{p_B} = \frac{r(b^6)}{p_B}$, $\{a^1, b^3\}$ and $\{a^2, b^6\}$ are "eqauals." At each decomposition, each agent is assigned to a group respecting priorities as in Table 1. To assign $\{a^1, b^3\}$ the same random assignment, we must have $p_1 + p_2 = p_0$, whereas to assign $\{a^2, b^6\}$ the same random assignment, we must have $p_2 = 0$. They imply $(p_0, p_1, p_2) = (0.5, 0.5, 0)$, i.e., $3p_0 + 2p_1 + p_2 = 2.5$, but it violates (eq.1*). This approach violates (p_A, p_B)-equal treatment of equal relative positions. $\square$

We refer to non-wastefulness as NW, respect of priorities as RP, and equal treatment of equal absolute positions (relative positions) as ETA (ETR).

Proof of Proposition 1.
($\Leftarrow$) Fix $(\omega, \rhd)$. NW is immediate because the best $|\bigcup_{c\in C} S_c|$ seats are assigned. RP is immediate because, within each category, a better student is assigned earlier than a worse student.
($\Rightarrow$) Fix an assignment μ that satisfies NW and RP. For each student $s \in S_c$, let $\omega_s = |\mu(s) \cap S_c|$ and for each pair s, s', let $s \rhd s' \Leftrightarrow \mu(s) \le \mu(s')$. Then, $A(\omega, \rhd)(P) = \{\{s \in \bigcup_{c\in C} S_c : (k-1)/\bar{k} < \sum_{s';a(s')\le a(s)} (1/\omega_{s'}) \le k/\bar{k}\}\}_{k\in K} = \mu$. $\square$

We use the following lemmas.

Lemma 1. *If a protocol satisfies (i) non-wastefulness, (ii) respect of priorities, and (iii-1) equal treatment of equal absolute positions, then, it satisfies (iv) consistency.*

Proof. To lead a contradiction, suppose that φ satisfies (i)-(iii-1) but is inconsistent. Let $s \in S_c \subsetneq \bigcup_{c'\in D} S_{c'}$ be a student at the smallest absolute position with $\varphi_s(D, \{S_{c'}\}_{c'\in D}, K, (q^k_{D,\mu})) = k' \neq k = \varphi_s(C, \{S_{c'}\}_{c'\in C}, K, q)$. Denote $P = (C, \{S_{c'}\}_{c'\in C}, K, q)$ and $P_D = (D, \{S_{c'}\}_{c'\in D}, K, (q^k_{D,\mu}))$. By ETA, for each $s' \in \bigcup_{c'\in C} S_{c'}$ with $a(s') = a(s)$, $\varphi_{s'}(P) = \varphi_s(P) = k$ and for each $s' \in \bigcup_{c'\in D} S_{c'}$ with $a(s') = a(s)$, $\varphi_{s'}(P_D) = \varphi_s(P_D) = k'$. Thus, by RP, $a(s) = 1$ implies $k = 1$ and $k' = 1$. Since $k \neq k'$, we have $a(s) > 1$.

Case 1: $k' < k$. Recall that μ on which φ puts a positive probability at P is NW. Thus, at μ, each group $k'' < k$ is filled up to its capacity $q^{k''}$. Also recall that μ is RP and thus at μ, each group $k'' < k$ is filled with students with absolute positions strictly smaller than $a(s)$. Pick μ' on which φ puts a positive probability at P_D. Since μ' is NW, each group $k'' < k$ is filled up to its capacity $q^{k''}_{D,\mu}$. Since $k' = \varphi_{s'}(P_D) < k$, and by ETA, at least $|s' \in \bigcup_{c' \in D} S_{c'} : a(s') = a(s)|$ seats for group k is occupied with students $\underline{S}$ with absolute positions strictly smaller than $a(s)$. When $\underline{S}$ contains a student whose category has at least $a(s)$ students, RP is directly violated for her category. When $\underline{S}$ contains no such students, by ETA, there is μ'' on which φ puts a positive probability at P_D and a student whose category has at least $a(s)$ students are assigned group k. Because students at absolute position $a(s)$ are assigned group $k' < k$ at μ'', RP is violated.

