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Quota Adjustment Process

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【要旨】

In designing priority-based matching problems, such as school choice, the quota of each school is, to some extent, a design variable for a market designer. We define an ex-post student-optimal stable matching (ESOSM) as a matching that is stable at some implementable quota distribution, and is not Pareto dominated by any stable matching among all implementable quota distributions. A simple question is whether the total resources subject to several constraints are optimally distributed in the above sense. As optimal quota distributions vary depending on students' preferences realized, no predetermined quota distribution is optimal for all of the students' preferences. This requires a new matching mechanism design for the variable quota distributions. We propose the novel mechanism called quota adjustment process (QAP), which endogenously finds an ESOSM and the corresponding optimal quota distribution for any preferences and any initial quota distribution. To put this into practice, we proposed the QAP in the process of admission reform at the University of Tsukuba in Japan. Our proposal was officially approved and implemented at the University of Tsukuba in 2021.

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Quota Adjustment Process*

Taro Kumano† Morimitsu Kurino‡

November 18, 2022

Abstract

In designing priority-based matching problems, such as school choice, the quota of each school is, to some extent, a design variable for a market designer. We define an **ex-post student-optimal stable matching (ESOSM)** as a matching that is stable at some implementable quota distribution, and is not Pareto dominated by any stable matching among all implementable quota distributions. A simple question is whether the total resources subject to several constraints are optimally distributed in the above sense. As optimal quota distributions vary depending on students' preferences realized, no predetermined quota distribution is optimal for all of the students' preferences. This requires a new matching mechanism design for the variable quota distributions. We propose the novel mechanism called **quota adjustment process (QAP)**, which endogenously finds an ESOSM and the corresponding optimal quota distribution for any preferences and any initial quota distribution.

To put this into practice, we proposed the QAP in the process of admission reform at the University of Tsukuba in Japan. Our proposal was officially approved and implemented at the University of Tsukuba in 2021.

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1 Introduction

Market design has found various applications to improve existing markets in the past few decades. Although applications are increasing year after year, design requests are becoming more demanding. Matching theory traditionally assumes that the quotas or capacities of schools (a quota distribution) are exogenous. The literature has successfully developed a theory of stable matching, guiding a market designer to manage a matching market. Remarkable examples include entry-level medical labor markets in the US (Roth, 1984) and school choice programs in many countries (Abdulkadiroğlu and Sönmez, 2003).

However, it is often the case that a market designer can actively control quota distribution.[1] Prominent applications include a resident-hospital matching with regional caps (Kamada and Kojima, 2015), and school choice with affirmative actions (Ehlers et al., 2014). In addition, there are several important real-life situations in which the market designer should determine the quota distribution:

- **Program selection at the University of Tsukuba**

  The first example is undergraduate students' selection of their programs at the University of Tsukuba – one of the largest national universities in Japan. Each academic department offers one or more educational programs. The university president decided to introduce a matching mechanism where students can "choose" their programs at the end of their first year. A distinctive feature is that the university desired quotas of programs to be adjusted depending on their popularity among students. However, the range of total quotas of all programs within a department is not allowed to change due to government regulations. Thus, the university faces a distributional constraint where some specific quota distributions are allowed.

- **Vaccine distribution**

  Another example is vaccine distribution.[2] As vaccines are shot by doctors, each state prepares inoculation venues. People who are willing to take a shot need to reserve a venue. Subject to the aggregate supply, the government needs to distribute vaccines to venues. People have preferences over venues but they are unknown ex-ante. In practice, the government predicts the demand at each venue and distributes vaccines accordingly.

[1] There is existing literature that determines quotas endogenously in two-sided matching markets (Sönmez, 1997). It investigates whether a hospital (or school) truly reports its quota. Unlike the market designer in our study, the quotas are determined through a mechanism by hospitals.

[2] This is not the reservation but the distribution. See Hakimov et al. (2021) for a desirable reservation system.

However, it often happens that some places are short of vaccines though some places are left with surplus.

- **College admission in Ukraine**

  Ukraine's college admission since 2008 is another example of quota adjustments in terms of the matching problem (Kiselgov, 2011). Public universities offer two types of seats: state-financed and open-enrollment. The former has no tuition fees and the latter does. The quotas of state-financed seats are adjusted depending on popularity. Although this policy affects students, it is not known how an "optimal" quota distribution can be found.

From the above examples, we can derive the result that a quota distribution is rather a control variable for market designers.

This study extends the matching problem by allowing a quota distribution to be a variable. The extended problem consists of the set of students, students' preferences for programs, the set of programs with the upper quotas, priority rankings of programs over students, and the set of departments with fixed quotas. Each program belongs to one department, and the sum of the quotas of programs in some department is equal to its fixed quota. Thus, as long as each program does not exceed its upper limit, the quotas of programs within a department are variable. A quota distribution is an element of the product of the set of quota distributions of all departments. Hence, such a quota distribution can be implemented by a market designer. Given a quota distribution, each student is matched with at most one program, and each program can accept students up to the quota of the given quota distribution.

The literature has successfully developed the theory of stable matchings when a quota distribution is given exogenously. A matching is stable if no pair of a student and a program finds it preferable to deviate from the current matching. Gale and Shapley (1962) introduced the deferred acceptance (DA) mechanism which finds a stable matching. Moreover, the DA mechanism finds a stable matching that is not Pareto dominated by any other stable matching, and Dubins and Freedman (1981) and Roth (1982) pointed out its strategy-proofness in that it is immune to strategic manipulation by a single student. However, when a quota distribution is variable, it is important to note that the literature has never had any guidance as to which quota distribution (and thus, which matching) is "good" or "bad" in any sense.

We say that a matching is ex-post student-optimal stable if given a student's preference profile, it is stable at some implementable quota distribution and is not Pareto dominated by any other stable matching at any quota distribution. We abbreviate ex-post student-optimal stable matching as ESOSM. It is clear that an ESOSM always exists. This solution concept mainly has the two interpretations: Intuitively, the ESOSM coincides with one of the matchings obtained via DA mechanism at various quota distributions as if a market designer knew a quota distribution which maximizes the welfare of students among all stable matchings. Another interpretation is that it is more theoretical. Among all matchings at all quota distributions, no group of students and programs can find it preferable to deviate from an ESOSM, except for a case in which a student and a program rematch while some other student desires that program and has a higher priority. We say that a quota distribution is optimal if it achieves an ESOSM.

While keeping the resulting matching stable, the choice of quota distribution matters. Given a fixed total quotas, it is not optimal to choose a quota distribution arbitrarily; that is, some stable matching at some quota distribution, even though it is not Pareto dominated by any other stable matching at that quota distribution, may be Pareto dominated by another stable matching at the other quota distribution. The following example clarifies this point:

Example 1. There are 100 agents, $i_1, i_2, \cdots, i_{100}$, and 100 venues $x_1, x_2, \cdots, x_{100}$ and each agent i_k lives in place of x_k. The aggregate number of vaccines is 101, and each venue prepares at least one vaccine. In each venue, the highest priority is given to an agent living in the area. Thus, the government must distribute one additional shot to one of the 100 venues.

Now suppose that each agent, except i_{100}, likes to receive a shot at the venue near their area (in ascending order of proximity) and at the venue in her place next, that is, i_k ($k \in \{1, 2, \cdots, 99\}$) prefers x_{k+1} the most and x_k the second most, and i_{100} likes x_{100}. As the government is not aware of their preferences when determining where to distribute the extra shot to, it may distribute it to x_1. Then, it is easy to see that each agent ends up taking a shot at the venue in which they live near. However, if the government distributes one more shot to x_{100}, then all the agents are weakly better off. In particular, 99% of all the agents are strictly better off. A similar argument applies to a case where all but i_1 likes to take a shot at the venue next to their place (in a descending manner) and the government distributes an extra shot to x_{100}. $\diamond$

This simple example shows that an arbitrary quota distribution does not achieve ESOSM. More importantly, there is no unique optimal quota distribution for all the preference profiles. Rather, the optimal quota distribution is preference-dependent. The example also indicates that even a small difference leads to a large welfare improvement. Although market designers can improve welfare through an optimal quota distribution, they do not know the students' preferences in advance. Hence, it is challenging to determine an optimal quota distribution endogenously. In summary, we address the following new concern in designing a matching market:

> *Among all implementable quota distributions, does a market designer endogenously find an optimal quota distribution (and achieve an ESOSM) or not?*

This study explores a method to find an optimal quota distribution (and achieve an ESOSM). The simplest way to do so is to compare all outcomes of the DA mechanism at all quota distributions in terms of students' welfare. However, the number of implementable quota distributions becomes exponential, and this procedure is computationally difficult. We alternatively propose a new algorithm, the quota adjustment process (QAP), which is a two-step algorithm:

The first step calculates the matching of the DA mechanism for an arbitrary quota distribution. The second step updates a matching and quota distribution such that an updated matching is stable at an updated quota distribution, and that the matching Pareto dominates the updating matching. We call the second-step algorithm the quota-adjustment stable improvement cycle (QASIC). We show that a stable matching at some quota distribution is an ESOSM if and only if there is no QASIC. Therefore, the set of ESOSMs was fully characterized by the QAP. Notably, the computational complexity of QAP is polynomial.

We also show that the QAP is immune to strategic manipulation. A single-valued mechanism is strategy-proof if the truth-telling strategy is a dominant strategy in its induced preference revelation game. As previously mentioned, the DA mechanism is strategy-proof. As the QAP is not necessarily single-valued, we employ the extended notion of strategy-proofness, called strategy-resistance (Jackson, 1992). It states that any deviation from the truth-telling strategy does not result in a matching that is strictly better than all the matching obtained by the truth-telling strategy. We show the strategy-resistance of the QAP.

We then conducted computer simulations to evaluate the performance of the QAP by varying the upper bounds of the quotas. In the simulation, we proportionally increased the upper bounds of all quotas by γ, starting from an initial quota distribution. When there is no correlation among students' preferences and programs' priorities, QAP improves by about 50 students out of 2,000, even when $\gamma = 0.05$. Comparatively, we observe that with more correlation, the number of students better off is high; for instance, the number is twice when $\gamma = 0.05$ and ten times when $\gamma = 0.5$. This is remarkable because more correlation means more popularity of students and programs (i.e., more conflicts among them), thus implying less room for welfare gains. The simulation also shows that in any environment, the increase in the flexibility rate γ increases the number of students who are better off, which is consistent with Proposition 2. These results show considerable welfare gains from small quota distribution changes.[3]

We emphasize that the QAP is applied to the real-life matching market. During the process of admission reform at the University of Tsukuba in Japan, we were asked to advise on how to solve the program selection problem (the second example mentioned above). We proposed the QAP in 2018, which has been implemented since 2021.

Finally, we conclude the introduction by briefly discussing how it relates to previous literature. Notably, Kamada and Kojima (2017) proposed two stability notions – strong and weak stability – in a similar framework. Strong stability is immune to any pairwise deviation, but does not necessarily exist. Weak stability is immune to restricted pairwise deviation. The difference from ESOSM comes from the deviation by group. Our ESOSM is shown to be immune

[3]We share the same spirit with several papers on a matching problem with indifferences in the sense that indifferences matter for a stable matching realized when student preferences are unknown ex-ante (Erdil and Ergin, 2008; Abdulkadiroğlu et al., 2009). The critical problem is how to break ties in indifferent priorities. If we know the students' preferences, then we know that there exists a specific tie breaking; thus, DA achieves an optimal stable matching for students.

to a large class of group deviations. Thus, the set of ESOSMs is a proper subset of weakly stable matchings and there is no logical relationship between ESOSM and strong stable matching. We will discuss the related literature in more details in Section 5. The proofs of all results are provided in the Appendix.

2 Model

First, we describe a matching problem without any quota adjustment $(N, X, q, R, \succ)$, which is the standard school choice problem (Abdulkadiroğlu and Sönmez, 2003). Here, N is a finite set of students and X is a finite set of programs. The quota or capacity of the program $x \in X$ is the maximum number of students that program x can accept. We call the vector of quotas of all programs $q := (q_x)_{x \in X}$ a **quota distribution**. An outside option, denoted by $\varnothing$, is available to each student. Each student $i \in N$ has a complete, transitive, and antisymmetric preference relation R_i over $X \cup \{\varnothing\}$. Let P_i denote the strict part of R_i and $\mathcal{R}$ be the set of all preference relations. Let $R = (R_i)_{i \in N}$ be the preference profile. Each program $x \in X$ is endowed with a complete, transitive, and antisymmetric priority order $\succ_x$ over N. We denote the priority profile as $\succ = (\succ_x)_{x \in X}$.

