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An Example from Accounting Rules

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1 Introduction

This paper proposes a model to explain why accounting rules have been accepted by stakeholders of firms by using cooperative game theory. Accounting rules, called Generally Accepted Accounting Principles (GAAP), are originally based on social norms developed during transactions, accepted as fair and useful, and gradually established as a common practice (Saito 2011 [18]). Stakeholders who make various decisions concerning firms use the accounting information generated by the rules. In particular, earnings, which are also called net income, are used as not only a signal of firms' financial conditions but also as a proxy for the amount available for distribution by firms to their stakeholders. In this paper, we focus on the latter role of accounting information and extract the conditions that accounting rules are accepted.

As a starting point, we deal with depreciation that has been used all over the world for a long time. Depreciation is defined as the rational and systematic allocation of the original cost of an asset over the expected useful life of that asset (Hendriksen and Breda 2001 [12], p. 523). There are many possible ways to allocate investment costs over time, but in practice, only certain methods, such as the straight-line method or the declining-balance method [1], are used.

Our goal is not to explain how these depreciation methods had come up but to clarify why they have been accepted. Therefore, we adopt cooperative game theory which assumes that players can talk to each other and make agreements that are binding on later play (Heap and Varoufakis 2004 [11]). Furthermore, cooperative game theory has many solution concepts, such as the core, the Shapley value, the nucleolus, and so on. They can be linked with norms, such as "benefit should be shared according to their marginal contribution (the Shapley value) " and "benefit is shared subjected to minimize maximum excess (the nucleolus)." Therefore, we considered that cooperative game theory was suitable for axiomatizing and analyzing accounting rules.

The rest of this paper proceeds as follows: Section 2 reviews the previous research that uses cooperative game theory in accounting-related fields and others. In Section 3, after we provide an overview of cooperative game theory applied to cost-sharing problems, we develop our model, i.e. depreciation game, and show the properties of its cost function. In Section 4,

[1] The decreasing method supposes that asset depreciation occurs quickly in the beginning and then decreases over time

we adopt the core, which offers a class of solutions that players accept, as a solution concept, and discuss whether depreciation methods used in practice can be explained by the core. Furthermore, in Section 5, we introduce the Shapley value and the nucleolus to examine the condition that the depreciation methods which are given by them are the same as the straight-line method. Finally, we present our concluding remarks in Section 6.

2 Literature Review

In this section, we review prior research from three viewpoints: 1) analytical research for depreciation in accounting, 2) related research using cooperative game theory, and 3) research on depreciation using cooperative game theory.

In the field of accounting, there were many analytical discussions of accounting rules in the 1960s. Of course, it was conducted on depreciation methods as well. According to Wright (2006) [22] which summarizes the discussion in the 1960s, it was "the normative kind-arguments about what accounting ought to be doing." They discussed the ideal depreciation methods, which were different from the straight-line method traditionally used. However, their methods have never been adopted, and certain methods such as the straight-line method and the declining balance method are still used [2].

In the field of game theory, researchers often apply cooperative game theory to analyze cost allocation of joint costs such as airport landing fees (e.g., Littlechild and Owen 1973[14]) or division of an estate among creditors (e.g., Aumann and Maschler 1985 [3]). Accounting researchers began frequently applying cooperative game theory from the late 1970s to the early 1980s. Hamlen et al. (1977)[9] examined the allocation of joint costs using core theory, and Callen (1978)[7], Roth and Verrecchia (1979)[17], Hamlen et al. (1980) [10] and Balachandran and Ramakrishnan (1981)[5] discussed allocation using the Shapley value. These studies focused on cross-sectional cash flow allocation, that is, how to allocate joint costs among departments within a firm.

On the other hand, some studies made normative judgments of depreciation methods using cooperative game theory in other research areas. Ben-Shahar and Sulganik (2009)[6] adopted the Shapley value as a solution concept because allocation under the Shapley value reflects the pattern in which firms consume their assets' economic benefits. Aparicio and Sanchez-Soriano

[2] Since the 1970s, these normative studies have disappeared and accounting studies have shifted to empirical studies that examine the usefulness of accounting information.

(2008) [1] built an innovative ”depreciation game” in which players are fiscal years of the asset’s useful life and indicated that payoff vectors given by the conventional methods do not always belong to the core. They proposed a new depreciation method that gives a payoff vector belonging to the core and that reflects the asset’s market value.

Most of the studies (except for Auman and Maschler, 1985) surveyed here are those that seek the ”best” way. However, our interest is not in clarifying the best accounting method, but in clarifying why the rules currently in use continue to be used.

Therefore, we modified Aparicio and Sanchez-Soriano model, hereinafter referred to as AS model, to analyze existing accounting standards under practical assumptions. We believe the modification is essential because there is semantical insufficiency in their model. Their conclusions and those of us are consistent in a limited domain, but the cost function they define has practical meaning only in that domain. In this paper, we restructure the model by introducing a new concept of ”sequentiality” so that the cost function has economic meaning in all domains, i.e., coalitions. Then we establish clear connections between practical decision-making and the depreciation game so that we can validate current depreciation methods. We thus contribute to game theory research in the field of social system engineering and introduce this new perspective to financial accounting research.

3 The Model

3.1 Overview of Cooperative Game Theory

We first provide an overview of the basic notions of cooperative game theory as applied to the cost-sharing problem. A transferable utility cost game is a pair (N, c), where $N = \{1, 2, \ldots, n\}$ is the set of players and c is the cost function from 2^N to $\mathbb{R}$ where $c(\emptyset) = 0$. For any coalition $S \in 2^N \backslash \{\emptyset\}$, $c(S)$ refers to the cost that players in S pay when they cooperate.

A payoff vector $\mathbf{x} = (x_i)_{i \in N} \in \mathbb{R}^N$ indicates the allocation to each player for a given (N, c) and define $\mathbf{x}(S) = \sum_{i \in S} x_i$ for each $S \in 2^N \backslash \{\emptyset\}$.

Definition 1 *Subadditivity*
A cost function is subadditive if

$$c(S) + c(T) \geq c(S \cup T) \text{ for all } S, T \in 2^N \backslash \{\emptyset\} \quad s.t. S \cap T = \emptyset.$$

Definition 2 *Imputation*
An imputation for a given cost game (N, c) *is a vector* $\mathbf{x} = (x_i)_{i \in N}$ *satisfying*

$$x_i \leq c(\{i\}) \ \forall i \in N, \tag{1}$$
$$\mathbf{x}(N) = c(N). \tag{2}$$

Let A *denote the set of all imputations of the game.*

The first condition (1) is individual rationality, meaning that no player pays more than they can pay on their own. The second condition (2) is called group rationality, meaning that an imputation is Pareto efficient.

Definition 3 *Domination*
We say an imputation vector $\mathbf{x} = (x_i)_{i \in N}$ *is dominated by the imputation vector* $\mathbf{y} = (y_i)_{i \in N}$ *with respect to coalition* S *if*

$$y_i < x_i \text{ for all } i \in S \tag{3}$$

and

$$c(S) \leq \mathbf{y}(S) \tag{4}$$

are satisfied. We express this as $\mathbf{y} \succ_s \mathbf{x}$*.*

Condition (3) indicates that the imputation $\mathbf{y}$ brings lower a cost for each element in S than the imputation $\mathbf{x}$. Condition (4) indicates that the elements in S can achieve imputation $\mathbf{y}$; in other words, S can improve the imputation, moving from $\mathbf{x}$ to $\mathbf{y}$.

