Uncertain Quality Evaluation in Procurement Auctions

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April 14, 2026

Abstract

This study examines bidder competition over price and quality in procurement auctions in which subjective quality evaluation creates uncertainty. We derive the symmetric Bayesian Nash equilibrium for a first-score auction, in which bidders submit cautious bids because the determination of a winner depends on chance. In a second-score auction, a truth-telling equilibrium exists and evaluation uncertainty does not affect bidding behavior. As the quality evaluation becomes more precise, the expected score and quality improve and expected price decreases. Moreover, the second-score auction is more efficient and achieves a better expected score, price, and quality than the first-score auction.

Keywords: multidimensional bidding, scoring auctions, procurement, quality evaluation

JEL codes: D44, H57, L13

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1 Introduction

In public-works procurement which amounts to tens of billions of dollars or more annually, governments are concerned not only about project cost but also nonmonetary attributes including noise reduction, time to completion, design, and quality of materials[1]. Consequently, an innovative contract design called a *scoring auction* has started to take hold worldwide to deal with these complex conditions. In a scoring auction, the government first announces the scoring rule by which they will rank the offers made by bidders. Bidders then submit not only price bids but also quality proposals, which include information on the technology used to perform the work, time to completion, and other work performance attributes. In the auction, the government evaluates the proposals made by bidders and assigns quality scores. Then, the winner is the bidder with the best combination of price and quality score. This type of multidimensional-bid auction is also used in cash-royalty auctions for oil lease contracts and bank resolution auctions and so has broader applicability.

The outcomes of scoring auctions are heavily influenced by the auctioneer's subjective quality evaluation. For example, in public procurement, suppose that a municipality desires an expedited completion of a street project that requires working at night. In this case, bidders might be asked to submit ideas that not only accelerate the completion timeline but also ensure pedestrian safety. To prevent pedestrians from falling, one firm might suggest laying steel plates on the ground, while another might suggest installing lighting equipment. Since there is no objective criterion to determine which idea is superior and to what degree, these ideas must be evaluated subjectively by reviewers. From the bidder perspective, this subjective evaluation generates uncertainty regarding the final evaluation score, which causes the allocation of contracts to occur at least to some degree by chance (Takahashi, 2018). Similar phenomena arise within markets other than public procurement (Krasnokutskaya, Song and Tang, 2020, Kong, Perrigne

[1]Public-works spending typically accounts for 2.3 percent of GDP annually on average in the US, 5 percent in European countries, and about 8 percent in China. See `https://www.brookings.edu/articles/how-federal-infrastructure-investment-can-put-america-to-work`.

and Vuong, 2022, and Allen, Clark, Hickman and Richert, 2023), so it is important to incorporate such uncertainty into the analysis of scoring auctions.

In this study, we theoretically analyze scoring auctions in which quality evaluation is uncertain, a situation that we formalize by introducing random noise into the assessment of quality bids. Building on Che (1993), we consider multi-dimensional-bid procurement auctions in which bidders select their price and quality in the face of this uncertainty and examine bidding behavior and auction outcomes for two auction mechanisms: first-score (FS) and second-score (SS) auctions. These auction formats are analogous to first- and second-price auctions in standard price-only auctions. In an FS auction, the winning bidder delivers quality at the price specified in its bid. In an SS auction, the winning bidder delivers the quality promised in its bid at the price that matches the score of the most-competitive rival.[2]

The contribution of our paper is two-fold. First, we characterize the symmetric Bayesian Nash equilibrium in both FS and SS auctions in the presence of uncertain quality evaluation. We show that the bidders' optimal quality bid is determined independently of the auction formats and score, which is consistent with Che (1993). We then show that under fairly general conditions, a truth-telling equilibrium exists in the SS auction so that uncertain quality evaluation does not influence the bidding incentive. As for the FS auction format, we present an explicit equilibrium under certain specifications for quality evaluation uncertainty and find that when evaluation uncertainty exists, bidders shade their score bids for all types, including the least efficient type. This is because the determination of the winner depends to some extent on chance, which reduces the incentive for bidders to submit aggressive bids and which, in turn, weakens competition among bidders.

The second contribution of our paper is to demonstrate that the equivalence theorem between FS and SS auctions fails even though our model has an independent private value setting. Specifically, we show that the SS auction performs better than the FS

[2]The SS auction here is an identical mechanism to the generalized second-price auction employed in sponsored search advertising.

auction in terms of efficiency, expected score, expected quality, and expected price. To show this, we first examine the comparative statics of the auction performance regarding the precision of the quality evaluation. In the FS auction, as the evaluation of quality becomes more precise, bidders become more aggressive and submit lower price bids. This occurs because the increased precision reduces the degree of chance involved in determining the winner, which increases bidding competition. Additionally, increased precision in the evaluation of quality increases the expected social surplus and the expected score (the buyer's payoff). Although the quality bids are not directly affected by evaluation uncertainty, the expected quality also increases with precision because the more precise evaluation chooses the efficient bidder as the winner, which improves the expected quality.