Next, we introduce a matching problem with a quota adjustment that adds the variation of q to the matching problem without any quota adjustment $(N, X, q, R, \succ)$. A finite set of departments exist, denoted by K. Each program $x \in X$ belongs to exactly one department and $k(x)$ denotes the department to which x belongs. Thus, letting X_k be the set of programs in department k, we have

$$\forall k, k' \neq k, \; X_k \cap X_{k'} = \emptyset, \; X = \bigcup_{k \in K} X_k.$$

In other words, the set of programs X is partitioned into different departments. Each department $k \in K$ has a fixed quota, $\bar{q}_k$, which is the maximum number of students who belong to the programs in department k. The quota adjustment of programs is expressed by the assumption that the quota q_x of program x can take integer values from 0 to $\bar{q}_x$. Thus, $\bar{q}_x$ denotes the upper bound of the quota at x. However, the sum of the quotas of the programs in department k should be equal to the quota of department k. In other words, $\sum_{x \in k} q_x = \bar{q}_k$. Let $\bar{q} := (\bar{q}_x)$. In summary, a market designer can control the quota distribution within the set $Q := \prod_{k \in K} Q^k$ where

$$Q^k := \left\{ (q_x)_{x \in k} \in \mathbb{Z}_+^{|X_k|} \;\middle|\; \sum_{x \in k} q_x = \bar{q}_k, 0 \leq q_x \leq \bar{q}_x \right\}.$$

We state that a **quota distribution** q **is implementable** when q is in Q.

A **matching** is a function $\mu : N \to X \cup \{\varnothing\}$ that assigns a program or an outside option, $\mu(i)$, to every student $i \in N$. A matching is **feasible** at quota distribution q if, for each $x \in X$,

$|\mu^{-1}(x)| \leq q_x$. Let $\mathcal{M}(q)$ be the set of all the matchings feasible at q and $\mathcal{M} = \bigcup_{q \in Q} \mathcal{M}(q)$ be the set of all feasible matchings at implementable quota distributions. Matching $\mu \in \mathcal{M}$ is **individually rational** at R if, for each $i \in N$, $\mu(i)\ R_i\ \varnothing$. We say that a program $x \in X$ is **acceptable** to student $i \in N$ if x is preferred over the outside option $\varnothing$ for student i, i.e., $x\ P_i\ \varnothing$. Thus, for an individual rational matching μ, the matched program $\mu(i)$, if not the outside option, is acceptable to every student i. Given a matching $\mu \in \mathcal{M}(q)$, pair (i, x) of a student and a program is called a **blocking pair** of μ at (R, q) if $x\ P_i\ \mu(i)$ and $[|\mu^{-1}(x)| < q_x$ or for some $j \in \mu^{-1}(x)$, $i \succ_x j]$. The Gale and Shapley's (1962) stability is standard in literature, and is also important in both theory and practice (Roth, 2002). However, this notion is applicable when there is no quota adjustment. As our model allows for quota adjustment, we rename it distribution-specific stability to avoid confusion: a matching μ is **distribution-specific stable** at (R, q) if $\mu \in \mathcal{M}(q)$, μ is individually rational at R, and there is no blocking pair of μ at (R, q). A matching μ **Pareto dominates** another matching ν at R if for all $i \in N$, $\mu(i)\ R_i\ \nu(i)$, and there exists $j \in N$ such that $\mu(j)\ P_j\ \nu(j)$. Note that Pareto domination does not depend on a quota distribution q. Thus, the efficiency consideration itself is independent of q, so we say that μ is **efficient** at R if no matching in $\mathcal{M}$ Pareto dominates μ at R. We say that matching μ is **distribution-specific student-optimal stable** at (R, q) if μ is distribution-specific stable at (R, q) and no other distribution-specific stable matching Pareto dominates μ at (R, q). The distribution-specific student-optimal stable matching is obtained via Gale and Shapley's (1962) student-proposing deferred acceptance (DA) algorithm:

DA algorithm

- $q = (q_x)_{x \in X} \in Q$, $R = (R_i)_{i \in N}$, and $\succ = (\succ_x)_{x \in X}$ are given.

Step 1 All students apply to their most preferred program among acceptable ones. Each program x tentatively fills its quota q_x by accepting students among the applicants according to its priority $\succ_x$, and the others are rejected.

Step s ($s \geq 2$) Students who are rejected at Step $s - 1$ apply to their next preferred program among acceptable ones. Each program x tentatively fills its quota q_x by accepting students among those tentatively accepted in the step $s - 1$ and the new applicants according to its priority $\succ_x$, and the others are rejected.

END If each student is either tentatively accepted at some program or rejected by all acceptable programs, then the algorithm ends and outputs that matching.

Using this algorithm, we obtain a feasible matching at $q \in Q$. We call this **DA matching**. The DA matching is distribution-specific student-optimal stable (Gale and Shapley, 1962), which implies the existence of a distribution-specific student-optimal stable matching in a problem for each quota distribution. For the matching problem with and without quota adjustment, R is primitive, so we drop R and simply use q to express the properties of the matchings as long as no confusion arises.

We now turn to mechanisms and their incentive properties. A **mechanism** is a function (not necessarily single valued) f from the set of preference profiles $\mathcal{R}^N$ to the set of matchings, $\mathcal{M}$. Accordingly, the efficiency and stability are extended to a mechanism. Mechanism f is **efficient** if for all $R \in \mathcal{R}^N$, each matching in $f(R)$ is efficient at R. We say that mechanism f is **stable** if for all $R \in \mathcal{R}^N$, each matching in $f(R)$ is distribution-specific stable at (R, q) for some $q \in Q$. When mechanism f is single-valued, it is a selection of matchings for each R. Therefore, we interpret $f(R)$ to be a matching and we write i's match at $f(R)$ as $f_i(R)$. We also say that f is **strategy-proof** if for all $i \in N$, $R \in \mathcal{R}^N$ and $R'_i \in \mathcal{R}$, we have $f_i(R_i, R_{-i})\ R_i\ f_i(R'_i, R_{-i})$. Strategy-proofness simply states that stating true preferences is a dominant strategy for everyone. We also use an extended notion of strategy-proofness, called **strategy-resistance**, for a correspondence (Jackson, 1992). We say that f is strategy-resistant if there exists no $i \in N$, $R \in \mathcal{R}^N$, $R'_i \in \mathcal{R}$, $\nu \in f(R'_i, R_{-i})$ such that $\nu(i)\ P_i\ \mu(i)$ for all $\mu \in f(R)$. Strategy-resistance states that regardless of others' strategies there is no misrepresentation of preferences such that one of the resulting matchings Pareto dominates all matching under truth telling. Thus no one can certainly be better off because of misrepresentation.

For convenience, keeping the quota distribution q fixed, we denote DA^q as the single-valued mechanism that always selects the DA matching as the output.

3 Results

This section consists of four subsections: The first subsection introduces ESOSM and proves its existence. The second subsection verifies the method of finding an ESOSM endogenously in polynomial time. Here, we introduce our main mechanism: QAP. The third subsection analyzes the welfare properties of ESOSM. The last subsection presents the incentive properties of QAP.

3.1 Ex-post student-optimal stable matching

Here, we introduce our main solution concept.

Definition 1. For a matching problem with quota adjustment, a matching μ is an **ex-post student-optimal stable matching** (ESOSM) at R if μ is distribution-specific stable at some $q \in Q$ and there is no $p \in Q$ and stable matching ν at p such that ν Pareto dominates μ. We say that a quota distribution q is **optimal** if there is an ESOSM μ such that $\mu \in \mathcal{M}(q)$.

Note that multiple ESOSMs exist (See Example 2 in the next subsection). In addition, depending on the preferences realized, the optimal quota distribution varies. There is an important relationship between the set of ESOSMs (denoted by $ESOSM(R)$) and the set of DA matchings at all implementable quota distributions: for all preference profiles $R \in \mathcal{R}^N$,

$$ESOSM(R) \subset \bigcup_{q \in Q} \{DA^q(R)\}$$

In general, the left-hand side is a proper subset of the right-hand side. As stated in the introduction, depending on R, the optimal quota distribution varies. Hence, an arbitrary choice of quota distribution always has room for weak welfare improvement, and such improvement becomes possible after the students' preferences are realized.

First, we demonstrate the existence of ESOSM.

Theorem 1. *For any matching problem with quota adjustment, there exists an ESOSM.*

Given our definition of ESOSM, one may wonder if there are many deviations across the quota distributions. To clarify this, we introduce another stability concept that is immune to deviations across the quota distributions.

Definition 2. A matching $\mu \in \mathcal{M}$ is **stable across distributions** if whenever $x \; P_i \; \mu(i)$, the following three conditions hold.

(1) For each $j \in \mu^{-1}(x)$, $j \succ_x i$.

(2) $\sum_{y \in X_{k(x)}} |\mu^{-1}(y)| = \bar{q}_{k(x)}$.

(3) Let ν be a matching such that all but i are the same as μ and i is matched with x instead of $\mu(i)$. If $\nu \in \mathcal{M}$, then there is $j \in N$ such that $x \; P_j \; \mu(j)$ and $j \succ_x i$.

This notion states that with conditions (1) and (2), the matching μ is distribution-specific stable at its corresponding quota distribution. With condition (3), even if it is feasible to move student i to their preferred program x with adjusted quotas, we can find another agent j preferring x to his matched program with higher priority.

Proposition 1. *If a matching μ is an ESOSM, then it is stable across distributions.*

Hence, although the ESOSM is defined on distribution-specific stable matchings, it prevents deviations across quota distributions. The relationship to stability notions in the literature is discussed later in Section 5.

3.2 Quota adjustment process

As we showed the existence of ESOSM for any matching problem, we will explore how to find it. As discussed in the introduction, the simplest way to find an ESOSM is to run the DA algorithm for all the quota distributions. However, this approach is computationally infeasible. Alternatively, we propose a novel procedure, QAP, to find an ESOSM, and a corresponding optimal quota distribution.

QAP is a two-step algorithm. The first step runs the DA algorithm for an arbitrary quota distribution to find a distribution-specific stable matching. The second step updates the obtained distribution-specific stable matching and the quota distribution through exchanges of programs among students who prefer one of those to their matched programs, while preserving distribution-specific stability at an updated quota distribution. Note that several types of welfare improvement cycles are present in the literature. As we allow changes in quota distributions, the main difficulty here is that, because our exchanges consist of not only students but also vacant seats, we need to control the balance in quotas between departments. To this end, we introduce two types of chains.

Definition 3. Given is that R and q. For each $x \in X$, a **unit chain** for stable matching μ at (R, q) is a vector $(x, i) \in X \times N$ such that $x \; P_i \; \mu(i)$ and $i \succ_x j$ for each $j \in \{j \in N \backslash \{i\} \mid x \; P_j \; \mu(j)\}$; that is, student i prefers program x to the matched program $\mu(i)$, and is the highest priority among those who prefer x to their matched programs.

We denote the sets of students and programs involved in a set of unit chains by N^μ and X^μ, respectively. Given matching μ, for each department k, if it has a vacant seat, we introduce one dummy student d_k regardless of the number of vacant seats. We denote the set of dummy students as D (perhaps empty). We match each dummy student with a program with a vacant seat in the department to which she belongs. Thus, we extend the matching μ to $\bar{\mu}$ with domain $N \cup D$ such that $\bar{\mu}|_N = \mu|_N$ and for each $d_k \in D$ and some $z \in X_k$ with $|\mu^{-1}(z)| < q_z$, we have $\bar{\mu}(d) = z$.

Definition 4. Let matching $\bar{\mu}$ and a set of unit chains for μ be given. For each $d \in D$, a **unit chain by dummy student** d is a vector $(x, d) \in (X \cup \{\varnothing\}) \times D$ such that x is assigned to some student $i \in N^\mu$ and is neither in X^μ nor $X_{k(d)}$.

To avoid confusion, we sometimes refer to a unit chain as a unit chain by a real student in Definition 3. Note that there could be multiple unit chains for a specific dummy student. This occurs in the following situations. Given unit chains and department k with vacant seats, there are multiple programs assigned in unit chains where those programs are not in unit chains nor in department k. Then, dummy student d_k is paired with all programs for unit chains by a dummy student. We use ι for a real or dummy student; that is, $\iota \in N \cup D$.

