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Inter-attribute equity in assignment problems: Leveling the playing field by priority design

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【要旨】

Priorities over agents are crucial primitives in assignment problems of indivisible objects without monetary transfers. In this paper, we introduce a prioritization problem: there are several exogenous attributes and each agent is equipped with one such attribute, and the priority is initially determined only for agents with the same attribute. This leads to a partial priority order. Our problem is to construct a complete priority, which is needed to implement a known mechanism such as the serial dictatorship. We propose a simple prioritization rule called the relative position rule. We formulate three equity axioms and an invariance property; the priority preservation law, the equal treatment of equal positions, the equal split, and the attribute-wise consistency. We show that the relative position rule is characterized by these equity axioms. The result is applicable to general assignment problems with partial priorities. In the context of college students' exchange programs, the rule levels the playing field in the sense that inequality across attributes is partially reduced.

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Inter-attribute equity in assignment problems: Leveling the playing field by priority design*

Tetsutaro Hatakeyama† Morimitsu Kurino‡

June 14, 2022

Abstract

Priorities over agents are crucial primitives in assignment problems of indivisible objects without monetary transfers. In this paper, we introduce a prioritization problem: there are several exogenous attributes and each agent is equipped with one such attribute, and the priority is initially determined only for agents with the same attribute. This leads to a partial priority order. Our problem is to construct a complete priority, which is needed to implement a known mechanism such as the serial dictatorship. We propose a simple prioritization rule called the relative position rule. We formulate three equity axioms and an invariance property; the priority preservation law, the equal treatment of equal positions, the equal split, and the attribute-wise consistency. We show that the relative position rule is characterized by these equity axioms. The result is applicable to general assignment problems with partial priorities. In the context of college students' exchange programs, the rule levels the playing field in the sense that inequality across attributes is partially reduced.

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Keywords: Priority-based assignment; Equity in attributes; The relative position rule; Market design

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1 Introduction

Indivisible goods allocation problems without monetary transfers are often encountered in real life markets such as on-campus housing (Abdulkadiroğlu and Sönmez, 1999; Kurino, 2014) and school choice problems (Abdulkadiroğlu and Sönmez, 2003). The mechanisms used in these markets often need a priority, or a complete and transitive order over agents. In general, the priority is formed by a law or exogenous rule, which the literature has focused on.[1] However, there is the case where priority has to be endogenously formed for a mechanism. In this paper, we study how to construct a priority fairly.

Universities all over the world have exchange programs in which millions of students study abroad at their partner universities.[2] Allocation of programs fairly to applicants at a low operating cost is crucial for universities. One university in Japan, which consulted the authors, has such a process of assigning about 400 students to hundreds of programs every year. Let us call this university "University X".[3] Due to the advantage of strategy-proofness and Pareto efficiency, it uses an allocation mechanism of serial dictatorship: given priority over students, the highest-priority student is assigned her most preferred program, the second-highest priority student her most preferred among the remaining programs, and so on.[4] However, before applying this mechanism, it is necessary to construct a complete priority order. One way is to use the grade point average (GPA). However, because every student belongs to one of several faculties and then takes courses offered by that faculty with its own grading policy, the simple use of GPA would not be fair to all faculties.[5] For this reason, University X held face-to-face interviews to have a complete priority. However, conducting such interviews has become costly due to an increase in the number of applicants. Therefore, University X decided to reform this prioritization process to a centralized rule. The problem that needs to be solved is how to construct a complete priority from partial priority where priority is available based on student GPAs derived from different grading policies of different faculties. Then there is a conflict among faculties, because every faculty would want the priority to favor its own students. Hence, University X faced a prioritization problem of setting a fair rulebecause a priority ordering of students from various faculties is not complete but partial.

Motivated by University X's example, we formulate the prioritization problem as an assignment problem of assigning exogenous K priority groups to agents when preferences are common and

[1]To allocate dormitory rooms to students, many universities in the United States use seniority-based priority. There are two priority groups, higher-priority group of non-freshmen and lower-priority group of freshmen. In Boston public high school matching, there are typically three priority groups: the first-priority group of students who have siblings attending the school, the second-priority group of students who live in the school's walk zone, and the last-priority group of remaining students.

[2]For example, see the webpage https://www.statista.com/chart/3624/the-countries-with-the-most-students-studying-abroad/ for details (accessed March 25, 2022).

[3]We are keeping its name secret due to a confidentiality agreement. We can provide the name upon request.

[4]For example, see Svensson (1994; 1999) for the serial dictatorship.

[5]For example, suppose there are two faculties, one with a mild grading policy and the other with a harsh grading policy. A complete priority only by GPA favors the students of the former.

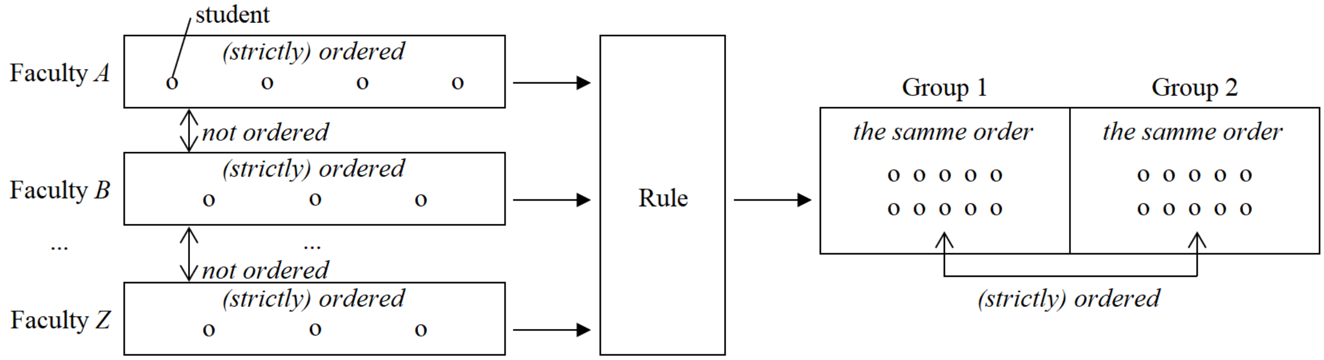

Figure 1: Prioritization problem

Note: Each "o" is a student. At first (left of the rule), students are ordered only within the same faculties. After applying a rule (right of the rule), all students are weakly ordered. Students within the same group have the same priority, while group 2's students have strictly higher priority than group 1's.

priorities are partial. Preferences here are over priority groups. Since it is natural to think that every agent prefers higher groups, we take agents' preferences as common. Each agent is equipped with one exogenous attribute, and the priority is determined for any two agents with the same attribute, but not for different attributes. In other words, we cannot compare any two agents across different attributes. The solution to this problem is a prioritization rule of finding a complete and transitive (not necessarily linear) priority over agents, equivalently an assignment of agents K priority groups.

To find a fair prioritization rule, we take an axiomatic approach as follows. We first state how agents should be compared as three new axioms called the *priority preservation law* (PPL), the *equal treatment of equal positions* (ETP), and the *equal split* (ES). The priority preservation law requires that, for any two agents of the same attribute, an agent of a lower position in a partial priority should not obtain a higher position in the resulting complete priority. The equal treatment of equal positions, the equal treatment of equals in our context, compares students of distinct attributes by their *relative positions*, i.e., ranks in an attribute scaled to an interval (0,1]. It requires that two agents of the same relative positions should obtain exactly the same priority group in the resulting complete priority. The equal split is a requirement regarding the equity in different relative position levels. It says that for each attribute, if possible, the number of agents that fall into one priority group should be equal to that fall into another. In addition, we also require one invariance property for the rule, called the *attribute-wise consistency* (AC). While consistency axioms for assignment problems have been studied in literature (e.g., Ergin, 2000), the attribute-wise consistency requires that, when agents of some attribute leave with their assigned objects, it does not affect the assignment of remaining agents.

We introduce a prioritization rule, called the *relative position rule*. This rule assigns each agent to a corresponding priority group with respect to her relative position in her own attribute. The main features of this rule are: (i) it determines the assignment only by the information on each agent's own attribute, i.e., independently from the one on the others', and (ii) its resulting assignment is calculated for each agent in a very simple formula. We show in Theorem 1 that this rule is characterized by the above equity axioms of PPL, ETP, and ES, together with an invariance property of AC.

Because the number of priority groups are taken to be exogenous, we further analyze how many priority groups would be appropriate for each prioritization problem. We first point out that a freely chosen and fixed number of priority groups may cause two distinct problems, unbalanced treatment of attributes and position-wise envy of students. In response to a negative result that it is generally difficult to perfectly resolve both problems, we propose two solution concepts, called the envy- and null-lexicographic optimum. These solutions are based on the spirit of minimizing two kinds of negative excess in a problem, namely by lexicographic manner.

Our prioritization problem seems narrow at first, but is quite applicable to a general assignment problem where priority is partial. The idea is to have the two stages of implementation. The first stage is to determine a complete and fair priority over agents. The second stage is to allocate real objects (e.g., exchange programs) to agents based on their preferences and the priority constructed in the first stage. Since the second stage is a one-sided assignment problem with complete and transitive (not necessarily linear) priority, we can achieve a strategy-proof and efficient mechanism such as serial dictatorship.

Contacted by the International Center of University X, we recommended the relative position rule for exchange programs, which was adopted and will be implemented from 2022. In exchange programs, the relative position rule is to "level the playing field." When a faculty's grading policy is relatively harsh, the rule protects its students from being given tremendously low priorities. Under the rule, each student only needs to care about her position within a faculty. This effect is similar to that of affirmative-action-oriented quota setting in school choice (Hafalir et al., 2013) that protects minority students. However, unfortunately, we also obtain a similar result to Kojima (2012) that improvement in priority may end up with worse assigned object.

1.1 Related Literature

Indivisible goods allocation problems without monetary transfers were first studied by Shapley and Scarf (1974). Since then, in many studies, such as those on house allocation problems (Abdulkadiroğlu and Sönmez, 1999; Kurino, 2014) and school choice problems (Abdulkadiroğlu and Sönmez, 2003), exogenous priority over agents arises in natural ways. Among a few, Álvarez and Medina (2020) consider a school choice problem with students' transferable characteristics, which are the source of higher priority. They introduce an algorithm which determines both priority and student-school matching simultaneously. More similarly, Greenberg et al. (2021) characterize the cumulative offer mechanism together with a priority (or more precisely a choice rule) by five axioms, while we characterize a priority alone by four axioms. While our model considers partial priorities, in school choice literature, there are several papers in which complete but weak priority is considered such as Erdil and Ergin (2008) and Erdil and Kumano (2019). Balbuzanov and Kotowski (2019) study exchange economy in which priority may be partial, but their approach is different from ours; they define new concepts of cores based on property rights and characterize them by generalized top trading cycles algorithm.

Equity concepts in assignment problems have been studied in the school choice context. Hafalir et al. (2013) explore how to implement an affirmative action policy respecting its spirit. Kojima (2012) shows impossibility results for two natural affirmative action policies. Though our study is not for affirmative action objective, it is technically related with the above-mentioned papers (i.e., we wish to help students of faculties with harsh grading policy like they aim to help minority students). However, these papers treat exogenous quota (and reserve) as primitive, while our approach is to find equitable quota endogenously by axioms. To our knowledge, there are a few papers in which quota is determined within models. Kamada and Kojima (2015) and Kumano and Kurino (2022) adjust initially given quota by their proposing algorithms, so that the resulting allocations Pareto improve. Our approach is different from theirs in that we do not require initial quota as primitive and determine the desirable quota by axioms.

The structure of this paper is as follows. Section 2 introduces the model and three equity axioms. Section 3 characterizes the prioritization rule. Section 4 is dedicated to two considerations about the rule. Section 5 extends the model to general assignment problems. Section 6 concludes the paper. All the proofs can be found in the appendix.

2 Model

We consider the problem of assigning ordered groups to agents who are partially ordered. Each agent who initially belongs to an attribute is assigned a group. To obtain a rule that works well in any population domain, we employ a variable population setting. We set a rule as a function of population in the next section.

There are K (priority) groups to be assigned to agents. We abuse the notation as $K = \{1, \ldots, K\}$. We assume that each group has no exogenous capacity restriction so that it can be assigned to any number of agents.[6] An agent has one attribute. Let $\mathcal{F}$ be a set of potential attributes, called **faculties**. In the case of universities, an attribute of a student is the department to which she belongs. We use f or g as a generic element of $\mathcal{F}$. Let $\mathcal{S}_f$ be a set of potential agents, or **students**, belonging to faculty $f \in \mathcal{F}$. Each potential student belongs to only one faculty, that is, for all $f, f' \in \mathcal{F}$ with $f \neq f'$, we have $\mathcal{S}_f \cap \mathcal{S}_{f'} = \emptyset$. Within each faculty $f \in \mathcal{F}$, its students are ordered by a linear order $\succeq_f$ on $\mathcal{S}_f$, but across distinct faculties students are not ordered. We

[6]Our problem can also be interpreted as endogenous selection of capacity, or quota, for each combination of a group and an attribute. Equity requirements determine desirable quota.

call $\succeq_f$ a **priority**. Since a priority $\succeq_f$ is a linear order, there is no tie among distinct students. Thus, $s \succeq_f s'$ if and only if $s \succ_f s'$ or $s = s'$. For simplicity, we assume that $\mathcal{F}$ and each $\mathcal{S}_f$ are countable sets.

Let $F \subsetneq \mathcal{F}$ and $S_f \subsetneq \mathcal{S}_f$ be finite sets of faculties and students in faculty f. We denote by n_f the number of students in S_f, i.e., $n_f = |S_f|$. Once S_f is fixed, we can define an **absolute position** of student $s \in S_f$ in faculty f as $a(s|S_f) = |\{s' \in S_f \mid s \succeq_f s'\}|$. This represents the rank of student s from the *bottom*. For example, when $S_f = \{s, s', s''\}$ and $s \succ_f s' \succ_f s''$, we have $(a(s|S_f), a(s'|S_f), a(s''|S_f)) = (3, 2, 1)$. Thus we have $\{a(s|S_f)\}_{s \in S_f} = \{n_f, n_f - 1, ..., 2, 1\}$. On the other hand, we define a **relative position** of student s in S_f as $\frac{a(s|S_f)}{n_f}$.

A **prioritization problem**, or a problem, is represented by $(F, \{S_f\}_{f \in F})$ where $F \subsetneq \mathcal{F}$, $S_f \subsetneq \mathcal{S}_f$. For notational simplicity, we denote it by $(F, \{S_f\})$. The problem is to assign K groups to students who are prioritized only in their own faculties. For each problem, we assume that $|F| \geq 2$ and $K \geq 2$. Even though we do not include students' preferences in primitives explicitly, we assume that, in each problem, all students have the common linear preference in which K is the most preferred, $K-1$ the second most preferred, $K-2$ the third most preferred, and so on. This restriction is to focus on the preferences over priority groups, in which every agent prefers a "higher" group. As described in Section 5, this restriction is also a natural one when preferences for real objects are heterogeneous.