Definition 4 *The core*
The set of all undominated imputations for a game (N, c) *is called the core, which is denoted by* $Core(c)$*.*

Proposition 1 *The core of a cost game*
If c *is subadditive, then* $Core(c)$ *is*

$$\{\mathbf{x} \in A : \mathbf{x}(S) \leq c(S), \forall S \in 2^N \backslash \{\emptyset\}\}. \tag{5}$$

Proof 1 [3]. *If* $\mathbf{x}$ *is an element of (5), then satisfies* $\mathbf{x}(S) \leq c(S)$*. We assume that* $y_i < x_i$ *for all* $i \in S$*. This means that*

$$\mathbf{y}(S) < c(S),$$

[3] This proof is based on Owen (1995) [15]

and it is not possible that $\mathbf{y} \succ_s \mathbf{x}$ *because of (4). Therefore* $\mathbf{x} \in Core(c)$.

Conversely, suppose that $\mathbf{y} \in A$ *does not satisfy* $\mathbf{y}(S) \leq c(S)$ *for* $\exists S$. *This implies that*

$$\mathbf{y}(S) = c(S) + \epsilon \quad \textit{for } \exists \epsilon > 0.$$

Let

$$\alpha = \sum_{i \in N-S} c(\{i\}) - \{c(N) - c(S)\}.$$

By subadditivity,

$$\begin{aligned} & c\,(N-S) + c(S) \geq c(N) \\ \leftrightarrow\quad & c(N-S) \geq c(N) - c(S) \end{aligned} \tag{6}$$

and

$$\sum_{i \in N-S} c(\{i\}) \geq c(N-S). \tag{7}$$

From (6) and (7), $\alpha \geq 0$. *Then, let* s *be the number of elements in* S. *Now define* $\mathbf{z}$ *by*

$$z_i = \begin{cases} y_i - \epsilon/s & (i \in S) \\ c(i) - \alpha/(n-s) & (i \notin S) \end{cases}$$

First,

$$z_i \leq \begin{cases} y_i \leq c(\{i\}) & (i \in S) \quad \because \mathbf{y} \textit{ is an imputation.} \\ c(i) - \alpha/(n-s) & (i \notin S) \end{cases} \tag{8}$$

Then, $\mathbf{z}$ *satisfies (1).*
Second,

$$\begin{aligned} \mathbf{z}(N) &= \mathbf{z}(S) + \mathbf{z}(N-S) \\ &= \mathbf{y}(S) - \epsilon + \sum_{i \in N-S} c(\{i\}) - \alpha \\ &= c(S) + c(N) - c(S) \\ &= c(N) \end{aligned}$$

Then, $\mathbf{z}$ *satisfies (2). Therefore* $\mathbf{z}$ *is an imputation.*

Form (8) and $\mathbf{z}(S) = c(S)$, $\mathbf{z} \succ_s \mathbf{y}$. *Hence* $\mathbf{y} \notin Core(c)$. $\square$

3.2 Depreciation Game

In this section, we develop our model [4] to analyze the rationality of depreciation methods.

We assume that a firm plans to use an asset whose useful life is n years and its market value is decreasing from C to zero over n years. Our depreciation game is denoted by (N, d): N is the set of players and d is the cost function defined from 2^N to $\mathbb{R}$ where $d(\emptyset) = 0$. The set of players consists of the fiscal years during the asset's useful life. A player represents the firm's stakeholders in each fiscal year. A payoff vector $\mathbf{x} \in \mathbb{R}^N$ is a distribution of cost among the players, which indicates the amount each player (fiscal year) should pay for a given (N, d)[5].

For any coalition $S \in 2^N \backslash \{\emptyset\}$, $d(S)$ is the cost that the players in S should pay for use of the asset. Depreciation is an accounting procedure for a purchased asset that corresponds to the asset gained by the grand coalition in this game. We assume that other coalitions gain the asset through leases because they use the asset during only a part of its economic useful life. When no cooperation exists, the firm invests in the asset annually through a one-year lease contract. We assume that players enter a coalition to reduce investment costs. When a "sequential" coalition exists[6], such as $S = \{2, 3, 4\}$, the firm makes a multi-year lease contract (for this example, a three-year contract). Therefore, $d(S)$ could be represented by a combination of lease payments. We discuss the definition of $d(S)$ in detail below.

First, we introduce some concepts necessary for our discussion, i.e. sequentiality, length, market value function, and restricted cost function.

Definition 5 *Sequentiality*

Let $N = \{1, 2, \ldots, n\}$*. A coalition* $S \in 2^N \backslash \{\emptyset\}$ *is sequential if*

$$(\forall a, b \in S)(\forall c \in N)(a < c < b \rightarrow c \in S)$$

holds.

Seq^N denotes the set of all sequential subsets of N.

[4] The cost function is partially the same as AS model. As we mentioned in the previous section, we explain the background of a firm's decision-making and define the cost function for the non-restricted domain.

[5] AS model does not consider such background of a firm's decisions.

[6] Sequetiality is precisely defined below.

Definition 6 *Length of* $S \in Seq^N$
Given N*, we define the length of* $S \in Seq^N$ *to be the cardinality of* S*. We adopt the natural notation* $l_S \in \{0, 1, 2, \ldots, n\}$ *for the value.*
Note that for S *and* $T \in Seq^N$ *such that* $S \cup T \in Seq^N$*, we have that* $T \cup S \in Seq^N$ *and* $l_{S \cup T} = l_{T \cup S}$*.*

For example, when $S = \{4, 5, 6\} \in Seq^N, l_S = 3$.

Definition 7 *Market value function*
We define f*, which denotes the market value of the asset, as follows:*

$$f : \{0, 1, \ldots, n\} \to \mathbb{R} \quad s.t.\ C = f(0) > f(1) > f(2) > \cdots > f(n) = 0 \quad (9)$$

Consider a leasing company that leases the asset to the firm. Assume that the leasing company has access to the second-hand market while the firm does not. The leasing company can predict its market value $f(k)$ on the secondhand market, which indicates the value of the asset used for k-years. We also assume that the leasing company can recover the fall in its market value $(= C - f(k))$ during the lease term by lease payments. Then we define the lease payment function as follows:

Definition 8 *Restricted cost function (Lease payment function)*
We define $d^* : Seq^N \to \mathbb{R}$ *as*

$$d^* : S \mapsto C - f(l_S) \qquad \forall S \in Seq^N. \quad (10)$$

d^* indicates the amount of lease payments, i.e. cost, for members in Seq^N, therefore it is termed a restricted cost function. For example, when $S = \{2, 3, 4\}$, $d^*(S) = C - f(3)$.

In this function, we do not count the interest costs included in actual lease payments because we also exclude the interest costs when the asset purchase is accompanied by debt.

In reality, a multi-year lease contract is more cost-saving than a combination of short-term lease contracts because continuous lease contracts can reduce transaction costs. For a firm that uses an asset for $k + t$ (k and t are integers.) years, it is rational to choose one lease contract for $k + t$ years rather than the combination of lease contracts for k years and t years. From this property of lease contracts, we then induce the following condition of our market value function.

