Dual Monotonic Consistency and Deferred Acceptance

Manabu Toda
School of Social Sciences, Waseda University

Abstract

This paper studies axiomatic characterizations of the student-proposing deferred acceptance mechanism in two-sided matching markets with variable populations. For one-to-one matching, I introduce a consistency-type requirement, called *dual monotonic consistency*. It requires that, after passing to a subproblem induced by the selected matching, the remaining students should not be worse off and the remaining colleges should not be better off, with respect to the usual student–college lattice order over matchings. The main result shows that the student-proposing deferred acceptance mechanism is the unique mechanism satisfying individual rationality, mutual-best, and dual monotonic consistency. A parallel characterization is obtained by replacing this consistency requirement with a dual version of weak Maskin monotonicity. The paper further develops the same approach for many-to-one matching problems, where colleges may have responsive choice functions and capacities greater than one. The many-to-one extension clarifies how the dual order and the consistency requirement should be formulated when colleges compare sets of students rather than individual partners.

Keywords: deferred acceptance; stability; consistency; Maskin monotonicity; many-to-one matching; variable population.
JEL classification: C78, D47, D71.

1 Motivation

Deferred acceptance (DA) is one of the central mechanisms in matching theory. A large literature characterizes DA by combining stability with additional requirements such as strategy-proofness, weak consistency, or weak Maskin monotonicity. Under a fixed population, Alcalde and Barberà (1994) characterize DA by stability and strategy-proofness. Afacan and Dur (2017) show an incompatibility between stability and consistency, while Chen (2017) studies weaker consistency requirements for DA. Morrill (2013) provides a characterization based on stability and weak Maskin monotonicity.

The present paper takes a different route. It allows the population of agents to vary across problems and asks whether weaker requirements than stability can still identify the student-proposing DA mechanism. The first contribution is a characterization of DA by individual rationality, a mutual-best requirement, and a new consistency axiom. The second contribution is a parallel characterization based on a dual weak Maskin monotonicity condition. It also reports extending these results from one-to-one matching to many-to-one matching, which is the natural environment for college admissions and school choice.

2 One-to-one matching model

Let S be a finite set of students and C a finite set of colleges. Each agent has a strict preference relation over agents on the opposite side and the option of remaining unmatched, denoted by $\varnothing$. A matching is a function μ on $S \cup C$ such that $\mu(a)$ is either an agent on the opposite side of a or $\varnothing$, and $\mu(\mu(a)) = a$ whenever $\mu(a) \neq \varnothing$.

A matching is *individually rational* if every agent weakly prefers her assignment to remaining unmatched. A student–college pair (s, c) blocks μ if $c \succ_s \mu(s)$ and $s \succ_c \mu(c)$. A matching is stable if it is individually rational and has no blocking pair. The student-proposing DA mechanism selects the student-optimal stable matching whenever a stable matching exists.

A mechanism φ assigns a matching $\varphi(S, C, \succ_{S\cup C})$ to every problem. The domain considered here is a variable-population domain: there is no fixed upper bound on the number of students or colleges. This feature is essential for the consistency-based characterization below.

3 Student-monotonic and dual monotonic consistency

For a problem $(S, C, \succ_{S\cup C})$ and a matching μ, a subproblem $(S', C', \succ_{S'\cup C'})$ is *induced by* μ if each remaining agent in $S' \cup C'$ is matched either to another remaining agent or to $\varnothing$ under μ. Let $\mu_{S'\cup C'}$ denote the restriction of μ to the subproblem.

Student-monotonic consistency requires that, after some agents leave and the induced subproblem is considered, no remaining student becomes worse off. Together with stability, this property characterizes student-proposing DA on the variable-population domain. However, the same statement fails on the fixed-population domain.

The main new axiom strengthens this idea by using the familiar duality between the student side and the college side. Define the order $\succeq_{S\cup C}$ over matchings by

$$\mu' \succeq_{S\cup C} \mu \quad \Longleftrightarrow \quad \left[\mu'(s) \succeq_s \mu(s) \text{ for all } s \in S\right] \text{ and } \left[\mu(c) \succeq_c \mu'(c) \text{ for all } c \in C\right].$$

Thus μ' is above μ in the student order if students weakly prefer μ' to μ and colleges weakly prefer μ to μ'. This is the standard order underlying the lattice structure of stable matchings.