An (priority) **assignment** at a problem $(F, \{S_f\})$ is represented by a collection $\{S^k\}_{k \in K}$ of subsets of the set of all students, $\cup_{f \in F} S_f$, such that $\cup_{k \in K} S^k = \cup_{f \in F} S_f$ and for all $k, k' \in K$ with $k \neq k'$, we have $S^k \cap S^{k'} = \emptyset$. Here the collection is composed of K groups and each student s is assigned one group, say $k \in K$, i.e., belongs to set S^k. In other words, a student in S^k is assigned group k. Each S^k is called the ***k*-th worst group** (in the sense of the common preferences). Note that the k-th worst group S^k may consist of students from different faculties, and thus S^k is partitioned to $\{S^k_f\}_{f \in F}$ where S^k_f is the set of students in faculty f who belongs to the k-th worst group. Thus, we have $\{S^k\}_{k \in K} = \{\cup_{f \in F} S^k_f\}_{k \in K}$ and $S_f = \bigcup_{k \in K} S^k_f$. We call each S^k_f a **subgroup** of S^k. Note that $s \in S^k_f$ stands for (i) s is a student of faculty f and (ii) s is assigned the k-th worst group.

We introduce several equity axioms for an assignment. The first one is on the equity within a faculty.

Definition 1 (Priority preservation law)**.** An assignment $\{S^k\}_{k \in K}$ satisfies the **priority preservation law (PPL)** if for each faculty $f \in F$, when a student $s \in S_f$ has lower priority than a student $s' \in S_f$, then the assigned group for s is not better than the one for s'. More precisely, for each $f \in F$, for any $s, s' \in S_f$ with $a(s|S_f) \leq a(s'|S_f)$, when $s \in S^k$ and $s' \in S^{k'}$, we have $k \leq k'$.

The priority preservation law says that the priority of two students within a faculty is transformed to the resulting assignment in the sense that for any two students in any faculty, the lower-priority student is assigned a worse group.

Next, we formalize an inter-attribute equity by applying the idea of equal treatment of equals to our problem. We consider two students to be "equal" if they have the same relative positions. Thus, our axiom requires equal students to be assigned the same group. Note that we compare students across faculties by their relative positions.

Definition 2 (Equal treatment of equal positions)**.** An assignment $\{S^k\}_{k\in K}$ satisfies the **equal treatment of equal positions (ETP)** if, for any two faculties $f, f' \in F$, when two students are of the same relative position then these two students are assigned the same group. More precisely, for any $f, f' \in F$, for any $s \in S_f$ and any $s' \in S_{f'}$, when $\frac{a(s|S_f)}{n_f} = \frac{a(s'|S_{f'})}{n_{f'}}$, for some $k \in K$, we have $\{s, s'\} \subseteq S^k$.

In a situation like University X's case, we can compare students only within a faculty by using priority. Since two students of different faculties are not comparable, we compare them by using the information of relative positions; we regard them "equals" if the relative positions are the same. We note that this requirement implies anonymity of faculties in the sense of population; when two faculties accommodate the same number of students, the assigned group for the best student in one faculty is the same as that for the best student in another, the assigned group for the next student in one faculty is the same as that for the next student in another, and so on. More precisely, when f, f' satisfy $n_f = n_{f'}$, it requires that for each $i = 1, 2, ..., n_f$, $a(s|S_f) = a(s'|S_{f'}) = i$ implies for some $k(i)$, $\{s, s'\} \subseteq S^{k(i)}$.

The following is the last requirement regarding the equity in different relative position levels.

Definition 3 (Equal split)**.** An assignment $\{S^k\}_{k\in K}$ satisfies the **equal split (ES)** if, for each faculty $f \in F$, when K divides the number of students n_f, then all of the assigned group in faculty f have the same number of students. More precisely, when some number $m \in \mathbb{N}$ satisfies an equation $n_f = mK$ for some $f \in F$, then, for each $k \in K$, we have $|S^k_f| = m$.

The equal split is a requirement which regards equity in positions in a faculty. Let us consider the case in which each student can improve her priority within a faculty by effort investment. If the group size differs tremendously, one student (in the bottom of a small group) may easily improve her priority while another student (in the bottom of a large group) may not. For example, when we divide 100 students to 30:5:5:60, a student at #68 (from the bottom) may easily improve his assigned group while a student at #38 may not. When ES binds, we can see that the inequality in this sense is partially eased.

Example 1. List of faculties, population, and the number of groups are $F = \{A, B\}$, $(n_A, n_B) = (6, 8)$, and $K = 3$.

The priority preservation law excludes the case in which student $s \in S_A$ with $a(s|S_A) = 2$ (lower position) is assigned a group S^2 but student $s' \in S_A$ with $a(s'|S_A) = 3$ (higher position) is assigned a group S^1.

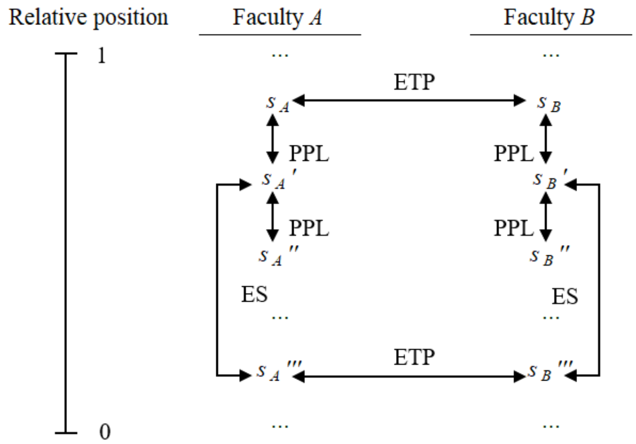

Figure 2: Relation of three axioms.

Since students $s \in S_A$ with $a(s|S_A) = 3$ and $s' \in S_B$ with $a(s'|S_B) = 4$ obtain the same relative positions $\frac{a(s|S_A)}{n_A} = \frac{a(s'|S_B)}{n_B} = 0.5$, the equal treatment of equal positions requires them to be assigned the same group. Similarly, students $s'' \in S_A$ with $a(s''|S_A) = 6$ and $s''' \in S_B$ with $a(s'''|S_B) = 8$ (top students in faculties) are required to be assigned the same group.

Since the number of students in faculty A is divided by the number of groups $K = 3$, the equal split requires that the number of faculty A's students who are assigned the same group is exactly two for each of $k = 1, 2, 3$.[7]

3 Prioritization Rules

Next, we formulate assignment rules in our context. A **prioritization rule** is a function φ that associates a priority assignment $\varphi(F, \{S_f\}) = \{S^k\}_{k \in K}$ to each problem $(F, \{S_f\})$. For each problem $(F, \{S_f\})$, we denote by $\varphi_g^k(F, \{S_f\})$ the set of students in faculty g for group k under rule φ, i.e., if $\varphi(F, \{S_f\}) = \{S^k\}_{k \in K}$, for each $g \in F$, $\varphi_g^k(F, \{S_f\}) = S_g^k$. Moreover, we denote by $\varphi_s(F, \{S_f\})$ the assigned group of student s under rule φ. We say that a prioritization rule satisfies the priority preservation law, the equal treatment of equal positions, or the equal split when its resulting assignment satisfies it for any problems.

We now introduce a condition for rules that requires no externality from other attributes.

Definition 4 (Irrelevance of other attributes)**.** A prioritization rule φ satisfies the **irrelevance of other attributes (IOA)** if for each faculty $g \in F$, when the numbers of students of g are the same in two population profiles $\{S_f\}_{f \in F}, \{\hat{S}_f\}_{f \in F}$, then, for each group, the number of g's students

[7]When we require both the priority preservation law and the equal split, only one assignment is possible for faculty A's students (i.e., students with absolute positions of 6 and 5 are assigned the best group, those with 4 and 3 are assigned the next group, and the remaining are assigned the worst group).

are the same for these two profiles. More precisely, for each $g \in F$, for each $\{S_f\}_{f\in F}, \{\hat{S}_f\}_{f\in F}$, when $n_g = \hat{n}_g$, for each $k \in K$, we have $|\varphi_g^k(F, \{S_f\})| = |\varphi_g^k(F, \{\hat{S}_f\})|$.

The irrelevance of other attributes requires that increase or decrease in population of a faculty should not affect the assignment of other faculties. For an agent, even if reassignment due to her own attribute's population change may be understandable, that due to other attribute's population change may not. The irrelevance of other attributes protects agents from irrelevant changes of population.

Since the irrelevance of other attributes seems quite restrictive in some situation, we compare it with several invariance properties. To this end, we adapt the standard axioms of population-monotonicity, consistency, and converse consistency in our prioritization problem. See Thomson (2011) for these axioms in various models. For the population-monotonicity, we consider the following definition.

Population-monotonicity. A prioritization rule φ is **population-monotonic (PM)** if, when a student is added in the worst position, assignments for existing students are never worse off. More precisely, for each problem $(F, \{S_f\})$, for each faculty $g \in F$, for each $S'_g = S_g \cup \{s'\}$ such that $a(s'|S'_g) = 1$, we have, for each $s \in \cup_{f\in F} S_f$, $\varphi_s(F, \{S'_g\} \cup \{S_f\}_{f\in F\setminus\{g\}}) \geq \varphi_s(F, \{S_f\})$.

When a student is added in an arbitrary position other than the worst, e.g., in the middle, its effect on other students may not be in one direction; students at better positions than the added student may be better off but the remaining ones may be worse off. Therefore, in the definition above, we only consider the cases where the effect is obviously in one direction. For the next two axioms, we need the concept of reduced problems. For each problem $(F, \{S_f\})$, for each subset of faculties $G \subset F$, we call $(G, \{S_g\}_{g\in G})$ a **reduced problem** of $(F, \{S_f\})$.[8]

Attribute-wise consistency. A prioritization rule φ is **attribute-wise consistent (AC)** if, for each reduced problem, each student is assigned the same group as the one assigned in the original problem. More precisely, for each problem $(F, \{S_f\})$ with $|F| \geq 3$, for each subset of faculties $G \subset F$ with $|G| \geq 2$, we have, for each student $s \in \cup_{g\in G} S_g$, $\varphi_s(G, \{S_g\}_{g\in G}) = \varphi_s(F, \{S_f\}_{f\in F})$.

Attribute-wise converse consistency. A prioritization rule φ is **attribute-wise conversely consistent (ACC)** if, for each problem, each student is assigned the same group as the one assigned in reduced problems. More precisely, for each problem $(F, \{S_f\})$ with $|F| \geq 2$, when for each subset of faculties $G \subset F$ with $|G| = 2$, $\varphi(G, \{S_g\}_{g\in G}) = \{\cup_{g\in G} S_g^k\}_{k\in K}$, we have $\varphi(F, \{S_f\}) = \{\cup_{f\in F} S_f^k\}_{k\in K}$.

The attribute-wise consistency requires that, when students of some faculties leave with their assigned groups, it does not affect the remaining students' assigned groups. The attribute-wise converse consistency is namely its converse. We only consider reduced problems where all students

[8] We note that this definition is a bit restrictive when compared to those in literature. Only the cases where a whole students of a faculty leave are considered as reduced problems here.

of a faculty simultaneously leave because, when an arbitrary subset of students leaves, requiring no effect on remaining students seems too much restrictive. For instance, consider that a faculty's students except its bottom one "s" leave. It is not obvious that we should keep s in the worst group, i.e., allowing him to be reassigned a better group may be reasonable.

The relations between the irrelevance of other attributes and above invariance properties are as follows.

Lemma 1. *Suppose that a prioritization rule φ satisfies the priority preservation law, the equal treatment of equal positions, and the equal split.*
(i) *A rule φ satisfies the irrelevance of other attributes if and only if it is attribute-wise consistent.*
(ii) *A rule φ is attribute-wise conversely consistent if it satisfies the irrelevance of other attributes. However, φ may not satisfy the irrelevance of other attributes if it is attribute-wise conversely consistent.*
(iii) *A rule φ is population-monotonic if it satisfies the irrelevance of other attributes. However, φ may not satisfy the irrelevance of other attributes if it is population-monotonic.*

The next rule is of our main interest. For each $x \in \mathbb{R}$, let $[x] \in \mathbb{Z}$ be the largest integer that does not exceed x. For instance, $[3.14] = 3$, $[2.71828] = 2$, and $[3] = 3$.

Definition 5 (Relative position rule)**.** The **relative position rule** φ^* is a prioritization rule such that for each problem $(F, \{S_f\})$, for each student $s \in \cup_{f\in F} S_f$, her assigned group $\varphi^*_s(F, \{S_f\})$ satisfies

$$\left[\frac{n_f\{\varphi^*_s(F,\{S_f\}) - 1\}}{K}\right] < a(s|S_f) \leq \left[\frac{n_f \varphi^*_s(F,\{S_f\})}{K}\right].$$

The above formula determines a unique assigned group $\varphi^*_s(F, \{S_f\}) = k \in K$ for each student $s \in \cup_{f\in F} S_f$, i.e., if some $k \in K$ satisfies both inequality for a student, any other $k' \in K$ do not satisfy them. When K divides n_f, all groups accommodate the same number of students from this faculty ($|\varphi^k_f(F, \{S_f\})| = \frac{n_f}{K} \in \mathbb{N}$ for each $k \in K$). This is clearly what the equal split requires. In other cases, the relative position rule results in "approximately" equal group sizes (i.e., $|\varphi^k_f(F, \{S_f\})| \in (\frac{n_f}{K} - 1, \frac{n_f}{K} + 1)$ for each $k \in K$).

Example 2. List of faculties, population, and the number of groups are $F = \{A, S\}$, $(n_A, n_S) = (5, 8)$, and $K = 5$. For student $s \in S_A$ with $a(s|S_A) = 3$, the relative position rule assigns her group S^3 since $a(s|S_A) = 3 = [\frac{n_A\times 3}{K}]$. On the other hand, for student $s' \in S_S$ with $a(s'|S_S) = 5$, the relative position rule assigns him group S^4 since $[\frac{n_S\times 3}{K}](= 4.8) < a(s|S_S) = 5 \leq [\frac{n_S\times 4}{K}](= 6.4)$.

Using Lemma 2 below, we can see this rule actually meets the requirements of all other axioms above (Proposition 1).

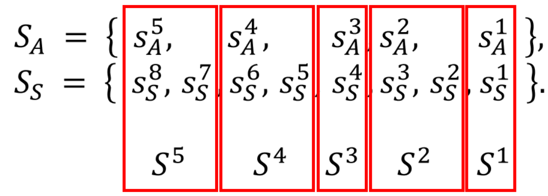

Figure 3: The assignment by φ^* in Example 2.

Lemma 2. *Suppose that an assignment $\{S^k\}_{k \in K}$ is generated by the relative position rule. Then, for each $s \in S_f$, $s \in S^k$ if and only if*

$$\frac{k-1}{K} < \frac{a(s|S_f)}{n_f} \leq \frac{k}{K}.$$

Proposition 1. *The relative position rule satisfies* (i) *the priority preservation law,* (ii) *the equal treatment of equal positions,* (iii) *the equal split,* (iv) *the irrelevance of other attributes,* (v) *the attribute-wise consistency,* (vi) *the attribute-wise converse consistency, and* (vii) *the population-monotonicity.*

Moreover, as in Theorem 1 below, we can characterize the relative position rule φ^* by the combination of four axioms.