We are now ready to define our improvement cycle.

Definition 5. A **quota-adjustment stable improvement cycle** (QASIC) for R, μ, q and $\bar{\mu}$ is an ordered set of unit chains by real or dummy students,

$$\langle (x_\ell, \iota_\ell) \rangle_{\ell=0}^{m}, \; mod(m)$$

($m \geq 1$), such that

1. real students and dummy students involved are distinct,
2. at least one real student is involved, i.e., $\iota_\ell \in N$ for some $\ell \in \{1, \dots, m\}$,
3. for all $\ell \in \{1, \dots, m\}$,
   (a) $\mu(\iota_{\ell-1}) \neq \varnothing$ implies $\mu(\iota_{\ell-1}) \in X_{k(x_\ell)}$,
   (b) $\mu(\iota_{\ell-1}) = \varnothing$ implies $x_\ell = \varnothing$, and
4. for all $\ell \in \{1, \dots, m\}$, $\mu(\iota_{\ell-1}) \neq x_\ell$ implies $q_{x_\ell} < \bar{q}_{x_\ell}$.

We sometimes explicitly denote a QASIC as follows:

$$(x_0, \iota_0) \longleftarrow (x_1, \iota_1) \longleftarrow \cdots \longleftarrow (x_{m-1}, \iota_{m-1}) \longleftarrow (x_m, \iota_m) = (x_0, \iota_0),$$

where each unit chain (x_ℓ, ι_ℓ) points to the next unit chain $(x_{\ell-1}, \iota_{\ell-1})$. In the traditional cycle, such as in Gale's top trading cycle (Shapley and Scarf, 1974), we carry out trades in a cycle such that student ι_ℓ receives the next student $\iota_{\ell-1}$'s program $\mu(\iota_{\ell-1})$, that is, $x_\ell = \mu(\iota_{\ell-1})$. However, because we have a quota adjustment, we must allow $x_\ell \neq \mu(\iota_{\ell-1})$. In this case, as we have to maintain the department quota, we require student ι_ℓ's receiving school, x_ℓ, to be in the same department as that of $\iota_{\ell-1}$'s assigned program $\mu(\iota_{\ell-1})$. This is why we have conditions (3)-(a) and (4). This argument is applicable only when student $\iota_{\ell-1}$'s assigned program $\mu(\iota_{\ell-1})$ is not an outside option. However, when quotas are adjusted to increase, some students may want an increased program from their assigned outside options. To accommodate this, we used a dummy student's unit chain. In particular, when program $\mu(\iota_{\ell-1})$ is the outside option, because no real student prefers the outside option to the matched program owing to the individual rationality of μ, some unit chain by a dummy student, $(\varnothing, d)$, points to the unit chain $(x_{\ell-1}, \iota_{\ell-1})$ so that student $\iota_{\ell-1}$ can move from the outside option to program $x_{\ell-1}$. Therefore, we have condition (3)-(b).

Given unit chains by real or dummy students, because ESOSMs are on the Pareto frontier of distribution-specific stable matchings and the Pareto order is partial, more than one QASIC might exist (See Example 2). As we will show, QASICs fully characterize the set of ESOSMs which Pareto dominates a starting distribution-specific stable matching.

We define the main algorithm as follows:

Quota Adjustment Process (QAP)

- Given $R, \succ, Q$ and arbitrary $q \in Q$

Step 0 Set $q^0 = q$. Run the DA algorithm at $(R, \succ, q^0)$ and obtain a matching μ^0.

Step u ($u \geq 2$).

Step u.1 Find unit chains for μ^{u-1}, $\langle (x_\ell^{u-1}, i_\ell^{u-1}) \rangle$.

- If D for μ^{u-1} is non-empty, then construct $\bar{\mu}^{u-1}$ in a way that for each $d \in D$, choose program $z \in X_{k(d)}$ which has a vacant seat randomly, and $\bar{\mu}(d) = z$. Then find unit chains by dummy students for $\bar{\mu}^{u-1}$, and proceed.
- Otherwise proceed.

Step u.2 Given unit chains by students and dummy students, $\langle (x^{u-1}, \iota^{u-1}) \rangle$,

- if there is no QASIC, then terminate the process and output (μ^{u-1}, q^{u-1}).
- Otherwise, we have a QASIC for matching μ^{u-1}, $\langle (x_\ell, \iota_\ell) \rangle_{\ell=0}^{m-1}$ ($m \geq 1$). If there are multiple QASICs, then pick one randomly. Construct (μ^u, q^u) as follows:

$$\mu^u(j) = \begin{cases} x_\ell & \text{if } j = \iota_\ell, \text{ for some } \ell \in \{0, \ldots, m-1\} \text{ and } \iota_\ell \text{ is a student} \\ \varnothing & \text{if } j = \iota_\ell, \text{ for some} \ell \in \{0, \ldots, m-1\} \text{ and } \iota_\ell \text{ is a dummy student} \\ \mu^{u-1}(j) & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} q_{x_\ell}^u &= q_{x_\ell}^{u-1} + 1, && \text{for some } \ell \in \{0, \ldots, m-1\} \\ q_{\mu(\iota_\ell)}^u &= q_{\mu(\iota_\ell)}^{u-1} - 1, && \text{for some } \ell \in \{0, \ldots, m-1\} \\ q_x^u &= q_x^{u-1}, && \text{otherwise} \end{aligned}$$

Go to Step $u + 1$. ◇

In other words, when there is a QASIC, we update the matching such that a student receives the program in the unit chain, which is made possible by updating the quotas. We refer to the resulting matching as **QAP matching**. Figure 1 illustrates the way in which QAP works.

Example 2 (Multiple QASICs and ESOSMs, and execution of QAP)**.** Four students, $\{i_1, i_2, i_3, i_4\}$ and four programs, $\{x_1, x_2, x_3, x_4\}$ with $(\bar{q}_{x_1}, \bar{q}_{x_2}, \bar{q}_{x_3}, \bar{q}_{x_4}) = (1, 2, 2, 2)$. The four programs are sorted into two departments: $X_k = \{x_1\}$ and $X_{k'} = \{x_2, x_3, x_4\}$ with $\bar{q}_k = 1$ and $\bar{q}_{k'} = 3$. The preferences and priorities are as follows:

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}
x_3	x_2	x_2	x_3
x_2	x_3	$\varnothing$	$\varnothing$
any	any		

,

$\succ_{x_1}$	$\succ_{x_2}$	$\succ_{x_3}$	$\succ_{x_4}$
any	i_1	i_2	any
	i_3	i_4	
	i_2	i_1	
	any	any	

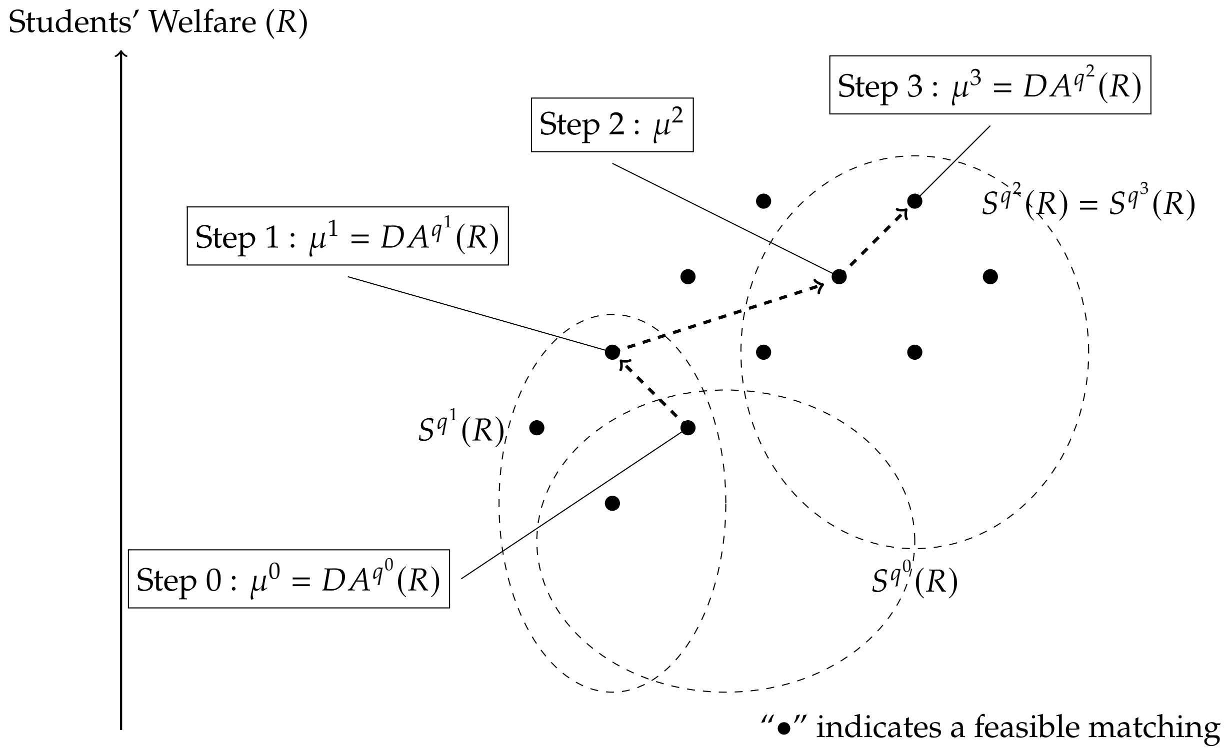

Figure 1: Illustration of the QAP

Suppose that a market designer sets the initial quota distribution to $(1, 1, 1, 1)$. Then

$$DA^{(1,1,1,1)}(R) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_2 & x_3 & \varnothing & \varnothing \end{pmatrix}.$$

For this matching, there are two unit chains, $(x_2, i_3), (x_3, i_4)$, and two unit chains by dummy students, $(\varnothing, d_1), (\varnothing, d_2)$ where $\mu(d_1) = x_1$ and $\mu(d_2) = x_4$. It is easy to verify that both $\{(x_2, i_3), (\varnothing, d_2)\}$ and $\{(x_3, i_4), (\varnothing, d_2)\}$ are QASICs. If we implement the former, we obtain

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_2 & x_3 & x_2 & \varnothing \end{pmatrix}$$

with $q_1 = (1, 2, 1, 0)$. It can be verified that ν is an ESOSM. If we implement the latter, we obtain

$$\nu' = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_2 & x_3 & \varnothing & x_3 \end{pmatrix}$$

with $q_1 = (1, 1, 2, 0)$. It can be verified that ν' is also an ESOSM. $\diamond$

Our main theorem states that if a distribution-specific stable matching is an ESOSM, then we do not find a QASIC any more; conversely, if it is not an ESOSM, there exists a QASIC. Thus, this fully characterizes the relationship between ESOSM and QASIC, where QAP always stops in finite steps and reaches an ESOSM.

Theorem 2. *For a matching problem with quota adjustment, a distribution-specific stable matching is an ESOSM if and only if there is no QASIC for the matching.*

Theorem 2 directly implies the following.

Corollary 1. *QAP matching is an ESOSM, regardless of which QASIC selections are made during the process.*

For practical use, the QAP algorithm terminates in a polynomial time. The computational complexity of DA is known to be $O(|N|^2)$ and that of a QASIC search is $O((|N|+|K|)(|X|+1))$.[4] In each step of the QASIC at least one student becomes better off. Thus, the number of search steps of QASICs in the second stage is at most $|N||X|$ times.[5] However, if we run the DA algorithm for all implementable quota distributions, we can encounter computational difficulties.[6] Therefore, it is obvious that the QAP is fast, and thus an applicable algorithm in practice. As noted in the introduction, QAP was implemented at University of Tsukuba in 2021.

3.3 Welfare

Here, we provide welfare analysis. First, expanding the set of implementable quota distributions increases student welfare. The simplest way to expand such a set is to increase all of the upper bounds weakly. This corresponds to an increase in the total supply. More importantly, it is also possible to reduce the number of departments. If we reduce the number of departments, while the sum of the upper bound of the departmental quota does not change, and none of them decrease, then the total supply does not change, but the set of implementable quota distributions weakly increase. In both cases, we conclude that students' welfare increases weakly.