Case 2: $k' > k$. Since μ is NW, each group $k'' < k$ is filled up to its capacity $q^{k''}$ at μ. If group k has vacant seats at μ, by NW, no students are assigned group $k' > k$ at μ. $\varphi_s(P_D) = k'$ contradicts NW. Thus, group k is also filled up to its capacity q^k at μ. Since μ is RP, each student assigned group $k'' > k$ at μ is with absolute position strictly larger than $a(s)$. Pick μ' on which φ puts a positive probability at P_D. Since μ' is NW, each group $k'' < k'$ is filled up to its capacity $q^{k''}_{D,\mu}$. Since $k' = \varphi_{s'}(P_D) > k$, and by ETA, at least $|s' \in \bigcup_{c' \in D} S_{c'} : a(s') = a(s)|$ seats for group k is occupied with students $\overline{S}$ with absolute positions strictly larger than $a(s)$. Because for each $s'' \in \overline{S}$, there is a student s' with $a(s') = a(s) < a(s'')$, $\varphi_{s'}(P_D) = k' > k$, and whose category is the same as the one for s'. RP is violated. $\square$

Lemma 2. *If a protocol satisfies (i) non-wastefulness, (ii) respect of priorities, and either (iii-2) equal treatment of equal relative positions, (iii-3) p-equal treatment of equal absolute positions, or (iii-4) p-equal treatment of equal relative positions, it may not satisfy (iv) consistency.*

Proof. Consider problem P with $C = \{a, b, c\}$, $(S_a, S_b, S_c) = (\{a^1, a^2, a^3\}, \{b^1, b^2\}, \{c^1, c^2\})$, $K = 2$, and $(q^1, q^2) = (2, 5)$. Let the priorities be $\succeq_a$: a^1, a^2, a^3, $\succeq_b$: b^1, b^2, and $\succeq_c$: c^1, c^2. Let $\varphi(P) = \mu$ for sure, where a^1 and a^2 are assigned group 1. Consider a reduced problem $P_{\{a,b\}} = (\{a, b\}, (S_a, S_b), K, (2, 3))$. Let $\varphi(P_{\{a,b\}}) = \mu'$ for sure, where a^1 and b^1 are assigned group 1. Both μ and μ' are NW, RP, and ETR. Also, for $p = (p_a, p_b, p_c) = (3, 2, 2)$ and $p' = (p'_a, p'_b, p'_c) = (1, 1, 1)$, both μ and μ' satisfy p-ETA and p'-ETR. However, φ is not consistent because $\varphi_{b^1}(P) = 2 \neq 1 = \varphi_{b^1}(P_{\{a,b\}})$. $\square$

Proof of Theorem 1. NW and RP are immediate from Proposition 1.

ETA: Fix P and pick $\{s_1, ..., s_N\} \subsetneq \bigcup_{c \in C} S_c$ with $a(s_1) = ... = a(s_N) =: a$. For any tie-breaker, the following are all true:

(a) The students are assigned at step a, that is, $s_1, ..., s_N \in S(a)$, $e(a) = (0, ..., 0)$, and they all finish at the same time $t'(a) = t(a) + \frac{1}{\omega_{s_1}} = ... = t(a) + \frac{1}{\omega_{s_N}}$. Thus, $s_1, ..., s_N \in S'(a)$.

(b) The groups they are assigned are the same, that is, $q(a)$ is independent from tie-breaker.

(c) Tie-breakers applied to $s_1, ..., s_N$ are perfectly randomized.