The condition of the market value function
Given N, we choose coalitions $S, T \in Seq^N$ such that $S \cap T = \emptyset$ and $S \cup T \in Seq^N$. When a firm engages in a lease transaction, the following inequality holds:

$$\begin{aligned} & d^*(S) + d^*(T) \geq d^*(S \cup T) \\ \rightleftharpoons \; & C - f(l_S) + C - f(l_T) \geq C - f(l_{S \cup T}) \\ \rightleftharpoons \; & C - f(l_S) \geq f(l_T) - f(l_{T \cup S}) \\ \rightleftharpoons \; & C(= f(l_0)) - f(l_S) \geq f(l_T) - f(l_T + l_S). \end{aligned} \tag{11}$$

We can rewrite (11) more simply as below:

$$C - f(t) \geq f(k) - f(k + t) \tag{12}$$

where k and t are integers from 1 to n and $2 \leq k + t \leq n$ [7].
From (11) and (12) we can see that a firm engages in a multi-year lease contract if the asset value declines rapidly in the early times.

In general, coalitions are not always sequential. To generalize, we introduce a "maximally sequential coalition."

Definition 9 *Maximally sequential coalition*
Given N, let us choose a coalition $S \in 2^N \backslash \{\emptyset\}$. A sequential subset $S' \subseteq S$ is called a maximally sequential coalition in S if,

$$\begin{aligned} & \textit{for any} \quad a \in N, \\ & \quad \textit{if} \quad S' \cup \{a\} \textit{is sequential}, \\ & \quad \textit{then} \quad S' \cup \{a\} = S' \quad \textit{or} \quad S' \cup \{a\} \not\subseteq S. \end{aligned}$$

Definition 10 *Set of maximally sequential coalitions*
Let $S \in 2^N \backslash \{\emptyset\}$ be given, the set $(S)^m$ of maximally sequential coalitions of S is defined as follows:

$$(S)^m =_{def} \{S_j \subseteq S | S_j \textit{ is maximally sequential in} \, S.\}.$$

[7] When its equality holds, the choice between purchasing and leasing is indifferent and then the essentiality of this cost function does not hold. In a real situation, however, there could be such a case, we consider the equality in (11) and (12).

When $S = \{2,3,6,7,8\}$, the coalition S consists of two subsets of maximally sequential coalitions, $(S)^m = \{\{2,3\},\{6,7,8\}\}$. $|X|$ denotes the *cardinality* for a given set X.

Finally, we define the cost function.

Definition 11 *Cost function*

$$d : 2^N \backslash \{\emptyset\} \to \mathbb{R} \qquad s.t.\ d(S) = \sum_{S_j \in (S)^m} d^*(S_j)$$

From Definition 10, the coalition S is divided into sequential subsets. During one of the sequential subsets, the firm uses a continuous multi-year lease contract. When S consists of several sequential subsets, the cost function is the sum of the cost of multiple lease contracts. For example, when $S = \{2,3,6,7,8\}$, we determine the cost function as follows:

$$\begin{aligned} d(\{2,3,6,7,8\}) &= d^*(\{2,3\}) + d^*(\{6,7,8\}) \\ &= (C - f(l_{\{2,3\}})) + (C - f(l_{\{6,7,8\}}) = 2C - f(2) - f(3) \end{aligned}$$

In case of a grand coalition, that is, the firm purchases the asset, the cost function is

$$d(N) = d^*(N) = C - f(l_{\{N\}}) = C - f(n) = C.$$

The cost function d is well-defined.

Proposition 2

$$d|_{Seq^N} = d^*.$$

Proof 2 *It is trivial.* □

3.3 Properties of our cost function

Our cost function has the following properties:

Proposition 3 *Given a cost game (N, d), if S and $T (\in Seq^N)$ are disjoint, then*

$$d^*(S) + d^*(T) \geq d(S \cup T) \tag{13}$$

always holds.

Proof 3 *(case 1)* $S \cup T \in Seq^N$*:*

$$d(S \cup T) = d^*(S \cup T) = C - f(l_{S \cup T});$$

therefore,

$$\begin{aligned} d^*(S) + d^*(T) - d(S \cup T) &= 2C - (f(l_S) + f(l_T)) - (C - f(l_{S \cup T})) \\ &= C - (f(l_S) + f(l_T) - f(l_{S \cup T})) \\ &= C - (f(l_S) + f(l_T) - f(l_T + l_S)) \ \cdots \ (a) \end{aligned}$$

From (11),

$$C - f(l_S) \geq f(l_T) - f(l_T + l_S) \quad \rightleftharpoons \quad C - (f(l_S) + f(l_T) - f(l_T + l_S)) \geq 0,$$

hence

$$(a) \geq 0.$$

Then, case 1's proof is complete.

(case 2) $S \cup T \notin Seq^N$ *:*

$$d(S \cup T) = d^*(S) + d^*(T),$$

which completes the proof of the statement. □

Lemma 1 *The cost function* d *is subadditive. That is,*

$$d(S) + d(T) \geq d(S \cup T) \quad \forall S, T \in 2^N \backslash \{\emptyset\} \text{ such that } S \cap T = \emptyset. \tag{14}$$

Proof 4 *By induction with respect to* $|(S)^m| + |(T)^m|$*. Let* S *and* $T \in 2^N \backslash \{\emptyset\}$ *be non-empty.*
(Base:) $|(S)^m| + |(T)^m| = 2$*, in which case that* S *and* $T \in Seq^N$*,*

$$d(S) + d(T) = d^*(S) + d^*(T),$$

and Proposition 3 ensures the required result.

(Step:) Suppose $d(S)+d(T) \geq d(S \cup T)$, *where* $|(S)^m|+|(T)^m| = k$. *Let* T' *be* $T \cup V$, *where* $V \in Seq^N$, $V \cap S = \emptyset$ *and* $T_j \cup V \notin Seq^N$ *for all* $T_j \in (T)^m$. *In this case,* $|(S)^m|+|(T')^m| = k+1$. *There are three possible cases:*

(Case 1) $|(S \cup T')^m| = |(S \cup T)^m|+1$. *In this case,* $S_j \cup V \notin Seq^N$ *for all* $S_j \in (S)^m$. *Therefore, using the definition of* d^*,

$$\begin{aligned} d(S)+d(T')-d(S \cup T') &= d(S)+d(T)+d^*(V)-(d(S \cup T)+d^*(V)) \\ &= d(S)+d(T)-d(S \cup T) \\ &\geq 0 \text{ (because of the induction hypothesis)} \end{aligned}$$

(Case 2) $|(S \cup T')^m| = |(S \cup T)^m|$, *if there is only one* $S_j \in (S)^m$ *such that* $S_j \cup V \in Seq^N$. *Also we use* S^- *to denote* $S \backslash S_j$. *Then,*

$$\begin{aligned} d(S)+d(T')-d(S \cup T') &= d(S)+d(T)+d^*(V)-(d(S^- \cup T)+d^*(V \cup S_j)) \\ &\geq d(S \cup T)+d^*(V)-(d(S^- \cup T)+d^*(V \cup S_j)) \\ &\quad \text{(because of the induction hypothesis)} \\ &= d((S^- \cup T) \cup S_j)-d(S^- \cup T)+(d^*(V)-d^*(V \cup S_j)) \\ &= d^*(S_j)+d^*(V)-d^*(V \cup S_j) \\ &\quad \text{(because of the assumption of } S_j\text{)} \\ &\geq 0 \\ &\quad \text{(because of the induction hypothesis)} \end{aligned}$$