The effect of evaluation uncertainty on the expected price is characterized under an additional condition. More precise evaluations cause auctions to be more competitive which lowers price bids, but the win by a more efficient bidder increases payments as quality improves. The relative magnitude of these two effects determines the effect on the expected price. We show that when the virtual cost, which is the cost taking into account the bidder's optimal quality and information rent, is monotone in their type, then the expected price decreases as the efficiency of allocation improves. These properties can be similarly established in the SS auction.

Based on the comparative statics on uncertainty in quality evaluation, we show that under a regularity condition, the SS auction improves social surplus, expected score, expected price, and expected quality as compared with the FS auction. The intuition behind this is that while bidders bid truthfully in the SS auction, they shade score bids in the FS auction. Under the specification in which we obtain an explicit equilibrium for the FS auction, the bids are closer to each other in the FS auction than in the SS auction. The closer bids imply that the determination of the winner is affected more by evaluation uncertainty, and the allocation is more random in the FS auction than the SS auction. Thus, even with the same evaluation uncertainty, the SS auction is more efficient than the FS auction in equilibrium. Hence, analogous to the comparative statics regarding the precision of quality evaluation, the SS auction achieves a higher expected score and quality than the FS auction. The expected price is also lower in the SS than the FS auction under the condition that virtual cost function is monotone. These results imply that when uncertainty in quality evaluation is inevitable, the SS auction is the better mechanism for both social welfare and buyer payoff.

1.1 Related Literature

This paper contributes to the theoretical analysis on scoring auctions introduced by Che (1993) which to date has focused on quasilinear scoring auctions in which price and quality are additively separable and the scoring rule is linear in price. While Che (1993)'s seminal work has been extended to cases of interdependent cost (Branco, 1997), multidimensional signals (Asker and Cantillon, 2008), multidimensional quality (Nishimura, 2015), and non-quasilinear scoring rules (Dastidar, 2014; Hanazono, Hirose, Nakabayashi and Tsuruoka, 2020, Hanazono, Nakabayashi, Sano and Tsuruoka, 2024), and researchers have compared the equilibrium outcomes of scoring auctions and alternative mechanisms (see, for example, Asker and Cantillon, 2008, 2010; Awaya, Fujiwara and Szabo, 2025; and Sano, 2023), all these studies assume quality evaluation that is certain, through "known scoring rules". Despite the importance of uncertain scoring rules in real-world situations, the theoretical literature remains scarce to date. To the best of our knowledge, this is the first study to characterize the equilibrium of scoring auctions when quality evaluation is uncertain and then compare the performance between FS and SS auctions.

There are a few studies on scoring auctions with uncertain quality evaluation. A closely related study is Takahashi (2018), who develops a structural scoring auction model where bidders face uncertainty through noises on quality bids. The paper empirically examines procurement auctions that utilize a price per quality ratio as the scoring rule and, using scoring auction data from the Florida Department of Transportation (FDOT), evidence is shown of substantial differences in the quality scores among reviewers for a given quality bid, which implies the existence of evaluation uncertainty.

A numerical exercise shows that as the degree of evaluation uncertainty increases, the equilibrium price and quality bids rise on average. In another related paper, Ortner, Chassang, Kawai and Nakabayashi (2025) consider repeated procurement auctions and examine the effect of subjective quality evaluation in scoring auctions on bidder collusion.

Unlike standard auctions, the bidder with the highest bid does not always win in our model, which makes it difficult to provide a general characterization of the FS auction equilibrium. Despite the vast literature on auction theory, there remains very little theoretical analysis of cases in which the highest bidder may not win. A notable exception is that of an average-bid auction in which the bidder closest to the average bid wins. In this setup, there exists an equilibrium in which all bids are tied, and the winner is chosen randomly (Decarolis, 2018). Another auction where random allocation occurs is a standard auction with a ceiling price, and Lopomo, Persico and Villa (2023) demonstrate its optimality in a procurement auction with adverse selection. Board (2007) and Engelmann, Frank, Koch and Valente (2023) analyze situations where the highest bidder in a standard auction defaults, causing the good to be sold to the second-highest bidder, and Board (2007) shows that a second-price auction performs better than a first-price auction under a certain specification. Beyond these auction examples, random determination of a winner is often analyzed in the contest theory literature, and the formulation of the winning probability employed in our FS auction is identical to the success function analyzed in Che and Gale (2000).

In contrast to theoretical treatments, empirical research on scoring auctions with uncertain scoring rules is growing, ranging from Florida DOT public-work auctions (Takahashi, 2018), to Federal Deposit Insurance Corporation (FDIC) bank resolution procedures (Allen, Clark, Hickman and Richert, 2023) , procurement auctions for computer programming services (Krasnokutskaya, Song and Tang, 2020), and cash-royalty auctions for oil lease contracts (Kong, Perrigne and Vuong, 2022).[3] Kong, Perrigne

[3] Previous empirical research on scoring auctions using known scoring rules includes Lewis and Bajari (2011), Andreyanov (2018), Huang (2019), and Andreyanov, Decarolis, Pacini and Spagnolo (2024).

and Vuong (2022) consider unknown allocation rules, while Allen, Clark, Hickman and Richert (2023), using FDIC bank resolution data, structurally analyze auctions where weights on bid components are not known to bidders. Their findings suggest that the FDIC reduces its resolution cost by alleviating uncertainty about the scoring rule. Similar to Allen, Clark, Hickman and Richert (2023), Krasnokutskaya, Song and Tang (2020) and Takahashi (2018) allow for uncertain scoring rules, with Krasnokutskaya, Song and Tang (2020) considering the quality bid to be exogenous whereas Takahashi (2018) treats it as endogenous. The counterfactual simulations presented by Takahashi (2018) suggest that a sharp increase in the number of reviewers leads to a lower winning price and quality. The broad scope of these empirical studies motivates us to deepen our theoretical understanding of uncertain quality evaluation in scoring auctions.