From above, we can see that for any $s, s' \in \{s_1, ..., s_N\}$, distributions over assigned groups are the same. $\square$

Consistency: Fix P with $|C| \geq 3$, assignment μ that φ chooses with positive probability, and $D \subsetneq C$ with $|D| \geq 2$. Consider $P_D = (D, \{S_c\}_{c \in D}, K, \{q^k_{D,\mu}\})$. Pick $s \in \bigcup_{c \in D} S_c$ with $\mathcal{U}_s(P) = \mu(s) =: k$. We show $\mathcal{U}_s(P_D) = k$, i.e., for each $\hat{\mu}$ which φ chooses for P_D with positive probability, s is assigned k. Because $\mathcal{U}$ satisfies ETA, for each $s' \in \bigcup_{c' \in C} S_{c'}$ with $a(s') = a(s)$, $\mathcal{U}_{s'}(P)$ is deterministic.
Case 1: For each $s' \in \bigcup_{c' \in C \setminus D} S_{c'}$ with $a(s') < a(s)$, $\mathcal{U}_{s'}(P)$ is deterministic. At each step of the algorithm until step $a(s)$, each $s' \in \bigcup_{c' \in D} S_{c'}$ is assigned the same group in P and in P_D. s is assigned k at P_D.
Case 2: There is $s' \in \bigcup_{c' \in C \setminus D} S_{c'}$ with $a(s') < a(s)$ and $\mathcal{U}_{s'}(P)$ is non-deterministic. Because $\mathcal{U}$ satisfies ETA, for each $s'' \in \bigcup_{c' \in C} S_{c'}$ with $a(s'') = a(s')$, $\mathcal{U}_{s''}(P)$ is non-deterministic. At each step of the algorithm until step $a(s') - 1$, each $s'' \in \bigcup_{c' \in D} S_{c'}$ is assigned the same group in P and in P_D. At step $a(s')$, while each student may be assigned different groups in P and in P_D, total capacity assigned to them is the same, i.e., $q(a(s')) - q(a(s') + 1)$ in P_D is the same as $q(a(s')) - q(a(s') + 1) - (\mu(s''))_{s'' \in \bigcup_{c' \in C \setminus D} S_{c'}}$ in P. s is assigned k at P_D. $\square$

Least-randomization/Uniqueness: Suppose, by way of contradiction, that there is $\varphi \neq \mathcal{U}$ that satisfies (i)-(iv) where $\mathcal{U}$ is not less randomized than φ. Fix P in which the number of students assigned non-deterministic groups is at least as large in $\mathcal{U}(P)$ as in $\varphi(P)$ and $\varphi(P) \neq \mathcal{U}(P)$. Because both $\varphi(P)$ and $\mathcal{U}(P)$ satisfy ETA, and

$$
\begin{aligned}
&|\{s \in \bigcup_{c \in C} S_c : \varphi_s(P) \text{ is non-deterministic}\}| \\
&= \sum_a |\{s \in \bigcup_{c \in C} S_c : a(s) = a\}| \mathbb{1}\{\{\varphi_s(P)\}_{s; a(s) = a} \text{ is non-deterministic}\} \\
&\leq \sum_a |\{s \in \bigcup_{c \in C} S_c : a(s) = a\}| \mathbb{1}\{\{\mathcal{U}_s(P)\}_{s; a(s) = a} \text{ is non-deterministic}\} \\
&= |\{s \in \bigcup_{c \in C} S_c : \mathcal{U}_s(P) \text{ is non-deterministic}\}|
\end{aligned}
$$

there is a such that $\{\varphi_s(P)\}_{s; a(s) = a}$ is deterministic and $\{\mathcal{U}_s(P)\}_{s; a(s) = a}$ is non-deterministic. Pick the smallest such a. Either case 1 or 2 should hold.
Case 1: There is $s \in \bigcup_{c \in C} S_c$ with $a(s) = a$ and μ where $\mathcal{U}$ places positive probability on it (write $\mu(s) = k$), satisfying $k < \varphi_s(P)$. Since $\varphi(P)$ is NW, there is $s' \in \bigcup_{c \in C} S_c$ with $a(s') = a' > a$ and at some $\hat{\mu}$ where φ places positive probability on it (write $\hat{\mu}(s') = k'$), $k' = k < \varphi_s(P)$. That is, $a(s) < a(s')$ and $\hat{\mu}(s) > \hat{\mu}(s')$. Since φ satisfies RP, there are $c, c' \in C$ with $c \neq c'$, $s \in S_c$, and $s' \in S_{c'}$.
Case 2: There is $s \in \bigcup_{c \in C} S_c$ with $a(s) = a$ and μ where $\mathcal{U}$ places positive probability on it (write