Proposition 2. *For any R, $\succ$, Q, Q' and $q \in Q$, if $Q \subset Q'$, then there exists no QAP matching for Q such that it Pareto dominates any QAP matching for Q'.*

Proposition 2 implies the following.

Corollary 2. *If the number of K decreases while the sum of the upper bound of the departmental quota does not change, and each of them do not decrease, then there is no QAP matching for the problem before decreasing K, which Pareto dominates any QAP matching for the problem after decreasing K.*

[4]To search a QASIC, consider a directed graph $G = (V, E)$ such that V is a set of unit chains by students and dummy students and $E \subset V \times V$ is a set of connected unit chains. Thus, $|V| \leq |N| + |K|$ and $|E| \leq (\max_{k\in K} |k|)(|N| + |K|)$ because each unit chain can connect to another unit chain; that is, $k(\mu(\iota)) = k(x)$. In such a directed graph, a cycle is found, for example, by a depth-first search method in $O(|E| + |V|)$.

[5]In contrast with matching with indifferences, cycle searches in Erdil and Ergin's (2008) stable improvement cycle (SIC) are $\frac{1}{2}|N|(|X| - 1)$ because a shortest SIC consists of two students. However, in the present case, the shortest QASIC consists of one student. Furthermore, a student who is assigned nothing obtains a program.

[6]For instance, if $\bar{q}_x = \bar{q}_k$ for all $x \in X$, the number of non-zero quota distributions is $|Q| = \prod_{k\in K} \left({}_{\bar{q}_k - 1}C_{|X_k|-1}\right)$.

3.4 Incentives

We turn to the strategic properties of QAP to see whether each student truthfully states their preferences. When employing QAP, students face a multiplicity of ESOSMs. Thus, we examine strategy-resistance, requiring that no student should improve his/her match by misrepresenting his/her preferences.

Theorem 3. *Let R, $\succ$, Q and $q \in Q$ be given. Then the QAP is strategy-resistant.*

Therefore, manipulation is limited to a certain extent ex-ante.

Traditionally, the literature has explored whether a matching mechanism is strategy-proof. When a quota distribution is exogenously given, DA and Gale's top trading cycles algorithm are single-valued and strategy-proof. Unfortunately, there is no strategy-proof selection for QAP, as shown below.

Theorem 4. *No single-valued ex-post student-optimal stable function is strategy-proof.*

Under quota adjustment, we share similarity to a matching problem with indifferences. This is a matching problem with a fixed quota distribution and only one department. However, the priorities of the programs are assumed to be complete and transitive (not necessarily antisymmetric). Therefore, some students were ranked equally in some programs, which did not differentiate between those students. In such cases, Erdil and Ergin (2008) and Abdulkadiroglu et al. (2009) highlight a trade-off between distribution-specific student-optimal stability and strategy-proofness. In our environment, Kamada and Kojima (2018) show that there is a strategy-proof selection that is compatible with their stability. Together with Theorem 4, we can conclude that there is a trade-off between ESOSM and strategy-proofness. Hence, given stability as one of the design goals, a market designer must choose between efficiency and strategic robustness.[7]

4 Case Study

4.1 The background and the implementation of QAP

The University of Tsukuba (henceforth Tsukuba) is one of the largest national universities in Japan. In 2017, the implementation of QAP was initiated by the university president. Tsukuba decided to reform its admission process, which necessitated the introduction of a matching mechanism. The second author of this paper was a faculty member at Tsukuba and was invited to lead a matching task force which took charge of the admission reform. With much discussion of the task force, because the university wanted to have quota flexibility while maintaining the

[7]The University of Tsukuba understood such a trade-off. As QAP is strategy-resistant and they assume that students are not fully aware of the complex procedure of QAP, they would like to achieve the highest level of welfare among distribution-specific stable matchings.

Table 1: The current school-college structure at the University of Tsukuba

School	College	Quota	School	College	Quota
Humanities and culture	Humanities	120	Science and engineering	Mathematics	40
	Comparative culture	80		Physics	60
	Japanese language and culture	40		Chemistry	50
Social and international studies	Social sciences	80		Engineering sciences	120
	International studies	80		Engineering systems	130
Human sciences	Education	35		Policy and panning sciences	120
	Psychology	50	Informatics	Information science	80
	Disability sciences	35		Media, arts, science, and technology	50
Life and environmental sciences	Biological sciences	80		Knowledge and library sciences	100
	Agro-biological resource sciences	120	Medicine and medical sciences	Medicine	98
	Geoscience	50		Nursing	70
Physical education, health and sport sciences		240		Medical sciences	37
Art and design		100			

stability and improving the efficiency, the authors of this paper invented QAP, which has been officially approved for implementation in 2021. In this section, we discuss the background of the implementation of the QAP as well as the simulation result.

Unlike most universities in Europe and the USA, in Japan, students must choose one of the programs in their application for admission.[8] It is very difficult for a student to change their program during the enrollment process in college. A number of universities, such as the University of Tokyo and Hokkaido University, did not adopt this admission system. However, in their systems a student can "choose" one program at the end of their first or second year. Thus, they used a matching mechanism to match students with programs. Tsukuba also wanted to adopt this system as part of the reform.

Every year, approximately 2,000 students are enrolled in Tsukuba. The university has eight schools, each consisting of several colleges. Note that the terms of schools and colleges correspond to that of the departments and the programs in our matching problem with quota adjustment. Each college has a quota, and the quota of a school is just the sum of the quotas of all affiliate colleges. See Table 1 for the structure of schools and colleges with these capacities for 2020.

The reform necessitated a matching mechanism for students at the end of their first year. With such a mechanism, each student submits a preference ranking for colleges and each college has a priority ranking determined by GPA or other criteria. This can be modeled as the Abdulkadiroğlu and Sönmez' (2003) school choice problem. However, Tsukuba wished for

[8]The matching process for admissions in Japan is not centralized but decentralized. See Hafalir et al. (2018) for an analysis.

quota adjustment, which could reflect the students' popularity, to make the reform seem more attractive. At the same time, there are two constraints on its quota distribution: (1) Each school must keep the initial quota – the sum of all the quotas of its colleges, according to the Ministry of Education, Culture, Sports, Science and Technology; (2) Flexibility of colleges' quotas are not welcome, because they are related to their budgets. This resulted in the adoption of the upper bounds of quotas for colleges.

Tsukuba desired stability as the most important property of its mechanism, followed by efficiency. As efficiency and strategy-proofness is incompatible (Theorem 4), its choice is efficiency-oriented or strategy-proof among stable mechanisms. Tsukuba was less anxious about the strategic properties of the QAP: Even if the QAP is only strategy-resistant, its strategic manipulation is not as obvious as in the Boston mechanism. For these reasons, the task force team chose QAP with an emphasis on efficiency. Tsukuba has approved the use of QAP for implementation in 2021.[9]

4.2 Simulation

Our theoretical results show that the QAP reaches the Pareto frontier among all distribution-specific stable matchings (Theorem 2), and improves all students' welfare unambiguously, as flexibility increases (Proposition 2). In this subsection, we describe a computer simulation, to quantify welfare gains under QAP compared to DA, and those from the quota flexibility in the Tsukuba environment.[10]

In the simulation, we set the number of students to $n = 2,065$, the number of schools is $|K| = 9$, and the number of colleges is $|X| = 25$. See Table 1 for the quotas of each college and other details. The upper bounds of the quotas of colleges are set as +5%, +10% $\cdots$ +50% uniformly (rounding up to decimal places). We use γ as the parameter of the quota flexibility such that for each college $x \in X$,

$$\bar{q}_x = (1 + \gamma)q_x^0.$$

We vary the parameter γ from 0.05 to 0.50 in increments of 0.05.

Following the specifications of Erdil and Ergin (2008), we randomly create preferences and priorities based on utility functions of students and colleges. The utilities of student $i \in N$ are

[9]For instance, the University of Tokyo uses the three rounds of matching process: The first round uses the variant of Boston (immediate acceptance) algorithm for 70% of its total quota; the second round uses DA algorithm for the rest of the quota, and a scrambled market opens in the final round. The resulting matching becomes unfair in some situations, when a student finds other colleges more preferable than her matched college, and there is another student who is less prioritized but still matched with that college, and these mechanisms can be manipulated.

[10]In the case of Tsukuba, the number of colleges and their quotas are asymmetric across schools. Therefore, the simulation results may be biased, and hence affect the performance of the QAP. Thus, we also set up a symmetric environment artificially, in which the number of students was 1,000, the number of schools was five, each school had four colleges, and the initial quota for each college was 50. All simulation results are available upon request.

generated as

$$U_{ix} = \alpha Z_x^* + (1 - \alpha) Z_{ix},$$

where Z_x^*, Z_{ix} are independent, identical and normally distributed random variables, with a mean of 0 and a variance of 1. In particular, Z_x^* reflects common preferences among students. Then, the preference R_i is obtained as $x \ P_i \ y \Leftrightarrow U_{ix} > U_{iy}$. In case $U_{ix} = U_{iy}$, we randomly break the ties. On the other hand, the utilities of college $x \in X$ are generated as:

$$V_{xi} = \beta Z_{ki}^* + (1 - \beta) Z_{xi}$$

where Z_{ki}^*, Z_{xi} are independent, identical and normally distributed random variables, with a mean of 0 and a variance of 1. In particular, Z_{ik}^* reflects the common priority among colleges belonging to department $k(x)$. Then, the priority $\succ_x$ of college x belonging to school $k \in K$ is obtained as $i \succ_x j \Leftrightarrow V_{xi} > V_{xj}$. In the case of $V_{xi} = V_{xj}$, we randomly break ties.

Parameters α and β control how correlated students' preferences or college priorities are respectively. Thus, all parameters of zero indicate no correlation, where preferences and priorities are independently generated. Note that, even when $\beta = 1$, colleges belonging to different departments had different priorities.

We compared the matchings of QAP and DA at the initial distribution. As QAP is identified by its flexibility, we denote QAP as $QAP(\gamma)$, whose flexibility is bounded above at $1 + \gamma$ where $\gamma \in \{0.05, 0.1, \ldots, 0.45, 0.5\}$. Table 2 shows a comparison between DA and $QAP(\gamma)$ for $\gamma = 0.05, 0.1, 0.2, 0.5$, with respect to the average number of students matched with their first and second college of preference; the average of the mean of the rank matched for all students, and the mean of the number of students better off from the DA matching. Figure 2 shows the average number of students better off as a function of flexibility rate γ.

These results show that, when preferences and priorities are uncorrelated, that is, $(\alpha, \beta) = (0, 0)$, approximately one-third of the students are matched with their first preference under DA. In this case, because the preferences generated are so diverse, there is little conflict amongst the students, which means there is little room for improvement, even though flexibility is sufficiently high (50% of all college quotas). By contrast, when preferences and priorities are positively correlated, for example as in $(\alpha, \beta) = (0.5, 0.5)$, there are relatively popular colleges among students, and relatively preferable students from colleges belonging to the same school. This induces conflict among the students, and QAP improves many students' situation with an updated quota distribution. Thus, it is difficult for policymakers to set an initial quota distribution well, and QAP can help them significantly. In particular, the more flexible the quota distributions are, the more the students' situations improve, which is consistent with Proposition 2.