(Case 3) In case of $|(S \cup T')^m| = |(S \cup T)^m|-1$ *: there are some* $S_{j1}, S_{j2} \in (S)^m$ *with the condition* $S_{j1} \cup S_{j2} \cup V \in Seq^N$. *Let us use the same notation of case 2) :* $\tilde{S}$ *to denote* $S \backslash (S_{j1} \cup S_{j2})$. *The calculation is almost the same as the case 2, but let us describe it below.*

$$\begin{aligned} d(S)+d(T')-d(S \cup T') &= d(S)+d(T)+d^*(V)-(d(\tilde{S} \cup T)+d^*(V \cup (S_{j1} \cup S_{j2})) \\ &\geq d(S \cup T)+d^*(V)-(d(\tilde{S} \cup T)+d^*(V \cup (S_{j1} \cup S_{j2})) \\ &= d(S \cup T)-d(\tilde{S} \cup T)+(d^*(V)-d^*(V \cup (S_{j1} \cup S_{j2})) \\ &= d((\tilde{S} \cup T) \cup (S_{j1} \cup S_{j2}))-d(\tilde{S} \cup T)+(d^*(V)-d^*(V \cup S_{j1} \cup S_{j2})) \\ &= (d^*(S_{j1})+d^*(S_{j2})+d^*(V))-d^*(V \cup S_{j1} \cup S_{j2}) \\ &\geq 0. \end{aligned}$$

Just in case : the last inequality $\geq$ *is jusified (using Proposition 3) as :*

$$d^*(S_{j1})+d^*(S_{j2}) \geq d(S_{j1} \cup S_{j2})$$
$$d^*(S_{j1})+d^*(S_{j2})+d^*(V) \geq d(S_{j1} \cup S_{j2})+d^*(V) = d^*(V \cup S_{j1} \cup S_{j2}).$$

□

Definition 12 *Diminishing*
The market value function is said to be diminishing when

$$f(k) - f(k+u) \geq f(t) - f(t+u) \tag{15}$$

holds, where k, t *and* u *are integers from* 1 *to* n*,* $k < t$*, and* $2 \leq k+u, u+t \leq n$.

Proposition 4 *The cost function* $d(S)$ *is concave if the market value function is decreasing and diminishing with respect to time.*

Proof 5 *The cost function is said to be concave* [8] *if*

$$d(S) + d(T) \geq d(S \cup T) + d(S \cap T) \quad \forall S, T \in 2^N \backslash \{\emptyset\}. \tag{16}$$

We use the induction along with $|(S)^m|$ *and* $|(T)^m|$.
(base:) $|(S)^m| = |(T)^m| = 1$.

(case1) $S \cap T = \emptyset$. *Lemma 1 ensures*

$$d(S) + d(T) = d(S \cup T) \tag{17}$$

and from our assumption $S \cap T = \emptyset$, $d(S \cap T) = 0$ *is the fact and completes the proof of this case.*

(case 2)$S \cap T \neq \emptyset$. *If* $S \subseteq T$ *(or* $T \subseteq S$*), then this case is trivial. Suppose* $S \not\subseteq T$, $T \not\subseteq S$ *and also first element of* S *is smaller than that of* T*. Then,*

$$\begin{aligned} d(S) + d(T) &= d^*(S) + d^*(T) = C - f(l_S) + C - f(l_T) \\ &= 2C - f(l_S) - f(l_T) \end{aligned} \tag{18}$$

$$d(S \cup T) + d(S \cap T) = 2C - f(l_{S \cup T}) - f(l_{S \cap T}) \tag{19}$$
$$(\textit{because } S \cup T \textit{ and } S \cap T \in Seq^N)$$

To show$(18) - (19) \geq 0$*, first we immediately have that if* $U = T - S$,

[8]Driessen (1992), [8]

$$f(l_S) - f(l_{S\cup U}) \le f(l_{T-U}) - f(l_{(T-U)\cup U}), \tag{20}$$

because of the assumption of f to be diminishing. Then,

$$\begin{aligned}(20) \Leftrightarrow & f(l_S) + f(l_T) \le f(l_{S\cup T}) + f(l_{S\cap T}) \\ & (\textit{notice that } T - U = S\cap T \textit{ and } (T-U)\cup U = T) \\ \Leftrightarrow & 2C - (f(l_S) + f(l_T)) - (2C - (f(l_{S\cup T}) + f(l_{S\cap T}))) \ge 0, \end{aligned} \tag{21}$$

which is the result we wanted.
(Step:)

Let us assume : $(S)^m = \{S_1, \cdots, S_m\}$, $(T)^m = \{T_1, \cdots, T_n\}$, $R \in Seq^N$ and $S_j \cap R = \emptyset$ for all $j = 1, \cdots, m$. If $R\cap T = \emptyset$, the statement is trivially true. Assume $R \cap T \ne \emptyset$. Note that $|(T \cup R)^m|$ does not increase in this case. Then from the induction hypothesis :

$$\begin{aligned} & d(S) + d(T\cup R) \ge d(S\cup T\cup R) + d(S\cap (T\cup R)) \\ \Leftrightarrow\ & d(S) + d(T\cup R) \ge d(S\cup T\cup R) + d((S\cap T)\cup (S\cap R)) \\ \Leftrightarrow\ & d(S) + d(T\cup R) \ge d(S\cup T\cup R) + d((S\cap T)) \quad (\textit{ because } S\cap R = \emptyset\ ). \\ \Leftrightarrow\ & d(S) - d(S\cap T) \ge d(S\cup T\cup R) - d(T\cup R) \end{aligned} \tag{22}$$

Also, the induction hypothesis yields :

$$\begin{aligned} & d(R) + d(T) \ge d(R\cup T) + d(R\cap T) \\ \Leftrightarrow\ & d(R\cap T) \le d(T) + d(R) - d(R\cup T) \end{aligned} \tag{23}$$

Let $S' = S\cup R$. (Note that $|(S')^m| = |(S)^m| + 1$ and $d(S') = d(S) + d(R)$.) Our goal is :

$$\begin{aligned} & d(S') + d(T) \ge d(S'\cup T) + d(S'\cap T) \\ \Leftrightarrow\ & d(S) + d(R) + d(T) \ge d(S\cup T\cup R) + d((S\cup R)\cap T) \\ \Leftrightarrow\ & d(S) + d(R) + d(T) \ge d(S\cup T\cup R) + d((S\cap T)\cup (R\cap T)) \\ \Leftrightarrow\ & d(S) + d(R) + d(T) \ge d(S\cup T\cup R) + d(S\cap T) + d(R\cap T) \ (\textit{because } R\cap S = \emptyset\ ) \\ \Leftrightarrow\ & d(S) - d(S\cap T) \ge d(S\cup T\cup R) - (d(T) + d(R) - d(R\cap T)) \end{aligned} \tag{24}$$

Now (24) is ensured from (22) and (23), and proof is done. □

Basically, as shown above, concavity is ensured only if property of "diminishing" is supposed, which seems to be quite essential.

4 The core of the depreciation game

4.1 The core

In section 3.1, we formulated conditions of the core for a cost game. Here, we identify the core of our depreciation game (N, d).