The remainder of the paper is organized as follows. Section 2 formulates a scoring auction model in which subjective quality evaluation is incorporated as a noisy scoring rule. Then, in Section 3, we transform multi-dimensional-bid auctions into a unidimensional score-bid auction, following Che (1993). In Section 4, we present the main results, characterizing the symmetric equilibria of FS and SS auctions, examining the effect of evaluation precision on equilibrium outcomes, and comparing the performance of the two auction formats. Section 5 discusses the robustness of our results, and Section 6 concludes the paper.

2 Model

A procurement buyer auctions off a procurement contract to 2 risk-neutral bidders who are ex ante symmetric. A type of bidder $i \in \{1, 2\}$ is denoted by θ_i and is independently and identically drawn from a distribution over $[\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$. Let F be the cumulative distribution of θ_i with density $f > 0$. Non-monetary attributes (*quality*) of the good

Further, to develop structural auction models in which firms submit unit price bids for each item needed to carry out a project, Bajari, Houghton and Tadelis (2014), Bolotnyy and Vasserman (2023), and Luo and Takahashi (2025) use an approach analogous to the theoretical analyses of scoring auctions in Che (1993) and Asker and Cantillon (2008).

are represented by a unidimensional variable $q \in \mathbb{R}_+$. A contract is a price-quality pair, (p, q). When the procurement buyer signs a contract (p, q), they earn a payoff

$$q - p.$$

When bidder i wins the auction and signs a contract (p, q), their payoff is given by

$$p - C(q, \theta_i),$$

where $C(q, \theta_i)$ is their production cost. The losing bidder's payoff is zero.

We assume that the cost function C is strictly increasing in both q and θ ($C_q > 0$, $C_\theta > 0$), strictly convex in quality ($C_{qq} > 0$), and exhibits non-decreasing differences ($C_{q\theta} \geq 0$).[4] Both the production cost and marginal cost are increasing in type so that a bidder of a lower type is more efficient.

A procurement contract is determined through a *scoring auction*, in which each bidder i submits a proposal (p_i, q_i), where $p_i \leq \bar{p}$ is a price bid and $q_i \geq \underline{q}$ is a quality bid, with reserve price and minimum quality denoted by $\bar{p} > 0$ and $\underline{q} > 0$. For a bidder's quality bid q_i, the buyer reviews the proposal and assigns a quality score. The evaluated quality is given by $\hat{q}_i = q_i + \varepsilon_i$, where ε_i represents the buyer's noisy subjective evaluation. We assume that for each $i \in \{1, 2\}$, ε_i is a random variable independently distributed from (p_1, q_1, p_2, q_2).[5] There is a pre-announced scoring rule $S : [0, \bar{p}] \times [\underline{q}, \infty) \to \mathbb{R}$, which maps price and evaluated quality bids $(p_i, \hat{q}_i)$ into a unidimensional score $\hat{s}_i = S(p_i, \hat{q}_i)$. We focus on a simple quasilinear scoring rule

$$S(p_i, \hat{q}_i) = \hat{q}_i - p_i,$$

where the bidder with the highest score wins.

We examine *first-score (FS)* and *second-score (SS)* auctions, which are the scoring auction counterparts of the familiar first-price and second-price auctions. In both FS and SS auctions, the bidder with the highest evaluated score $\hat{s}_i = \hat{q}_i - p_i$ wins. In an FS auction, the winner's proposal (p_i, q_i) is finalized as a contract. In an SS auction,

[4]Note that $C_\theta = \partial C / \partial \theta$ and that the other subscripts are defined in the same manner.

[5]We allow ε_1 and ε_2 to be correlated.

however, the winner is required to meet the loser's evaluated score. Because it is difficult in practice to change quality after the auction, we suppose that quality is fixed and only the price is adjusted in order to fulfill the loser's score. Suppose that bidder i submits a bid (p_i, q_i) and wins at the score $\hat{s}_i$, and that bidder j's bid is scored as $\hat{s}_j < \hat{s}_i$. Then, the price p^* that gives score $\hat{s}_j$ given the evaluated quality $\hat{q}_i$ is

$$\hat{q}_i - p^* = \hat{s}_j \;\Leftrightarrow\; p^* = \hat{q}_i - \hat{s}_j.$$

Thus, bidder i signs a contract $(\hat{q}_i - \hat{s}_j, q_i)$ in an SS auction.[6]

Remark 1 This remark about quality bids has two components. First, in scoring auctions for public-works procurement, the evaluation by procurers of the various non-monetary attributes included in a quality bid is inherently subjective and thus affected by chance, creating uncertainty for bidders at the time of bidding. This uncertainty is verified by Takahashi (2018), who finds a substantial difference in the reviewers' scores of a given quality bid by using the Florida Department of Transportation (FDOT) public-works data.[7] A variance decomposition of the reviewers' quality evaluation scores finds that the within-bidder between-reviewer variation accounts for about half of the total variation, and the within-auction between-bidder variation also contributes substantially to the total variation. Thus the presence of uncertainty in the evaluation of quality bids from the perspective of bidders is confirmed empirically.[8]

[6]In reality, bidders submit not a number q_i but contract details, i.e., specific construction plans and details of goods specifications. When we write a contract (p_i, q_i), this represents the contract as perceived from the bidder's point of view. See Remark 1 as well.