Table 2: Comparison among DA and $QAP(\gamma)$

$(\alpha,\beta)=(0,0)$	DA	$QAP(0.05)$	$QAP(0.1)$	$QAP(0.2)$	$QAP(0.5)$
1st ranked	763.04	780.29	782.61	783.93	784.22
2nd ranked	482.34	486.57	487.13	487.37	487.40
Average rank	2.68	2.63	2.62	2.62	2.62
Better off	0	44.47	50.28	53.46	54.19

$(\alpha,\beta)=(0.5,0.5)$	DA	$QAP(0.05)$	$QAP(0.1)$	$QAP(0.2)$	$QAP(0.5)$
1st ranked	160.80	167.62	172.56	182.81	213.16
2nd ranked	171.14	177.90	182.90	193.07	223.41
Average rank	8.15	7.91	7.74	7.41	6.68
Better off	0	96.11	160.75	276.50	517.89

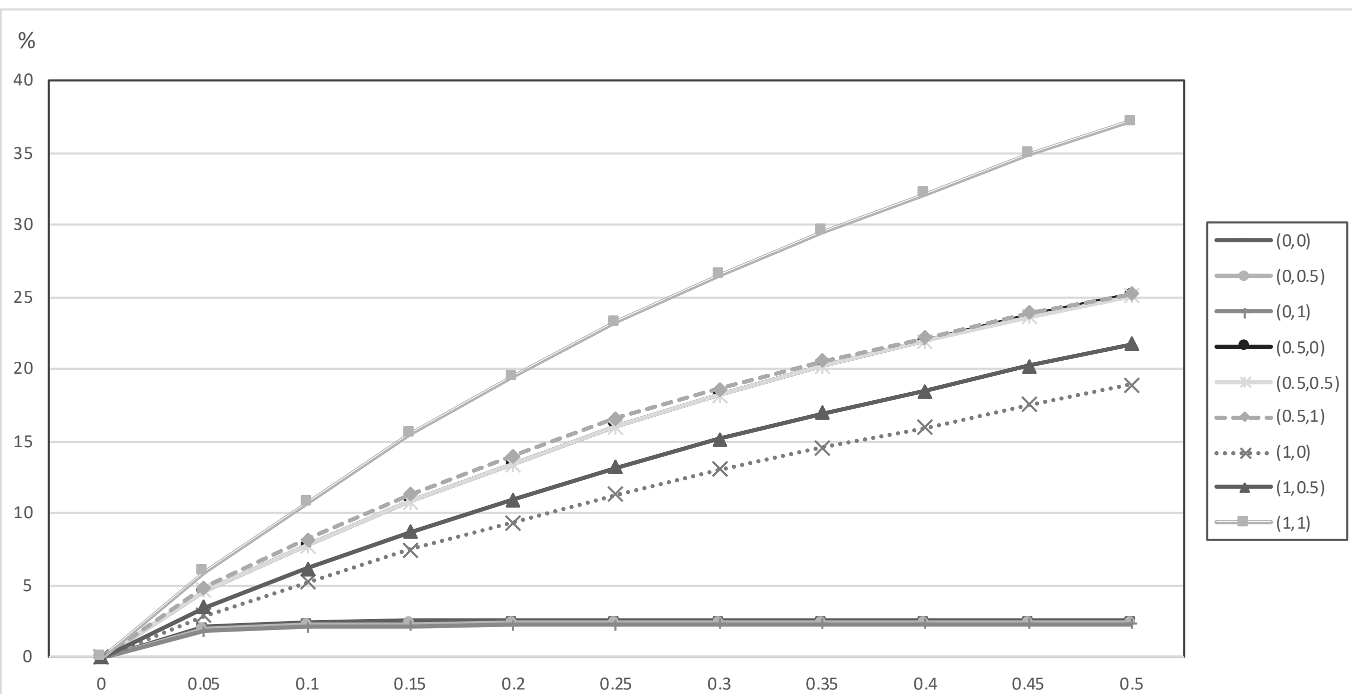

Figure 2: Percent of improving students as a function of flexibility γ. The vertical axis is the average number of students better off, while the horizontal axis is the flexibility rate γ.

5 Discussion

5.1 Stability concepts

This study refines the notion of stability and introduces ex-post student-optimal stability. In an independent work, Dur and Van der Linden (2022) also introduced the same notion of stability. Our framework includes departments, but theirs do not, which can be viewed as their model having only one department. Whereas we provide QAP, they provide a different matching algorithm to achieve an ESOSM. As the two papers differs in terms of motivation, the main algorithms have different features. We were initially requested the development of a matching procedure for the University of Tsukuba. Their aim is to improve students' welfare through quota adjustments from the original quota distribution while maintaining stability and departmental quota constraints. Hence, the QAP starts from an arbitrary implementable quota distribution, which is different from the algorithm proposed by Dur and Van der Linden (2022).

As our model contains departments, it constrains implementable quota distributions. Thus, our research contributes to part of literature which investigates matching under these constraints. Thus, we have compared our stability concept with that of Kamada and Kojima (2017), who first introduced matching under constraints. In their model, the implementable quota distributions are more general than ours, and we let $g : Z_+^{|X|} \to \{0, 1\}$ be feasibility constraints, such that for all $w, w' \in Z_+^{|X|}$, $g(w) \geq g(w')$ when $w \leq w'$ and $g(0) = 1$. A variable w is said to be feasible if $g(w) = 1$, and not feasible if $g(w) = 0$. We use a notation $w(\mu)$ as a vector of the number of matched students at μ, which is $w(\mu) = (|\mu^{-1}(x_1)|, |\mu^{-1}(x_2)|, \cdots, |\mu^{-1}(x_{|X|})|)$. Then in our model,

$$g(w) = \begin{cases} 1 & \text{if } w \leq q,\ \forall q \in Q \\ 0 & \text{otherwise} \end{cases}$$

Now, we rephrase their notions of strong stability and weak stability. The first is strong stability.

Definition 6. A matching μ is strongly stable if it is feasible and individually rational, and if there exists a pair $(i, x) \in N \times X$ such that

$$xP_i\mu(i) \text{ and } \left[\exists j \in \mu^{-1}(x)(i \succ_x j) \text{ or } |\mu^{-1}(x)| < \overline{q}_x\right]$$

then it must be that

$$g(w(\mu) + e_x - e_{\mu(i)}) = 0 \text{ and } \forall j \in \mu^{-1}(x)(j \succ_x i)$$

where e_y is the yth unit vector.

They showed that a strongly stable matching does not necessarily exist in their environment. Neither does it exist within our restricted environment, however, an ESOSM exists in any case (Theorem 1). Strong stability requires strict conditions, and would be regarded as a plausible concept if it did exist. However, it fails to be immune to group-wise deviation, even if it exists. This is shown below.

Example 3. There are two students, i_1, i_2 and two departments, k_1, k_2 with two programs for each $X_{k_1} = \{x, y\}$ and $X_{k_2} = \{z, w\}$. The upper bound for each program is one, and the quotas of the departments are $(q_{k_1}, q_{k_2}) = (1, 1)$. The preferences and priorities are given as follows:

i_1	i_2		k_1		k_2	
w	y		x	y	z	w
x	z	,	i_1	i_2	i_2	i_1
any	any		i_2	i_1	i_1	i_2

The possible quota distributions are $Q = \{(0,1,0,1), (1,0,1,0)\}$. Thus, $f(w) = 1$ if $w \leq q$ for any $q \in Q$. Now, we consider the following matching.

$$\mu = \begin{pmatrix} i_1 & i_2 \\ x & z \end{pmatrix}$$

Clearly, μ is strongly stable because $g(w(\mu)) = g((1,0,1,0)) = 0$ and if i_1 wants to be better off, then it requires i_1 being matched with w and such a move violates the feasibility constraint $g(w(\mu) + e_w - e_x) = g((0,0,1,1)) = 1$, and the symmetric argument is applied to i_2. By contrast, μ is not ex-post student-optimal stable, because the following matching ν

$$\nu = \begin{pmatrix} i_1 & i_2 \\ w & y \end{pmatrix}$$

is distribution-specific stable at $q = (0,1,0,1)$ (thus it is feasible) and Pareto dominates μ. ◇

Therefore, there is no logical inclusion relation between ESOSM and strong stable matching. Note that the matching ν above is also strongly stable. One may think of the stability notion as the combination of both ex-post student-optimal stability and strong stability, while keeping in mind that a strongly stable matching does not necessarily exist.

Kamada and Kojima (2017) also proposed a weaker notion called weak stability.

Definition 7. A matching μ is weakly stable if it is feasible and individually rational, and if there exists a pair $(i, x) \in N \times X$ such that

$$xP_i\mu(i) \text{ and } \left[\exists j \in \mu^{-1}(x)(i \succ_x j) \text{ or } |\mu^{-1}(x)| < \overline{q}_x\right]$$

then it must be that

$$g(w(\mu) + e_x) = 0 \text{ and } \forall j \in \mu^{-1}(x)(j \succ_x i).$$

Note that strong stability implies weak stability because $g(w) \geq g(w')$ when $w \leq w'$.

Claim 1. If a matching μ is an ESOSM, then it is weakly stable.

As an ESOSM exists for any problem and is immune to some group deviations, it can be seen as a sharper notion than weak stability. Alternatively, we can say that the set of ESOSMs lay on the Pareto frontier of the set of all weakly stable matchings. We end up with logical relations among ex-post student-optimal stability, weak stability, and strong stability, as depicted in Figure 3.

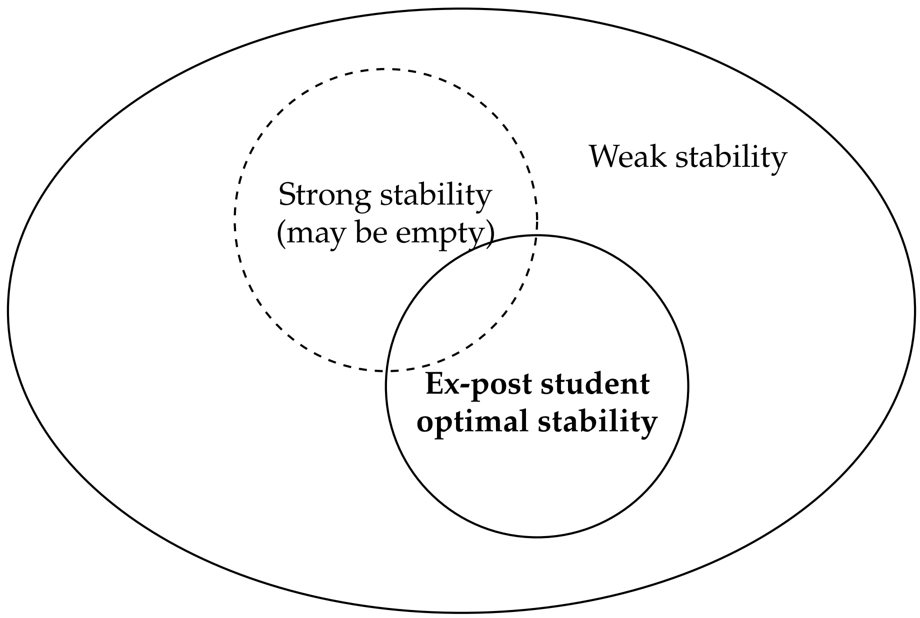

Figure 3: Logical relations among several stability notions

5.2 Affirmative action

Our model is related to matching models with affirmative action policies. In this model, students are separated into types. There are upper and lower bounds for each type in each school. If we translate a type into a program and a school as a department (each department has the same number of programs as the types, and students' preferences are defined not over programs but over departments), such a model falls in our environment. This type of model is not distinguished in terms of diversity. There are two distinct approaches that satisfy type-specific constraints. Ehlers et al. (2014) consider this kind of matching problems with an affirmative action policy. They developed a student exchange algorithm to ensure stability-constrained efficient matching among those that satisfy their notion of stability and type-specific constraints. Their algorithm works close to our QAP. The critical difference is that when we see a department as a school that has type-specific quotas, their model allows all students to apply to departments, but ours allows all students to apply to several programs within departments.

For other models with an affirmative action policy, there are sometimes ideal type-specific quota distributions. The goal of these situations is to find the matching close to the ideal quota distribution, while preserving stability as defined in each study. For example, see Hafalir et al. (2013), Echenique and Yenmez (2015), or Aygün and Turhan (2020). These studies do not treat quota distributions equally; thus, our model does not apply to every study. However, it is possible to incorporate an affirmative action policy into our model in such a way that each program is subject to type-specific constraints, which would enrich the model. This may be an interesting topic for future research.

5.3 Priorities with ties

Often, some students are ranked indifferently. In other words, priorities are allowed to include ties. As noted earlier, there is a similar relationship between the matching problem with quota adjustment and indifferences. Here, we discuss a unified model. As ESOSM is defined on distribution-specific stable matchings and when a quota distribution is fixed, distribution-specific student-optimal stable matching is equivalent to ESOSM. Our model incorporates ties in straightforward manner. For each quota distribution, one can extend distribution-specific student-optimal stable matching, as in Erdil and Ergin (2008). The QAP shares the idea of stable improvement cycle (SIC) of Erdil and Ergin (2008) when a quota distribution is fixed, and updates a quota distribution independent of SIC. Thus, with a small modification, QAP is still valid for priorities with ties. Future research should incorporate to be able to solve more realistic matching problems.