Theorem 1. *A prioritization rule satisfies* (i) *the priority preservation law,* (ii) *the equal treatment of equal positions,* (iii) *the equal split, and* (iv) *the attribute-wise consistency if and only if it is the relative position rule.*

We note that, by Lemma 1, the relative position rule is also characterized by the combination of (i)-(iii) above and (iv') the irrelevance of other attributes. Though the relative position rule satisfies the attribute-wise converse consistency and the population-monotonicity, because they do not imply the irrelevance of other attributes, we cannot characterize the rule by the combination of (i)-(iii) and (iv") the attribute-wise converse consistency or (iv"') the population-monotonicity.

The logical independence of four axioms is shown in the following examples.

Example 3. Consider the "reversed" version of the relative position rule φ^1 such that

$\varphi^1_s(F, \{S_f\}) = (K+1) - \varphi^*_s(F, \{S_f\})$. We can verify that this rule satisfies ETP, ES, and AC, but violates PPL. $\Diamond$

Example 4. Consider the "faculty-dependent" rule φ^2 such that, for each $s \in \cup_{f \in F} S_f$,

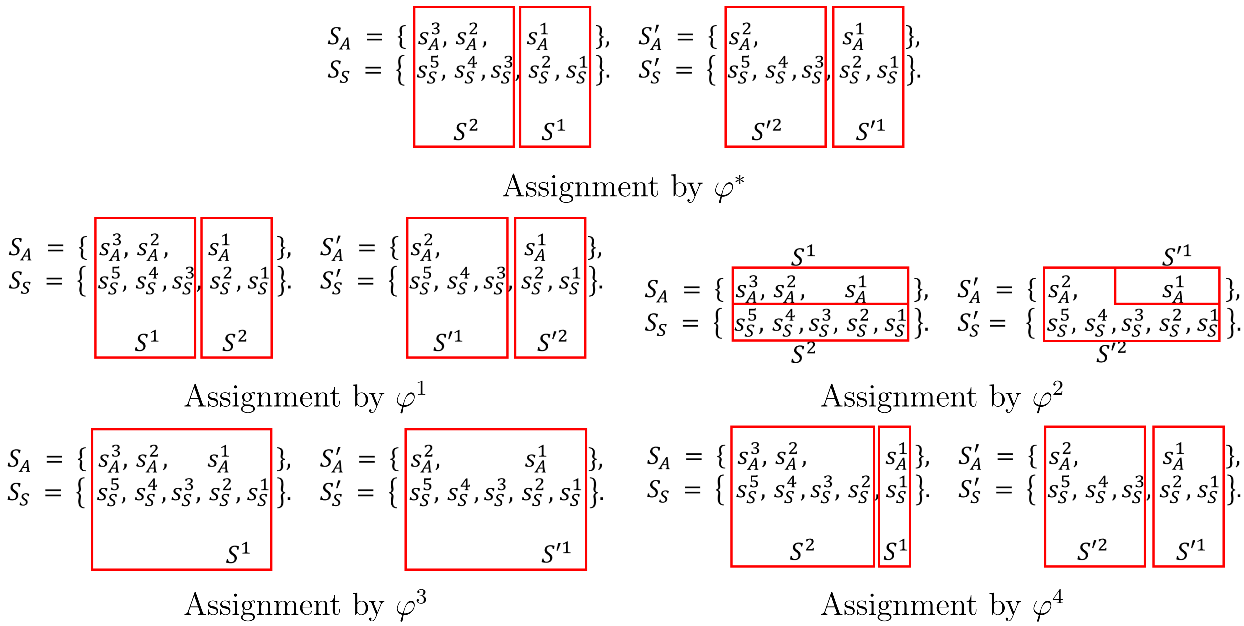

Figure 4: The assignment by each rule in Example 7.

$$\varphi_s^2(F, \{S_f\}) = \begin{cases} \varphi_s^*(F, \{S_f\}) & [\frac{n_f}{K} \in \mathbb{N}] \\ h(s) & otherwise \end{cases}$$

where $h(s) = h(s')$ if and only if for some $f \in F$, $\{s, s'\} \subseteq S_f$. We can verify that this rule satisfies PPL, ES, and AC, but violates ETP. ♢

Example 5. The "constant" rule φ^3 such that $\varphi_s^3(F, \{S_f\}) = 1$ satisfies PPL, ETP, and AC, but violates ES. ♢

Example 6. Consider the "domain-conditional" rule φ^4 such that

$$\varphi_s^4(F, \{S_f\}) = \begin{cases} \varphi_s^*(F, \{S_f\}) & [\exists s' \in S_{f'}, s'' \in S_{f''} \ s.t. \ \frac{a(s'|S_{f'})}{n_{f'}} = \frac{a(s''|S_{f''})}{n_{f''}} \neq 1] \\ & \vee [\exists f' \in F \ s.t. \ \frac{n_{f'}}{K} \in \mathbb{N}] \\ \min\{K, a(s|S_f)\} & \text{otherwise.} \end{cases}$$

We can verify that this rule satisfies PPL, ETP, and ES, but violates AC. ♢

Example 7. List of faculties, population, and the number of groups are $F = \{A, S\}$, $(n_A, n_S) = (3, 5)$, $(n'_A, n'_S) = (2, 5)$, and $K = 2$. The relative position rule φ^* and other rules $\varphi^1, \varphi^2, \varphi^3, \varphi^4$ above assign students as in figure 4.

4 Discussion

4.1 Student-Optimal Group Number

Until now we have taken the number of groups, K, as exogenously given, and then discussed the desirability of the relative position rule. In this section, we discuss how to choose K endogenously. To see points inherent in this problem, let us consider the following example.

Example 8 (Queuing problem[9])**.** There are one facility (an amusement park or a museum) and two types of students (customers), eight with normal tickets (type N) and four with special tickets (type S, old, handicapped, invitee etc). To avoid crowds, this facility divides students to K priority groups $\{S^k\}_{k\in K}$ (students in S^{k-1} can enter several minutes/hours after those in S^k). Within types N and S, students are ordered as queues $s_N^8 \succ_N \ldots \succ_N s_N^1$ and $s_S^4 \succ_S \ldots \succ_S s_S^1$ on a first-come-first-served basis. When the facility owner uses the relative position rule to treat two types fairly based on the relative positions in queues, the problem that the owner faces is the selection of the number of priority groups K. Consider the following two cases and the results of the relative position rule (See Figure 5.).

Case 1: When $K = 2$, the rule gives $S^2 = \{s_N^8, s_N^7, s_N^6, s_N^5, s_S^4, s_S^3\}$ and $S^1 = \{s_N^4, s_N^3, s_N^2, s_N^1, s_S^2, s_S^1\}$.
Case2: When $K = 8$, the rule gives $S^k = \{s_N^k\}$ for odd k's; $S^k = \{s_N^k, s_S^{k/2}\}$ for even k's. ◊

In this example, the parameter K is not primitive, but is actually a design variable. This leads to the questions: (i) if there exists the optimal K (in some sense), and (ii) how we can find it (if exists). From this point on, we will search for answers maintaining the following assumption.

Assumption 1 (Preference domain)**.** Each student prefers (i) to be assigned a higher priority group, and (ii) less number of students for her own assigned group.

The first assumption is the same as that of previous sections. The second one, a negative externality of students, is not obvious but is plausible when customers hate crowds, and/or objects are scarce.

Back to Case 1 ($K = 2$) in Example 4. In type N, s_N^4 is assigned to the lower group S^1, while her upper-priority neighbor, or immediate predecessor at $\succeq_N$, s_N^5 is assigned to the higher group S^2. The difference in their relative positions is $\frac{5-4}{8} = 0.125$. In type S, s_S^1 is assigned to S^1, the same group as the neighbor s_S^2, while the difference in their relative positions is $\frac{2-1}{4} = 0.25$. Hence, there is a gap in treatments of s_N^4 and s_S^1: Even though the former is closer to his upper-priority neighbor than the latter to her neighbor, only the latter is successfully assigned the same group as his neighbor. We can view that, when K=2, s_N^4 "envies" s_S^1 in the sense of position differences. In

[9]This example is different from generally considered queuing problems (e.g., Maniquet (2003)) in that (i) more than two agents can be served simultaneously, and (ii) no monetary compensations are allowed. Bloch (2017) considers a queuing model without monetary transfers but he applies a stochastic assignment rule.

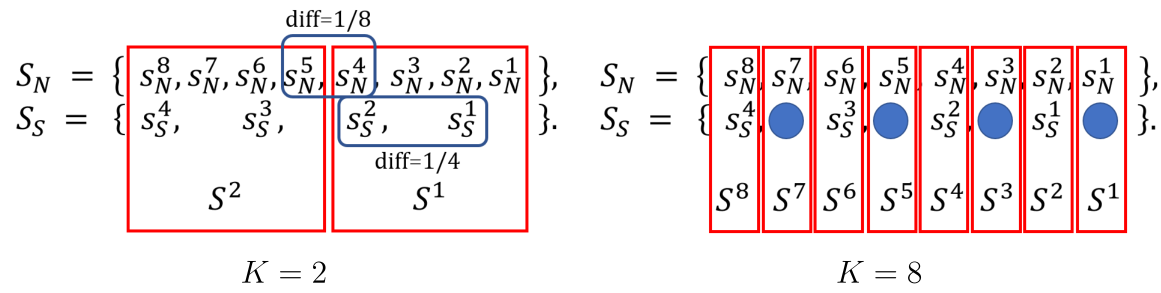

Figure 5: Two problems in Example 8.

this case, there is also a gap in treatment of s_N^5 and s_S^2; The former is closer to his lower-priority neighbor, or immediate successor at $\succeq_N$, than the latter to her neighbor while only the latter is assigned the same group as the neighbor. We can also view that when K=2, s_S^2 "envies" s_N^5 in the sense of position differences. We note that s_N^4's envy is by (i) of Assumption 1, while s_S^2's is by (ii) of Assumption 1.

In Case 2, no students envy others but another problem occurs. In type S, no students are assigned to four out of eight priority groups, i.e., S_S^1,S_S^3,S_S^5,S_S^7 are all null. We can see that s_S^i is in the same group as s_N^{2i}, strictly higher group than s_N^{2i-1}'s. The result that seems slightly preferable to type S occurs because we permit each subgroup to be null, i.e., to be assigned to no students. In this case, among $K \times |F| = 8 \times 2 = 16$ subgroups, S_S^1,S_S^3,S_S^5,S_S^7 are null.

The above discussion shows that our next goal is to eliminate envious students and null subgroups. We can see that neither of them exist only when $K=4$. Therefore, we view $K=4$ as the optimum in this example.

Definition 6 (Position-wise envy)**.** Consider a problem $(F, \{S_f\})$ and an assignment $\{S^k\}$ with a fixed K.

1. A student s has a **lower-side envy** at an assignment $\{S^k\}$ if she is assigned the strictly worse group than her upper-priority neighbor and some student has an upper-priority neighbor in the same priority group as his, even though the difference between his relative position and his neighbor's is strictly larger than that of s and her neighbor's. That is, for $s \in S_f^k$, $a^{-1}(a(s|S_f)+1|S_f) \notin S_f^k$; and for some $f' \in F\backslash\{f\}$ with $n_{f'} < n_f$, for some $k' \in K$, there is $s' \in S_{f'}^{k'}$ such that $a^{-1}(a(s'|S_{f'})+1|S_{f'}) \in S_{f'}^{k'}$.

2. A student s has an **upper-side envy** at an assignment $\{S^k\}$ if she is assigned the same group as her lower-priority neighbor and some student has a lower-priority neighbor in the strictly worse priority group than his, even though the difference between his relative position and his neighbor's is strictly smaller than that of s and her neighbor's. That is, for $s \in S_f^k$, $a^{-1}(a(s|S_f)-1|S_f) \in S_f^k$; and for some $f' \in F\backslash\{f\}$ with $n_{f'} > n_f$, for some $k' \in K$, there is $s' \in S_{f'}^{k'}$ such that $a^{-1}(a(s'|S_{f'})-1|S_{f'}) \notin S_{f'}^{k'}$.

3. A student s is **position-wise envious** at an assignment $\{S^k\}$ when she has either lower-side or upper-side envy.

In reality, the numbers of students (purchasing tickets or applying to the exchange programs) arise in a random manner. Thus, in most cases, we can expect that two distinct faculties have different numbers of students ($n_f \neq n_{f'}$ if $f \neq f'$) and each faculty has at least two students. We call such a case the **regular case**. Unfortunately, the next proposition shows negative results. We omit the proof because it is almost straightforward from Definition 6.[10]

Proposition 2. *In the regular case, given the relative position rule, the following hold.*
(i) *No students are position-wise envious if and only if K is not less than the second largest number in $\{n_f\}_{f\in F}$.*
(ii) *No subgroups are null if and only if K is not greater than the smallest number in $\{n_f\}_{f\in F}$.*
(iii) *When $|F| \geq 3$, there are no K's in which both envious students and null subgroups never exist.*

Therefore, we next search for the "second-best" solution concepts. For each problem $(F, \{S_f\})$, given the relative position rule φ^*, we introduce two definitions of optimal K's as follows. The formulation is based on the nucleolus of cooperative games, that is similar to our solution in the spirit of minimizing (a kind of) negative excess in a model.[11]

Definition 7 (Lexicographic optima)**.** For each problem $(F, \{S_f\})$ and each prioritization rule φ, denote the number of students with lower-side envy *plus* those with upper-side envy under K by $\theta^e(K|F, \{S_f\}, \varphi)$.[12] In the same manner, denote the number of null subgroups under K by $\theta^{\emptyset}(K|F, \{S_f\}, \varphi)$. That is,

$$\begin{aligned}\theta^e(K|F, \{S_f\}, \varphi) \;&:=\; |\cup_{f\in F}\{s \in S_f \mid s \text{ is lower-side envious at } (F, \{S_f\}, K, \varphi)\}| \\ &\quad\; +|\cup_{f\in F}\{s \in S_f \mid s \text{ is upper-side envious at } (F, \{S_f\}, K, \varphi)\}|, \\ \theta^{\emptyset}(K|F, \{S_f\}, \varphi) \;&:=\; |\cup_{f\in F}\{S^k_f \subseteq S_f \mid S^k_f = \emptyset \text{ at } (|F, \{S_f\}, K, \varphi)\}|.\end{aligned}$$

1. K is the **envy-lexicographic optimum** for $(F, \{S_f\}, \varphi^*)$ if K minimizes $\theta^{\emptyset}(\,\cdot\,|F, \{S_f\}, \varphi^*)$ among the minimizers of $\theta^e(\,\cdot\,|F, \{S_f\}, \varphi^*)$. That is, we have
   (a) for each $K' \geq 2$, $\theta^e(K|F, \{S_f\}, \varphi^*) \leq \theta^e(K'|F, \{S_f\}, \varphi^*)$, and
   (b) for each $K' \geq 2$ with $\theta^e(K|F, \{S_f\}, \varphi^*) = \theta^e(K'|F, \{S_f\}, \varphi^*)$, $\theta^{\emptyset}(K|F, \{S_f\}, \varphi^*) \leq \theta^{\emptyset}(K'|F, \{S_f\}, \varphi^*)$.