$\mu(s) = k$), satisfying $k > \varphi_s(P)$. Since $\mathcal{U}(P)$ is NW, $\varphi_s(P)$ is occupied by other $s' \in \bigcup_{c \in C} S_c$ with $a(s') < a$. At any $\hat{\mu}$ where φ places positive probability on it (write $\hat{\mu}(s') = k'$), $k' > \varphi_s(P)$. That is, $a(s) > a(s')$ and $\hat{\mu}(s) < \hat{\mu}(s')$. Since φ satisfies RP, there are $c, c' \in C$ with $c \neq c'$, $s \in S_c$, and $s' \in S_{c'}$.

We see below that Case 1 violates either of (i)-(iv). Case 2 also violates either of (i)-(iv) in the same manner.

Let $\bar{P}$ be $\bar{C} = C \cup \{\bar{c}\}$, $|S_{\bar{c}}| \geq a'$, and P is a reduced problem of $\bar{P}$. Since φ satisfies consistency, and $\varphi_s(P) = k$ is deterministic, we have $\varphi_s(\bar{P}) = \varphi_s(P) = k$. By ETA of φ, for $\bar{s}, \bar{s}' \in S_{\bar{c}}$ with $(a(\bar{s}), a(\bar{s}')) = (a, a')$, $\varphi_{\bar{s}}(\bar{P}) = \varphi_s(\bar{P}) = k$ (deterministic) and $\varphi_{\bar{s}'}(\bar{P}) = \varphi_{s'}(\bar{P})\ldots(*)$. Since there is positive probability for assignment μ^1 with $\mu^1(s') < k$, to satisfy (*), there should be positive probability for μ^2 with $\mu^2(\bar{s}') < k$, that is, $\mu^2(\bar{s}') < \mu^2(\bar{s})$. Since $a(\bar{s}) < a(\bar{s}')$, RP is violated for $\bar{s}, \bar{s}' \in S_{\bar{c}}$. □

Proof of Theorem 2. NW and RP are immediate from Proposition 1.

ETR: Fix P and pick $\{s_1, ..., s_N\} \subsetneq S$ with $r(s_1) = ... = r(s_N) =: r$. For any tie-breaker, the following are all true:

(a) The students are assigned at the same step, call j, that is, they all finish at the same time $t'(j) = a(s_1) \times \frac{1}{\omega_{s_1}} = ... = a(s_N) \times \frac{1}{\omega_{s_N}}$. Thus, $s_1, ..., s_N \in S'(j)$.

(b) The groups they are assigned are the same, that is, $q(j)$ is independent from tie-breaker.

(c) Tie-breakers applied to $s_1, ..., s_N$ are perfectly randomized.

From above, we can see that for any $s, s' \in \{s_1, ..., s_N\}$, distributions over assigned groups are the same. □

Consistency: Fix P with $|C| \geq 3$, $\{S^k\}$ that φ chooses with positive probability, and $D \subsetneq C$ with $|D| \geq 2$. Consider $P_D = (D, \{S_c\}_{c \in D}, K, \{q^k_{D,\{S^k\}}\})$. Pick $s \in \bigcup_{c \in D} S_c$ with $\mathcal{P}_s(P) = k(s) =: k$. We show $\mathcal{P}_s(P_D) = \mathcal{P}_s(P) = k$, i.e., for each $\{\hat{S}^k\}$ which φ chooses for P_D with positive probability, s is assigned k. Because $\mathcal{P}$ satisfies ETR, for each $s' \in \bigcup_{c' \in C} S_{c'}$ with $r(s') = r(s)$, $\mathcal{P}_{s'}(P)$ is deterministic.

Case 1: For each $s' \in \bigcup_{c' \in C \setminus D} S_{c'}$ with $r(s') < r(s)$, $\mathcal{P}_{s'}(P)$ is deterministic. At each step of the algorithm until the step s is assigned, each $s' \in \bigcup_{c' \in D} S_{c'}$ is assigned the same group in P and in P_D. s is assigned k at P_D.