[7]FDOT scoring auction data used by Takahashi (2018) was mostly for procuring large-scale projects including bridge and building construction projects, and FDOT reviewers were chosen from among a pool of skilled civil engineers in the department. In the auctions, upon receiving price bids and quality bids, each reviewer independently assessed and assigned a score to each quality bid, and the final quality bid score was the average of the scores of all reviewers.

[8]Another interpretation of uncertainty in quality evaluation is the incompleteness of scoring rules. It is difficult for reviewers to anticipate all possible proposals for various non-monetary attributes and to prescribe scores for all of them *a priori*. Hence, the scoring rule must be incomplete and thus, there is much room for reviewer discretion.

The second point about quality bids is that we interpret q_i as a *true* quality whereas the evaluated quality $\hat{q}_i$ is a noisy and imprecise evaluation. In reality, bidders submit not a number q_i but contract details, i.e., specific construction plans and details of goods specifications. Bidder i knows that the true quality of its own contract is q_i, whereas the buyer evaluates it as $\hat{q}_i$. Note that in the procurement of public works or high-tech equipment, the buyer may be relatively less informed about quality than the suppliers. Hence, we suppose that ε_i is just an error. Based on this interpretation, the buyer's payoff for a contract (p, q) is given by $q-p$, not by $\hat{q}-p$, and the buyer observes the true quality q after the delivery of the good. This interpretation is consistent with previous studies such as Takahashi (2018) and Ortner, Chassang, Kawai and Nakabayashi (2025).

Remark 2 Although we generally follow Che (1993), we define the SS auction in a slightly different manner. Under the original SS auction rule in Che (1993), the winner is free to choose a contract (p, q) such that the associated score fulfills the highest rejected score; that is, the winner can change both price and quality. Instead, we suppose that quality cannot be changed after bidding because of the nature of the quality score which depends on noisy subjective evaluation. This assumption is reasonable in public procurements such as construction projects due to the adjustment costs associated with changing construction plans in a short time. Moreover, given the presence of noisy quality perception, it may be difficult for the buyer and the winning bidder to reach an agreement on how to adjust project attributes after bidding. Che (1993) shows that the optimal quality is independent of score and that bidders have no incentive to change quality after winning under the original SS auction rule, so the equilibrium properties provided in Che (1993) do not change even if quality is fixed ex post. Our analysis in the next section also implies that bidders have no incentive to change quality after winning even if evaluation uncertainty exists and quality were adjustable. The assumption that quality cannot be changed after bidding with only the price being adjusted is particularly appropriate in this context.

Remark 3 To the best of our knowledge, the FS auction is the prevailing format in public procurement, and there are no cases in which the SS auction has been employed.

However, the SS auction is an identical mechanism to the generalized second-price auction, which is often used in sponsored search advertising on the web (Edelman, Ostrovsky and Schwarz, 2007). A mechanism that adjusts only the price according to the second-highest score is straightforward to implement and therefore warrants analytical attention.

3 Score-Bid Auctions

Che (1993) shows that scoring auctions are analyzed by transforming a multidimensional-bidding game into a unidimensional score-bidding game. In this section, we show that scoring auctions are transformed into a score-bid auction form even with the presence of evaluation uncertainty.

Let A be the cumulative distribution of $\varepsilon \equiv \varepsilon_2 - \varepsilon_1$. To maintain symmetry between the two bidders, we suppose that A is symmetrically distributed with zero mean. Thus, we suppose $A(0) = 1/2$ and $A(x) + A(-x) = 1$ for all $x \in \mathbb{R}$.

Bidder i's *strategy* is a mapping from each θ_i to a proposal $(p_i(\theta_i), q_i(\theta_i))$, and the bidder's strategy generates its score strategy, which is denoted by $s_i(\theta_i) \equiv q_i(\theta_i) - p_i(\theta_i)$. Suppose that bidder 1 is of type θ_1 and submits a proposal (p_1, q_1). Let $s_1 \equiv q_1 - p_1$ be the bidder's *intended score*, whereas the *evaluated score* made by the buyer is $\hat{s}_1 = \hat{q}_1 - p_1 = s_1 + \varepsilon_1$. Given a score strategy by bidder 2, let $\pi^I(p_1, q_1, \theta_1)$ be bidder 1's expected payoff in an FS auction when submitting (p_1, q_1). Then, we have

$$\begin{aligned} \pi_1^I(p_1, q_1, \theta_1) &= \Pr\{s_1 + \varepsilon_1 \geq s_2(\theta_2) + \varepsilon_2\}(p_1 - C(q_1, \theta_1)) \\ &= E_{\theta_2}[A(s_1 - s_2(\theta_2))](p_1 - C(q_1, \theta_1)). \end{aligned}$$

The winning probability is a function of s_1, and given s_1, it is independent of the choice of (p_1, q_1).[9] Hence, given a bidder's intended score s, they determine an optimal contract

[9] As we will assume in Assumption 1, we ignore the possibility that the evaluated quality $\hat{q}_i$ falls below $\underline{q}$ and the proposal is rejected.