[10] We comment only about (ii). "Only if" is immediate from contrapositive. "If" is shown as follows. Suppose $n_f \geq K$. Then, for each $k \in K$, $[\frac{n_f k}{K}] - [\frac{n_f(k-1)}{K}] \geq [\frac{n_f k}{K} - \frac{n_f(k-1)}{K}] = [\frac{n_f}{K}] \geq 1$. Therefore, for each $k \in K$, there is $s \in S_f$ such that $\varphi^*_s(F, \{S_f\}) = k$. Statement (ii) also holds in non-regular cases.

[11] For a modern definition and the nature of the nucleolus, for example, see Chapter 5 of Moulin (1991).

[12] When a student has both lower-side and upper-side envies, we count them as two. Thus, $\theta^e(K|F, \{S_f\}, \varphi)$ is not always the same as the number of position-wise envious students at $(F, \{S_f\}, K, \varphi)$.

2. K is the **null-lexicographic optimum** for $(F, \{S_f\}, \varphi^*)$ if K minimizes $\theta^e(\cdot|F, \{S_f\}, \varphi^*)$ among the minimizers of $\theta^\emptyset(\cdot|F, \{S_f\}, \varphi^*)$. That is,

   (a) for each $K' \geq 2$, $\theta^\emptyset(K|F, \{S_f\}, \varphi^*) \leq \theta^\emptyset(K'|F, \{S_f\}, \varphi^*)$, and

   (b) for each $K' \geq 2$ with $\theta^\emptyset(K|F, \{S_f\}, \varphi^*) = \theta^\emptyset(K'|F, \{S_f\}, \varphi^*)$, $\theta^e(K|F, \{S_f\}, \varphi^*) \leq \theta^e(K'|F, \{S_f\}, \varphi^*)$.

The idea of two lexicographic optima is to prioritize two policy objectives, reducing envy and null, in a lexicographic manner. The envy (null) optimum mainly focuses on reducing the number of envious students (null subgroups) and supplementary reduces the other. Roughly speaking, larger (smaller) K induces more null subgroups (envious students) but it does less envious students (null subgroups). The next proposition helps finding these optima as well as assuring the existence.[13]

Proposition 3. *For any problems, the following two hold.*
(i) *In the regular case,* $K = \max_{f \in F}\{n_f \mid n_f < \max_{g \in F}\{n_g\}\}$ *is the unique envy-lexicographic optimum. Otherwise, either* $\max_{f \in F}\{n_f \mid n_f < \max_{g \in F}\{n_g\}\}$ *or* $\max_{f \in F}\{n_f\}$ *is (possibly one of) the envy-lexicographic optimum.*
(ii) *In the regular case,* $K = \min_{f \in F}\{n_f\}$ *is the unique null-lexicographic optimum. Otherwise, either* $\min_{f \in F}\{n_f\}$ *or 2 is (possibly one of) the null-lexicographic optimum.*

The following example illustrates the result of Proposition 3.

Example 9. Add faculty C (children) and students $S_C = \{s_C^7, ..., s_C^1\}$ to Example 8. Since n_N, n_S, and n_C are distinct, this is a regular case of $|F| = 3$. See Table 1. $K = 7$ is the unique envy-lexicographic optimum because $\theta^e(K)$ is minimized where $K \geq 7$, and among them, $\theta^\emptyset(K)$ is minimized at $K = 7$. $K = 4$ is the unique null-lexicographic optimum because $\theta^\emptyset(K)$ is minimized where $K \leq 4$, and among them, $\theta^e(K)$ is minimized at $K = 4$.

[13] By this proposition, we can actually construct a lattice of desirable group numbers. Technically, its structure is similar to that of stable matchings in marriage markets (Adachi, 2000), yielding the men- and women-optimal matching as maximum and minimum of the defined order. Let $(F, \{S_f\})$ be an arbitrarily regular case. Let $K^*(F, \{S_f\})$ be a set of natural number K's satisfying $\min_{f \in F}\{n_f\} \leq K \leq \max_{f \in F}\{n_f \mid n_f < \max_{g \in F}\{n_g\}\}$. Let $\succeq_{K^*}$ be a relation on $K^*(F, \{S_f\})$ such that, for any $K, K' \in K^*(F, \{S_f\})$, $K \succeq_{K^*} K'$ if and only if $\theta^e(K|F, \{S_f\}, \varphi^*) \leq \theta^e(K'|F, \{S_f\}, \varphi^*)$ and $\theta^\emptyset(K|F, \{S_f\}, \varphi^*) \geq \theta^\emptyset(K'|F, \{S_f\}, \varphi^*)$. The pair $(K^*(F, \{S_f\}), \succeq_{K^*})$ forms a lattice. Within $K^*(F, \{S_f\})$, the maximal (minimal) element of $\succeq_{K^*}$ is the best (worst) in the number of position-wise envious students, while being the worst (best) in the number of null subgroups.

Table 1: a regular case of $|F| = 3$

K	lower-side envious students	upper-side envious students	null subgroups	$\theta^e(K)$	$\theta^\emptyset(K)$
2	s_N^4, s_C^3	$s_C^7, s_C^6, s_C^5, s_C^3, s_C^2, s_S^4, s_S^2$	-	9	0
3	$s_N^5, s_N^2, s_C^4, s_C^2$	$s_C^7, s_C^6, s_C^4, s_C^2, s_S^4$	-	9	0
4	s_N^6, s_N^4, s_N^2	s_C^7, s_C^5, s_C^3	-	6	0
5	$s_N^6, s_N^4, s_N^3, s_N^1$	s_C^7, s_C^4	S_S^1	6	1
6	$s_N^6, s_N^5, s_N^4, s_N^2, s_N^1$	s_C^7	S_S^4, S_S^1	6	2
7	-	-	S_S^5, S_S^3, S_S^1	0	3
8	-	-	$S_C^1, S_S^7, S_S^5, S_S^3, S_S^1$	0	5
9	-	-	$S_N^1, S_C^5, S_C^1, S_S^8, S_S^6, S_S^4, S_S^2, S_S^1$	0	8

4.2 Modeling Options

In the analysis in the previous sections, the concept of students' absolute and relative positions played a key role. We refer them to define an axiom of the equal treatment of equal positions and the relative position rule that implements the axioms.

Recall that the absolute position is defined as the rank of each student from the *bottom* within a faculty. So, one question here is: if we modify the definition to "the rank of each student from the *top*," does the relative position rule still meet the equal treatment of equal "modified" (relative) positions? In other words, when two students are of the same relative position from the top, are these two students assigned the same group? This question arises because there are many ways of defining absolute positions.[14] Unfortunately, we obtain a negative result for this question.

To see the result, we first define the *inverse* absolute position, each student's rank in her faculty from the *top*. For each problem $(F, \{S_f\})$, for each student $s \in \cup_{f\in F} S_f$, the **inverse absolute position** is given by $a^{inv}(s|S_f) := |\{s' \in S_f \mid s' \succeq_f s\}|$. For example, when $S_f = \{s, s', s''\}$ and $s \succ_f s' \succ_f s''$, we have $(a^{inv}(s|S_f), a^{inv}(s'|S_f), a^{inv}(s''|S_f)) = (1, 2, 3)$. We observe the relation for each student $s \in \cup_{f\in F} S_f$, $a^{inv}(s|S_f) = n_f + 1 - a(s|S_f)$. Similarly, we define an **inverse relative position** of student s in S_f as $\frac{a^{inv}(s|S_f)}{n_f}$. Then, we consider the following example.

Example 10. List of faculties, population, and the number of groups are $F = \{A, S\}$, $(n_A, n_S) = (3, 6)$, and $K = 2$. In figure 7, s_f^i is the student of an absolute position $a(s_f^i|S_f) = i$. The assignment of the left image, (S^2, S^1), is given by the relative position rule. Since the rule satisfies the equal treatment of equal positions, each pair of equal relative position students, (s_A^3, s_S^6) at $\frac{a(s_A^3|S_A)}{n_A} = \frac{3}{3} = \frac{6}{6} = \frac{a(s_S^6|S_S)}{n_S}$, (s_A^2, s_S^4) at $\frac{a(s_A^2|S_A)}{n_A} = \frac{2}{3} = \frac{4}{6} = \frac{a(s_S^4|S_S)}{n_S}$, or (s_A^1, s_S^2) at $\frac{a(s_A^1|S_A)}{n_A} = \frac{1}{3} = \frac{2}{6} = \frac{a(s_S^2|S_S)}{n_S}$, is assigned the same group ($\{s_A^3, s_S^6\} \subset S^2$, $\{s_A^2, s_S^4\} \subset S^2$, or $\{s_A^1, s_S^2\} \subset S^1$).

[14] Other than the original and this subsection's formulations, the following definition may also be natural. For each problem $(F, \{S_f\})$, for each student $s \in \cup_{f\in F} S_f$, let $\hat{a}(s|S_f) := |\{s' \in S_f \mid s \succ_f s'\}| = a(s|S_f) - 1$. When $S_f = \{s, s', s''\}$ and $s \succ_f s' \succ_f s''$, we have $(\hat{a}(s|S_f), \hat{a}(s'|S_f), \hat{a}(s''|S_f)) = (2, 1, 0)$. This is also the rank of a student from the bottom but the bottom student's absolute position is normalized to be zero.

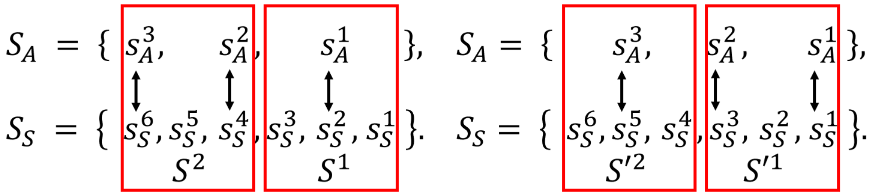

Figure 6: Two different assignments in Example 10.

Each s_f^i is of an inverse absolute position $a^{inv}(s|S_f) = n_f + 1 - i$, i.e.,

$$(a^{inv}(s_A^3|S_A), a^{inv}(s_A^2|S_A), a^{inv}(s_A^1|S_A)) = (1,2,3), \text{ and}$$
$$(a^{inv}(s_S^6|S_S), a^{inv}(s_S^5|S_S), a^{inv}(s_S^4|S_S), a^{inv}(s_S^3|S_S), a^{inv}(s_S^2|S_S), a^{inv}(s_S^1|S_S)) = (1,2,3,4,5,6).$$

For inverse relative positions, pairs (s_A^3, s_S^5), (s_A^2, s_S^3), and (s_A^1, s_S^1) are of the same inverse relative positions because $\frac{a^{inv}(s_A^3|S_A)}{n_A} = \frac{1}{3} = \frac{2}{6} = \frac{a^{inv}(s_S^5|S_S)}{n_S}$, $\frac{a^{inv}(s_A^2|S_A)}{n_A} = \frac{2}{3} = \frac{4}{6} = \frac{a^{inv}(s_S^3|S_S)}{n_S}$,and $\frac{a^{inv}(s_A^1|S_A)}{n_A} = \frac{3}{3} = \frac{6}{6} = \frac{a^{inv}(s_S^1|S_S)}{n_S}$. We notice that pairs of the same inverse relative positions differ from those of the same relative positions (e.g., s_A^3 is of the same relative position as s_S^6, but is of the same inverse relative position as s_S^5). Then, a pair (s_A^2, s_S^3) is not assigned the same group ($s_A^2 \in S^2$ but $s_S^3 \in S^1$).

In figure 7, in the assignment of the right image (S'^2, S'^1), each pair of equal inverse relative position students is assigned the same group, e.g., $(s_A^2, s_S^3) \subset S'^1$. However, a pair (s_A^2, s_S^4) is of the same relative position but is not assigned the same group ($s_A^2 \in S'^1$ but $s_S^4 \in S'^2$). ♢

Formally, we state the inverse version of the equal treatment of equal positions as follows.[15][16] As is shown in Example 9, the relative position rule does not satisfy this requirement.

Equal treatment of equal inverse positions. An assignment $\{S^k\}_{k \in K}$ satisfies the **equal treatment of equal inverse positions (ETIP)** if, for any two faculties $f, f' \in F$, when two students are of the same inverse relative position then these two students are assigned the same group. More precisely, for any $f, f' \in F$, for any $s \in S_f$ and any $s' \in S_{f'}$, when $\frac{a^{inv}(s|S_f)}{n_f} = \frac{a^{inv}(s'|S_{f'})}{n_{f'}}$, for some $k \in K$, we have $\{s, s'\} \subseteq S^k$.

The following rule, the inverse version of the relative position rule, satisfies the equal treatment of equal inverse positions above. The assignment (S'^2, S'^1) in Example 9 is given by this rule.

[15]The priority preservation law, also depending on the formulation of absolute positions, is rewrite as follows: for each $f \in F$, for any $s, s' \in S_f$ with $a^{inv}(s|S_f) \geq a^{inv}(s'|S_f)$, when $s \in S^k$ and $s' \in S^{k'}$, we have $k \leq k'$. This is completely the same requirement as the original definition because $a^{inv}(s|S_f) \geq a^{inv}(s'|S_f)$ is equivalent to $a(s|S_f) \leq a(s'|S_f)$.

[16]Using an alternatively formulated absolute position on a footnote above, we can change the statement as follows: for any $f, f' \in F$, for any $s \in S_f$ and any $s' \in S_{f'}$, when $\frac{\hat{a}(s|S_f)}{n_f} = \frac{\hat{a}(s'|S_{f'})}{n_{f'}}$, for some $k \in K$, we have $\{s, s'\} \subseteq S^k$. This requirement is completely the same as the equal treatment of equal *inverse* positions because $\frac{\hat{a}(s|S_f)}{n_f} = \frac{\hat{a}(s'|S_{f'})}{n_{f'}}$ is equivalent to $1 - \frac{\hat{a}(s|S_f)}{n_f} = 1 - \frac{\hat{a}(s'|S_{f'})}{n_{f'}}$, or $\frac{n_f + 1 - a(s|S_f)}{n_f} = \frac{n_{f'} + 1 - a(s'|S_{f'})}{n_{f'}}$, which means $\frac{a^{inv}(s|S_f)}{n_f} = \frac{a^{inv}(s'|S_{f'})}{n_{f'}}$.

Thus, this rule does not satisfy the equal treatment of equal positions.