5.4 Lower bounds

As QAP adjusts a quota distribution subject to the upper bounds, holding department quotas constant, it may theoretically occur that under slack upper bounds, some programs lose all of their quota. To avoid such a problem, it is possible to incorporate lower bounds into QAP as in Fragiadakis et al. (2016) and Fragiadakis and Troyan (2017). The lower bounds work because every program must have students at least equal to its lower bound. As a matching model with lower bounds often faces the non-existence of distribution-specific stable matchings, we should assume that all students prefer any program to the outside option, as in the literature on a matching model with lower bounds. Then, we restrict QAP such that we exclude QASIC which binds to the lower bounds. It is clear that this modification does not hinder QAP because the process works and weakly improves student welfare compared to distribution-specific stable matching in an original quota distribution. However, the improvement from introducing lower bounds is less than the one without the lower bounds, because QAP finds an ESOSM via exchanges following the QASIC, which is restricted by the lower bounds. However, regardless of the lower bounds, as long as quota distributions are variable, it is possible to weakly improve students' welfare using QAP.

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A Appendix: Proofs

A.1 Proof of Theorem 1

Proof. For a fixed $q \in Q$ and $R \in \mathcal{R}^N$, there is a unique student-optimal stable (SOS) matching. As the set Q of quota distributions is finite, the set of distribution-specific SOS at all quota distributions, $\bigcup_{q \in Q} S^q(R)$, is also finite. Here, $S^q(R)$ is the set of all distribution-specific stable matchings at (R, q). Hence, there is an ESOSM because it is a maximal element of the finite set of $\bigcup_{q \in Q} S^q(R)$ in the partial order of Pareto domination. □

A.2 Proof of Proposition 1

Proof. Let μ be an ESOSM and q be the corresponding quota distribution. Suppose that at least one of the three conditions does not hold true. If (1) fails, this violates the distribution-specific stability of μ at q, which is a contradiction. If (2) fails, then $\sum_{y \in X_{k(x)}} |\mu^{-1}(y)| < \bar{q}_{k(x)}$. Thus, there is a vacant seat in some program $y \in X_{k(x)}$. Then, choose the student, denoted by j, with the highest priority among those who prefer x to the matched program in μ. Let a new matching ν be one in which only j's position changes to x from μ. Let p be a quota distribution, in which $p_x = q_x + 1$ and $p_y = q_y - 1$. Then, $\mu \in \mathcal{M}$ and is distribution-specific stable at p. Clearly, ν Pareto dominates μ, which is a contradiction. If (3) fails, then new matching ν is stable and Pareto dominates μ, which is a contradiction. □

A.3 Proof of Theorem 2

We use the following two propositions to prove Theorem 2.

Proposition 3. *Suppose that a matching μ is distribution-specific stable at a quota distribution q. If there is a QASIC for μ, $\langle (x_\ell, \iota_\ell) \rangle_{\ell=0}^{m}$, and construct a matching ν and quota distribution p in a way that*

$$\nu(j) = \begin{cases} x_\ell & \text{if } j = \iota_\ell, \forall \ell \in \{0, \cdots, m-1\} \text{ and } \iota_\ell \text{ is a student} \\ \varnothing & \text{if } j = \iota_\ell, \forall \ell \in \{0, \cdots, m-1\} \text{ and } \iota_\ell \text{ is a dummy student} \\ \mu(j) & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} p_{x_\ell} &= q_{x_\ell} + 1, && \forall \ell \in \{0, \cdots, m-1\} \\ p_{\mu(\iota_\ell)} &= q_{\mu(\iota_\ell)} - 1, && \forall \ell \in \{0, \cdots, m-1\} \\ p_x &= q_x, && \text{otherwise} \end{aligned}$$

then ν is distribution-specific stable at p.

Proposition 4. *Suppose that μ and ν are distribution-specific stable matchings at q and p, respectively. If ν Pareto dominates μ, then there is a QASIC for μ.*

The propositions will be proved in the next subsections. Proposition 3 says that a QASIC preserves stability. More important is Proposition 4 which is about whether we could always reach an ESOSM. Note that Proposition 4 does not say that even if ν Pareto dominates μ, applying a QASIC for μ results in ν.

By applying the above propositions, we prove Theorem 2. Let μ be distribution-specific stable at q.

(⇒) Suppose that there is a QASIC for μ. Then it follows from Proposition 3 that the constructed matching ν is distribution-specific stable at p, and it is obvious that ν Pareto dominates μ. Hence, μ is not an ESOSM.

($\Leftarrow$) Suppose that μ is not an ESOSM. Then there exist $p \in Q$ and a distribution-specific stable matching $\nu \in \mathcal{M}(p)$ at p such that ν Pareto dominates μ. Thus it follows from Proposition 4 that there exists a QASIC for μ.

A.3.1 Proof of Proposition 3

By definition of QASICs, if q is implementable, then p is straightforwardly implementable. Thus it suffices to show that ν is distribution-specific stable at p. Suppose by contradiction that ν is not distribution-specific stable at p. Then there is a blocking pair (j, x) for ν at p, so $x \; P_j \; \nu(j)$. Since ν Pareto dominates μ, we have $x \; P_j \; \mu(j)$. Thus, since μ is distribution-specific stable at q, we have $|\mu^{-1}(x)| = q_x$. Suppose that x is involved in the QASIC. Then an increase, decrease, or no change of q_x is made in association with a transfer of a student, so $|\nu^{-1}(x)| = p_x$. Suppose that x is not involved in the QASIC. Then $|\nu^{-1}(x)| = |\mu^{-1}(x)| = q_x = p_x$. Hence, since (j, x) is a blocking pair for ν at p, there is $h \in \nu^{-1}(x)$ such that $j \succ_x h$. There are two cases.

(Case 1) x is a program involved in the QASIC.

If $h \in \mu^{-1}(x) \cap \nu^{-1}(x)$, then it follows from the previous paragraph that $x \; P_j \; \mu(j)$ and $j \succ_x h$, i.e., μ is not distribution-specific stable at q, a contradiction. On the other hand, if $h \notin \mu^{-1}(x) \cap \nu^{-1}(x)$, then h is a member of some unit chain (x, h) in the QASIC. Since $x \; P_j \; \mu(j)$ by the previous paragraph, $j \in \{i \mid x \; P_i \; \mu(i)\}$, and thus $h \succ_x j$, a contradiction.

(Case 2) x is a program not involved in the QASIC.

Then $\nu^{-1}(x) = \mu^{-1}(x)$ and $p_x = q_x$. Thus $h \in \mu^{-1}(x)$ and $j \succ_x h$, which means that (j, x) is a blocking pair of μ at q. This contradicts the distribution-specific stability of μ. □

A.3.2 Proof of Proposition 4

We begin to introduce the following notion to prove the proposition.

Definition 8. We say that a distribution-specific stable matching ν at $p \in Q$ **minimally Pareto dominates** another distribution-specific stable matching μ at $q \in Q$ if ν Pareto dominates μ and there is no distribution-specific stable matching η at some $q' \in Q$ such that ν Pareto dominates η and η Pareto dominates μ.

Let μ and ν be distribution-specific stable matchings at $q \in Q$ and $p \in Q$, respectively. We assume, without loss of generality, that ν minimally Pareto dominates μ. We then extend matchings μ and ν to $\bar{\mu}$ and $\bar{\nu}$ to the following extended problem so that we add the set of dummy students D and treat $\varnothing$ as a real program.

The extended problem for μ is

$$\big(N \cup D, X \cup \{\varnothing\}, (R_\iota)_{\iota \in N \cup D}, (\succsim_x^\mu)_{x \in X \cup \{\varnothing\}}, Q \times \{\bar{q}_{k_\varnothing}\}, K \cup \{k_\varnothing\}\big)$$

where

- D is a set of dummy students and D satisfies

$$|D| = \sum_{k \in K} \max \left\{ \sum_{x \in k} \left(|\nu^{-1}(x)| - |\mu^{-1}(x)| \right), 0 \right\},$$

- $\varnothing$ is treated as a real program and its quota satisfies $q_\varnothing = |\mu^{-1}(\varnothing)|$, and it belongs to department $k_\varnothing$ with $\bar{q}_{k_\varnothing} = q_\varnothing$ which consists of only one program, that is, $X_{k_\varnothing} = \{\varnothing\}$,
- for all $x \in X$, $\succsim_x^\mu$ is a complete and transitive order over $N \cup D$ in which for all $i, j \in N$,

$$i \succ_x j \Rightarrow i \succ_x^\mu j,$$

for all $i \in N$ and $d \in D$,

$$i \succ_x^\mu d$$

for all $d, d' \in D$,

$$d \sim_x^\mu d'$$

and $\succsim_\varnothing^\mu$ is a complete and transitive order over $N \cup D$ in which for all $i \in N$ and all $d \in D$,

$$i \succ_\varnothing^\mu d,$$ [11]

and

- for all $d \in D$, R_d is complete and transitive preferences over $X \cup \{\varnothing\}$ in which for all $x, y \in X \cup \{\varnothing\}$,

$$x I_d y.$$ [12]

In this problem, we extend μ and ν as follows:

$\bar{\mu}$: for all $i \in N$, $\bar{\mu}(i) = \mu(i)$ and each dummy student is arbitrarily matched with a program in which $|\nu^{-1}(x)| \neq |\mu^{-1}(x)|$ among departments such that $\sum_{x \in X_k} \left(|\nu^{-1}(x)| - |\mu^{-1}(x)| \right) > 0$.

$\bar{\nu}$: for all $i \in N$, $\bar{\nu}(i) = \nu(i)$ and each dummy student is arbitrarily matched with a program in which $|\nu^{-1}(x)| \neq |\mu^{-1}(x)|$ among departments such that $\sum_{x \in X_k} \left(|\nu^{-1}(x)| - |\mu^{-1}(x)| \right) < 0$ and vacant seats of program $\varnothing$ if any.

Thus all students and dummy students are matched with some program and all seats are fulfilled both in $\bar{\mu}$ and $\bar{\nu}$ at $(q, q_\varnothing)$ and $(p, q_\varnothing)$ respectively. Note that if for some k, $\sum_{x \in X_k} \left(|\nu^{-1}(x)| - |\mu^{-1}(x)| \right) > 0$, then since $q_k = \sum_{x \in X_k} q_x = \sum_{x \in X_k} p_x \geq \sum_{x \in X_k} |\nu^{-1}(x)| > \sum_{x \in X_k} |\mu^{-1}(x)|$, there must exist vacant seats at μ. Note also that no dummy student is matched with a program in the same department between $\bar{\nu}$ and $\bar{\mu}$.

[11] We write $i \succ_x^\mu j$ if $i \succsim_x^\mu j$ and not $j \succsim_x^\mu i$, and $i \sim_x^\mu j$ if $i \succsim_x^\mu j$ and $j \succsim_x^\mu i$, respectively.

[12] We write $x I_d y$ if $x R_d y$ and $y R_d x$.

Lemma 1. *$\bar{\mu}$ and $\bar{\nu}$ are distribution-specific stable at $(q, q_\varnothing)$ and $(p, q_\varnothing)$, respectively. Furthermore, $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$.*[13]

Proof. As noted above, $\bar{\mu}$ is feasible at $(q, q_\varnothing)$. Since no student $i \in N$ prefers $\varnothing$ to $\mu(i)$ by individual rationality of μ, no pair of $i \in N$ and $x \in X \cup \{\varnothing\}$ blocks $\bar{\mu}$ for R at $(q, q_\varnothing)$. For all $d \in D$, d finds any program indifferent. Thus $\bar{\mu}$ is distribution-specific stable at $(q, q_\varnothing)$. Similar discussion applies to $\bar{\nu}$ at $(p, q_\varnothing)$.

