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Tetsutaro Hatakeyama

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Institute for Economic Studies, Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan
ies-office@adst.keio.ac.jp
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Tetsutaro Hatakeyama
Graduate School of Economics, Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan
t.hatakeyama@keio.jp

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We study sequential assignment systems in which objects are assigned to agents in multiple stages. While such systems are prevalent in real-life school choice and college admissions, Dur and Kesten (2019) show that these systems are neither non-wasteful nor straightforward in general. To overcome this negative observation, we consider a model in which the mechanism designer chooses an allocation schedule, i.e., in which stage to allocate each object, as well as the allocation mechanisms it uses within a system. Our analysis newly reveals that (i) in general, no allocation schedules avoid wastefulness/non-straightforwardness and (ii) a non-wasteful/straightforward allocation schedule exists if and only if the preference domain is "tiered." This result supports practices in which the tiered domain naturally arises (e.g., Chinese college admissions practice). However, this also highlights the difficulty of sequential assignments in more diverse preference domains.

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1 Introduction

In recent decades, market design has been successful in analyzing and improving real-life resource allocations where monetary transfers are not allowed, for example, school choice (Abdulkadiroğlu and Sönmez, 2003) and college admissions (Balinski and Sönmez, 1999). While much of the existing literature concentrates on (and contributes by) designing *static* allocation systems, several systems

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†Graduate School of Economics, Keio University, 2-15-45 Mita, Minato-ku, Tokyo 108-8345, Japan. E-mail: t.hatakeyama@keio.jp

Order	Position category	Example(s)	First exam	Announcement of result
1	National ministries	Ministry of Finance	April 9	June 8
2	Metropolitan government	Tokyo	April 30	July 14
3	Prefectural governments	Osaka, Kyoto	June 18	August
4	City offices	many local cities	September 19	depends on cities

Table 1: 2023 exam timeline in Japan

in reality allocate resources *sequentially* in the sense that some are allocated first (e.g., this month) and the remaining are next (e.g., the next month).

In many regions of the United States, for example, Boston, Chicago, and New York, student enrollment systems for exam, charter, magnet, private, and specialized schools are operated separately from the enrollment system for regular (mainstream) schools with different timelines. Every student can participate in multiple systems where different systems adopt different mechanisms (e.g., student-proposing deferred acceptance or serial dictatorship) and different priority structures (e.g., lottery numbers or exam scores). Therefore, one student may participate in the exam school system and proceed to the regular school system only when she is not assigned to any exam school, while another student may only participate in the regular school system.

In China, the college admissions system comprises several stages where admissions for top-tier (the most prestigious) colleges are followed by admissions for lower-tier (less prestigious) colleges. In Japan, a similar sequential system is used in employment exams for public officers, that is, exams for the most prestigious national employee positions are followed by those for less prestigious prefecture/city-level employee positions (see Table 1).[1] Although extensively practiced, as shown in Dur and Kesten (2019), a sequential enrollment/admission has deficiencies, including inefficiency of allocations (wastefulness) and student incentives in reporting true preferences (non-straightforwardness).[2]

These examples pose a puzzle: a sequential assignment is not desirable in theory, but it is so prevalent in practice. In fact, static resource allocation is not that easy in some practices. According to Manjunath and Turhan (2016), a journalist says that it is difficult to unify public and private school enrollment systems in Milwaukee because they are in competition for the same students and have no incentives for coordination. Toward addressing this puzzling observation, we ask if there are circumstances when these deficiencies can be avoided. We show that despite the general result, under a restrictive but realistic condition (the *tiered* preference domain), a sequential assignment overcomes the above deficiencies. More interestingly, outside the tiered domain, the general impossibility result always holds (Proposition 2). Thus, our results also highlight the difficulty of sequential assignments in general diverse domains.

By slightly modifying the setting of Dur and Kesten (2019) and Andersson et al. (2018), this

[1]Table 1 is based on the webpage https://90r.jp/schedule.html (written in Japanese, accessed March 26, 2023).
[2]See Section 4 of Dur and Kesten (2019).

paper introduces a problem of designing an *allocation schedule*: Given a fixed number of stages (available dates) in which objects can be allocated, a mechanism designer (a local government) chooses in which stage to allocate each object to agents. The designer has two standard objectives: non-wastefulness (no object should be disposed if there is someone who wishes it) and straightforwardness (no lie should be profitable). We assume that objects (schools) are not strategic and they can coordinate enough to follow a fixed schedule. To focus on the design of the allocation schedule, we assume that each mechanism used at each stage satisfies certain desirable properties.[3] If we pick one stage and allocate all the objects in that stage, because it is a static system, we can readily achieve both non-wastefulness and straightforwardness. While we wish to allocate objects in two or more stages, Proposition 1 shows a negative result in general domains. This means that there are cases in which any schedule comprising multiple stages leads to wastefulness and non-straightforwardness. Afterward, we search for a condition for a positive result, that is, a condition under which non-wasteful and straightforward allocation schedules exist. The main result is that a special type of domain, the so-called "tiered domain," is necessary and sufficient for a possibility (Proposition 2). Roughly speaking, this domain contains two types of profiles: the one with all agents partially agreeing on which objects are good (bad) and the other with agents preferring only specific objects to being unassigned.