Inverse relative position rule. For each $k \in K$, let $\pi(k) = K + 1 - k$. The **inverse relative position rule** φ^{*inv} is a prioritization rule such that for each problem $(F, \{S_f\})$, for each student $s \in \cup_{f \in F} S_f$, her assigned group $\varphi_s^{*inv}(F, \{S_f\})$ satisfies

$$\left\lceil \frac{n_f\{\pi(\varphi_s^{*inv}(F, \{S_f\})) - 1\}}{K} \right\rceil < a^{inv}(s|S_f) \leq \left\lceil \frac{n_f \pi(\varphi_s^{*inv}(F, \{S_f\}))}{K} \right\rceil .$$

This rule is characterized by the equal treatment of equal inverse positions with other three axioms (Proposition 4. (c)). This result is obtained by renaming positions in Theorem 1 in a straightforward way. We omit the complete proof for the following proposition because (a) is by Proposition 1 and Example 9, (b) is by the above argument and Example 9, (c) is by the above argument, and (d) is by Theorem 1 and (c).

Proposition 4. (a) *The relative position rule satisfies the equal treatment of equal positions but it may not satisfy the equal treatment of equal inverse positions.*
(b) *The inverse relative position rule satisfies the equal treatment of equal inverse positions but it may not satisfy the equal treatment of equal positions.*
(c) *A prioritization rule satisfies* (i) *the priority preservation law,* (ii') *the equal treatment of equal inverse positions,* (iii) *the equal split, and* (iv) *the attribute-wise consistency if and only if it is the inverse relative position rule.*
(d) *There are no prioritization rules that satisfy* (i) *the priority preservation law,* (ii) *the equal treatment of equal positions,* (ii') *the equal treatment of equal inverse positions,* (iii) *the equal split, and* (iv) *the attribute-wise consistency.*

Thus, when we require the axioms of (i), (iii), and (iv) above, there are at least two reasonable prioritization rules: the relative and the inverse relative position rules. The judgment of which rule to implement depends on which axiom out of (ii) equal treatment of equal positions and (ii') equal treatment of equal inverse positions, we wish to emphasize.

Finally, we additionally note the following conditions that are actually incompatible if we choose the relative or the inverse relative position rule.

Definition 8 (Top/Bottom-agent invariance)**.** (i) A prioritization rule φ is **top-agent invariant** if each student who is top-ranked within her faculty is always assigned the best group. That is, for each $(F, \{S_f\})$, for each $g \in F$, for each $s \in S_g$ such that $a(s|S_g) = n_g$, $\varphi_s(F, \{S_f\}) = K$.
(ii) A prioritization rule φ is **bottom-agent invariant** if each student who is bottom-ranked within her faculty is always assigned the worst group. That is, for each $(F, \{S_f\})$, for each $g \in F$, for each $s \in S_g$ such that $a(s|S_g) = 1$, $\varphi_s(F, \{S_f\}) = 1$.

As the following proposition shows it, the relative position rule satisfies only the former while the inverse relative position rule does only the latter. We omit the straightforward proof.[17] In reality, University X adopts the relative position rule, instead of the inverse relative position rule, considering the top-agent invariance being a desirable feature of a prioritization rule.

Proposition 5. (i) *The relative position rule is top-agent invariant but is not bottom-agent invariant.*
(ii) *The inverse relative position rule is bottom-agent invariant but is not top-agent invariant.*

5 Extension: Priority Design for General Assignment Problems

In the example of University X, our prioritization problem is followed by an assignment problem of exchange programs, real indivisible objects. In this section, we apply the argument of previous sections to a general assignment problem where priority is partial. The idea is to have the two stages of implementation which are in line with University X's practice. In the first stage, we determine a complete, transitive, and fair priority. We apply the relative position rule to achieve equity requirements such as the equal treatment of equal positions. Hence, this stage is a prioritization problem of previous sections. The second stage consists of allocating real objects (e.g., exchange programs) to agents. Allocation is determined based on agents' preferences and the priority formed in the first stage, i.e., this is an assignment problem of indivisible objects with single priority.

Because the priority is complete and transitive but is weak, i.e., students in the same priority group are of the same priority, a tie-breaking rule is required to allocate objects. A **tie-breaking rule** is a rule that transforms each weak priority into a linear priority that is consistent with the weak priority. In this paper, we assume an exogenous rule following a common formulation in the literature (e.g., Abdulkadiroğlu et al., 2009).[18]

Assumption 2 (Tie-breaking rule)**.** For each prioritization problem $(F, \{S_f\})$, there is a linear order $\succeq_0 (F, \{S_f\})$ in the set of all students $\cup_{f \in F} S_f$ that agrees with each priority within a faculty $\succeq_f |_{S_f}$, i.e., for each $f \in F$, $\succeq_0 (F, \{S_f\})|_{S_f} = \succeq_f |_{S_f}$.

[17]The part (i) is shown as follows. For $s \in S_g$ such that $a(s|S_g) = n_g$, since $a(s|S_g) = \left\lceil \frac{n_g K}{K} \right\rceil$, $\varphi^*_s(F, \{S_f\}) = K$ by the definition of φ^*. When $(n_g, K) = (2, 4)$, for $s' \in S_g$ such that $a(s'|S_g) = 1$, since $a(s'|S_g) = \left\lceil \frac{n_g \times 2}{K} \right\rceil$, $\varphi^*_{s'}(F, \{S_f\}) = 2 (> 1)$. The part (ii) can be shown in the same manner. For $s'' \in S_g$ such that $a(s''|S_g) = 1$, since $a^{inv}(s''|S_g) = n_g = \left\lceil \frac{n_g K}{K} \right\rceil$, and $K = \pi(1)$, $\varphi^{*inv}_{s''}(F, \{S_f\}) = 1$ by the definition of φ^{*inv}. When $(n_g, K) = (2, 4)$, for $s''' \in S_g$ such that $a(s'''|S_g) = n_g$, since $a^{inv}(s'''|S_g) = 1 = \left\lceil \frac{n_g \times 2}{K} \right\rceil$, and $2 = \pi(3)$, $\varphi^{*inv}_{s'''}(F, \{S_f\}) = 3 (= K - 1)$.

[18]One practical interpretation of this rule is the descending order of GPA or a lottery result.

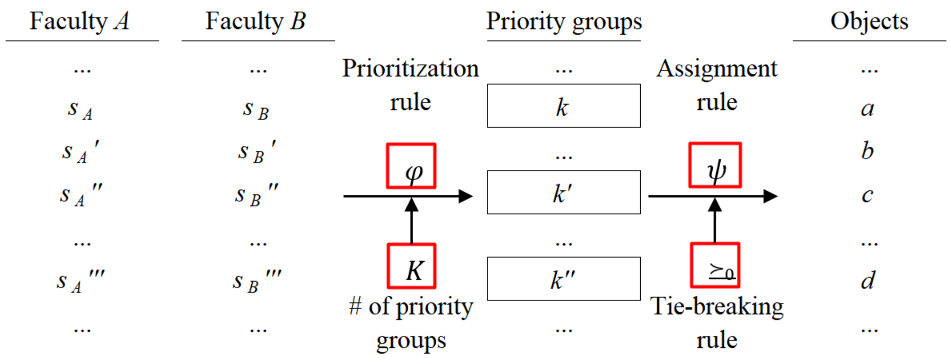

Figure 7: The approach to exchange program allocation.

Our objective is, where the tie-breaking rule is *not* necessarily fair, to observe the effect of having the first stage above. Thus, we do not require any assumptions or axioms for the tie-breaking rule other than above. When we fix a problem, we refer a tie-breaking rule as $\succeq_0$.

The following example, based on the practice at University X, illustrates this idea.

Example 11 (Two-stage implementation)**.** Consider a university composed of two faculties, schools of arts (A) and sciences (S). This university provides students six different exchange programs $\{a, b, c, d, e, f\}$, while each student can take part in at most one program, and each program accommodates at most one student. Let the number of students be six for faculty A and four for faculty S. Assume that students are initially ordered within each faculty as $s_A^6 \succ_A ... \succ_A s_A^1$ and $s_S^4 \succ_S ... \succ_S s_S^1$.

Stage 1: Prioritization problem. We apply a prioritization rule to combine the priorities above to a complete one, so that an assignment rule can be applied on the subsequent stage. The university chooses the *relative position rule* with $K = 4$ (envy- and null-lexicographic optimum). Then, we obtain a priority assignment

$$(S^4, S^3, S^2, S^1) = (\{s_A^6, s_A^5, s_S^4\}, \{s_A^4, s_S^3\}, \{s_A^3, s_A^2, s_S^2\}, \{s_A^1, s_S^1\}).$$

Stage 2: Assignment problem. We apply an assignment rule to allocate exchange programs to students. We apply the following exogenous tie-breaker, which is consistent with $\succeq_A$ and $\succeq_S$ but is seemingly preferable for faculty A's students

$$\succeq_0 \colon s_A^6, s_A^5, s_A^4, s_A^3, s_S^4, s_A^2, s_S^3, s_S^2, s_A^1, s_S^1.$$

We apply the tie-breaking rule to (S^4, S^3, S^2, S^1) (i.e., first, for S^4, three students are ordered by $\succeq_0$ as s_A^6, s_A^5, s_S^4, next, for S^3, two students are ordered by $\succeq_0$ as s_A^4, s_S^3, and so on) and obtain

$$\succeq^* \colon s_A^6, s_A^5, s_S^4, s_A^4, s_S^3, s_A^3, s_A^2, s_S^2, s_A^1, s_S^1.$$

As the assignment rule, the university chooses the *serial dictatorship rule* induced by $\succeq^*$. It proceeds in the order of $\succeq^*$, as s_A^6 takes her best program (e.g., d) first, s_A^5 takes his best among the remaining (e.g., a), and so on. $\Diamond$

We first observe the effect of having Stage 1, i.e., the effect of the relative position rule, to the resulting linear priority $\succeq^*$ under an exogenous tie-breaking rule $\succeq_0$. When Stage 1 is not implemented, the priority the university can use is $\succeq_0$ itself. Thus, the effect of Stage 1 is the difference between $\succeq_0$ and $\succeq^*$. We observe that, from $\succeq_0$ to $\succeq^*$,
(i) Average rank (from the bottom) of A's students changes from 41/6=(10+9+8+7+5+2)/6 to 37/6=(10+9+7+5+4+2)/6,
(ii) Average rank of S's students changes from 14/4=(6+4+3+1)/4 to 18/4=(8+6+3+1)/4,
(iii) The difference of average ranks in faculties reduced from 10/3 to 5/3, and
(iv) Each A's student is weakly worse off in her rank, while each S's student is weakly better off.

This example indicates that the relative position rule partially decreases the feature of $\succeq_0$, even if $\succeq_0$ is used to tie-breaking within each group. Thus, the rule is beneficial for students of faculties like S. Especially when the feature of $\succeq_0$ is not desirable, the rule helps "leveling the playing field."[19] It means that initially disadvantaged students can improve their ranks.

We next state a proposition that formalizes the intuition from the example. Let $\succeq_0$ be an arbitrary tie-breaking rule. Let $\succeq^*$, or more precisely $\succeq^* (F, \{S_f\}, K, \succeq_0)$, be a linear order in $\cup_{f\in F} S_f$ with application of the relative position rule φ^*, such that for any distinct students $s, s' \in \cup_{f\in F} S_f$, $s \succeq^* s'$ if and only if (i) $\varphi^*_s(F, \{S_f\}) > \varphi^*_{s'}(F, \{S_f\})$ or (ii) $\varphi^*_s(F, \{S_f\}) = \varphi^*_{s'}(F, \{S_f\})$ and $s \succeq_0 s'$. We denote each student s's absolute positions in all students by $a_0(s|\{S_f\}) := |\{s' \in \cup_{f\in F} S_f \mid s \succeq_0 s'\}|$ (in $\succeq_0$) or $a^*(s|\{S_f\}) := |\{s' \in \cup_{f\in F} S_f \mid s \succeq^* s'\}|$ (in $\succeq^*$) respectively.

Proposition 6. (i) $|F| = 2$. *If a student $s \in \cup_{f\in F} S_f$ obtains a relative position $\frac{a_0(s|\{S_f\})}{\sum_{f\in F} n_f}$ in $\succeq_0$ lower than the one in his own faculty $\frac{a(s|S_f)}{n_f}$, his absolute position weakly improves by the use of the relative position rule φ^*. That is, for each $(F, \{S_f\})$ with $|F| = 2$, for each $s \in \cup_{f\in F} S_f$ such that $\frac{a_0(s|\{S_f\})}{\sum_{f\in F} n_f} < \frac{a(s|S_f)}{n_f}$, $a^*(s|\{S_f\}) \geq a_0(s|\{S_f\})$.*
(ii) $|F| \geq 2$. *If a student $s \in \cup_{f\in F} S_f$ obtains a relative position $\frac{a_0(s|\{S_f\})}{\sum_{f\in F} n_f}$ in $\succeq_0$ lower than $\frac{a(s|S_f)}{n_f} - \{\frac{1}{K} + \frac{|F|}{\sum_{f\in F} n_f}\}$, his absolute position strictly improves by the use of the relative position rule. That is, for each $(F, \{S_f\})$, for each $s \in \cup_{f\in F} S_f$ such that $\frac{a_0(s|\{S_f\})}{\sum_{f\in F} n_f} < \frac{a(s|S_f)}{n_f} - \{\frac{1}{K} + \frac{|F|}{\sum_{f\in F} n_f}\}$, $a^*(s|\{S_f\}) > a_0(s|\{S_f\})$.*

This proposition says that, when $|F| = 2$, a student who is more disadvantaged in all faculties than in his own faculty, measured by the relative positions, improves his absolute position in all

[19]The phrase "leveling the playing field" here has a slightly different meaning than that in a well-known study (Pathak and Sönmez, 2008). While they help strategically naive households by introducing the deferred acceptance mechanism, we help students of faculties with harsh grading policies using the relative position rule. If we apply the rule to student prioritization for school choice problems, it is understood as a kind of affirmative action policy to balance the treatment of students with different backgrounds.

faculties by the use of the relative position rule. However, when $|F| \geq 3$, since such students may exist in several faculties, not all these students but those who are sufficiently disadvantaged can improve their absolute positions in all faculties.

Next, we move on to the assignment of real objects.[20] In contrast to above, the result is not so optimistic. Let H be a finite set of real objects (e.g., exchange programs in Example 10) and "self-match" $\{\emptyset\}$. We denote by h a generic element of H. We assume that each real object can be assigned to at most one student, and each student prefers at most one object. Self-match yields no capacity restrictions. Let $\succsim_s$ be a preference of student $s \in \cup_{f\in F} S_f$ on H. We assume each $\succsim_s$ is linear, i.e., for any $a, b \in H$, $a \succsim_s b$ and $b \succsim_s a$ if and only if $a = b$. An assignment $\{h_s\}_{s\in\cup_{f\in F} S_f}$ is a list such that for each $s \in \cup_{f\in F} S_f$, $h_s \in H$, and for each $h \in H \backslash \{\emptyset\}$, $|\{s \in \cup_{f\in F} S_f \mid h_s = h\}| \leq 1$. An assignment rule is a function ψ that associates with each preference profile $\succsim = \{\succsim_s\}_{s\in\cup_{f\in F} S_f}$ an assignment $\psi(\succsim) = \{h_s\}_{s\in\cup_{f\in F} S_f}$. We denote assigned object for student s by $\psi_s(\succsim) = h_s$. Among possible rules, the following one is of our interest here.