Definition 13 *Imputation of the depreciation game*
For depreciation game (N, d)*, the* imputation *(denoted by A) is the subspace of* $\mathbb{R}^n$ *satisfying:*

$$x_i \leq d(\{i\}) \ \forall i \in N, \tag{25}$$

$$\mathbf{x}(N) = d(N). \tag{26}$$

From Proposition 1 and Lemma 1, the *core* is defined as follows:

Definition 14 *The core of depreciation game* (N, d)
Given depreciation game (N, d)*,* $Core(d)$ *is defined as:*

$$Core(d) = \{\boldsymbol{x} \in A \mid \mathbf{x}(S) \leq d(S) \quad \forall S \in 2^N \backslash \{\emptyset\}\} \tag{27}$$

Definition 15 *Straight-line method*
The payoff vector given by the straight-line method *is*

$$\mathbf{SL} = (C/n, C/n, \cdots, C/n).$$

Proposition 5 **SL** *is an element of A.*

Proof 6 *From the condition of the market value function (12),*

$$\begin{aligned}
C - f(1) &\geq C - f(1) \\
C - f(1) &\geq f(1) - f(2) \\
C - f(1) &\geq f(2) - f(3) \\
&\vdots \\
C - f(1) &\geq f(n-1) - f(n)
\end{aligned}$$

Summing all observations, we obtain $n(C - f(1)) \geq C - f(n) = C$*. (25) is satisfied because* $d(\{i\}) = C - f(1)$ *and* $x_i = C/n$ *for* **SL***. Furthermore,*

$$\mathbf{SL}(N) = nC/n = C.$$

Thefore **SL** *also satisfies (26).* □

Theorem 1 *The payoff vector SL is an element of Core(d).*

Proof 7 *For an arbitrary* $S \in 2^N \backslash \{\emptyset\}$*, let* $|S| = k$*,*

$$\mathbf{SL}(S)(= \sum_{i \in S} SL_i) = kC/n;$$

Therefore, we must derive

$$\frac{kC}{n} \leq d(S) \quad \cdots \quad (b).$$

We use induction with respect to $|S|$*.*
(base:) When $|S| = 1$*,*
Proposition 5 indicates that

$$d(\{i\}) = d^*(\{i\}) = C - f(1) \geq \frac{C}{n},$$

which is the result.
(steps:) There are three cases, and we demonstrate only the following case: for some $i \in N$*,*

$$S = \bigcup_{S_j \in (S)^m} S_j \ s.t. \ |S| = k,$$
$$S_1, S_2 \in (S)^m \text{ and } S_1 \cup \{i\} \cup S_2 \in Seq^N.$$

Suppose (b) holds for all cases of $|S| \leq k$*. Let* $S' = S \cup \{i\}$*, then*

$$d(S') = d(S \cup \{i\}) = d(\tilde{S}) + d^*(S_1 \cup \{i\} \cup S_2),$$

where $\tilde{S} \cup S_1 \cup S_2 = S$*. Then the induction hypothesis gives:*

$$d(\tilde{S}) - \frac{\tilde{k}}{n} C \geq 0 \quad \text{and} \quad d(S_1 \cup \{i\} \cup S_2) - \frac{k^{1,2,i}}{n} C \geq 0 \ \cdots (c),$$

(here, we assume $\tilde{k} = |\tilde{S}|$ *and* $k^{1,2,i} = |S_1 \cup \{i\} \cup S_2|$*.) Note that*

$$d^*(S_1 \cup \{i\} \cup S_2) = d(S_1 \cup \{i\} \cup S_2) \quad \text{and} \quad \tilde{k} + k^{1,2,i} = k + 1.$$

We add each side of the two inequalities of (c) to obtain

$$d(\tilde{S}) + d^*(S_1 \cup \{i\} \cup S_2) - \left(\frac{\tilde{k}}{n} C + \frac{k^{1,2,i}}{n} C \right) \geq 0$$
$$\Leftrightarrow d(S') - \frac{k+1}{n} C \geq 0,$$

to reach our conclusion. □

4.2 Implications for practice

In this section, we discuss the practical implications of the core for our depreciation game. To make the discussion easier, we consider only a subset $\mathcal{N} = \{\{1\}, \{1,2\}, \cdots, \{1,2,\cdots,n-1\}, \{N\}\}$ of Seq^N as domain of the restricted cost function $d^*(S)$. In the situation where all players are participating in the game, that is, where it is supposed to use an asset for n years, it is not necessary to assume the investment by non-sequential years (e.g. $S = \{1, 2, 5\}$). The following provides a conceptualization of the core for practical situations. We can rewrite the three conditions in which the payoff vector belongs to the core as below and make implications for practice.

$$\text{Group rationality } \mathbf{x}(N) = d^*(N) = C \tag{28}$$

$$\text{Individual rationality } x_i \leq d^*(\{i\}) \tag{29}$$

$$\text{Coalitional rationality } \mathbf{x}(S) \leq d^*(S) \text{ for all } S \in 2^N \backslash \{\emptyset\} \tag{30}$$

Because depreciation is defined as the systematic allocation of the original cost of an asset (C) over the expected useful life of it (n years), a payoff vector given by depreciation always satisfies group rationality (28). Individual rationality (29) means that the depreciation amount in each year should be equal to or less than the amount of the lease payments in a one-year contract. Coalitional rationality (30) means that the amount of accumulated depreciation for l_s years, $\mathbf{x}(S) = d^*(S)$, should be equal to or less than the amount of lease payments in an s-year contract, that is, $C - f(l_S)$. As we are discussing only $S \in \mathcal{N}$, the length of a lease contract is equal to the cardinality of S. Then from here, we denote it just as s instead of l_S.

We have already confirmed mathematically that **SL** is an element of the core. Figure 1 summarizes some practical implications of the result. In Figure 1, the horizontal axis represents year t and the vertical axis represents the asset's value. Now, we assume that our market value function is continuous, and simply denoted as

$$V = f(t). \tag{31}$$

We draw this as curve A in Figure 1. When $t = 0$, that is, at the time of acquisition, the asset's market value is equal to the acquisition cost C. Curve A represents changes in the asset's market value as it ages.

We call a line that connects $(0, C)$ and $(n, 0)$ as a depreciation line and any depreciation line satisfies group rationality. If the slope of a depreciation

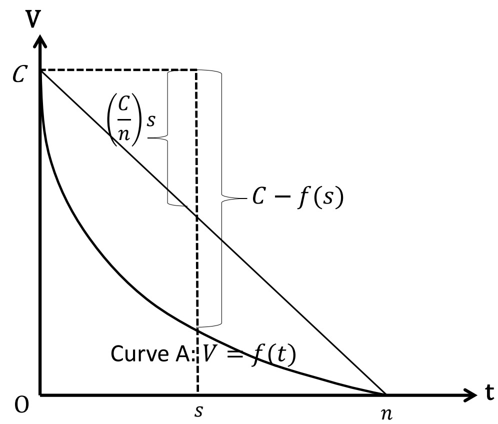

Figure 1: Straight line method

line is gentler than that of curve A in the first year, individual rationality is satisfied. From the condition of the market value function (12), we can confirm that the slope of the straight-line method is always gentler than that of curve A in the first year. We can also visually verify the coalitional rationality of the straight-line method. From Figure 1, we can confirm that the amount of accumulated depreciation by the straight-line method for s-years, that is, $\mathbf{x}(S) = \frac{C}{n}s$, is less than the total lease payment for s-years, or $d^*(S) = C - f(s)$.

Next, we draw the core of the game for $\mathcal{N}$. While coalitional rationality gives us an upper limit of accumulated depreciation amounts, a lower limit could exist.