Proposition 2 gives us an implication on when sequential assignment costs (does not cost) wastefulness or non-straightforwardness. The tiered domain is highly restricted but it naturally arises under specific circumstances. In Chinese college admissions (Japanese employment exams), it is commonly presumed that everyone prefers a more prestigious college (position). As long as these presumptions are realistic, prestige-based preference tiers naturally arise, and thus, non-wastefulness and straightforwardness are both achievable via a sequential assignment. Interestingly, the allocation schedule in reality is actually "better tiers in earlier stages," which achieves non-wastefulness and straightforwardness.[4] This observation indicates that the schedule in reality may not be problematic in theory. In contrast, under more general, diverse preference profiles where no tiers exist, such a positive result does not hold.[5]

Related literature. After pioneering works by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003), many papers study real-life school choice and college admissions systems. Among them, Abdulkadiroğlu et al. (2005) document a concern on that two-stage implementation of stable mechanisms for specialized and regular schools possibly violates stability. Recently, several

[3]The properties are described in the next section. Specific mechanisms that satisfy these properties appear in Corollary 2.

[4]See Example 1 for this type of schedule. In cases (ii) and (iii), $\{a, b\}$ are better objects for everyone and thus they are allocated earlier than other objects.

[5]The example of school choice is a bit more complicated. If we just focus on regular and specialized schools, it is not implausible to assume that students who apply to specialized schools prefer them to any regular schools and the other majority students prefer being unassigned to any specialized schools. Under this assumption, regular/specialized-based preference tier arises. However, it is ambiguous if we can treat other (e.g., private) schools in the same manner. Preference tier may not exist in this example.

papers study sequential assignment problems similar to ours. Ekmekci and Yenmez (2019) study school incentives to join the common enrollment system and explain why exam and regular school markets operate separately. Dur and Kesten (2019) show that sequential allocation systems in practice suffer from several deficiencies, including wastefulness and non-straightforwardness, and analyze equilibria of associated preference revelation games. Andersson et al. (2018) and Doğan and Yenmez (2023) also study the properties of equilibria in sequential preference revelation games. The former finds a straightforward allocation system that minimizes wasted seats, while the latter finds a condition under which an additional stage improves student welfare. Our approach is different from these papers' in two directions: in this study, (i) we view the allocation schedule as a design variable, i.e., chosen by the mechanism designer, instead of a primitive, and (ii) to overcome deficiencies of the system, we restrict the preference domain instead of weakening normative requirements or restricting the priority structure. A similar problem is also studied under different commitment assumptions (Manjunath and Turhan, 2016; Turhan, 2019; Doğan and Yenmez, 2019). In their models, multiple mechanisms allocate school seats to students simultaneously, and students choose the most preferred school seat from assigned ones. That is, students participate in all the stages and can dispose assigned seats. Since this assumption makes the problem essentially different from ours, the solution to recover non-wastefulness is also different from ours. For instance, in Manjunath and Turhan (2016), a sufficient number of iterative re-matching processes recover non-wastefulness.

The paper is structured as follows. Section 2 describes the model. Section 3 establishes the results. Section 4 concludes. Proofs are in the Appendix.

2 Model

Let I and H be finite sets of **agents** and **objects** with $|I|, |H| \geq 2$. For each object $h \in H$, $q_h \in \mathbb{N}$ is its **capacity,** or the number of available copies. Each agent $i \in I$ has a linear **preference relation** P_i over $H \cup \{\emptyset\}$ and each object $h \in H$ has a linear **priority** $\succeq_h$ over $I \cup \{\emptyset\}$, where for both agents and objects, $\emptyset$ represents "being unassigned." Let R_i denote the associated "at least as good as" relation of agent i. We denote a **preference profile** by $P = (P_i)_{i \in I}$ and the sets of all the possible preferences and preference profiles by $\mathcal{P}$ and $\mathcal{P}^I$. To exclude trivial cases, we assume for each $P \in \mathcal{P}^I$, for each $h \in H$, there is $i \in I$ such that $h P_i \emptyset$.[6] We call a tuple $(I, H, P, q, \succeq)$, where $q = (q_h)_{h \in H}$ and $\succeq = (\succeq_h)_{h \in H}$, an **(assignment) problem.** Throughout the paper, we fix $(I, H, q, \succeq)$ and denote a problem by a preference profile P.

A **matching** $\mu : I \to H \cup \{\emptyset\}$ is a function such that the number of agents assigned an object does not exceed its capacity, that is, for each $h \in H$, $|\mu^{-1}(h)| \leq q_h$. A matching μ is **individually**

[6] Technically, we can add any number of objects that are not preferred to being unassigned by any agents. Such objects can be assigned at any stage (defined in Section 3), but they are not to be assigned to any agent as long as (φ^s) are individually rational. Therefore, they do not affect the outcome of mechanisms/systems.

rational at P if for each $i \in I$, $\mu(i) R_i \emptyset$ and $i \succeq_{\mu(i)} \emptyset$. That is, no agent is assigned an object that is worse than being unassigned nor he is unacceptable for. A matching μ is **non-wasteful** at P if there is no agent-object pair (i,h) such that $|\mu^{-1}(h)| < q_h$, $i \succeq_h \emptyset$, and $h P_i \mu(i)$. That is, there is no object being unassigned, even though an acceptable agent prefers it to what she is assigned.