Serial dictatorship rule. The serial dictatorship (SD) rule $\psi^{\succeq}$ induced by a (linear) priority $\succeq$ chooses the assignment so that for each student $s \in \cup_{f\in F} S_f$, $\psi^{\succeq}_s(\succsim)$ is her most preferred object in the set of real objects and self-match H, excluding the real objects assigned to students with higher priority than her. More precisely, for each $s \in \cup_{f\in F} S_f$,

$$\psi^{\succeq}_s(\succsim) = \arg\max \left\{ \succsim_s \mid h \in \left[H \backslash \bigcup_{s' : a(s'|\{S_f\}) > a(s|\{S_f\})} \psi^{\succeq}_{s'}(\succsim) \right] \cup \{\emptyset\} \right\}.$$

It is well known that the serial dictatorship rules induced by any priorities are strategy-proof and Pareto efficient (Svensson, 1994). Moreover, since serial dictatorship rule is a specific case of student-proposing deferred acceptance rules, it respects improvements (Balinski and Sönmez, 1999). It means that, if only one student improves in the priority, her assigned object is not worse than the one before the improvement, when evaluated by her own preference. However, when multiple students improve their positions in the priority, its effect is not obvious. It means that, if more than two students improve in the priority, some of them may obtain worse object than the one before the improvement. Because the relative position rule adjusts several students' positions in the priority simultaneously, improvement in priority (from $\succeq_0$ to $\succeq^*$) does not always imply improvement of assigned object. That is, the implementation of Stage 1, the relative position rule, may hurt some initially disadvantaged students, when evaluated by their own preferences.

Proposition 7. *Even if a student obtains a better position in $\succeq^*$ than in $\succeq_0$, her assigned object may be worse in $\succeq^*$-based serial dictatorship than in $\succeq_0$-based serial dictatorship, in the sense of*

[20] We formulate the situation as a one-to-one assignment problem for notational simplicity. The same argument can be naturally applied to many-to-one assignments.

her preference. More precisely, there exists $(F, \{S_f\})$ *such that for some* $s \in \cup_{f\in F} S_f$, *for some* $\succsim$, $a^*(s|\{S_f\}) \geq a_0(s|\{S_f\})$ *but* $\psi_s^{\succeq_0}(\succsim) \succsim_s \psi_s^{\succeq^*}(\succsim)$.

6 Conclusion

In this paper, we examine a fair prioritization problem of agents, given a set of partial priorities and a notion of agents' relative positions. Theorem 1 shows that the relative position rule φ^* is a unique rule that satisfies all of our requirements, which are: the priority preservation law, the equal treatment of equal positions, the equal split, and the attribute-wise consistency. In more general assignment problems with partial priorities, the rule φ^* has a feature that helps leveling the playing field; agents of specific attributes, e.g., faculty S of Example 11, are better off.

At the end of the paper, we note on some limitations of our model and possibilities of further development of the rule in general assignment problems.

Faculties as strategic agents. In some cases, it is natural for faculties to want their own students to benefit more - this is actually the case for University X. In this situation, faculties may try to strategically manipulate the resulted priority assignment. For instance, if each priority is private information held by each faculty, and the university collects the information to form a complete priority, faculties may have incentive to misreport the information. Suppose that there are two faculties A and B with two students each and two priority groups. We consider true ranking results $s_B^2, s_A^2, s_B^1, s_A^1$. If faculty A wishes to maximize average rank (from the bottom) of its own students, it will possibly benefit by reporting $\succ'_A$: s_A^1, s_A^2 instead of $\succ_A$: s_A^2, s_A^1, to obtain $s_B^2, s_A^1, s_A^2, s_B^1$. As a result of this misreporting, the average rank of students in A improves from $2(= (3 + 1)/2)$ to $2.5(= (3 + 2)/2)$. However, it hurts s_A^2, whose absolute position changes from three to two.

Student-specific evaluation. If we wish to achieve the "true" ranking of students, our model may not be appropriate. For example, consider that two faculties A and B are clearly different in the level of students, i.e., the bottom student in A is better (in any sense) than the top one in B. Under the objective of finding a true ranking, our rule's resulting priority assignment may not be acceptable: another formulation and another rule are required.

Endogenous or exogenous quota. In reality, it is common to set exogenous quota for each attribute and each group or object, but our approach is to endogenously determine it using equity axioms. Comparing exogenous and endogenous quotas may give practical implications.

7 Appendix

We first prove useful lemmas.

Lemma 3. *The following two statements are equivalent.*
(a) φ *satisfies PPL.*
(b) *For each* $g \in F$, *for each* $s \in S_g$, $\varphi_s(F, \{S_f\}) = k$ *where* $\sum_{j \leq k-1} |\varphi_g^j(F, \{S_f\})| < a(s|S_g) \leq \sum_{j \leq k} |\varphi_g^j(F, \{S_f\})|$.

Proof. (b)⇒(a): Suppose (b) is satisfied. For $s \in S_g^k$ and $s' \in S_g^{k'}$ with $a(s|S_g) \leq a(s'|S_g)$, $\sum_{j \leq k-1} |\varphi_g^j(F, \{S_f\})| < a(s|S_g) \leq \sum_{j \leq k} |\varphi_g^j(F, \{S_f\})|$ and $\sum_{j \leq k'-1} |\varphi_g^j(F, \{S_f\})| < a(s'|S_g) \leq \sum_{j \leq k'} |\varphi_g^j(F, \{S_f\})|$ imply $\sum_{j \leq k-1} |\varphi_g^j(F, \{S_f\})| < \sum_{j \leq k'} |\varphi_g^j(F, \{S_f\})|$, i.e., $k \leq k'$.
(a)⇒(b): Suppose (a) is satisfied. Assume, by way of contradiction, that $s \in S_g^k$ but $a(s|S_g) \leq \sum_{j \leq k-1} |\varphi_g^j(F, \{S_f\})|$. Then, $a(s|S_g) - 1 < \sum_{j \leq k-1} |\varphi_g^j(F, \{S_f\})|$ implies that there exists $s'' \in S_g$, such that $a(s''|S_g) > a(s|S_g)$ and $s'' \in S_g^{k'}$ where $k' \leq k$. In the same manner, if we assume that $s \in S_g^k$ but $\sum_{j \leq k} |\varphi_g^j(F, \{S_f\})| < a(s|S_g)$, $\sum_{j \leq k} |\varphi_g^j(F, \{S_f\})| \leq a(s|S_g) - 1$ implies that there exists $s''' \in S_g$, such that $a(s'''|S_g) < a(s|S_g)$ and $s''' \in S_g^{k'}$ where $k' > k$. These contradict to PPL. □

Lemma 4. *Suppose that* φ *satisfies PPL. The following two statements are equivalent.*
(a) φ *satisfies IOA.*
(b) *For each* $g \in F$, *for each* $\{S_f\}_{f \in F}, \{\hat{S}_f\}_{f \in F}$, *when* $n_g = \hat{n}_g$, *for any* $s \in S_g$ *and* $\hat{s} \in \hat{S}_g$ *with* $a(s|S_g) = a(\hat{s}|\hat{S}_g)$, $\varphi_s(F, \{S_f\}) = \varphi_{\hat{s}}(F, \{\hat{S}_f\})$.

Proof. (b)⇒(a) is immediate. We show (a)⇒(b). Suppose (a) is satisfied. Take $\{S_f\}_{f \in F}, \{\hat{S}_f\}_{f \in F}$ such that for $g \in F$, $n_g = \hat{n}_g$. By IOA, for each $k \in K$, $|\varphi_g^k(F, \{S_f\})| = |\varphi_g^k(F, \{\hat{S}_f\})|$. Take $s \in S_g$ and $\hat{s} \in \hat{S}_g$ with $a(s|S_g) = a(\hat{s}|\hat{S}_g)$. We must show $\varphi_s(F, \{S_f\}) = \varphi_{\hat{s}}(F, \{\hat{S}_f\})$. It follows from Lemma 3 that for $k = \varphi_s(F, \{S_f\})$, $\sum_{j \leq k-1} |\varphi_g^j(F, \{S_f\})| < a(s|S_g) \leq \sum_{j \leq k} |\varphi_g^j(F, \{S_f\})|$, and for $\hat{k} = \varphi_{\hat{s}}(F, \{\hat{S}_f\})$, $\sum_{j \leq \hat{k}-1} |\varphi_g^j(F, \{\hat{S}_f\})| < a(\hat{s}|\hat{S}_g) \leq \sum_{j \leq \hat{k}} |\varphi_g^j(F, \{\hat{S}_f\})|$. Since for each $j \in K$, $|\varphi_g^j(F, \{S_f\})| = |\varphi_g^j(F, \{\hat{S}_f\})|$, and $a(s|S_g) = a(\hat{s}|\hat{S}_g)$, we have $\varphi_s(F, \{S_f\}) = k = \hat{k} = \varphi_{\hat{s}}(F, \{\hat{S}_f\})$. □

Lemma 5. *Suppose that* φ *satisfies IOA. The following two statements are equivalent.*
(a) φ *satisfies PPL and ETP.*
(b) *For each* $f, f' \in F$, *for each* $s \in S_f$ *and* $s' \in S_{f'}$, *when* $\frac{a(s|S_f)}{n_f} \leq \frac{a(s'|S_{f'})}{n_{f'}}$, *we have* $\varphi_s(F, \{S_f\}) \leq \varphi_{s'}(F, \{S_f\})$.

Proof. (b)⇒(a): When (b) is true, PPL and ETP are given by restrictions $f = f'$ and $\frac{a(s|S_f)}{n_f} = \frac{a(s'|S_{f'})}{n_{f'}}$ respectively.
(a)⇒(b): Suppose (a) is true but (b) is not for $(F, \{S_f\}_{f \in F})$, where we denote $(n_f, n_{f'}) = (n^1, n^2)$. Let $s \in S_f$ and $s' \in S_{f'}$ be $\frac{a(s|S_f)}{n_f} \leq \frac{a(s'|S_{f'})}{n_{f'}}$, but $\varphi_s(F, \{S_f\}) = k > k' = \varphi_{s'}(F, \{S_f\})$. Call $(a(s|S_f), a(s'|S_{f'})) = (a, a')$ in advance.
(i) Consider $(F, \{S'_f\})$ where $(n'_f, n'_{f'}) = (n^1, n^1 n^2)$, and for each $g \in F \backslash \{f, f'\}$, $S'_g = S_g$. Lemma 4 implies by PPL and IOA that for $s \in S'_f$ such that $a(s|S'_f) = a$, $\varphi_s(F, \{S'_f\}) = k$. Since $\frac{a(s|S'_f)}{n'_f} = \frac{a}{n^1} = \frac{n^2 a}{n'_{f'}}$, ETP implies that for $s' \in S'_{f'}$ such that $a(s'|S'_{f'}) = n^2 a$, $\varphi_{s'}(F, \{S'_f\}) = \varphi_s(F, \{S'_f\}) = k$.

(ii) Consider $(F, \{S''_f\})$ where $(n''_f, n''_{f'}) = (n^2, n^2)$,and for each $g \in F\backslash\{f, f'\}$, $S''_g = S_g$. Lemma 4 implies that for $s'' \in S''_f$ such that $a(s''|S''_f) = a'$, $\varphi_{s''}(F, \{S''_f\}) = \varphi_{s'}(F, \{S'_f\}) = k'$.
(iii) Consider $(F, \{S'''_f\})$, where $(n'''_f, n'''_{f'}) = (n^2, n^1 n^2)$, and for each $g \in F\backslash\{f, f'\}$, $S'''_g = S_g$. Lemma 4 implies that for $s'''_1 \in S'''_f$ such that $a(s'''_1|S'''_f) = a'$, and for $s'''_2 \in S'''_{f'}$ such that $a(s'''_2|S'''_{f'}) = n^2 a$, $\varphi_{s'''_1}(F, \{S'''_f\}) = k'$ and $\varphi_{s'''_2}(F, \{S'''_f\}) = k$ respectively. Since $\frac{a'}{n'''_f} = \frac{a'}{n^2} = \frac{n^1 a'}{n'''_{f'}}$, ETP implies that for $s'''_3 \in S'''_{f'}$ such that $a(s'''_3|S'''_{f'}) = n^1 a'$, $\varphi_{s'''_3}(F, \{S'''_f\}) = \varphi_{s'''_1}(F, \{S'''_f\}) = k'$. Then, for $s'''_2, s'''_3 \in S'''_{f'}$, $\frac{n^2 a}{n'''_{f'}} = \frac{a}{n^1} \leq \frac{a'}{n^2} = \frac{n^1 a'}{n'''_{f'}}$ but $\varphi_{s'''_2}(F, \{S'''_f\}) = k > k' = \varphi_{s'''_3}(F, \{S'''_f\})$, a contradiction to PPL within f'. □

PROOF OF LEMMA 1. Suppose that φ satisfies PPL, ETP, and ES.
(i) ($\Leftarrow$) Suppose that φ satisfies AC. Let $|F| \geq 2$. Let $f \in F$. Let $\{S_g\}_{g \in F}$ and $\{\hat{S}_g\}_{g \in F}$ with $n_f = \hat{n}_f$. Take $f' \in \mathcal{F} \setminus F$, and $S_{f'}, \hat{S}_{f'}$ with $n_{f'} = \hat{n}_{f'} = n_f K$. We denote $P = (F, \{S_g\}_{g \in F})$, $P' = (F \cup \{f'\}, \{S_g\}_{g \in F \cup \{f'\}})$, $\hat{P} = (F, \{\hat{S}_g\}_{g \in F})$, and $\hat{P}' = (F \cup \{f'\}, \{S_g\}_{g \in F \cup \{f'\}})$. It is sufficient to show that for each $s \in S_f$, and each $\hat{s} \in \hat{S}_f$ with $a(s|S_f) = a(\hat{s}|\hat{S}_f)$, we have $\varphi_s(P) = \varphi_{\hat{s}}(\hat{P})$. Let $i \in \{1, \ldots, n_f\}$, $s \in S_f$, and $\hat{s} \in \hat{S}_f$ with $a(s|S_f) = a(\hat{s}|\hat{S}_f) = i$.

1. There is $s' \in S_{f'}$ such that $a(s'|S_{f'}) = a(s|S_f)K$. Since $\frac{a(s|S_f)}{n_f} = \frac{Ka(s|S_f)}{Kn_f} = \frac{a(s'|S_{f'})}{n_{f'}}$, by ETP, $\varphi_s(P') = \varphi_{s'}(P')$.

2. Similarly, by ETP, there is $\hat{s}' \in \hat{S}_{f'}$ such that $a(\hat{s}'|\hat{S}_{f'}) = a(\hat{s}|\hat{S}_f)K$. Thus, $\varphi_{\hat{s}}(\hat{P}') = \varphi_{\hat{s}'}(\hat{P}')$.