By construction of the preferences of dummy students, each of them finds her assignment at $\bar{\mu}$ indifferent to that at $\bar{\nu}$. Furthermore, for all $x \in X$, q_x and p_x are unchanged in the extended problem from the original problem, and no student i prefers $\varnothing$ to $\mu(i)$ (thus $\nu(i)$). From these facts, since ν minimally Pareto dominates μ, $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. □

Let $N' = \{i \in N | \bar{\mu}(i) \neq \bar{\nu}(i)\}$.[14] Denote a pair of programs at $\bar{\nu}$ and $\bar{\mu}$ of student $i \in N'$ by

$$(\bar{\nu}(i), \bar{\mu}(i))$$

and a pair of programs at $\bar{\nu}$ and $\bar{\mu}$ of dummy student $d \in D$ by

$$(\bar{\nu}(d), \bar{\mu}(d)).$$

Consider a directed graph (V, E) where a set of vertices $V = K \cup \{k_\varnothing\}$ and a set of directed edges

$$E = \{(k', k'') \in [K \cup \{k_\varnothing] \times [K \cup \{k_\varnothing] \mid k(\bar{\nu}(\iota)) = k' \text{ and } k(\bar{\mu}(\iota)) = k'', \text{ for some } \iota \in N' \cup D\}.$$

That is, for all $\iota \in N' \cup D$, and for all pairs $(\bar{\nu}(\iota), \bar{\mu}(\iota))$ such that $k(\bar{\nu}(\iota)) = k'$ and $k(\bar{\mu}(\iota)) = k''$, we write a directed edge k'' to k'. We say that an ordered set of pairs $\langle(\bar{\nu}(\iota_s), \bar{\mu}(\iota_s))\rangle_{s=0}^{t}$ $(t \geq 1)$ constitutes a cycle if $\iota_0 = \iota_t$ and $k(\bar{\mu}(\iota_s)) = k(\bar{\nu}(\iota_{s+1}))$ for all $s \in \{0, 1, \ldots, t-1\}$. We denote a cycle by C which is formed visually as follows:

$$C : \bar{\nu}(\iota_0) \longleftarrow \underbrace{\bar{\mu}(\iota_0), \bar{\nu}(\iota_1)}_{k} \longleftarrow \underbrace{\bar{\mu}(\iota_1), \bar{\nu}(\iota_2)}_{k'} \longleftarrow \ldots \longleftarrow \underbrace{\bar{\mu}(\iota_{t-1}), \bar{\nu}(\iota_t)}_{k''} \longleftarrow \underbrace{\bar{\mu}(\iota_t), \bar{\nu}(\iota_0)}_{k'''} \longleftarrow \bar{\mu}(\iota_0).$$

Since we add dummy students in a way that the total numbers of matched students and dummy students at each k are constant across $\bar{\mu}$ and $\bar{\nu}$, it is obvious by this construction that each $k \in K$ and $k_\varnothing$ have the same numbers of incoming and outgoing edges (possibly none). Hence there must exist cycles $\{C^m\}$, $m \in \{1, 2, \ldots, n\}$ in which for each $\iota \in N' \cup D$, ι is involved in the exactly one cycle.

Note that there may be multiple cycles at this stage, but it is important that all students and

[13] We adopt the definition of distribution-specific stability in the extended problem as before.
[14] Since $\bar{\nu}$ Pareto dominates $\bar{\mu}$, this set is equivalent to $\{i \in N | \bar{\nu}(i) P_i \bar{\mu}(i)\}$.

dummy students are involved in the exactly one cycle. For the sake of exposition, for each C^m, the set of programs which are involved in $\bar{\nu}$ and $\bar{\mu}$ are denoted by $C^m_{\leftarrow}$ and $C^m_{\rightarrow}$, respectively.

There are potentially multiple ways to construct such cycles, but the following lemma tells that we can find cycles in which if $\bar{\mu}(\iota_s) \neq \bar{\nu}(\iota_{s+1})$ then $\bar{\mu}^{-1}(\bar{\nu}(\iota_{s+1})) < \bar{q}_{\bar{\nu}(\iota_{s+1})}$. Note that no cycle is constituted only of dummy students since in each k if there are outgoing edges by dummy students then there are no incoming edges by dummy students, and vice versa.

Lemma 2. *Among the sets of cycles constructed above, there must exist the set of cycles $\{C^m\}$, $m \in \{1, 2, \dots, n\}$ such that if $\bar{\mu}(\iota_s) \neq \bar{\nu}(\iota_{s+1})$ then $\bar{\mu}^{-1}(\bar{\nu}(\iota_{s+1})) < \bar{q}_{\bar{\nu}(\iota_{s+1})}$.*

Proof. Since all of $N' \cup D$ are involved in the exactly one cycle, if all $\iota \in N' \cup D$ exchange their matched programs at $\bar{\mu}$ along with $(\bar{\nu}(\iota), \bar{\mu}(\iota))$ then we obtain $\bar{\nu}$. The fact that $\bar{\nu}$ is feasible at $(p, q_{\varnothing})$ implies that the desired cycles exist. □

We then claim that the set of cycles satisfying the above condition consists of the exactly one cycle.

Lemma 3. *The set of cycles found in Lemma 2 is constituted of the exactly one cycle.*

Proof. Suppose not. If there are more than one distinct cycles, then we construct new matchings for each of cycles as follows: for each $m \in \{1, 2, \dots, n\}$,

$$\eta^{C^m}(\iota) = \begin{cases} \bar{\nu}(\iota) & \iota \in C^m \\ \bar{\mu}(\iota) & otherwise \end{cases}$$

$$\begin{array}{ll} p^m_{\nu(\iota)} = q_{\nu(\iota)} + 1 & \iota \in C^m \\ p^m_{\mu(\iota)} = q_{\mu(\iota)} - 1 & \iota \in C^m \\ p^m_x = q_x \ (x \in X \cup \{\varnothing\}) & otherwise \end{array}$$

Since exchanges along with the all cycles at $\bar{\mu}$ preserve feasibility and those cycles are distinct, $\{\eta^{C^m}\}$ are all feasible at $(p^m, q_{\varnothing})$, respectively. If there exists some m such that η^{C^m} is distribution-specific stable at $(p^m, q_{\varnothing})$, then it contradicts to the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. Thus we assume that all $\{\eta^{C^m}\}$ are not distribution-specific stable. That is, since η^{C^m} is individually rational, for each η^{C^m} there exists a pair of j^m and y^m that blocks η^{C^m}. Since $\bar{\mu}$ is distribution-specific stable at $(q, q_{\varnothing})$, $y^m \in C^m_{\leftarrow}$.

For all $y^m \in C^m_{\leftarrow}$, let j^{y^m} be of the highest priority among those who desire y^m at $\bar{\mu}$ if such student exists, and be a dummy student involved in $(y^m = \bar{\nu}(d), \bar{\mu}(d))$ if there is no student who desires y^m at $\bar{\mu}$. Denote them by $\{j^{y^m_0}, \dots, j^{y^m_{t-1}}\}$, possibly $j^{y^m_s} = j^{y^m_{s'}}$ for some s, s' with $s \neq s'$. For any C^m, if j^{y^m} is not a dummy student, then $j^{y^m} \in N'$, for otherwise $j^{y^m} \in N \setminus N'$ whose matched program does not change among $\bar{\mu}$, $\eta^{C^1}, \dots, \eta^{C^n}$, and $\bar{\nu}$, which contradicts to distribution-specific stability of $\bar{\nu}$. Since $\bar{\nu}$ is distribution-specific stable, for all m,

$$\bar{\nu}(j^m) R_{j^m} y^m P_{j^m} \bar{\mu}(j^m).$$

For each m, if $j^{y_0^m}, \ldots, j^{y_{t-1}^m}$ are all involved in C^m, we find a shorter cycle in a way that we add $(y_s^m, \bar{\mu}(j^{y_s^m}))$ and replace the original $(y_s^m, \bar{\mu}(\iota))$ by $(y_s^m, \bar{\mu}(j^{y_s^m}))$ in a sequential manner. That is,

- we begin with y_s^m, and replace $(y_s^m, \bar{\mu}(\iota))$ by $(y_s^m, \bar{\mu}(j^{y_s^m}))$.
- Then we consider pair $(y_{s'}^m = \nu(\iota'), \mu(\iota'))$ next to pair $(\bar{\nu}(j^{y_s^m}), \bar{\mu}(j^{y_s^m}))$ in C^m.
  - If ι' is a dummy student or is of the highest priority among those who desire $\nu(\iota')$ at $\bar{\mu}$, namely $\iota' = j^{y_{s'}^m}$, then we go to the pair next to $(\bar{\nu}(\iota'), \bar{\mu}(\iota'))$. Note that $\iota' \neq j^{y_s^m}$.
  - Otherwise $\iota' \neq j^{y_{s'}^m}$ then we replace $(y_{s'}^m, \mu(\iota'))$ by $(y_{s'}^m, \bar{\mu}(j^{y_{s'}^m}))$. Note that $j^{y_{s'}^m} \neq j^{y_s^m}$.
- Then we consider pair $(y_{s''}^m = \nu(\iota''), \mu(\iota''))$ next to pair $(\bar{\nu}(j^{y_{s'}^m}), \bar{\mu}(j^{y_{s'}^m}))$ in C^m.
  - If ι'' is a dummy student or is of the highest priority among those who desire $\nu(\iota'')$ at $\bar{\mu}$, namely $\iota'' = j^{y_{s''}^m}$, then we go to the pair next to $(\bar{\nu}(\iota''), \bar{\mu}(\iota''))$. Note that $\iota'' \neq j^{y_{s''}^m}$.
  - If $\iota'' \neq j^{y_{s''}^m}$ and $j^{y_{s''}^m} = j^{y_s^m}$, then we find a shorter cycle such that

$$y_{s'}^m \leftarrow \underbrace{\bar{\mu}(j^{y_{s'}^m}), y_{s''}^m}_{k'} \leftarrow \underbrace{\bar{\mu}(j^{y_s^m}), y_{s'}^m}_{k''} \leftarrow \bar{\mu}(j^{y_{s'}^m}).$$

  - Otherwise $\iota'' \neq j^{y_{s''}^m}$ and $\iota'' \neq j^{y_s^m}$ then we replace $(y_{s''}^m, \mu(\iota''))$ by $(y_{s''}^m, \bar{\mu}(j^{y_{s''}^m}))$. Note that $j^{y_{s''}^m} \neq j^{y_{s'}^m}$.

Since all of $j^{y_1^m}, \ldots, j^{y_t^m}$ are involved in C^m and $C_{\leftarrow}^m$ is finite, this process finds a shorter cycle than C^m. Then a matching induced by exchanging programs at $\bar{\mu}$ along with the above shorter cycle is feasible, distribution-specific stable and Pareto dominates $\bar{\mu}$, which contradicts to the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. Hence, for each m, there exists $y_s^m \in C_{\leftarrow}^m$ and $j^{y_s^m}$ who desires y_s^m at $\bar{\mu}$, is of the highest priority among those desire y_s^m at $\bar{\mu}$ and is not involved in C^m.

Now we consider $y_s^m \in C_{\leftarrow}^m$ for all m, s. As above for all m there must exist j_s^m involving in C^m who is of the highest priority among those who desire $y_{s'}^{m'} \in C_{\leftarrow}^{m'}$ for some $m' \neq m$. Then we find a new cycle such that students who are of the highest priority at $\bar{\mu}$ and are involved in different cycles. By construction, exchanges along with this new cycle induces a matching which is not equal to $\bar{\nu}$, feasible, distribution-specific stable, and Pareto dominates $\bar{\mu}$, which violates the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. Therefore, the set of cycles found in Lemma 2 is constituted of the exactly one cycle. □

From the above lemmata, we denote C by the cycle found above.

Lemma 4. *No $x \in C_{\leftarrow} \cap X$ appears more than once.*

Proof. Let C be $\langle (\bar{\nu}(\iota_s), \bar{\mu}(\iota_s)) \rangle_{s=0}^{t}$. Suppose that there exists $x \in X$ which appears more than once in the cycle. That is, there are at least two distinct students $\iota_{s'}$ and $\iota_{s''}$ (we assume without loss of generality that $s' < s''$). We know by Lemma 2 that C satisfies that $k(\bar{\mu}(\iota_s)) = k(\bar{\nu}(\iota_{s+1}))$, $\forall s \in \{0, 1, \ldots, t-1\}$ and if $\bar{\mu}(\iota_s) \neq \bar{\nu}(\iota_{s+1})$ then $\bar{\mu}^{-1}(\bar{\nu}(\iota_{s+1})) < \bar{q}_{\bar{\nu}(\iota_{s+1})}$.