A **mechanism** φ is a function that associates a matching to each problem, where its **domain** is denoted by $\mathcal{D} = \times_{i \in I} \mathcal{D}_i \subseteq \mathcal{P}^I$. We denote the matching that φ selects for problem P by $\varphi(P)$ and the object that φ assigns to $i \in I$ at P by $\varphi_i(P)$. We say that a mechanism φ is individually rational (non-wasteful) on $\mathcal{D}$ if the matching $\varphi(P)$ is individually rational (non-wasteful) for each problem $P \in \mathcal{D}$. A mechanism φ is **strategy-proof on** $\mathcal{D}$ if for each $i \in I$, for each $P \in \mathcal{D}$ and each $P'_i \in \mathcal{D}_i$, $\varphi_i(P) R_i \varphi_i(P'_i, P_{-i})$. That is, truthfully reporting his preference is always a (weakly) dominant strategy for every agent. A mechanism φ is **minimally non-bossy on** $\mathcal{D}$ if for each $i \in I$, for each $P \in \mathcal{D}$ and each $P'_i \in \mathcal{D}_i$, if $\varphi_i(P) = \varphi_i(P'_i, P_{-i}) = \emptyset$, then $\{j \in I : \varphi_j(P) = \emptyset\} = \{j \in I : \varphi_j(P'_i, P_{-i}) = \emptyset\}$. That is, no unassigned agent can make other agent unassigned by misreporting without being assigned an object. Remark 1 below shows that this new property is weak enough in that it is implied by two general axioms. When φ is individually rational, non-wasteful, strategy-proof, or minimally non-bossy on $\mathcal{P}^I$, we may omit the part "on $\mathcal{P}^I$."

Remark 1. When φ is either non-bossy or stable, it is minimally non-bossy.

We consider a **(sequential assignment) system** $\Psi = (\varphi^1, ..., \varphi^{|S|}, \sigma)$, a combination of $|S| \geq 2$ mechanisms and a schedule function. For each $(I, H, q, \succeq)$, a **schedule function** σ assigns objects to **stages** $\{1, ..., |S|\}$. We denote the assigned stage for object h by $\sigma_h(I, H, q, \succeq) \in \{1, ..., |S|\}$, or simply σ_h. We note that the outcome of a schedule function does not depend on the preference profile. This is because, in many real-life situations, the allocation schedule is fixed before agents report their preferences and the designer commits to a predetermined schedule. To focus on the design of the schedule function, we fix mechanisms (φ^s) to be individually rational and non-wasteful ones. For notational simplicity, for each $I' \subseteq I$ and each $H' \subseteq H$, we denote by $P|I', H'$ a restriction of a problem P to agents in I' and objects in H' (i.e., $P|I', H'$ denotes $(I', H', (P_i|_{H'})_{i \in I'}, (q_h)_{h \in H'}, (\succeq_h |_{I'})_{h \in H'})$ to be more precise). For each problem P, a system Ψ associates a matching $\Psi(P)$ (object $\Psi_i(P) \in H \cup \{\emptyset\}$ for agent $i \in I$) in the following sequential procedure.[7]

Stage 1. Set $I_1 = I$ and $H_1 = \{h \in H : \sigma_h = 1\}$. For each $i \in I$, let $\mu^1(i) = \varphi^1_i(P|I_1, H_1)$.
Stages $s = 2, ..., |S|$. Set $I_s = \{i \in I : \mu^{s-1}(i) = \emptyset\}$ and $H_s = \{h \in H : \sigma_h = s\}$. For each $i \in I$, let

$$\mu^s(i) = \begin{cases} \varphi^s_i(P|I_s, H_s) & (i \in I_s) \\ \mu^{s-1}(i) & (i \in I \backslash I_s) \end{cases}.$$

[7]We allow at each stage s, $|I_s| \in \{0,1\}$ or $|H_s| \in \{0,1\}$. When $|I_s| = 1$ or $|H_s| = 1$, the mechanism φ^s works in the same way as $|I_s|, |H_s| \geq 2$. When $|I_s| = 0$ or $|H_s| = 0$, to be precise, the stage s works as follows. If $|I_s| = 0$, let $\mu^{|S|}(i) = \mu^{s-1}(i)$ for each $i \in I$ and proceed to stage $|S|+1$. If $|H_s| = 0$, let $\mu^s(i) = \mu^{s-1}(i)$ for each $i \in I$ and proceed to stage $s+1$.

Stage $|S|+1$. For each $i \in I$, let $\Psi_i(P) = \mu^{|S|}(i)$.

Following Dur and Kesten (2019), we employ a crucial assumption on agent commitment. (i) At each stage, every agent chooses whether to participate in that stage,[8] and (ii) when an object is assigned to an agent, she cannot dispose that object and proceed to the next stage. Although strong, such a commitment rule is actually adopted, for instance, in Turkish school choice system (Andersson et al., 2018). To benefit from using multiple stages, we restrict our attention on multi-shot schedule functions. A function σ is **multi-shot** if for each problem P, there are objects $h, h' \in H$ such that $\sigma_h \neq \sigma_{h'}$.[9] We say that a system Ψ is individually rational (non-wasteful) on $\mathcal{D}$ if the matching $\Psi(P)$ is individually rational (non-wasteful) for each problem $P \in \mathcal{D}$. As an incentive property in a sequential assignment, we consider straightforwardness, which is introduced by Andersson et al. (2018). A system Ψ is **straightforward on** $\mathcal{D}$ if for each $i \in I$, for each $P \in \mathcal{D}$ and each $P'_i \in \mathcal{D}_i$, $\Psi_i(P) R_i \Psi_i(P'_i, P_{-i})$. That is, reporting true preference at all stages is always a (weakly) dominant strategy for every agent.

3 Results

We first see the result on the full domain $\mathcal{D} = \mathcal{P}^I$.