Proposition 6 *When the payoff vector satisfies the three conditions, it also satisfies the inequation:*

$$\mathbf{x}(S) \geq f(n-s)$$

Proof 8 *From group rationality,*

$$x_1 + x_2 + \cdots + x_n = C \quad \cdots \quad (d)$$

From coalitional rationality,

$$x_2 + x_3 + \cdots + x_n \leq C - f(n-1) \quad \cdots \quad (e)$$

From (d) and (e), we derive

$$x_1 \geq C - \{C - f(n-1)\} = f(n-1) = f(n-1) - f(n),$$

because we have $f(n) = 0$.

Next, when $x_1 = f(n-1)$, *group rationality indicates*

$$x_2 + x_3 + \cdots + x_n = C - f(n-1) \quad \cdots \quad (f)$$

and coalitional rationality indicates

$$x_3 + x_4 + \cdots + x_n \leq C - f(n-2) \quad \cdots \quad (g)$$

From (f) and (g), we derive

$$x_2 \geq C - f(n-1) - \{C - f(n-2)\} = f(n-2) - f(n-1)$$

and so on. We can generalize the lower limit of each year as

$$x_i \geq f(n-i) - f(n-i+1)$$

We calculate the sum payoff of coalition S.

$$\mathbf{x}(S) \geq f(n-s) \quad \square$$

Then, we introduce curve B, which represents the function

$$V = C - f(n-t). \tag{32}$$

Figure 2 illustrates the core of the game for Seq^N. Curve B represents the lower limit of accumulated depreciation amounts. When a depreciation line is inside the shape of the lens surrounded by curve A and curve B, the method's payoff is an element of the core. We can then determine that stakeholders accept the firm's method. We have hints that the payoff vectors given by the declining-balance (decreasing) and sinking-fund (increasing) methods are also elements of the core on the condition that the depreciation line is inside the lens in Figure 2.

Given the factors discussed previously, how can players accept a fair value measurement in this game? International Accounting Standards Board also

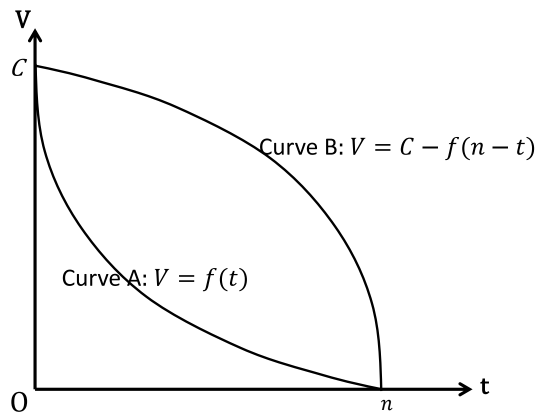

Figure 2: The core of the depreciation game

permits revaluations of tangible fixed assets at fair value, which is a broad concept that includes the market value (IASB 2013 [13]). Here, we define fair value as market value. Then we need to reform the restricted cost function, i.e. lease payment function for solving this problem.

We defined the lease payment function as restricted cost function in (10). Under real k-year lease contract, the leasing company predicts market value after k years, and the value of assets in the future is conservatively estimated to determine lease payments. Then we re-write the restricted function as follows:

$$d^*(S) = C - (1 - a_k)g(k) \quad s.t. \quad 0 < a_1 < a_2 < \cdots < a_{n-1} < 1. \tag{33}$$

where k is an integer from 1 to n, $g(k)$ denotes the leasing company's predicted market value after k years, a_k is the risk premium which increases over time.

Then, we define the payoff vector of the fair value measurement below.

Definition 16 *Fair value measurement*
The payoff vector of the fair value measurement is

$$FV = (g(0) - g(1), g(1) - g(2), ..., g(n-1) - g(n)).$$

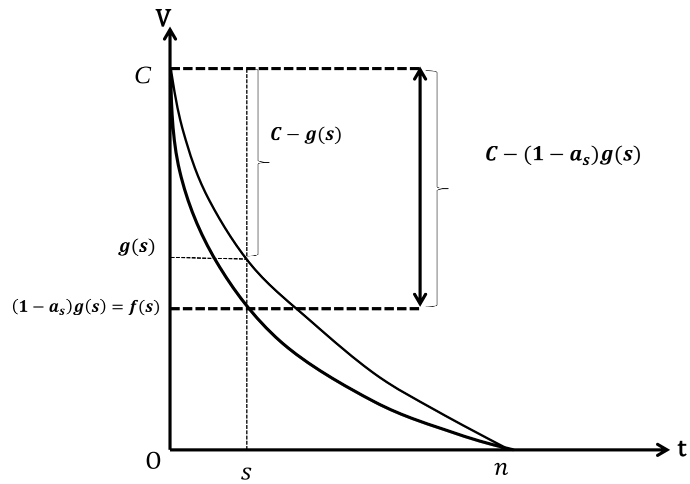

Figure 3: Fair value measurement

Recall that $f(k) = (1 - a_k)g(k)$, and $g(k)$ denotes the lease company's prediction of market value of the asset after k years of use and a_k is the risk premium. If the lease company's prediction comes true and the lessee can know the current market value, the payoff vector of the fair value measurement also belongs to the core because $C - (1 - a_s)g(s) \geq C - g(s)$ (See Figure 3).

However, the fair value measurement is accepted only if a firm can reliably obtain the market value of an asset; otherwise, the payoff vectors are not guaranteed to belong to the core.

5 Other solutions of this game

5.1 The Shapley value

In this section, we use the original Shapley value[20] as a solution concept and examine its properties.

Definition 17 *The Shapley value*
(N, v) *is a game in characteristic function form with transferable utility.*

The Shapley value of this game is as follows. For all $i \in N$,

$$\phi_i(v) = \sum_{S \subset N\ i \notin S} \frac{(s-1)!(n-s)!}{n!}\{v(S) - v(S - \{i\})\},$$

where s is the cardinality of a coalition S and n is the cardinality of N.

As our depreciation game is a cost game, we convert the cost function into a characteristic function.

$$v(S) = \sum_{i \in S} d^*(i) - d(S) = sd^*(1) - d(S). \tag{34}$$

After we obtain the set of the Shapley value, we re-convert it into that of cost burden.

$$x_i = d^*(1) - \phi_i(v). \tag{35}$$

[Example 1] When $N = \{1, 2, 3\}$ and $f(0) = 150 > f(1) = 60 > f(2) = 20 > f(3) = 0$, we obtain the Shapley values and the amount of cost burden vectors given by them as shown in Table 1[9].

Table 1: The Shapley values and cost allocation when $n = 3$.

	Player 1	Player 2	Player 3
$\phi_i(v)$	95/3	170/3	95/3
Cost burden	175/3	100/3	175/3

Shapley(1971) [21] shows the theorem that the core of every concave cost function is nonempty and contains the Shapley value. We showed a sufficient condition for the concavity of our cost function in Section 3.

Proposition 7 *In the case of $|N| = 3$, the Shapley value of this game is equal to the payoff vector given by the SL method if and only if $f(0) - f(1) = f(1) - f(2)$ holds.*

[9] See Appendix more details.