(p, q) that solves

$$\max_{(p,q)} p - C(q, \theta)$$
$$\text{s.t. } q - p = s, \tag{1}$$
$$p \le \bar{p},\ q \ge \underline{q}.$$

Throughout the analysis, we assume that the reserve price $\bar{p}$ is sufficiently high and not binding at (1). By substituting the score constraint into the objective function, the profit maximization problem is written as

$$\max_{q \ge \underline{q}} q - C(q, \theta) - s. \tag{2}$$

A similar argument applies to the SS auction. Suppose that bidder 2 submits an intended score s_2 and that the subjective evaluation is realized as $\varepsilon = \varepsilon_2 - \varepsilon_1$. When bidder 1 wins the auction with an intended score $s_1 > s_2 + \varepsilon$, their ex post payoff is

$$\hat{q}_1 - \hat{s}_2 - C(q_1, \theta_1) = q_1 - C(q_1, \theta_1) - (s_2 + \varepsilon).$$

Hence, given winning at an intended score s, the winner's profit is maximized by the optimal contract that solves (1) or (2).

We assume that the maximization problem (2) has an interior solution. That is, the optimal quality $q^*(\theta)$ is determined by the first order condition

$$C_q(q^*(\theta), \theta) = 1. \tag{3}$$

Note that $q^*(\theta)$ is non-increasing in θ by the non-decreasing differences $C_{q\theta} \ge 0$. We further assume that the optimal quality $q^*(\theta)$ is sufficiently higher than $\underline{q}$, so that the buyer never evaluates quality lower than the reservation $\underline{q}$ when a bidder submits a quality bid $q^*(\theta)$.

Assumption 1 It holds that $C_q(\underline{q}, \bar{\theta}) < 1$. In addition, $q^*(\bar{\theta})$ is sufficiently higher than $\underline{q}$ and $\Pr\{q^*(\bar{\theta}) + \varepsilon_i < \underline{q}\} = 0$.

Assumption 1 and the first order condition (3) show that the winning profit is maximized by making quality $q^*(\theta)$ regardless of the intended or evaluated score. Thus, this is the optimal quality bid in both FS and SS auctions. This is similar to Che (1993), in which there is no evaluation uncertainty, and a useful property of quasilinear scoring rules.

Lemma 1 *Under Assumption 1, it is optimal for each bidder to submit quality bid $q^*(\theta)$ in both FS and SS auctions.*

Let us define the *pseudo-valuation* $v(\theta)$ by

$$v(\theta) \equiv q^*(\theta) - C(q^*(\theta), \theta). \tag{4}$$

Note that v is decreasing in θ. The equilibrium of the scoring auction is derived by solving auctions in terms of score bid s, where each bidder has a valuation $v(\theta)$. Hence, it is sometimes convenient to treat the pseudo-valuation $v(\theta)$ rather than θ as a type.[10] Bidder i's pseudo-value is denoted by $v_i \equiv v(\theta_i) \in [\underline{v}, \bar{v}]$, where $\bar{v} \equiv v(\underline{\theta})$ and $\underline{v} \equiv v(\bar{\theta})$. Also, by abusing notations, bidder strategy is sometimes denoted by $s(v_i)$ rather than $s(\theta_i)$. Suppose that bidder i with a pseudo-value v_i submits a score bid s_i. Given the other bidder's score strategy $s_j(\cdot)$, the expected payoff $\pi^I(s_i, v_i)$ in an FS auction is given by

$$\pi^I(s_i, v_i) \equiv E_{v_j}[A(s_i - s_j(v_j))](v_i - s_i). \tag{5}$$

When bidder i wins an SS auction with intended score bids (s_i, s_j) and $\varepsilon = \varepsilon_j - \varepsilon_i$, their ex post payoff is given by

$$u^{II}(s_i, s_j, \varepsilon, v_i) = \begin{cases} v_i - s_j - \varepsilon & \text{if } s_i > s_j + \varepsilon \\ 0 & \text{if } s_i < s_j + \varepsilon \end{cases}. \tag{6}$$

When bidder i wins an FS auction, the proposal made by the winner is the final contract, so that the buyer's payoff is equal to s_i. In an SS auction, the winner i signs a contract $(\hat{q}_i - \hat{s}_j, q_i)$, so that the buyer's payoff is given by

$$q_i - (\hat{q}_i - \hat{s}_j) = s_j + \varepsilon.$$

[10]The pseudo-valuation is called *pseudotype* in Asker and Cantillon (2008) and *productive potential* in Che (1993).

Thus, the "score revenues" in FS and SS auctions correspond to the buyer's payoffs.