Case 2: There is $s' \in \bigcup_{c' \in C \setminus D} S_{c'}$ with $r(s') < r(s)$ and $\mathcal{P}_{s'}(P)$ is non-deterministic. Because $\mathcal{P}$ satisfies ETR, for each $s'' \in \bigcup_{c' \in C} S_{c'}$ with $r(s'') = r(s')$, $\mathcal{P}_{s''}(P)$ is non-deterministic. At each step of the algorithm until step $a(s') - 1$, each $s'' \in \bigcup_{c' \in D} S_{c'}$ is assigned the same group in P and in P_D. At step before s' is assigned, while each student may be assigned different groups in P and in P_D, total capacity assigned to them is the same. s is assigned k at P_D. □

Least-randomization/Uniqueness: Suppose, by way of contradiction, that there is $\varphi \neq \mathcal{P}$ that satisfies (i)-(iv) where $\mathcal{P}$ is not less randomized than φ. Fix P in which the number of students assigned non-deterministic groups is at least as large in $\mathcal{P}(P)$ as in $\varphi(P)$ and $\varphi(P) \neq \mathcal{P}(P)$. Because both $\varphi(P)$ and $\mathcal{P}(P)$ satisfy ETR, and

$$\begin{aligned}
&|\{s \in \bigcup_{c\in C} S_c : \varphi_s(P) \text{ is non-deterministic}\}| \\
&= \sum_r |\{s \in \bigcup_{c\in C} S_c : r(s) = r\}| \mathbb{1}\{\{\varphi_s(P)\}_{s;r(s)=r} \text{ is non-deterministic}\} \\
&\leq \sum_r |\{s \in \bigcup_{c\in C} S_c : r(s) = r\}| \mathbb{1}\{\{\mathcal{P}_s(P)\}_{s;r(s)=r} \text{ is non-deterministic}\} \\
&= |\{s \in \bigcup_{c\in C} S_c : \mathcal{P}_s(P) \text{ is non-deterministic}\}|
\end{aligned}$$

there is r such that $\varphi_s(P)\}_{s;r(s)=r}$ is deterministic and $\mathcal{U}_s(P)\}_{s;r(s)=r}$ is non-deterministic. Pick the smallest such r. Either case 1 or 2 should hold.

Case 1: There is $s \in \bigcup_{c\in C} S_c$ with $r(s) = r$ and μ where $\mathcal{P}$ places positive probability on it (write $\mu(s) = k$), satisfying $k < \varphi_s(P)$. Since $\varphi(P)$ is NW, there is $s' \in \bigcup_{c\in C} S_c$ with $r(s') = r' > r$ and at some $\hat{\mu}$ where φ places positive probability on it (write $\hat{\mu}(s') = k'$), $k' = k < \varphi_s(P)$. That is, $r(s) < r(s')$ and $\hat{\mu}(s) > \hat{\mu}(s')$. Since φ satisfies RP, there are $c, c' \in C$ with $c \neq c'$, $s \in S_c$, and $s' \in S_{c'}$.

Case 2: There is $s \in \bigcup_{c\in C} S_c$ with $r(s) = r$ and μ where $\mathcal{P}$ places positive probability on it (write $\mu(s) = k$), satisfying $k > \varphi_s(P)$. Since $\mathcal{P}(P)$ is NW, $\varphi_s(P)$ is occupied by other $s' \in \bigcup_{c\in C} S_c$ with $r(s') < r$. At any $\hat{\mu}$ where φ places positive probability on it (write $\hat{\mu}(s') = k'$), $k' > \varphi_s(P)$. That is, $r(s) > r(s')$ and $\hat{\mu}(s) < \hat{\mu}(s')$. Since φ satisfies RP, there are $c, c' \in C$ with $c \neq c'$, $s \in S_c$, and $s' \in S_{c'}$.

We see below that Case 2 violates either of (i)-(iv). Case 1 also violates either of (i)-(iv) in the same manner.