Proof 9 *If the payoff vector given by the Shapley value is equal to the payoff vector of the SL method, every element is the same amount. Therefore,*

$$\begin{aligned}
&(4d^*(1) + d^*(2) - 2d^*(3))/6 = (10d^*(1) - 2d^*(2) - 2d^*(3))/6 \\
\iff &4(f(0) - f(1)) + (f(0) - f(2)) - 2(f(0) - f(3)) \\
&= 10(f(0) - f(1)) - 2(f(0) - f(2)) - 2(f(0) - f(3)) \\
\iff &3f(0) - 4f(1) - f(2) + 2f(3) = 6f(0) - 10f(1) + 2f(2) + 2f(3) \\
\iff &3f(0) - 6f(1) + 3f(2) = 0 \\
\iff &f(0) + f(2) = 2f(1) \\
\iff &f(0) - f(1) = f(1) - f(2). \square
\end{aligned}$$

For the $N = \{1, 2, 3\}$ game, we have the idea that $x_1 = x_3 \geq x_2$, from (12). Intuitively, we can predict that the burden of player 2 is less than the burdens of players 1 and 3 because player 2 can make coalitions with both player 1 and player 3.

5.2 The nucleolus

Next, we use the nucleolus[19] as a solution concept and examine its properties. Although our game is a cost game, we consider the characteristic function shown in (34).

Definition 18 *The excess of S with respect to $\mathbf{x}$*
We define the excess of S with respect to $\mathbf{x}$ as $e(S, \mathbf{x}) = v(S) - \mathbf{x}(S)$, $S \subset N$ and $S \neq \emptyset, N$.

Definition 19 *The vector of the excess*
Let $\theta(\mathbf{x})$ be a vector in $\mathbb{R}^{2^n-2}$. The elements of $\theta(\mathbf{x})$ are the numbers $v(S) - \mathbf{x}(S)$, arranged according to their magnitude. We call this as the vector of the excess.

Definition 20 *The lexicographical order*
When we are given two vectors $\mathbf{x} = (x_1, ..., x_q)$ and $\mathbf{y} = (y_1, ..., y_q)$, we say that $\mathbf{x}$ is lexicographically smaller than $\mathbf{y}$ if there is some integer k, $1 \leq k \leq q$, such that

$$x_l = y_l \text{ for } 1 \leq l \leq k,$$
$$x_k < y_k.$$

We denote this relation as $\mathbf{x} <_L \mathbf{y}$.

Definition 21 *Acceptance*
If $\theta(\mathbf{x}) <_L \theta(\mathbf{y})$ *, we say that* $\mathbf{x}$ *is more acceptable than* $\mathbf{y}$*.*

Definition 22 *Nucleolus*
The nucleolus of the characteristic game (N, v) *is the set of imputations that satisfies:*

$$\mathcal{N}(v) = \{x \in A | \theta(\mathbf{x}) \leq_L \theta(\mathbf{y}), \ \forall \mathbf{y} \in A\} \tag{36}$$

[Example 2] We use the same case as Example 1, $N = \{1, 2, 3\}$ and $f(0) = 150 > f(1) = 60 > f(2) = 20 > f(3) = 0$. We seek an imputation that minimizes the maximum excess of all S except for $\emptyset$ and N. Then, we obtain $\mathbf{y}^* = (35, 50, 35)$ that is the nucleolus and the set of cost burden $(55, 40, 55)$[10]. In this case, the payoff vector given by the nucleolus is a menber of the core.

Proposition 8 *Under the condition* $|N| = 3$*, if* $f(1) - f(2) \geq C/3$ *holds, the payoff vector given by nucleolus is equal to that given by the SL method. If* $f(1) - f(2) < C/3$ *holds, on the other hand, the payoff vector given by nucleolus is not equal to that given by the SL method.*

Proof 10 *In general when* $|N| = 3$*, the market value function is* $f(0) = C > f(1) > f(2) > f(3) = 0$ *and it satisfies (12). Then we can re-write the characteristic function in general when* $|N| = 3$ *as follows:*

$v(\{1\}) = v(\{2\}) = v(\{3\}) = d^*(1) - d^*(1) = 0;$
$v(\{1, 2\}) = v(\{2, 3\}) = 2d^*(1) - d^*(2)$
$= 2(f(0) - f(1)) - (f(0) - f(2)) = C - 2f(1) + f(2);$
$v(\{1, 3\}) = 0;$
$v(\{1, 2, 3\}) = 3d^*(1) - d^*(3) = 3(f(0) - f(1)) - (f(0) - f(3)) = 2C - 3f(1).$

Therefore, we can also re-write the excess of $\mathbf{y}$ *for* $S \in 2^N$ *except for* $\emptyset$ *is as*

[10]See Appendix for more details.

follows:

$$
\begin{aligned}
&e(\{1\},\mathbf{y}) = -y_1;\\
&e(\{2\},\mathbf{y}) = -y_2;\\
&e(\{3\},\mathbf{y}) = -y_3;\\
&e(\{1,2\},\mathbf{y}) = v(\{1,2\}) - (y_1+y_2) = C - 2f(1) + f(2) - (v(\{1,2,3\}) - y_3)\\
&= C - 2f(1) + f(2) - (2C - 3f(1) - y_3) = -C + f(1) + f(2) + y_3;\\
&e(\{1,3\},\mathbf{y}) = v(\{1,3\}) - (y_1+y_3) = 0 - (v(\{1,2,3\}) - y_2) = -2C + 3f(1) + y_2;\\
&e(\{2,3\},\mathbf{y}) = v(\{2,3\}) - (y_2+y_3) = C - 2f(1) + f(2) - (v(\{1,2,3\}) - y_1)\\
&= C - 2f(1) + f(2) - (2C - 3f(1) - y_1) = -C + f(1) + f(2) + y_1.
\end{aligned}
$$

First, we make the lexicographical order of the payoff vector given by SL method, i.e. $y_1 = y_2 = y_3 = \frac{2C-3f(1)}{3} (= \frac{v(\{1,2,3\})}{3})$.

$$
\begin{aligned}
&e(\{i\},\mathbf{y}) = -\frac{2}{3}C + f(1)\\
&e(\{1,2\},\mathbf{y}) = e(\{2,3\},\mathbf{y}) = -C + f(1) + f(2) + \frac{2}{3}C - f(1) = -\frac{1}{3}C + f(2);\\
&e(\{1,3\},\mathbf{y}) = -2C + 3f(1) + \frac{2}{3}C - f(1) = -\frac{4}{3}C + 2f(1);
\end{aligned}
$$

Because $\mathbf{SL} \in Core(d)$*, we can induce* $e(S,\mathbf{y}) \le 0, \forall S$*. Then we make the lexicographical order of the excess.*
(case 1) When $f(1) - f(2) \ge C/3$,

$$e(\{i\},\mathbf{y}) > e(\{1,2\},\mathbf{y}) > e(\{1,3\},\mathbf{y}). \tag{37}$$

(case 2) When $f(1) - f(2) < C/3$,

$$e(\{1,2\},\mathbf{y}) > e(\{i\},\mathbf{y}) > e(\{1,3\},\mathbf{y}). \tag{38}$$

If the payoff vector given by $\mathbf{SL}$ *is not the nucleolus, there is some* $\epsilon > 0$ *which satisfies* $y_1 = y_3 = \frac{2C-3f(1)}{3} - \frac{1}{2}\epsilon$, $y_2 = \frac{2C-3f(1)}{3} + \epsilon$ *and make the highest excesses smaller* [11]*. Then we make the lexicographical order of the new payoff vector as follows:*

[11] We can assume that the nucleolus for $|N| = 3$ satisfies anonymity and player 1 and 3 are anonymous players. cf. Potters (1991)[16].