It is also convenient to consider the pseudo-valuation distribution instead of the type distribution. Let G be the cumulative distribution of pseudo-valuation; that is,

$$G(x) = \Pr\{v(\theta) \leq x\} = 1 - F(v^{-1}(x)).$$

Let g be the associated density of G. We impose a standard regularity condition of non-decreasing hazard rate.

Assumption 2 The distribution G possesses the non-decreasing hazard rate property: that is, $\frac{g(v)}{1-G(v)}$ is non-decreasing.[11]

4 Main Results

4.1 Equilibrium

First, we examine the equilibrium of SS and FS auctions. Despite the presence of evaluation uncertainty, in the SS auction, it is weakly dominant for each bidder to submit their truthful pseudo-value $v(\theta)$ as the score bid.

Theorem 1 *In the SS auction, it is a weakly dominant strategy for each bidder to submit a score bid $s^{II}(\theta_i) = v(\theta_i)$.*

All proofs are provided in the Appendix.

The truthfulness of the SS auction is shown in the following manner. Fix bidder 1's optimal quality bid $q_1^*(\theta)$, and consider an arbitrary intended score $s_2 = q_2 - p_2$ of bidder 2 and realized noise term ε. Then, bidder 1 wins when $s_1 \geq s_2 + \varepsilon$. When bidder 1 wins, their ex post payoff is given by $v(\theta_1) - (s_2 + \varepsilon)$. Thus, given s_2 and ε, bidder 1 wins the auction whenever their intended score is higher than a threshold $s_2 + \varepsilon$, and the associated "score cost" is equal to the threshold. The allocation and payment rules

[11] The associated condition on the type distribution F is that $f(\theta)/\left(C_\theta(q^*(\theta), \theta)F(\theta)\right)$ is non-increasing in θ.

are identical to the standard second-price auction and other strategy-proof allocation mechanisms. In other words, the noise term ε is a handicap or a fixed bid credit on bidders, and bidders play a second-price auction with this handicap. Because the handicap is calculated inside the mechanism, truth-telling is weakly dominant for any handicap so that the strategy-proofness still holds even when the handicap is random and hidden.

The dominant strategy property of the SS auction is identical to the case without evaluation uncertainty provided by Che (1993) and Asker and Cantillon (2008), and is also similar to the case of non-quasilinear scoring rules by Hanazono, Nakabayashi, Sano and Tsuruoka (2024). The uncertainty in quality evaluation does not influence bidders' incentives at all, and the equilibrium is invariant to arbitrary noise structure A and number of bidders.

As for the FS auction, however, the bidding incentive depends on the noise structure A, and in general it is difficult to have an explicit equilibrium because the winning probability in equilibrium is not reduced to the probability of order statistics of types. In what follows, we focus on a uniform distribution of ε under which we are able to obtain an explicit equilibrium of the FS auction. The specification of A below is the same as the success function used by Che and Gale (2000) in a contest model.

Assumption 3 There exists $\beta > 0$ and the subjective evaluation term $\varepsilon = \varepsilon_2 - \varepsilon_1$ is uniformly distributed over $[-1/2\beta, 1/2\beta]$; that is,

$$A(x) = \frac{1}{2} + \beta x$$

for $-1/2\beta \leq x \leq 1/2\beta$. Further, it holds that

$$\beta(v(\underline{\theta}) - v(\bar{\theta})) \leq 1. \tag{7}$$

Note that β represents the precision of the quality evaluation, with a higher β indicating that the buyer evaluates the quality more precisely. Condition (7) requires that the precision not be too high. This is imposed for $|s^I(\theta_1) - s^I(\theta_2)| \leq \frac{1}{2\beta}$ in equilibrium for all type profiles; that is, both bidders have a chance to win for all type profiles.

In addition, we impose the following technical condition which guarantees that any off-path score bid is not profitable.

Assumption 4 For all $x \in [v(\bar{\theta}), v(\underline{\theta})]$, it holds that

$$\frac{d}{dx} E[v(\theta_i) \mid v(\theta_i) < x] \leq 2.$$

Under these assumptions, the symmetric Bayesian Nash equilibrium of the FS auction is derived by the first-order approach.

Theorem 2 *Suppose that Assumptions 1, 3, and 4 hold. Then there exists a symmetric Bayesian Nash equilibrium of the FS auction in which the score-bid strategy is given by*

$$s^I(\theta_i) = \frac{1}{2}\left(v(\theta_i) + \mu - \frac{1}{\beta}\right), \tag{8}$$

where $\mu = E[v(\theta)]$.

Note that condition (7) in Assumption 3 implies $v(\underline{\theta}) - 1/\beta \leq v(\bar{\theta})$. Hence, by $\mu < v(\underline{\theta})$, the equilibrium score bid satisfies

$$s^I(\theta_i) < \frac{1}{2}\left(v(\theta_i) + v(\underline{\theta}) - \frac{1}{\beta}\right) \leq \frac{v(\theta_i) + v(\bar{\theta})}{2} \leq v(\theta_i).$$

This implies that even when bidders have the highest (the least efficient) type $\bar{\theta}$, they shade score bids from their true pseudo-valuation $v(\bar{\theta})$. This is because unlike standard auctions in which bidders of the worst type never win in equilibrium, bidders here even of the worst type can win due to the evaluation uncertainty. Hence, even these bidders shade their bids to enjoy a positive profit in case they do win.