Proposition 1. *Let (φ^s) be individually rational and non-wasteful mechanisms on $\mathcal{P}^I$. If σ is multi-shot, $((\varphi^s), \sigma)$ is neither non-wasteful nor straightforward on $\mathcal{P}^I$.*

Motivated by this impossibility result, we characterize a preference domain where a multi-shot system achieves non-wastefulness/straightforwardness. For $|T| \in \mathbb{N}$, a preference profile $P \in \mathcal{P}^I$ is ($|T|$-)**tiered** if objects H are partitioned to $\{H^P_t\}_{t \in \{1,\dots,|T|\}}$ such that for each $h \in H^P_t$ and each $h' \in H^P_{t'}$ with $t < t'$, for each $i \in I$ with $h, h' P_i \emptyset$, $h P_i h'$.[10] We call the set of all possible tiered preference profiles with $|T| \geq 2$ as a **tiered domain** and denote it by $\mathcal{P}^I(T)$.

Roughly speaking, 2-tiered preferences are more "similar to each other" than 1-tiered preferences, and 10-tier is much more than 2-tier. However, by definition, $\mathcal{P}^I(T)$ also includes preference profiles in which for each $h \in H^P_t$ and each $h' \in H^P_{t'}$ with $t < t'$, there is no $i \in I$ with $h, h' P_i \emptyset$. In such cases, tiered preferences are not similar to each other (see the next example).

Example 1. Agents and objects are $I = \{1, 2\}$ and $H = \{a, b, c, d\}$. Table 2 shows several preference profiles. In cases (vi) and (v), P_1 and P_2 are not similar to each other, but they are also tiered.

[8] An agent can choose not to participate in stage s by reporting that all objects to be allocated at stage s are worse than being unassigned.

[9] When this does not hold, at problem P, using system $((\varphi^s), \sigma)$ is equivalent to allocating all objects at some stage s via mechanism φ^s. Such an allocation is not what we intend.

[10] A tiered preference domain in a matching environment is also studied in Kesten (2010) and Kesten and Kurino (2019). The definition in the current study is slightly different from theirs in that we do not restrict the number of tiers to be two. Akahoshi (2014) and Kandori et al. (2010) also study similar domains.

case	P_1	P_2	$\|T\|$	$\{H_t^P\}$	σ
(i)	a, b, c, d	d, c, b, a	-	-	-
(ii)	a, b, c, d	b, a, d, c	2	$\{a, b\}, \{c, d\}$	$\sigma_a = \sigma_b = 1,\ \sigma_c = \sigma_d = 2$
(iii)	a, b, c, d	a, b, c, d	4	$\{a\}, \{b\}, \{c\}, \{d\}$	$\sigma_a = \sigma_b = 1,\ \sigma_c = \sigma_d = 2$
(vi)	a, b	c, d	2	$\{a, b\}, \{c, d\}$	$\sigma_a = \sigma_b = 1,\ \sigma_c = \sigma_d = 2$
(v)	a, b, c, d	a	2	$\{a\}, \{b, c, d\}$	$\sigma_a = 1,\ \sigma_b = \sigma_c = \sigma_d = 2$

Table 2: Example 1

Note: In columns 2 and 3, objects are ordered by preference relations. Objects below $\emptyset$ are omitted. Columns 3, 4, and 5 show examples of possible $|T|$'s, $\{H_t^P\}$'s, and σ's that attain non-wastefulness/straightforwardness when mechanisms are appropriately chosen (e.g., the serial dictatorship). Hyphen means no tier (nonexistence).

Our main result is the next proposition.

Proposition 2. *The following three are equivalent.*
(a) When (φ^s) are individually rational and non-wasteful, there is a multi-shot σ where Ψ is non-wasteful on $\mathcal{D} \subseteq \mathcal{P}^I$.
(b) When (φ^s) are strategy-proof, minimally non-bossy, individually rational, and non-wasteful, there is a multi-shot σ where Ψ is straightforward on $\mathcal{D} \subseteq \mathcal{P}^I$.
(c) The domain is a subset of a tiered domain, that is, $\mathcal{D} \subseteq \mathcal{P}^I(T)$.

As shown in the Appendix, the proof provides a specific class of schedule functions that maps better tiers to earlier stages without reversal. Using this class joint with well-designed mechanisms, the result can be extended as follows: when the designer wishes to use three or more stages, straightforwardness and non-wastefulness are both achievable under sufficiently fine tiers.

Corollary 1. *When $P \in \mathcal{P}^I(T)$ with $|T| \geq |S|$, there is a straightforward and non-wasteful system in which an object is allocated at each stage, that is, for each $s \in \{1, ..., |S|\}$, there is $h \in H$ such that $\sigma_h = s$.*

Finally, we relate our result with well-known mechanisms that are strategy-proof, individually rational, and non-wasteful, such as agent/object-proposing deferred acceptance (aDA/oDA), top trading cycles (TTC), and serial dictatorship (SD). Since oDA, TTC, and SD are non-bossy (Afacan and Dur, 2017; Pápai, 2000; Svensson, 1999) and aDA is stable, Remark 1 leads to the following.

Corollary 2. *When each φ^s is either aDA, oDA, TTC, or SD, there is a multi-shot σ where Ψ is straightforward and non-wasteful on $\mathcal{P}^I(T)$.*

4 Conclusion

This paper searches for a solution to a puzzle in which a sequential assignment is not desirable in theory but commonly used in practice. The main result gives a rationale for the practice of

Chinese college admissions but makes clear that a sequential assignment system is always wasteful and non-straightforward in more diverse preference domains. This result provides a practical insight on when (not) to use a sequential assignment system.

It may also be valuable to collect evidence on whether the tiered profile is realistic. A positive result further supports sequential assignment practice and vice versa. We leave it for future research.