$$e(\{1\}, \mathbf{y}') = e(\{3\}, \mathbf{y}') = -\frac{2}{3}C + f(1) + \frac{1}{2}\epsilon$$
$$e(\{2\}, \mathbf{y}') = -\frac{2}{3}C + f(1) - \epsilon$$
$$e(\{1,2\}, \mathbf{y}') = e(\{2,3\}, \mathbf{y}') = -\frac{1}{3}C + f(1) - \frac{1}{2}\epsilon;$$
$$e(\{1,3\}, \mathbf{y}') = -\frac{4}{3}C + 2f(1) + \epsilon;$$

(case 1) When $f(1) - f(2) \geq C/3$*,*

$$e(\{1\}, \mathbf{y}') = e(\{3\}, \mathbf{y}') > e(\{i\}, \mathbf{y})$$

always holds. This means that there is no ϵ *that make the highest excesses with (37) smaller, i.e., the payoff vector given by* **SL** *is the nucleolus.*

(case 2) When $f(1) - f(2) < C/3$*, we can find some* $\epsilon > 0$ *which satisfies*

$$e(\{1,2\}, \mathbf{y}) > e(\{1,2\}, \mathbf{y}').$$

This means that there is an ϵ *that makes the highest excesses with (38) smaller, i.e., the payoff vector given by* **SL** *is not the nucleolus.* □

Theorem 2 *In the case of* $|N| = 3$ *and* $f(0) - f(1) = f(1) - f(2)$*, the depreciation methods which are given by the Shapley value and the nucleolus are the same with the SL method.*

Proof 11 *When* $f(0) - f(1) = f(1) - f(2)$*, inequation* $f(1) - f(2) \geq C/3$ *in Proposition 8 is satisfied. Then, it is obvious from Proposition 7 and 8.* □

6 Conclusion

In this study, we develop the model by corporate game theory to analyze why stakeholders accept conventional depreciation methods. First, we considered the core concept as the solution to our game because it provides the scope of acceptable payoffs for the players. If a payoff vector of a certain accounting rule is an element of the core, we determine that the firm's stakeholders accept the rule. Depreciation methods given by the core suggest that stakeholders accept them because they reflect the cost savings of purchase compared to leases. We found that (1) the SL method is always accepted, (2) stakeholders' acceptance of the declining-balance method and sinking-fund method depends on the degree of decline in the fair value of the asset, and (3) stakeholders accept the fair value measurement when the firm can reliably measure the fair value of the asset.

Second, we examined the condition that the depreciation methods which are given by the Shapley value and the nucleolus are the same with the SL method. This finding shows the specialty of our game. This study's results can help accounting standards setters establish policies, especially the IASB, the body responsible for developing internationally accepted accounting standards. However, this area requires further study. As only a few alternatives have been used in real business, the core is too wide and the Shapley value and the nucleolus are too narrow for an explanation of depreciation.

In this paper, we borrow the existing solution concepts to analyze the accounting conventions. This acts as a stepping stone to seeking for original solution concept of accounting rules, which is a more narrow concept than the core.

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Appendix

Here is the details for examples.
[Example 1] When $N = \{1, 2, 3\}$ and $f(0) = 150 > f(1) = 60 > f(2) = 20 > f(3) = 0$, the characteristic function is as follows:

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = d^*(1) - d^*(1) = 0;$$
$$v(\{1, 2\}) = v(\{2, 3\}) = 2d^*(1) - d^*(2) = 50;$$
$$v(\{1, 3\}) = 0;$$
$$v(\{1, 2, 3\}) = 3d^*(1) - d^*(3) = 120.$$

The marginal contribution and the Shapley value of this case are shown in Table 2.

[Example 2] Then, we seek the payoff vector given by the nucleolus for our depreciation game. We use the same case with subsection 5.1, $N = \{1, 2, 3\}$ and $f(0) = 150 > f(1) = 60 > f(2) = 20 > f(3) = 0$.

The characteristic function of our depreciation game was as follows:

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = d^*(1) - d^*(1) = 0;$$
$$v(\{1, 2\}) = v(\{2, 3\}) = 2d^*(1) - d^*(2) = 50;$$
$$v(\{1, 3\}) = 0;$$
$$v(\{1, 2, 3\}) = 3d^*(1) - d^*(3) = 120.$$

Table 2: The Shapley values and cost allocation when $n = 3$.

Order	Player 1	Player 2	Player 3
(1,2,3)	0	$2d^*(1) - d^*(2) = 50$	$d^*(1) + d^*(2) - d^*(3) = 70$
(1,3,2)	0	$3d^*(1) - d^*(3) = 120$	0
(2,1,3)	$2d^*(1) - d^*(2) = 50$	0	$d^*(1) + d^*(2) - d^*(3) = 70$
(2,3,1)	$d^*(1) + d^*(2) - d^*(3) = 70$	0	$2d^*(1) - d^*(2) = 50$
(3,1,2)	0	$3d^*(1) - d^*(3) = 120$	0
(3,2,1)	$d^*(1) + d^*(2) - d^*(3) = 70$	$2d^*(1) - d^*(2) = 50$	0
$\phi_i(v)$	95/3	170/3	95/3
Cost burden	175/3	100/3	175/3

Here, we consider a benefit-based payoff vector $\mathbf{y} = (y_i)_{i \in N}$ not a cost-based one. Consider $\mathbf{y} \in A$. The excess of $\mathbf{y}$ for $S \in 2^N$ except for $\emptyset$ and N is as follows:

$$
\begin{aligned}
e(\{1\}, \mathbf{y}) &= v(\{1\}) - y_1 = -y_1 \\
e(\{2\}, \mathbf{y}) &= v(\{2\}) - y_2 = -y_2 \\
e(\{3\}, \mathbf{y}) &= v(\{3\}) - y_3 = -y_3 \\
e(\{1,2\}, \mathbf{y}) &= v(\{1,2\}) - (y1 + y_2) = 50 - (y_1 + y_2) \\
e(\{1,3\}, \mathbf{y}) &= v(\{1,3\}) - (y_1 + y_3) = -(y_1 + y_3) \\
e(\{2,3\}, \mathbf{y}) &= v(\{2,3\}) - (y_2 + y_3) = 50 - (y_2 + y_3)
\end{aligned}
$$

$\mathbf{y}$ is imputation, therefore $y_1 + y_2 + y_3 = v(\{1,2,3\}) = 120$. Then we rewrite the former three inequalities.

$$
\begin{aligned}
e(\{1,2\}, \mathbf{y}) &= y_3 - 70 \\
e(\{1,3\}, \mathbf{y}) &= y_2 - 120 \\
e(\{2,3\}, \mathbf{y}) &= y_1 - 70
\end{aligned}
$$

According to Definition 22, an imputation in the nucleolus minimizes the maximum excess of all S except for $\emptyset$ and N. Therefore, we can solve the linear programming problem as follows:

$$\begin{aligned} &\text{minimize} \quad \epsilon \\ &\text{subject to} - y_1 \leq \epsilon,\ -y_2 \leq \epsilon,\ -y_3 \leq \epsilon, \\ &\qquad\qquad y_3 - 70 \leq \epsilon,\ y_2 - 120 \leq \epsilon,\ y_1 - 70 \leq \epsilon \end{aligned}$$

The minimum value of ϵ of this problem is 35, then, $\mathbf{y}^* = (35, 50, 35)$ is the nucleolus.

As is the case with the Shapley value, we need to re-convert the payoff vector given by the nucleolus into the allocation.

$$x_i^* = d^*(1) - y_i^* \qquad (39)$$

where x_i^* is a payoff for i given by the nucleolus.

According to (39), we obtain the vector $(55, 40, 55)$ as the solution, which is a member of the core.