A similar property holds for the lowest (the most efficient) type $\underline{\theta}$. If there were no evaluation uncertainty and evaluated quality was always equal to the intended quality, or $\hat{q}_i = q_i$, then bidders would submit the expected pseudo-valuation of the rival given that the rival has a higher (worse) type. Hence, when a bidder has the lowest type $\underline{\theta}$, they submit a score bid equal to the average pseudo-valuation μ. In contrast, in equilibrium (8), the bidder of the lowest type submits

$$s^I(\underline{\theta}) = \frac{1}{2}\left(v(\underline{\theta}) + \mu - \frac{1}{\beta}\right) \leq \frac{v(\bar{\theta}) + \mu}{2} < \mu.$$

Thus, bidders have little incentive to submit aggressive bids, as the winner determination depends not only on the bid but also on chance. The presence of evaluation uncertainty thus leads to more cautious bids.

Assumption 4 is imposed for any off-path score bid being unprofitable. Although bidders have a standard quasilinear payoff structure, the first order condition and the monotonicity of the score bid function may not be sufficient for payoff maximization in an FS auction. This is because a score bid lower or higher than the equilibrium bid range may be profitable due to the evaluation uncertainty. This is in contrast with a standard first-price auction, in which a bid lower or higher than the equilibrium bid range is clearly unprofitable. Note that Assumption 4 is mild, and it holds when the density g of the pseudo-valuation does not drastically increase around some v. A sufficient condition for Assumption 4 is that G possesses a non-increasing reverse hazard rate. Indeed, let $\lambda(x) \equiv g(x)/G(x)$ be the reverse hazard rate of G, and suppose $\lambda' \leq 0$. Let v_i be distributed by G and let

$$\gamma(x) \equiv \frac{d}{dx} E[v_i \mid v_i < x] = \lambda(x)(x - E[v_i \mid v_i < x]).$$

By differentiation, we have

$$\gamma'(x) = \lambda'(x)(x - E[v_i \mid v_i < x]) + \lambda(x)(1 - \gamma(x)).$$

Hence, by $\lambda'(x) \leq 0$, we have $\gamma(x) \geq 1 \Rightarrow \gamma'(x) \leq 0$. Also, we should have $\gamma(\underline{v}) \leq 1$.[12] Hence, we conclude that $\gamma(x) \leq 1$ for all $x \in [\underline{v}, \bar{v}]$.

4.2 Quality Evaluation Precision

In the SS auction, bidders have a truthful dominant strategy and the random quality evaluation does not affect the bidding incentive but in the FS auction, the equilibrium score strategy is influenced by the quality evaluation. The equilibrium score bid function s^I is increasing in β, which indicates that bidders behave more competitively and submit

[12] If $\gamma(\underline{v}) > 1$, then for a sufficiently small $\delta > 0$, we have $\frac{E[v_i \mid v_i < \underline{v}+\delta] - \underline{v}}{\delta} > 1 \Leftrightarrow E[v_i \mid v_i < \underline{v} + \delta] > \underline{v} + \delta$, which is a contradiction.

bids aggressively as the quality evaluation gets more precise. Because the optimal quality bid is determined only by bidder type, this means that bidders lower their price bid as β becomes higher.

Corollary 1 *Bidders submit their score bid more aggressively in an FS auction as the precision of quality evaluation β becomes higher by lowering their price bid and not changing their quality bid. In contrast, the equilibrium bid is invariant to the precision of quality evaluation β in an SS auction.*

In an FS auction, an increase in evaluation uncertainty increases the element of chance in the determination of the winner and leads to less aggressive bidding. This is consistent with the revenue equivalence theorem. Under ex ante symmetry and Assumption 2, achieving an efficient allocation leads to increasing the auctioneer's expected revenue. Under random quality evaluation, efficient allocation is not realized but an inefficient bidder wins with a positive probability. Thus, the expected score with uncertainty in quality evaluation is lower than the mechanism that achieves efficient allocation without uncertainty.

Although the precision β of the quality evaluation does not affect the quality bids, it does affect the final quality because the random evaluation affects the choice of the winner in the auction. When the quality evaluation is highly random (β is low), the bidder of the higher (less efficient) type is more likely to win, which reduces the final quality. As precision rises (β becomes higher), the bidder of the lower type is more likely to win, which leads to the efficient quality level.

While the evaluation precision does not affect the bidding incentive in an SS auction, similar comparative statics arise because improved allocative efficiency leads to higher quality and scores. These observations are summarized in the following theorem.

Theorem 3 *In the FS auction under Assumptions 1, 3, and 4, the expected social surplus $q - C(q, \theta)$, expected score, and expected quality increase with the precision of quality evaluation β. Similarly, in the SS auction under Assumptions 1, 2, and 3, the expected social surplus, expected score, and expected quality increase with β.*

In contrast with the score and quality, the effect on the final price under random quality evaluation is relatively uncertain. In an FS auction, the price bid is decreasing in β. Thus, given a winner's type, the price decreases as the quality evaluation becomes more precise. However, higher precision leads to a more efficient winner being chosen, which increases average quality, so the associated price might increase due to this quality improvement. Thus, the effect of quality evaluation precision on the final price depends on a tradeoff between the effects of promotion on competition and quality improvement.