3. For each $k \in K$, it follows from ES that $|S^k_{f'}| = n_f$ as $n_{f'} = n_f K$; similarly, $|\hat{S}^k_{f'}| = n_f$, as $\hat{n}_{f'} = n_f K$. Thus, since $a(s'|S_{f'}) = a(\hat{s}'|\hat{S}_{f'}) = iK$, then by PPL, $\varphi_{s'}(P') = \varphi_{\hat{s}'}(\hat{P}')$.

4. Then, it follows from (1), (2), and (3) that $\varphi_s(P') = \varphi_{s'}(P') = \varphi_{\hat{s}'}(\hat{P}') = \varphi_{\hat{s}}(\hat{P}')$.

5. Moreover, by AC, $\varphi_s(P) = \varphi_s(P')$ and $\varphi_{\hat{s}}(\hat{P}) = \varphi_{\hat{s}}(\hat{P}')$.

Finally, by (4) and (5), we have $\varphi_s(P) = \varphi_s(P') = \varphi_{\hat{s}}(\hat{P}') = \varphi_{\hat{s}}(\hat{P})$.

($\Rightarrow$) Suppose, by way of contradiction, that φ satisfies IOA but it does not satisfy AC. Then, for some $(F, \{S_g\}_{g \in F})$ with $|F| \geq 3$ and for some $G \subset F$ with $|G| \geq 2$ there is $f \in F$ and $s \in S_f \subsetneq \cup_{g \in G} S_g$ such that $\varphi_s(G, \{S_g\}) \neq \varphi_s(F, \{S_g\})$. Let $\varphi_s(F, \{S_g\}) = k$ and $\varphi_s(G, \{S_g\}) = k'$ so that $k \neq k'$. Choose a faculty $f' \in G$ arbitrarily. Take a problem $(F, \{\hat{S}_g\}_{g \in F})$ such that $\hat{S}_f = S_f$ and $\hat{n}_{f'} = n_f K$. Let $s' \in \hat{S}_{f'}$ such that $a(s'|\hat{S}_{f'}) = a(s|S_f)K$.

1. By ES, since $\hat{n}_{f'} = n_f K$, for each $k \in K$, $|\varphi^k_{f'}(F, \{\hat{S}_g\})| = n_f$ and $|\varphi^k_{f'}(G, \{\hat{S}_g\})| = n_f$. Thus, since $s' \in \hat{S}_{f'}$ and $f' \in F \cap G$, it follows from PPL that $\varphi_{s'}(F, \{\hat{S}_g\}) = \varphi_{s'}(G, \{\hat{S}_g\})$.

2. It follows from IOA that since $\hat{S}_f = S_f$, we have $\varphi_s(F, \{\hat{S}_g\}) = \varphi_s(F, \{S_g\}) = k$ and $\varphi_s(G, \{\hat{S}_g\}) = \varphi_s(G, \{S_g\}) = k'$.

3. Since $\frac{a(s'|\hat{S}_{f'})}{\hat{n}_{f'}} = \frac{a(s|S_f)K}{n_f K} = \frac{a(s|S_f)}{n_f}$, it follows from ETP and (2) that $\varphi_{s'}(F, \{\hat{S}_g\}) = \varphi_s(F, \{\hat{S}_g\}) = k$ and $\varphi_{s'}(G, \{\hat{S}_g\}) = \varphi_s(G, \{\hat{S}_g\}) = k'$.

Therefore, by (1) and (3), we have $k = \varphi_{s'}(F, \{\hat{S}_g\}) = \varphi_{s'}(G, \{\hat{S}_g\}) = k'$. This contradicts our assumption that $k \neq k'$.

(ii) It is straightforward that IOA implies ACC. The following example shows the case in which φ satisfies ACC but it does not satisfy IOA. Let $F = \{f, g, h\}$ and $K = 2$. Consider two problems $(F, \{S_f\})$ and $(F, \{\hat{S}_f\})$, with $(n_f, n_g, n_h) = (2, 3, 4)$ and $(\hat{n}_f, \hat{n}_g, \hat{n}_h) = (3, 3, 3)$. We denote a student of i-th absolute position as s^i_f, s^i_g, or s^i_h. For $(F, \{S_f\})$, $(F, \{\hat{S}_f\})$, and their reduced problems, the assignments by φ are as follows.

Table 2: Assignments for (ii)

G	$\cup_{g\in G}\varphi^2_g(G, \{S_g\})$	$\cup_{g\in G}\varphi^1_g(G, \{S_g\})$	$\cup_{g\in G}\varphi^2_g(G, \{\hat{S}_g\})$	$\cup_{g\in G}\varphi^1_g(G, \{\hat{S}_g\})$
F	$\{s^2_f, s^2_g, s^3_g, s^3_h, s^4_h\}$	$\{s^1_f, s^1_g, s^1_h, s^2_h\}$	$\{s^2_f, s^3_f, s^2_g, s^3_g, s^2_h, s^3_h\}$	$\{s^1_f, s^1_g, s^1_h\}$
$\{f, g\}$	$\{s^2_f, s^3_g\}$	$\{s^1_f, s^1_g, s^2_g\}$	$\{s^2_f, s^3_f, s^2_g, s^3_g\}$	$\{s^1_f, s^1_g\}$
$\{g, h\}$	$\{s^2_g, s^3_g, s^3_h, s^4_h\}$	$\{s^1_g, s^1_h, s^2_h\}$	$\{s^2_g, s^3_g, s^2_h, s^3_h\}$	$\{s^1_g, s^1_h\}$
$\{f, h\}$	$\{s^2_f, s^3_h, s^4_h\}$	$\{s^1_f, s^1_h, s^2_h\}$	$\{s^2_f, s^3_f, s^2_h, s^3_h\}$	$\{s^1_f, s^1_h\}$

Since $\varphi^1_g(\{f, g\}, \{S_f\}) \neq \varphi^1_g(\{g, h\}, \{S_f\})$, ACC holds in $(F, \{S_f\})$. However, though $n_g = \hat{n}_g$, $|\varphi^1_g(\{f, g\}, \{S_f\})| = 2 \neq 1 = |\varphi^1_g(\{f, g\}, \{\hat{S}_f\})|$. IOA is violated.

(iii) It is straightforward that IOA implies PM. The following example shows the case in which φ satisfies PM but it does not satisfy IOA. Let $F = \{f, g\}$ and $K = 2$. Consider two problems $(F, \{S_f\})$ and $(F, \{\hat{S}_f\})$, with $(n_f, n_g) = (3, 5)$ and $(\hat{n}_f, \hat{n}_g) = (3, 6)$. φ assigns
$\varphi(F, \{S_f\}) = \{\{s^3_f, s^4_g, s^5_g\}, \{s^1_f, s^2_f, s^1_g, s^2_g, s^3_g\}\}$, and $\varphi(F, \{\hat{S}_f\}) = \{\{s^2_f, s^3_f, s^4_g, s^5_g, s^6_g\}, \{s^1_f, s^1_g, s^2_g, s^3_g\}\}$.
Since $\varphi_{s^2_f}(F, \{S_f\}) = 1 < 2 = \varphi_{s^2_f}(F, \{\hat{S}_f\})$, PM is satisfied but IOA is violated.

PROOF OF LEMMA 2. Let $s \in S_f$.
($\Rightarrow$) Let $s \in S^k$. Then, for $k = \varphi^*_s(F, \{S_f\})$, $[\frac{n_f(k-1)}{K}] < a(s|S_f) \leq [\frac{n_f k}{K}]$ holds from the definition. It is immediate that $a(s|S_f) \leq [\frac{n_f k}{K}] \leq n_f \frac{k}{K}$. On the other hand, because $[\frac{n_f(k-1)}{K}], a(s|S_f) \in \mathbb{N}$, $n_f \frac{k-1}{K} < [\frac{n_f(k-1)}{K}] + 1 \leq a(s|S_f)$. Therefore, $\frac{k-1}{K} < \frac{a(s|S_f)}{n_f} \leq \frac{k}{K}$ holds.
($\Leftarrow$) Suppose that $\frac{k-1}{K} < \frac{a(s|S_f)}{n_f} \leq \frac{k}{K}$. It is immediate that $[\frac{n_f(k-1)}{K}] \leq \frac{n_f(k-1)}{K} < a(s|S_f)$. If we assume $[\frac{n_f k}{K}] < a(s|S_f)$, since $[\frac{n_f k}{K}], a(s|S_f) \in \mathbb{N}$, it follows that $\frac{n_f k}{K} < [\frac{n_f k}{K}] + 1 \leq a(s|S_f)$, a contradiction. $[\frac{n_f(k-1)}{K}] < a(s|S_f) \leq [\frac{n_f k}{K}]$ leads $s \in S^k_f \subseteq S^k$. (Q.E.D.)

PROOF OF PROPOSITION 1.
PPL: This is clear from Lemma 3. The relative position rule is a specific case of rules in statement (b); $|\varphi^{*k}_f(F, \{S_f\})| = [\frac{n_f k}{K}] - [\frac{n_f(k-1)}{K}]$.
ETP: Suppose $s \in S_f$ and $s' \in S_{f'}$ with $\frac{a(s|S_f)}{n_f} = \frac{a(s'|S_{f'})}{n_{f'}}$. $\varphi_s(F, \{S_f\}) = \varphi_{s'}(F, \{S_f\})$ follows from Lemma 2.

ES: When $\frac{n_f}{K} \in \mathbb{N}$, for each $k \in K$, $|\varphi^k_f(F, \{S_f\})| = |S^k_f| = |\{s \in S_f \mid \frac{n_f}{K}\{\varphi^*_s(F, \{S_f\}) - 1\} < a(s|S_f) \leq \frac{n_f}{K}\varphi^*_s(F, \{S_f\})\}| = \frac{n_f}{K}$.
IOA, AC, and ACC: These are also clear from the formula of φ^*_s, only depending on the number of students in own faculty, not those in other faculties.
PM: Since population change does not affect other faculty's assignment under φ^*, we only check the effect within each faculty. Lemma 2 shows that $\varphi^*_s(F, \{S_f\}) = k$ is equivalent to $\frac{k-1}{K} < a(s|S_f) \leq \frac{k}{K}$. For $S'_f = S_f \cup \{s'\}$ such that $a(s'|S'_f) = 1$, for each $s \in S_f$, we have $a(s|S'_f) = a(s|S_f) + 1$. Because Lemma 2 also implies that $\varphi^*_s(F, \{\hat{S}_f\}) = k'$ is equivalent to $\frac{k'-1}{K} < a(s|S'_f) \leq \frac{k'}{K}$, $a(s|S'_f) > a(s|S_f)$ implies $k' \geq k$. (Q.E.D.)

PROOF OF THEOREM 1. "If" part is immediate from Proposition 1. We show "Only if" part. Suppose φ satisfies PPL, ETP, ES, and IOA.

Claim 1. For each $k \in K$, $\sum_{j \leq k} |\varphi^j_f(F, \{S_f\})| \geq [\frac{n_f k}{K}]$.

Proof. For $(F, \{S_f\})$, for $g \in F$, for $k_1 \in K$, let $\sum_{j \leq k_1} |\varphi^j_g(F, \{S_f\})| = [\frac{n_g k_1}{K}] - 1 =: a_1$. Note that for $s_1 \in S_g$ such that $a(s_1|S_g) = a_1 + 1$, by PPL, $\varphi_{s_1}((F, \{S_f\})) > k_1$. Pick another problem $(F, \{S'_f\})$ such that for $g' \in F$, $n'_{g'} = K$, and for $g \in F$, $S'_g = S_g$. ES implies that for $s'_1 \in S'_{g'}$ such that $a(s'_1|S'_{g'}) = k_1$, $\varphi_{s'_1}(F, \{S'_f\}) = k_1$ while Lemma 4 implies by PPL and IOA that $\varphi_{s_1}(F, \{S'_f\}) = \varphi_{s_1}(F, \{S_f\}) > k_1$. However, $\frac{a(s'_1|S'_{g'})}{n'_{g'}} = \frac{k_1}{K} \geq \frac{1}{n'_g}[\frac{n'_g k_1}{K}] = \frac{a_1+1}{n'_g}$ implies that statement (b) in Lemma 5 is violated, a contradiction. Applying the same argument shows that $\sum_{j \leq k_1} |\varphi^j_f(F, \{S_f\})| = [\frac{n_f k_1}{K}] - 2$, $[\frac{n_f k_1}{K}] - 3$, $[\frac{n_f k_1}{K}] - 4, \ldots$ also lead a contradiction. □

Claim 2. For each $k \in K$, $\sum_{j \leq k} |\varphi^j_f(F, \{S_f\})| \leq [\frac{n_f k}{K}]$.

Proof. For $(F, \{S_f\})$, for $g \in F$, for $k_2 \in K$, let $\sum_{j \leq k_2} |\varphi^j_g(F, \{S_f\})| = [\frac{n_g k_2}{K}] + 1 =: a_2$. Note that for $s_2 \in S_g$ such that $a(s_2|S_g) = a_2$, by PPL, $\varphi_{s_2}(F, \{S_f\}) \leq k_2$. Let $\epsilon := 1 - \{\frac{n_g k_2}{K} - [\frac{n_g k_2}{K}]\} \in (0, 1]$ and take a corresponding (sufficiently large) number $m^\epsilon := [\frac{n_g}{\epsilon K}] + 1$. Then, pick another problem $(F, \{S'_f\})$ such that for $g' \in F$, $n'_{g'} = m^\epsilon K$, and for $g \in F$, $S'_g = S_g$. ES implies that for $s'_2 \in S'_{g'}$ such that $a(s'_2|S'_{g'}) = m^\epsilon k_2 + 1$, $\varphi_{s'_2}(F, \{S'_f\}) = k_2 + 1$ while Lemma 4 implies by PPL and IOA that $\varphi_{s_2}(F, \{S'_f\}) = \varphi_{s_2}(F, \{S_f\}) \leq k_2$. However, $\frac{a(s'_2|S'_{g'})}{n'_{g'}} = \frac{m^\epsilon k_2 + 1}{m^\epsilon K} = \frac{k_2}{K} + \frac{1}{m^\epsilon K} \leq \frac{k_2}{K} + \frac{\epsilon}{n'_g} = \frac{1}{n'_g}\{[\frac{n'_g k_2}{K}] + 1\} = \frac{a_2}{n'_g} = \frac{a(s_2|S'_g)}{n'_g}$ implies that statement (b) in Lemma 5 is violated, a contradiction. Applying the same argument shows that $\sum_{j \leq k_2} |\varphi^j_f(F, \{S_f\})| = [\frac{n_f k_2}{K}] + 2$, $[\frac{n_f k_2}{K}] + 3$, $[\frac{n_f k_2}{K}] + 4, \ldots$ also lead a contradiction. □

By Claims 1 and 2, we characterize the relative position rule by (i) PPL, (ii) ETP, (iii) ES, and (iv) IOA. Since Lemma 1 shows that AC and IOA are equivalent under (i)-(iii), the relative position rule is now characterized by the combination of (i), (ii), (iii), and (iv') AC. (Q.E.D.)