Let $\bar{P}$ be $\bar{C} = C \cup \{\bar{c}\}$, $|S_{\bar{c}}| = |S_{c(s)}| \times |S_{c(s')}|$, and P is a reduced problem of $\bar{P}$. Since φ satisfies consistency, and $\varphi_s(P) = k$ is deterministic, we have $\varphi_s(\bar{P}) = \varphi_s(P) = k$. By ETR of φ, for $\bar{s}, \bar{s}' \in S_{\bar{c}}$ with $(r(\bar{s}), r(\bar{s}') = (r, r(s'))$, $\varphi_{\bar{s}}(\bar{P}) = \varphi_s(\bar{P}) = k$ (deterministic) and $\varphi_{\bar{s}'}(\bar{P}) = \varphi_{s'}(\bar{P})\ldots(**)$. Since there is positive probability for assignment μ^3 with $k < \mu^3(s')$, to satisfy (**), there should be positive probability for assignment μ^4 with $\mu^4(\bar{s}') > k$, that is, $\mu^4(\bar{s}') > \mu^4(\bar{s})$. Since $r(\bar{s}) > r(\bar{s}')$, RP is violated for $\bar{s}, \bar{s}' \in S_{\bar{c}}$. □

Proof of Proposition 3. The latter part is immediate because the best $|\bigcup_{c\in C} S_c|$ seats are assigned to students. For the first part, fix non-wasteful μ and set $(\omega, \triangleright)$ as follows: (i) for arbitrary $m > \bar{k}$, $\omega_s = m - 1$ if $\mu(s) = 1$, $\omega_s = m - 2$ if $\mu(s) = 2$, ..., $\omega_s = m - \bar{k}$ if $\mu(s) = \bar{k}$; (ii) fix any $\triangleright$. Clearly $\tilde{A}(\omega, \triangleright)(P) = \mu$. □

Proof of Proposition 4. We prove statement 1 below because the proof of statement 2 is parallel to it. NW is immediate from Proposition 3. We demonstrate the case with x: simple average.

Other cases are shown in the same logic.

RP: Take s, s' with (i) $\{c \in C : s \in S_c\} = \{c \in C : s' \in S_c\}$ and (ii) for each c with $s, s' \in S_c$, $s \succeq_c s'$. It suffices to show that, for each $\ell = 1, 2, \ldots, L$, s is assigned to a group at a strictly earlier step than a step s' is assigned. Since the step s (resp. s') is assigned ends at time $\frac{1}{\omega_s^\ell}$ (resp. $\frac{1}{\omega_{s'}^\ell}$), we must have $\frac{1}{\omega_s^\ell} - \frac{1}{\omega_{s'}^\ell} < 0$ where

$$\begin{aligned}
\frac{1}{\omega_s^\ell} - \frac{1}{\omega_{s'}^\ell} = \frac{1}{1/\mathbb{E}_c[a(s,c)]} - \frac{1}{1/\mathbb{E}_c[a(s',c)]} &= \mathbb{E}_c[a(s,c)] - \mathbb{E}_c[a(s',c)] \\
&= \mathbb{E}_c[a(s,c) - a(s',c)] \quad \because \text{(i)} \\
&< 0 \quad \because \text{(ii)}
\end{aligned}$$

ETA: Take s, s' with $\mathbb{E}_c[a(s,c)] = \mathbb{E}_c[a(s',c)]$. Since tie-breakers $(\rhd^\ell)_{\ell=1}^L$ are perfectly random, it suffices to show that, for each ℓ, s and s' are assigned a group at the same stage, i.e., $\frac{1}{\omega_s^\ell} - \frac{1}{\omega_{s'}^\ell} = 0$. We have

$$\frac{1}{\omega_s^\ell} - \frac{1}{\omega_{s'}^\ell} = \frac{1}{1/\mathbb{E}_c[a(s,c)]} - \frac{1}{1/\mathbb{E}_c[a(s',c)]} = \mathbb{E}_c[a(s,c)] - \mathbb{E}_c[a(s',c)] = 0$$

This completes the proof. □

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