Declarations

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Appendix

Proof of Remark 1. "Non-bossy⇒minimally non-bossy" is immediate from the definitions. We show "stable⇒minimally non-bossy." Let φ be a stable mechanism and fix $i \in I$, $P \in \mathcal{D}$, and $P'_i \in \mathcal{D}_i$ such that $\varphi_i(P) = \varphi_i(P'_i, P_{-i}) = \emptyset$. Denote $\varphi(P) = \mu$ and $\varphi(P'_i, P_{-i}) = \mu'$. Immediately from the definition of a stable matching, in a problem without i, i.e., $(I \backslash \{i\}, H, (P_j)_{j \in I \backslash \{i\}}, q, \succeq)$, μ and μ' are also stable. The rural hospital theorem (Roth and Sotomayor, 1990) implies that $\{j \in I \backslash \{i\} : \mu(j) \in H\} = \{j \in I \backslash \{i\} : \mu'(j) \in H\}$, i.e., $\{j \in I : \mu(j) = \emptyset\} = \{j \in I : \mu'(j) = \emptyset\}$. $\square$

Hereafter, we refer individual rationality as IR, non-wastefulness as NW, strategy-proofness as SP, minimal non-bossiness as MNB, and straightforwardness as SF. Also, for each tiered profile $P \in \mathcal{P}^I(T)$, for each object $h \in H$, we denote the tier which the object belongs to by $t(h) \in \{1, ..., |T|\}$.

Proof of Proposition 1. Consider a problem with agents $I = \{1, 2\}$, objects $H = \{h, \ell\}$, preference $P_1 : h, \ell, \emptyset$ and $P_2 : \ell, h, \emptyset$, capacity $q_h = q_\ell = 2$, and priority $\succeq_h, \succeq_\ell$: $1, 2, \emptyset$. Let $|S| = 2$ and (φ^1, φ^2) be IR and NW. It is sufficient to show that $\sigma_h \neq \sigma_\ell$ leads to violation of NW/SF.

<u>(i) $(\sigma_h, \sigma_\ell) = (1, 2)$</u>. $I_1 = \{1, 2\}$ and $H_1 = \{h\}$. Since there is a unique IR and NW matching, φ^1 selects $\varphi^1_1(P|\{1, 2\}, \{h\}) = \varphi^1_2(P|\{1, 2\}, \{h\}) = h$. Thus, $(\Psi_1(P), \Psi_2(P)) = \mu = (h, h)$. μ is wasteful because $|\mu^{-1}(\ell)| = 0 < 2 = q_\ell$, $2 \succeq_\ell \emptyset$, and $\ell P_2 \mu(2)$. Moreover, if agent 2 reports $P'_2 : \ell, \emptyset, h$, by IR and NW of (φ^1, φ^2), $(\varphi^1_1(P_1, P'_2|\{1, 2\}, \{h\}), \varphi^1_2(P_1, P'_2|\{1, 2\}, \{h\})) = (h, \emptyset)$ and $\varphi^2_2(P_1, P'_2|\{2\}, \{\ell\}) = \ell$. It follows $(\Psi_1(P_1, P'_2), \Psi_2(P_1, P'_2)) = (h, \ell)$ and $\Psi_2(P_1, P'_2) P_2 \Psi_2(P)$. Ψ is not SF.

(ii) $(\sigma_h, \sigma_\ell) = (2,1)$. $I_1 = \{1,2\}$ and $H_1 = \{\ell\}$. The same argument as case 1 leads $(\Psi_1(P), \Psi_2(P)) = \mu' = (\ell, \ell)$. μ' is wasteful because $|\mu'^{-1}(h)| = 0 < 2 = q_h$, $1 \succeq_h \emptyset$, and $hP_1\mu'(1)$. For $P_1' : h, \emptyset, \ell$, $(\Psi_1(P_1', P_2), \Psi_2(P_1', P_2)) = (h, \ell)$ and $\Psi_2(P_1', P_2) P_1 \Psi_2(P)$. Ψ is not SF. $\square$

Proof of Proposition 2.

(c)⇒(a): Suppose (c) is true and (φ^s) are IR and NW. Let σ be such that for any $t, t' \in \{1,...,|T|\}$ with $t \le t'$, for each $h \in H_t^P$ and each $h' \in H_{t'}^P$, $\sigma_h \le \sigma_{h'}$. Denote $\Psi(P) = \mu$. Take $h \in H$ such that $|\mu^{-1}(h)| < q_h$ and $i \in I$ such that $i \succeq_h \emptyset$ arbitrarily. If there are no such pairs, μ is NW. Suppose there is at least one such pair and show $\mu(i) R_i h$, i.e., $\mu(i) P_i h$ or $\mu(i) = h$.

case 1.1. $i \notin I_{\sigma_h}$. For some $s < \sigma_h$, i is assigned an object $\mu^s(i) = \varphi_i^s(P|I_s, H_s) \in H_s$. If $t(\mu^s(i)) \ge t(h)$, since $\sigma_{\mu^s(i)} = s < \sigma_h$, it contradicts to (c). Hence, $t(\mu^s(i)) < t(h)$ implies $\mu^s(i) P_i h$. We have $\mu(i) = \mu^{|S|}(i) = \mu^s(i)$ and $\mu(i) P_i h$.

case 1.2. $i \in I_{\sigma_h}$. Since φ^{σ_h} is NW, $\varphi_i^{\sigma_h}(P|I_{\sigma_h}, H_{\sigma_h}) R_i h$. If $\varphi_i^{\sigma_h}(P|I_{\sigma_h}, H_{\sigma_h}) = \emptyset$, $\emptyset P_i h$. Since $(\varphi^{\sigma_h+1}, ..., \varphi^{|S|})$ are all IR, $\mu(i) R_i \emptyset P_i h$. Otherwise, $i \notin I_{\sigma_h+1}$ implies $\mu^{\sigma_h+1}(i) = \mu^{\sigma_h}(i) = \varphi_i^{\sigma_h}(P|I_{\sigma_h}, H_{\sigma_h})$. Repeating this, we have $\mu(i) = \mu^{|S|}(i) = \varphi_i^{\sigma_h}(P|I_{\sigma_h}, H_{\sigma_h})$ and $\mu(i) R_i h$.