PROOF OF PROPOSITION 3. Throughout the proof, fix a problem $(F, \{S_f\})$ and the rule φ^*. Thus, we omit the notation of $\theta^e(\,\cdot\,|F, \{S_f\}, \varphi^*)$ and $\theta^\emptyset(\,\cdot\,|F, \{S_f\}, \varphi^*)$ as $\theta^e(\,\cdot\,)$ and $\theta^\emptyset(\,\cdot\,)$. We denote, for each K, the number of lower-side envious students and that of upper-side envious students by $\theta^l(K)$ and $\theta^u(K)$. Thus, $\theta^e(K) = \theta^l(K) + \theta^u(K)$.

(a) We first perform the following process to obtain $\{n^j, q^j\}_{j=1,\ldots,J}$, a *regular case representation* of $\{n_f\}_{f\in F}$. Align $\{n_f\}$ in *ascending* order with arbitrary tie-breaking.
Step 1. Pick the first (smallest) element from $\{n_f\}$ and name it n^1. Let $j(1) = 1$ and $q^0(1) = 1$.
Step $\boldsymbol{k \geq 2}$. Pick the k-th element from $\{n_f\}$. If the element is strictly larger than the element picked in step $k-1$, let $j(k) = j(k-1)+1$, $q^{j(k-1)} = q^0(k-1)$, $q^0(k) = 1$, and name the element $n^{j(k)}$. Otherwise, let $j(k) = j(k-1)$ and $q^0(k) = q^0(k-1)+1$. After step $|F|$, let $J = j(|F|)$, $q^{j(|F|)} = q^0(|F|)$, and terminate the process.

(b) Using the form $\{n^j, q^j\}_{j=1,\ldots,J}$, by induction of K, we next prove the relations
(1) $\theta^l(K) = \sum_{j=1}^{J-1}(K-1)q^{j+1}\mathbf{1}_{n^j>K}$, and (2) $\theta^u(K) = \sum_{j=1}^{J-1}(n^j - K)q^j\mathbf{1}_{n^j>K}$.
We note that, if $n_g = \min\{n_f\} = n^1$, since no $g' \in F$ such that $n_{g'} < n_g$ exist, no students in g have lower-side envy. Similarly, if $n_g = \max\{n_f\} = n^J$, since no $g' \in F$ such that $n_{g'} > n_g$ exist, no students in g have upper-side envy.
Step 1. Let $K = 2$. For $j \geq 2$, 1 out n^j students, top of the worse group, has lower-side envy if there is f with $n_f < n^j$, at least two students are assigned to one group, i.e., $\bigcup_{i\leq j-1}\{n^i > 2\} = \{n^{j-1} > 2\}$. Thus, we have $\theta^l(2) = \sum_{j=2}^{J} 1 \times q^j \times \mathbf{1}_{n^{j-1}>2} = \sum_{j=1}^{J-1}(2-1)q^{j+1}\mathbf{1}_{n^j>2}$.
For $j \leq J-1$, $n^j - 2$ out n^j students, all but bottom of each group, have upper-side envy if at least two students exist in each of this faculty and f with $n_f > n^j$, i.e., $\bigcup_{i\geq j}\{n^i > K\} = \{n^j > K\}$. Thus, we have $\theta^u(2) = \sum_{j=1}^{J-1}(n^j-2) \times q^j \times \mathbf{1}_{n^j>2} = \sum_{j=1}^{J-1}(n^j-2)q^j\mathbf{1}_{n^j>2}$.
Step $\boldsymbol{k \geq 2}$. Assume that (1) and (2) are true for $K = k-1$ and let $K = k$. Fluctuation from $\theta^l(K-1)$ to $\theta^l(K)$ is sum of a) decrease by $((K-1)-1)q^j$ if n^{j-1} is exactly equal to K, and b) increase by q^j if n^{j-1} is strictly larger than K. Thus,

$$
\begin{aligned}
\theta^l(K) &= \theta^l(K-1) - \sum_{j=2}^{J}((K-1)-1) \times q^j \times \mathbf{1}_{n^{j-1}>K-1 \cap n^{j-1}\leq K} + \sum_{j=2}^{J} 1 \times q^j \times \mathbf{1}_{n^{j-1}>K-1 \cap n^{j-1}>K} \\
&= \sum_{j=1}^{J-1}(K-1)q^{j+1}\mathbf{1}_{n^j>K}.
\end{aligned}
$$

Similarly, fluctuation from $\theta^u(K-1)$ to $\theta^u(K)$ is sum of a) decrease by $(n^j - (K-1))q^j$ if n^j is exactly equal to K, and b) decrease by q^j if n^j is strictly larger than K.

$$\theta^u(K) = \theta^u(K-1) - \sum_{j=1}^{J-1}(n^j - (K-1)) \times q^j \times \mathbf{1}_{n^j > K-1 \cap n^j \leq K} - \sum_{j=1}^{J-1} 1 \times q^j \times \mathbf{1}_{n^j > K-1 \cap n^j > K}$$
$$= \sum_{j=1}^{J-1}(n^j - K)q^j \mathbf{1}_{n^j > K}.$$

(c) Finally, we show the proposition as follows.

Envy-lexicographic optimum. Formulae (1) and (2) show $\theta^l(K) = \theta^u(K) = 0$ if and only if $K \geq n^{J-1}$. When $K = n^{L-1}$, null subgroups arise in faculties with $n_f \leq n^{J-2}$. As K grows, empty groups increase in faculties with $n_f \leq n^{J-1}$, and will arise in faculties with $n_f = n^J$ after $K \geq n^J$. Therefore, $K = n^{J-1}$ is the unique optimum in cases including regular ones. When n^{J-1} does not exist, i.e., $n_f = n_{f'}$ for any f, f', there are multiple optima including n_f.

Null-lexicographic optimum. By Footnote 8, it is sufficient to find the minimizer of $\theta^e(\cdot)$ subject to $K \leq n^1$. If $n^1 = 1$, since $K \geq 2$, $K = 2$ is the solution. Otherwise, where $K < n^1$,

$$\theta^l(K) + \theta^u(K) = \sum_{j=1}^{J-1}(K-1)q^{j+1} + (n^j - K)q^j = (n^1 - K)q^1 + \sum_{j=2}^{J-1}(n^j - 1)q^j + (K-1)q^J$$
$$= (q^J - q^1)K + n^1 q^1 + \sum_{j=2}^{J-1}(n^j - 1)q^j - q^J ...(c.1)$$

Where $K = n^1$,

$$\theta^l(K) + \theta^u(K) = \sum_{j=2}^{J-1}(K-1)q^{j+1} + (n^j - K)q^j = (1 - K)q^2 + \sum_{j=2}^{J-1}(n^j - 1)q^j + (K-1)q^J$$
$$= (q^J - q^2)K + q^2 + \sum_{j=2}^{J-1}(n^j - 1)q^j - q^J ...(c.2)$$

Immediately, if $q^1 = q^2 = q^J$, the statement is true. Since $(c.1)$ is a monotone function of K, the solution is either of corners $\{2, n^1\}$. $K = n^1$ is the minimizer if

$$\left[(q^J - q^1) + n^1 q^1 \geq (q^J - q^2)n^1 + q^2\right] \wedge \left[(q^J - q^1)(n^1 - 1) + n^1 q^1 \geq (q^J - q^2)n^1 + q^2\right]$$
$$\Leftrightarrow \left[q^1 + q^2 - q^J \geq 0\right] \wedge \left[q^1 + (n^1 - 1)q^2 - q^J \geq 0\right].$$

Since the former is binding, we obtain that $K = n^1$ is the optimum if $q^1 + q^2 - q^J \geq 0$ and $K = 2$ is the optimum if $q^1 + q^2 - q^J \leq 0$. (Q.E.D.)

PROOF OF PROPOSITION 4.

(i) Fix an arbitrary problem $(F, \{S_f\})$ with $F = \{f, g\}$, in which at least one student $s \in S_f$ with $a_0(s|\{S_f\}) < \frac{n_f + n_g}{n_f} a(s|S_f)$ exists. Name him s. Since the relative position rule satisfies PPL, $a^*(s|\{S_f\}) = a(s|S_f) + |\{s'' \in S_g \mid \varphi^*_{s''}(F, \{S_f\}) < \varphi^*_s(F, \{S_f\})\}| + |\{s'' \in S_g \mid [\varphi^*_{s''}(F, \{S_f\}) =$

$\varphi^*_s(F,\{S_f\})] \wedge [s \succ_0 s'']\}|$ ...(*).
(Case 1: $a_0(s|\{S_f\}) = a(s|S_f)$) It is immediate from (*) that $a^*(s|\{S_f\}) \geq a(s|S_f) = a_0(s|\{S_f\})$.
(Case 2: $a_0(s|\{S_f\}) < a(s|S_f)$) Take $s' \in S_g$ such that $a(s'|S_g) = a_0(s|\{S_f\}) - a(s|S_f)$. Then, by $a_0(s|\{S_f\}) < \frac{n_f+n_g}{n_f}a(s|S_f)$, $\frac{a(s'|S_g)}{n_g} < \frac{a(s|S_f)}{n_f}$. Lemma 5 implies that $\varphi_{s'}(F,\{S_f\}) \leq \varphi_s(F,\{S_f\})$. If $\varphi^*_{s'}(F,\{S_f\}) = \varphi^*_s(F,\{S_f\})$, by its construction, s' is included in the last term of RHS in (*). Otherwise, s' is included in the second term. PPL of φ^* implies that $a^*(s|\{S_f\}) \geq a(s|S_f) + a(s'|S_g) \geq a_0(s|\{S_f\})$.
The same result does not hold when $|F| \geq 3$, as shown in the next example. Let $F = \{f,g,h\}$, $(n_f,n_g,n_h) = (9,7,24)$, $K = 5$. $\succ_0$ is as follows (s^i_f stands for a student with $a(s^i_f|S_f) = i$).
$\succ_0$: ..., $s^3_f, s^2_f, s^1_f, s^7_g, s^6_g, s^5_g, s^4_g, s^3_g, s^2_g, s^1_g, s^3_h, s^2_h, s^1_h$. For , s^3_f, $\frac{a_0(s^3_f|\{S_f\})}{\sum_{f\in F} n_f} = \frac{13}{40} < \frac{3}{9} = \frac{a(s^3_f|S_f)}{n_f}$. However, $a^*(s^3_f|\{S_f\}) = a(s^3_f|S_f) + |\{s'' \in S_g \cup S_h \mid \varphi_{s''}(F,\{S_f\}) = 1\}| + |\{s'' \in S_g \cup S_h \mid [\varphi^*_{s''}(F,\{S_f\}) = 2] \wedge [s \succ_0 s'']\}| = 3 + (1+4) + (1+0) = 9 < 13 = a_0(s^3_f|S_f)$.
(ii) Fix an arbitrary problem $(F,\{S_f\})$, in which at least one student $s \in S_f$ with $\frac{a_0(s|\{S_f\})}{\sum_{f\in F} n_f} < \frac{a(s|S_f)}{n_f} - \{\frac{1}{K} + \frac{|F|}{\sum_{f\in F} n_f}\}$ exists. Name him s and his faculty f. The assumption $\frac{a_0(s|\{S_f\})}{\sum_{f\in F} n_f} < \frac{a(s|S_f)}{n_f} - \{\frac{1}{K} + \frac{|F|}{\sum_{f\in F} n_f}\}$ implies $a_0(s|\{S_f\}) < \{\frac{a(s|S_f)}{n_f} - \frac{1}{K}\}\sum_{f\in F} n_f - |F| \leq \{\frac{1}{n_f}[\frac{n_f\varphi^*_s(F,\{S_f\})}{K}] - \frac{1}{K}\}\sum_{f\in F} n_f - |F| \leq \frac{\varphi^*_s(F,\{S_f\})-1}{K}\sum_{f\in F} n_f - |F|$.
On the other hand, $a^*(s|\{S_f\}) = |\{s'' \in \cup_{f\in F} S_f \mid \varphi^*_{s''}(F,\{S_f\}) < \varphi^*_s(F,\{S_f\})\}| + |\{s'' \in \cup_{f\in F} S_f \mid [\varphi^*_{s''}(F,\{S_f\}) = \varphi^*_s(F,\{S_f\})] \wedge [s \succeq_0 s'']\}| \geq \sum_{f\in F}[\frac{n_f(\varphi^*_s(F,\{S_f\})-1)}{K}] \geq \sum_{f\in F}\{\frac{n_f(\varphi^*_s(F,\{S_f\})-1)}{K} - 1\} = \frac{\varphi^*_s(F,\{S_f\})-1}{K}\sum_{f\in F} n_f - |F|$. Thus, we obtain $a^*(s|\{S_f\}) > a_0(s|\{S_f\})$. (Q.E.D.)

PROOF OF PROPOSITION 5.

Let $F = \{f,g\}$, $(S_f,S_g) = (\{s^6,s^5,s^4\},\{s^3,s^2,s^1\})$ with $\succ_0$: s^6,s^5,s^4,s^3,s^2,s^1, $K = 3$, and $H = \{a,b,c,d,\emptyset\}$. $\succeq^*$ is determined as $\succ^*$: s^6,s^3,s^5,s^2,s^4,s^1. $\succsim$ is as follows.

Table 3: List of preferences

$\succsim_{s^6}$	$\succsim_{s^5}$	$\succsim_{s^4}$	$\succsim_{s^3}$	$\succsim_{s^2}$	$\succsim_{s^1}$
d	a	d	a	c	c
a	c	$\emptyset$	b	a	a
b	b	a	c	b	$\emptyset$
c	d	b	d	d	d
$\emptyset$	$\emptyset$	c	$\emptyset$	$\emptyset$	b

$\psi^{\succeq_0}(\succsim)$ is determined as $\psi^{\succeq_0}_{s^6} = d$, $\psi^{\succeq_0}_{s^5} = a$, $\psi^{\succeq_0}_{s^4} = \emptyset$, $\psi^{\succeq_0}_{s^3} = b$, $\psi^{\succeq_0}_{s^2} = c$, and $\psi^{\succeq_0}_{s^1} = \emptyset$. Similarly, $\psi^{\succeq^*}(\succsim)$ is determined as $\psi^{\succeq^*}_{s^6} = d$, $\psi^{\succeq^*}_{s^3} = a$, $\psi^{\succeq^*}_{s^5} = c$, $\psi^{\succeq^*}_{s^2} = b$, $\psi^{\succeq^*}_{s^4} = \emptyset$, and $\psi^{\succeq^*}_{s^1} = \emptyset$. We can see for s^2, $a^*(s^2|\{S_f\}) = 3 > 2 = a_0(s^2|\{S_f\})$ but $\psi^{\succeq_0}_{s^2}(\succsim) = c \succsim_{s^2} b = \psi^{\succeq^*}_{s^2}(\succsim)$. (Q.E.D.)

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