Then we decompose C into two cycles C^1 and C^2 as follows,

$$C^1 : x = \bar{\nu}(\iota_{s'}) \leftarrow \underbrace{\bar{\mu}(\iota_{s'}), \bar{\nu}(\iota_{s'+1})} \leftarrow \bar{\mu}(\iota_{s'+1}), \cdots, \bar{\nu}(\iota_{s''-1}) \leftarrow \underbrace{\bar{\mu}(\iota_{s''-1}), x = \bar{\nu}(\iota_{s'})}_{k'} \leftarrow \bar{\mu}(\iota_{s'})$$

and

$$C^2 : \bar{\nu}(\iota_0) \leftarrow \bar{\mu}(\iota_0), \cdots, \bar{\nu}(\iota_{s'-1}) \leftarrow \underbrace{\bar{\mu}(\iota_{s'-1}), x = \bar{\nu}(\iota_{s''})}_{k'} \leftarrow \bar{\mu}(\iota_{s''})), \cdots, \bar{\nu}(\iota_0) \leftarrow \bar{\mu}(\iota_0)$$

Then since the above two cycles C^1 and C^2 satisfy that $k(\bar{\mu}(\iota_{s''-1})) = k(\bar{\nu}(\iota_{s''})) = k(x) = k(\bar{\nu}(\iota_{s'}))$ and $k(\bar{\mu}(\iota_{s'-1})) = k(\bar{\nu}(\iota_{s'})) = k(x) = k(\bar{\nu}(\iota_{s''}))$, respectively, C^1 and C^2 are the set of cycles found in Lemma 2, which contradicts to Lemma 3. Therefore, $x \in C_{\leftarrow} \cap X$ appears just once. □

Lemmata 3 and 4 tell us that there exists the unique cycle and all programs in $C_{\leftarrow}$ but $\varnothing$ are distinct. We further see the students' priority in N'.

Lemma 5. *For each $x = \bar{\nu}(i_s) \in \nu(N')$, $i_s \succsim_x^{\mu} j$, $\forall j \in \{j \in N | x P_j \bar{\mu}(j)\}$*

Proof. As before, all students in N' are involved in C, that is $\bar{\nu}(N') = C_{\leftarrow}$. Suppose not, then there exists $x \in C_{\leftarrow}$ and j such that $j \succsim_x^{\mu} i_s$. Obviously, $j \notin D$. We know that $j \notin N \setminus N'$, otherwise violating distribution-specific stability of $\bar{\nu}$. Thus $j \in N'$. We then find a shorter cycle similar to the process in Lemma 3, and exchanges along with the shorter cycle at $\bar{\mu}$ induce a matching which is not equal to $\bar{\nu}$, feasible, distribution-specific stable and Pareto dominates $\bar{\mu}$, which contradicts the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. □

The next lemma allows us to consider at most one dummy student at each $k \in K$.

Lemma 6. *For each k involved in C, there is at most one d in X_k at $\bar{\mu}$*

Proof. Suppose that there are more than one dummy students who are matched with some program at k in C, namely,

$$(\bar{\nu}(d), \bar{\mu}(d)), \ldots, (\bar{\nu}(d'), \bar{\mu}(d')).$$

Since $\bar{\mu}(d), \bar{\mu}(d') \in k$, we have the shorter cycle which replaces pairs between $(\nu(d), \mu(d))$ and $(\nu(d'), \mu(d'))$ in C by $(\nu(d), \mu(d'))$.

Since there is only one cycle which involved with distinct programs and each student i involving $(\bar{\nu}(i), \bar{\mu}(i))$ is of the highest priority among those who desire $\bar{\nu}(i)$ at $\bar{\mu}$ by Lemma 5, a matching obtained by exchanges along with the shorter cycle above preserves feasibility and distribution-specific stability, and Pareto dominates $\bar{\mu}$, which violates the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. $\square$

Eventually, we obtain cycle $\langle(\bar{\nu}(\iota_s), \bar{\mu}(\iota_s))\rangle_{s=0}^{t}$ such that

- $\iota_0, \cdots, \iota_t$ are distinct, and at least one ι_s is a student,
- $\iota_s \in N' \Rightarrow \iota_s \succsim_{\bar{\nu}(\iota_s)}^{\mu} j, \; \forall j \in \{j \in N | \bar{\nu}(\iota_s) P_j \bar{\mu}(j)\}$,
- $\bar{\mu}(\iota_{s-1}) \neq \varnothing \Rightarrow \bar{\mu}(\iota_{s-1}) \in X_{k(\bar{\nu}(\iota_s))}$ and $\bar{\mu}(\iota_{s-1}) = \varnothing \Rightarrow \bar{\mu}(\iota_{s-1}) = \bar{\nu}(\iota_s)$,
  for the latter part, since $X_{k_\varnothing} = \{\varnothing\}$ and $\bar{\mu}(\iota_{s-1}) \in X_{k(\bar{\nu}(\iota_s))}$, $\bar{\mu}(\iota_{s-1}) = \bar{\nu}(\iota_s)$.
- $\bar{\mu}(\iota_{s-1}) \neq \bar{\nu}(\iota_s)$ implies $q_{\bar{\nu}(\iota_s)} < \bar{q}_{\bar{\nu}(\iota_s)}$.

Now we would like to show that the cycle above is indeed QASIC. For the original problem, the cycle we obtain can be represented as $\langle(x_s, \iota_s)\rangle_{s=0}^{t}$ in which $x_s = \bar{\nu}(\iota_s)$:

$$(x_0, \iota_0) \longleftarrow (x_1, \iota_1) \longleftarrow \cdots \longleftarrow (x_t, \iota_t) \longleftarrow (x_0, \iota_0)$$

such that

- students and dummy students involved are distinct, and at least one student is involved,
- if $\iota_s \in N$, then $x_s P_{i_s} \mu(i_s)$ and $i_s \succ_{x_s} j$ for all $j \in \{j \in N \backslash \{i_s\} | x P_j \mu(j)\}$,
- if $\iota_s \notin N$, then (x_s, ι_s) is a unit chain by a dummy student,
- for all s, $\bar{\mu}(\iota_{s-1}) \neq \varnothing \Rightarrow \bar{\mu}(\iota_{s-1}) \in X_{k(x_s)}$ and $\bar{\mu}(\iota_{s-1}) = \varnothing \Rightarrow \bar{\mu}(\iota_{s-1}) = x_s$,
- $\bar{\mu}(\iota_{s-1}) \neq x_s$ implies $q_{x_s} < \bar{q}_{x_s}$,

as is desired. $\square$

A.4 Proof of Proposition 2

Proof. Let f and g be the QAP mechanisms under Q and Q', respectively. Note that the QAP starts from the DA matchings in both distributions. If there is a matching $\mu \in f(R)$ such that μ Pareto dominates ν for some $\nu \in g(R)$. Let the corresponding quota distribution of μ be p. As $p \in Q$ and $Q \subset Q'$, μ is either an ESOSM or Pareto dominated by some $\eta \in g(R)$ under Q'. The former implies $\mu \in g(R)$, and the latter implies that η Pareto dominates ν. Thus, in both cases, $\nu \notin g(R)$, a contradiction. Therefore, all the students' welfare unanimously improves as flexibility increases. $\square$

A.5 Proof of Theorem 3

Proof. Let f be the QAP correspondence for the initial distribution q^0. Suppose not. Then there exists $i \in N$, R, R_i, $\nu \in f(R'_i, R_{-i})$ such that

$$\nu(i) P_i \mu(i), \forall \mu \in f(R).$$

Let the corresponding quota distributions for μ and ν be q and p, respectively. Since QAP begins with the DA matching, we know that for all $i \in N$,

$$DA_i^{q^0}(R)\ R_i\ DA_i^{q^0}(R'_i, R_{-i}).$$

If $p = q$, then $\mu = DA^q(R)$ and $\nu = DA^q(R'_i, R_{-i})$. Since DA mechanism is strategy-proof, we have

$$\mu(i) = DA_i^q(R)\ R_i\ DA_i^q(R'_i, R_{-i}) = \nu(i),$$

a contradiction.

If $p \neq q$, then there exists $DA^p(R)$ such that,

$$DA_i^p(R)\ R_i\ DA_i^p(R'_i, R_{-i}) = \nu(i).$$

Since $p \in Q$ and any ESOSMs which Pareto dominates $DA_i^{q^0}(R)$ are reached by QAP (Proposition 4), there exists $\xi \in f(R)$ such that $\xi(i)\ R_i\ DA_i^p(R)$.[15] Then we have

$$\xi(i)\ R_i\ DA_i^p(R)\ R_i\ \nu(i),$$

a contradiction. □

A.6 Proof of Theorem 4

Proof. By counterexample. Suppose that $N = \{i_1, i_2, i_3, i_4\}$, $X = \{x_1, x_2, x_3\}$, and $(\bar{q}_{x_1}, \bar{q}_{x_2}, \bar{q}_{x_3}) = (2, 2, 1)$. Moreover, $K = \{k, k'\}$ such that $k(x_1) = k(x_2) = k$, $k(x_3) = k'$, $(\bar{q}_k, \bar{q}_{k'}) = (3, 1)$, such that $Q = \{(1, 2, 1), (2, 1, 1)\}$. Preferences R and priorities $\succ$ are as follows:

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	$\succ_{x_1}$	$\succ_{x_2}$	$\succ_{x_3}$
x_2	x_1	x_1	x_2	i_1	i_3	
x_1	x_3	x_2	x_3	i_4	i_2	any
				i_2	i_4	
				i_3	i_1	

[15] Without Proposition 4, we do not guarantee ξ in the outcomes of QAP.

$$DA^{(1,2,1)}(R) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_1 & x_3 & x_2 & x_2 \end{pmatrix}, \quad DA^{(2,1,1)}(R) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_1 & x_1 & x_2 & x_3 \end{pmatrix}$$

are ESOSMs. Let f be an ex-post student-optimal stable and strategy-proof mechanism. Suppose the mechanism selects $DA^{(2,1,1)}(R)$ for R. If i_4 declares

$$R'_{i_4} : x_2\ x_1\ x_3,$$

the DA matchings for that preference profile and two quota distributions are

$$DA^{(1,2,1)}(R'_{i_4}, R_{-i_4}) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_1 & x_3 & x_2 & x_2 \end{pmatrix}, \quad DA^{(2,1,1)}(R'_{i_4}, R_{-i_4}) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_1 & x_3 & x_2 & x_1 \end{pmatrix}$$

respectively. As $DA^{(2,1,1)}(R'_{i_4}, R_{-i_4})$ is Pareto dominated by $DA^{(1,2,1)}(R'_{i_4}, R_{-i_4})$ at (R'_{i_4}, R_{-i_4}), $DA^{(1,2,1)}(R'_{i_4}, R_{-i_4})$ is an ESOSM at (R'_{i_4}, R_{-i_4}). Then, the mechanism must select

$$f(R'_{i_4}, R_{-i_4}) = DA^{(1,2,1)}(R'_{i_4}, R_{-i_4})$$

Obviously, i_4 has an incentive to misstate his/her preference, because

$$x_2 = f_{i_4}(R'_{i_4}, R_{-i_4})\ P_{i_4}\ f_{i_4}(R) = x_3$$

As i_2 and i_4 play a symmetric role, if $f(R)$ is $DA^{(1,2,1)}(R)$ then i_2 has an incentive to misreport his/her preferences. Therefore, no single-valued function is ex-post student-optimal stable and strategy-proof. □

A.7 Proof of Claim 1

Proof. Let μ be the ESOSM and q be the corresponding quota distribution. Suppose that it is not weakly stable. Then, because μ is feasible and individually rational, and if $yP_j\mu(j)$, then for all $\ell \in \mu^{-1}(y)$ with $\ell >_y j$, there exists a pair (i, x) such that $xP_i\mu(i)$ and $g(w(\mu) + e_x) = 1$.

As $g(w(\mu) + e_x) = 1$, $\sum_{y \in k(x)} |\mu^{-1}(y)| < q_k$, it implies that there is a vacant seat in some programs in department $k(x)$. Let the program be z. Then, quota distribution q' such that $q'_x = q_x + 1$ and $q'_z = q_z - 1$ while holding the other quotas unchanged, is also feasible ($q' \in Q$).

Thus, it is clear that a QASIC must exist. As i prefers x to $\mu(i)$, at least one student wants x. No matter who is of the highest priority at x among students who prefer x to their matched program at μ, because z has a vacant seat, we can find a QASIC. That is, starting from the unit chain (x, j), if there are unit chains involving students connected to (x, j), we are done. Otherwise a dummy student who is assigned z connects to (x, j). This result contradicts the fact that μ is an ESOSM. □