(a)⇒(c): We show its contrapositive "not (c)⇒not (a)." Suppose $P \notin \mathcal{P}^I(T)$. Fix a multi-shot σ and IR and NW (φ^s) arbitrarily. We construct a problem, i.e., q and $\succeq$, so that $\Psi(P)$ is wasteful.

case 2.1. Suppose that there are $s, s' \in \{1, ..., |S|\}$ with $s < s'$, $h, \ell \in H_s$, $h', \ell' \in H_{s'}$, and $i, i' \in I$ such that $h, \ell' P_i \emptyset$ and $\ell, h' P_{i'} \emptyset$. We do not exclude $h = \ell$ or $h' = \ell'$, but we exclude $h P_i \ell' P_i \emptyset$ and $\ell P_{i'} h' P_{i'} \emptyset$, i.e., since the case is symmetric, we specify the case to $\ell' P_i h P_i \emptyset$. This is because, if for any $h, \ell \in H_s$ and any $h', \ell' \in H_{s'}$, for any $i, i' \in I$ with $h, \ell' P_i \emptyset$ and $\ell, h' P_{i'} \emptyset$, $h P_i \ell'$ and $\ell P_{i'} h'$, we immediately have $P \in \mathcal{P}^I(T)$ by setting $\{H_t^P\} = \{H_s\}$, i.e., for each $t \in \{1, ..., |T|\}$, $H_t^P := \{h \in H : \sigma_h = t\} = H_t$. Let (i) $i \succeq_h \emptyset$ and $i \succeq_{\ell'} \emptyset$, (ii) for each $j \in I \backslash \{i\}$, $\emptyset \succeq_h j$ and $\emptyset \succeq_{\ell'} j$, and (iii) for each $k \in H \backslash \{h, \ell'\}$, $\emptyset \succeq_k i$. At each stage $r \in \{1, ..., s-1\}$, by (iii), φ^r, which is IR, never assigns any $k \in H_r \subsetneq H \backslash \{h, \ell'\}$ to i, thus $i \in I_s$. At stage s, by (ii), φ^s, which is IR, never assigns $h \in H_s$ to any $j \in I_s \backslash \{i\}$ and by (iii), it never assigns any $k \in H_s \backslash \{h\} \subsetneq H \backslash \{h, \ell'\}$ to i. Since $h P_i \emptyset$, φ^s, which is NW, assigns $h \in H_s$ to $i \in I_s$, thus, $\varphi_i^s(P|I_s, H_s) = h$. We have $\Psi_i(P) = h$. At stage s', by (ii), $\varphi^{s'}$, which is IR, never assigns $\ell' \in H_{s'}$ to any $j \in I_{s'} \subsetneq I \backslash \{i\}$, thus for each $j \in I_{s'}$, $\varphi_j^{s'}(P|I_{s'}, H_{s'}) \ne \ell'$, i.e., $\Psi_j(P) \ne \ell'$. $\Psi(P) = \mu$ is wasteful because $|\mu^{-1}(\ell')| = 0 < q_{\ell'}$, $i \succeq_{\ell'} \emptyset$, and $\ell' P_i \mu(i)$.

case 2.2. Suppose that there are no $s, s' \in \{1, ..., |S|\}$ with $s < s'$, $h, \ell \in H_s$, $h', \ell' \in H_{s'}$, and $i, i' \in I$ such that $h, \ell' P_i \emptyset$ and $\ell, h' P_{i'} \emptyset$, including the cases with $h = \ell$ or $h' = \ell'$. That is, for any $s, s' \in \{1, ..., |S|\}$ with $s < s'$, there is at most one $i(s, s') \in I$ such that $h, h' P_{i(s,s')} \emptyset$. If for any s, s', there is no $i(s, s')$, we immediately have $P \in \mathcal{P}^I(T)$ by setting $\{H_t^P\} = \{H_s\}$. Thus, we assume that there is at least one pair (s, s') such that $i(s, s') \in I$.

case 2.2.1. For each $|M| \in \{3, ..., |S|\}$, there are no $\{i_1, ..., i_{|M|}\} \subseteq I$ and $\{\{h_m, h_m'\}\}_{m=1}^{|M|} \subseteq H$ such that (i) for each $m \in \{1, ..., |M|\}$, $\sigma_{h_m} = \sigma_{h_m'}$ and for any $m, m' \in \{1, ..., |M|\}$, $\sigma_{h_m} \ne \sigma_{h_{m'}}$,