A mechanism satisfies *dual monotonic consistency* if, whenever $\mu = \varphi(S, C, \succ_{S\cup C})$ and $(S', C', \succ_{S'\cup C'})$ is an induced subproblem at μ, the matching $\mu' = \varphi(S', C', \succ_{S'\cup C'})$ satisfies

$$\mu' \succeq_{S'\cup C'} \mu_{S'\cup C'}.$$

In words, after taking an induced subproblem, remaining students should not become worse off and remaining colleges should not become better off. This additional college-side restriction is what turns a one-sided monotonicity idea into a dual monotonicity requirement.

4 Main results for one-to-one matching

The first main result replaces stability by two weaker properties. A mechanism is individually rational if it always selects an individually rational matching. It satisfies *mutual-best* if every pair consisting of a student and a college who rank each other first is matched by the mechanism.

Theorem 1 (One-to-one characterization)**.** *The student-proposing deferred acceptance mecha-*

nism is the unique mechanism satisfying individual rationality, mutual-best, and dual monotonic consistency on the variable-population domain of one-to-one matching problems.

This theorem shows that full stability need not be imposed directly. Instead, stability is recovered from individual rationality, mutual-best, and the dual consistency requirement. The dual part of the axiom plays an essential role: individual rationality, mutual-best, and student-monotonic consistency alone do not imply stability.

A parallel result is obtained by replacing consistency-type monotonicity with a Maskin-type monotonicity condition. Given a selected matching μ, a preference profile is a monotonic transformation at μ if, for every agent, the set of alternatives ranked above the assigned partner under the transformed preference is contained in the corresponding set under the original preference. Dual weak Maskin monotonicity requires that, after such transformations, the newly selected matching is weakly above the original one in the student–college order.

Theorem 2 (Dual weak Maskin monotonicity)**.** *The student-proposing deferred acceptance mechanism is the unique mechanism satisfying individual rationality, mutual-best, and dual weak Maskin monotonicity.*

5 Extension to many-to-one matching

The results in one-to-one matching problems are extended to many-to-one matching problems. In this model, each college c has a capacity q_c and may be matched with a set of students of size at most q_c. Students continue to have strict preferences over colleges and remaining unmatched. Colleges are represented by choice functions over sets of acceptable students. A natural benchmark is the responsive domain: each college has a strict ranking of individual students and chooses its most preferred acceptable students up to capacity.

The main conceptual issue is that the college side now compares sets of students rather than single partners. Therefore the dual order must be reformulated. On the student side, the order remains pointwise: students compare assigned colleges. On the college side, the order is induced by the college's choice function or, in the responsive case, by the responsive preference over sets. A many-to-one version of dual monotonic consistency then requires that, after taking an induced subproblem, remaining students should not be worse off and remaining colleges should not receive a strictly better set of students.

The many-to-one result is the following. On the responsive many-to-one domain with variable populations and capacities, the student-proposing DA mechanism is characterized by the many-to-one analogues of individual rationality, mutual-best, and dual monotonic consistency. A further question is whether the same approach can be pushed beyond responsiveness to substitutable choice functions, possibly under the law of aggregate demand. This is a promising but more delicate extension, because the lattice and comparative-statics properties used in the one-to-one and responsive many-to-one cases become more subtle.

References

Afacan, M. O. and U. Dur (2017). Incompatibility between stability and consistency. *Games and Economic Behavior*, 103, 26–36.

Alcalde, J. and S. Barberà (1994). Top dominance and the possibility of strategy-proof stable solutions to matching problems. *Economic Theory*, 4, 417–435.

Chen, Y. (2017). New axioms for deferred acceptance. *Social Choice and Welfare*, 48, 393–408.

Morrill, T. (2013). An alternative characterization of the deferred acceptance algorithm. *International Journal of Game Theory*, 42, 19–28.