and (ii) for each $m \in \{1, ..., |M|\}$, $h'_{m+1} P_{i_m} h_m$ where $h'_{m+1} = h'_1$. We do not exclude the cases where for each m, $h_m = h'_m$. To show that $P \in \mathcal{P}^I(T)$, we first see that there is $\bar{s} \in \{1, ..., |S|\}$ such that for each $h \in H_{\bar{s}}$, there is no $i \in I$ and no $h' \notin H_{\bar{s}}$ with $h' P_i h$. Suppose that there are no such $\bar{s}$, i.e., for each $s \in \{1, ..., |S|\}$, there is $h \in H_s$ such that there is $i \in I$ and $h' \notin H_s$ with $h' P_i h$. Then, we can construct $\{i_1, ..., i_{|M|}\} \subseteq I$ and $\{\{h_m, h'_m\}\}_{m=1}^{|M|} \subseteq H$ by the following steps.
Step 1. Pick $h \in H_1$ such that there is $i \in I$ and $h' \notin H_1$ with $h' P_i h$. Let $i_1 = i$, $h_1 = h$, and $h'_2 = h'$.
Steps $m = 2, ..., |S|$. Pick h from the same stage as h'_m's such that there is i and h' in another stage with $h' P_i h$. Let $i_m = i$, $h_m = h$, and $h'_{m+1} = h'$.
By assumption, at each step m, $\sigma_{h'_{m+1}} \notin \{\sigma_{h_1}, ..., \sigma_{h_{m-1}}\}$. However, since $|S|$ is finite, at step $|S|$, there is no $h'_{|S|+1}$ such that $\sigma_{h'_{|S|+1}} \notin \{\sigma_{h_1}, ..., \sigma_{h_{|S|-1}}\}$, i.e., either $\sigma_{h'_{|S|+1}} = \sigma_{h'_{|S|}}$ or $|S| = \bar{s}$. Hence, there is $\bar{s} \in \{1, ..., |S|\}$. A partition $\{H_1^P, H_2^P\} = \{H_{\bar{s}}, H \backslash H_{\bar{s}}\}$ is 2-tier preference, thus $P \in \mathcal{P}^I(T)$.

case 2.2.2. For some $|M| \in \{3, ..., |S|\}$, there are $\{i_1, ..., i_{|M|}\} \subseteq I$ and $\{\{h_m, h'_m\}\}_{m=1}^{|M|} \subseteq H$ described above. We show that $\Psi(P)$ is wasteful. By assumption, there is $m \in \{1, ..., |M|\}$ such that $\sigma_{h_m} < \sigma_{h'_{m+1}}$ and for $i_m \in I$, $h'_{m+1} P_{i_m} h_m$. Let (i) $q_{h_m} = q_{h'_{m+1}} = 1$, (ii) $i_m \succeq_{h_m} \emptyset$ and $i_m \succeq_{h'_{m+1}} \emptyset$, (iii) for each $j \in I \backslash \{i_m\}$, $\emptyset \succeq_{h_m} j$ and $\emptyset \succeq_{h'_{m+1}} j$, and (iv) for each $k \in H \backslash \{h_m, h'_{m+1}\}$, $\emptyset \succeq_k i_m$. By the same argument as case 2.1, we have $\Psi_{i_m}(P) = \varphi^{\sigma_{h_m}}(P | I_{\sigma_{h_m}}, H_{\sigma_{h_m}}) = h_m$, and $\Psi(P) = \mu$ is wasteful because $|\mu^{-1}(h'_{m+1})| = 0 < q_{h'_{m+1}}$, $i_m \succeq_{h'_{m+1}} \emptyset$, and $h'_{m+1} P_{i_m} \mu(i_m)$.

(c)⇒(b): Suppose (c) is true and (φ^s) are SP, MNB, IR, and NW. Let σ be such that for any $t, t' \in \{1, ..., |T|\}$ with $t \leq t'$, for each $h \in H_t^P$ and each $h' \in H_{t'}^P$, $\sigma_h \leq \sigma_{h'}$. Denote $\Psi(P) = \mu$. Fix a problem and take $i \in I$ arbitrarily. To lead a contradiction, suppose there is $P'_i \in \mathcal{P}(T)$ such that $\Psi_i(P'_i, P_{-i}) P_i \Psi_i(P)$. Denote the agents at stage s under profile P by I_s and under (P'_i, P_{-i}) by I'_s.
case 3.1. $\Psi_i(P) = \emptyset$, i.e., for each $s \in \{1, ..., |S|\}$, $\varphi_i^s(P | I_s, H_s) = \emptyset$. $\Psi_i(P'_i, P_{-i}) P_i \Psi_i(P)$ implies $\Psi_i(P'_i, P_{-i}) \neq \emptyset$, i.e., there is $s' \in \{1, ..., |S|\}$ such that for each $s'' < s'$, $\varphi_i^{s''}(P'_i, P_{-i} | I'_{s''}, H_{s''}) = \emptyset$ and $\Psi_i(P'_i, P_{-i}) = \varphi_i^{s'}(P'_i, P_{-i} | I'_{s'}, H_{s'}) \in H_{s'}$. Since for each $s'' < s'$, $\varphi_i^{s''}(P | I_{s''}, H_{s''}) = \varphi_i^{s''}(P'_i, P_{-i} | I'_{s''}, H_{s''}) = \emptyset$, MNB of $\varphi^{s''}$ implies $I_{s''} = I'_{s''}$, i.e., by induction, $I_{s'} = I'_{s'}$. $\Psi_i(P'_i, P_{-i}) P_i \Psi_i(P)$ implies $\varphi_i^{s'}(P'_i, P_{-i} | I_{s'}, H_{s'}) P_i \varphi_i^{s'}(P | I_{s'}, H_{s'})$, a contradiction to SP of $\varphi^{s'}$.
case 3.2. $\Psi_i(P) \neq \emptyset$, i.e., there is $s \in \{1, ..., |S|\}$ such that $\Psi_i(P) = \varphi_i^s(P | I_s, H_s) \in H_s$. If $\Psi_i(P'_i, P_{-i}) = \emptyset$, $\Psi_i(P'_i, P_{-i}) P_i \Psi_i(P)$ implies $\emptyset P_i \varphi_i^s(P | I_s, H_s)$, a contradiction to that φ^s is IR. Thus, $\Psi_i(P'_i, P_{-i}) \neq \emptyset$, i.e., there is $s' \in \{1, ..., |S|\}$ such that $\Psi_i(P'_i, P_{-i}) = \varphi_i^{s'}(P'_i, P_{-i} | I'_{s'}, H_{s'}) \in H_{s'}$.
case 3.2.1. $s < s'$. By the construction of σ, we have $t(\varphi_i^s(P | I_s, H_s)) < t(\varphi_i^{s'}(P'_i, P_{-i} | I'_{s'}, H_{s'}))$. Since each φ^s is IR, $\varphi_i^s(P | I_s, H_s), \varphi_i^{s'}(P'_i, P_{-i} | I'_{s'}, H_{s'}) P_i \emptyset$ implies $\varphi_i^s(P | I_s, H_s) P_i \varphi_i^{s'}(P'_i, P_{-i} | I'_{s'}, H_{s'})$, i.e., $\Psi_i(P) P_i \Psi_i(P'_i, P_{-i})$, a contradiction.
case 3.2.2. $s \geq s'$. Since for each $s'' < s'$, $\varphi_i^{s''}(P | I_{s''}, H_{s''}) = \varphi_i^{s''}(P'_i, P_{-i} | I'_{s''}, H_{s''}) = \emptyset$, MNB of $\varphi^{s''}$ implies $I_{s''} = I'_{s''}$, i.e., by induction, $I_{s'} = I'_{s'}$. $\Psi_i(P'_i, P_{-i}) P_i \Psi_i(P)$ implies $\varphi_i^{s'}(P'_i, P_{-i} | I_{s'}, H_{s'}) P_i \varphi_i^{s'}(P | I_{s'}, H_{s'})$, a contradiction to SP of $\varphi^{s'}$.

(b)⇒(c): We show its contrapositive "not (c)⇒not (b)." Suppose $P \notin \mathcal{P}^I(T)$. Fix a multi-shot σ

and (φ^s) satisfying SP, MNB, IR, and NW arbitrarily. We construct a problem, i.e., q and $\succeq$, so that Ψ is not SF. By the argument in (a)$\Rightarrow$(c), it is sufficient to see the following two cases.

case 4.1. Consider the same problem as case 2.1. That is, there are $s, s' \in \{1, ..., |S|\}$ with $s < s'$, $h, \ell \in H_s$, $h', \ell' \in H_{s'}$, and $i \in I$ such that $\ell' P_i h P_i \emptyset$. Let (i) $i \succeq_h \emptyset$ and $i \succeq_{\ell'} \emptyset$, (ii) for each $j \in I \backslash \{i\}$, $\emptyset \succeq_h j$ and $\emptyset \succeq_{\ell'} j$, and (iii) for each $k \in H \backslash \{h, \ell'\}$, $\emptyset \succeq_k i$. Since (φ^s) are IR and NW, it is obvious that $\Psi_i(P) = h$. By misreporting P'_i in which $\ell' P'_i \emptyset P'_i h$, i obtains $\varphi^s_i((P'_i, P_{-i})|I'_s, H_s) = \emptyset$ and $\varphi^{s'}_i((P'_i, P_{-i})|I'_{s'}, H_{s'}) = \ell'$, hence $\Psi_i(P'_i, P_{-i}) = \ell'$. Ψ is not SF.

case 4.2. Consider the same problem as case 2.2.2. That is, there is $\{i_1, ..., i_{|M|}\} \subseteq I$ and $\{\{h_m, h'_m\}\}_{m=1}^{|M|} \subseteq H$ such that (i) for each $m \in \{1, ..., |M|\}$, $\sigma_{h_m} = \sigma_{h'_m}$ and for any $m, m' \in \{1, ..., |M|\}$, $\sigma_{h_m} \neq \sigma_{h_{m'}}$, and (ii) for each $m \in \{1, ..., |M|\}$, $h'_{m+1} P_{i_m} h_m$ where $h'_{m+1} = h'_1$. There is $m \in \{1, ..., |M|\}$ such that $\sigma_{h_m} < \sigma_{h'_{m+1}}$ and for $i_m \in I$, $h'_{m+1} P_{i_m} h_m$. Let (i) $q_{h_m} = q_{h'_{m+1}} = 1$, (ii) $i_m \succeq_{h_m} \emptyset$ and $i_m \succeq_{h'_{m+1}} \emptyset$, (iii) for each $j \in I \backslash \{i_m\}$, $\emptyset \succeq_{h_m} j$ and $\emptyset \succeq_{h'_{m+1}} j$, and (iv) for each $k \in H \backslash \{h_m, h'_{m+1}\}$, $\emptyset \succeq_k i_m$. By the same argument as case 4.1, we have $\Psi_{i_m}(P) = \varphi^{\sigma_{h_m}}(P|I_{\sigma_{h_m}}, H_{\sigma_{h_m}}) = h_m$ and for misreporting $h'_{m+1} P'_{i_m} \emptyset P'_{i_m} h_m$, $\Psi_{i_m}(P'_{i_m}, P_{-i_m}) = \varphi^{\sigma_{h'_{m+1}}}((P'_{i_m}, P_{-i_m})|I'_{\sigma_{h'_{m+1}}}, H_{\sigma_{h'_{m+1}}}) = h'_{m+1}$. Ψ is not SF. $\square$

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