To see the effect of evaluation precision β on the final price, let us treat pseudo-valuation v as bidder type instead of θ, and let

$$C^*(x) \equiv C\left(q^*\left(v^{-1}(x)\right), v^{-1}(x)\right)$$

be the cost of supplying the optimal quality q^* for a bidder of pseudo-value $v = x$. The effect of evaluation precision on the expected price strongly depends on the form of C^*. When the total cost C^* of the optimal quality is non-increasing in pseudo-valuation v, the more efficient bidder supplies the optimal quality at a lower cost, which implies that the competition promotion effect dominates the quality improvement effect.

Let $k = I, II$ indicate the FS and SS auctions, respectively. Let $A_i^k(v_1, v_2)$ be the winning probability of bidder i for any pseudo-value pair (v_1, v_2) in equilibrium of auction $k \in \{I, II\}$, and $A^k(v) = E_{v_2}[A_1^k(v, v_2)]$. In addition, let $\Pi^k(v)$ be the equilibrium interim expected payoff of a bidder with pseudo-value v in auction k. Then, the expected revenue $P^k(v)$ of a bidder with pseudo-value v in the equilibrium of auction k is

$$P^k(v) = \Pi^k(v) + A^k(v)C^*(v).$$

By the standard envelope argument and calculations, the expected price for the buyer, denoted by P^k, is expressed by

$$\begin{aligned} P^k &= 2E\left[\Pi^k(v) + A^k(v)C^*(v)\right] \\ &= 2\underline{\Pi}^k + 2E\left[A^k(v)\left(C^*(v) + \frac{1-G(v)}{g(v)}\right)\right] \\ &= 2\underline{\Pi}^k + E\left[\sum_{i\in\{1,2\}} A_i^k(v_1, v_2)\psi(v_i)\right], \end{aligned} \tag{9}$$

where $\underline{\Pi}^k \equiv \Pi^k(\underline{v})$ and

$$\psi(v) \equiv C^*(v) + \frac{1 - G(v)}{g(v)} \tag{10}$$

is the *virtual cost function*. When C^* is non-increasing, the virtual cost ψ is non-increasing by the regularity Assumption 2. Hence, (9) indicates that the expected price decreases as the winner determination becomes more efficient. Thus, the more precise evaluation decreases the expected price.

Theorem 4 *Suppose that all the assumptions stated above hold and that C_q/C_θ is non-decreasing in q. Then, the expected price in both FS and SS auctions is decreasing in evaluation precision β.*

The condition that C_q/C_θ is non-decreasing in q guarantees that C^* is non-increasing. When the marginal cost is less sensitive to changes in type θ (i.e., $C_{q\theta}$ is small), the optimal quality is less sensitive to changes in type θ. In this situation, the cost reduction effect of type improvement is larger than the cost increase effect of quality improvement so that the cost of producing the optimal quality decreases as type θ improves; that is, $C(q^*(\theta), \theta)$ is increasing in type θ, which means C^* is decreasing.

Remark 4 Our comparative static of quality on β is not consistent with Takahashi (2018), who provides a numerical analysis of scoring auctions with random quality evaluation and finds that bidders' quality improves as evaluation uncertainty increases. This property may be due to the difference in the scoring rules between the two studies. Takahashi (2018) considers the *PQR scoring rule* in which scores are evaluated in terms of price per quality ratio p/q. Under that rule, bidders offer a higher quality bid as the score increases (becomes worse) (Hanazono, Nakabayashi, Sano and Tsuruoka, 2024). As Corollary 1 shows, as uncertainty in quality evaluation increases, the selection of the winning bidder becomes more random and thus competition among bidders decreases. Hence, under the PQR scoring rule, making a less aggressive score bid means proposing a higher quality at a higher price. In contrast, in our quasilinear scoring rule, the quality bid is independent of score and is determined solely by bidder type. Under our rule, an increase in evaluation uncertainty lowers expected quality because an inefficient bidder is more likely to be chosen as the winner.

4.3 Comparing SS and FS Auctions

Next, we compare the performance of SS and FS auctions. Although our model is built on a standard independent private value setting, the equivalence between the two auction formats fails to hold. Our first main result is that the SS auction achieves a higher expected social surplus and quality than the FS auction.

Theorem 5 *Suppose that all assumptions stated above hold. The expected social surplus and expected quality are higher in the SS auction than in the FS auction.*

The intuition behind the welfare ranking is simple. While bidders submit truthfully in the SS auction, they shade their bids in the FS auction, and the magnitude of the bid shading is larger for a higher valuation. The equilibrium score bid in the FS auction has the slope $ds^I/dv = 1/2 < 1$. Hence, given an arbitrary pseudo-valuation profile (v_1, v_2), bids are closer to each other in the FS than in the SS auctions, which makes the determination of the winning bidder more random in the FS auction. Hence, the SS auction is more efficient than the FS auction.

The derivation of the quality ranking is analogous to that of the social welfare. The optimal quality is determined solely by the winner's type, and a bidder with a better (lower) type supplies a weakly higher quality. Because the more lower-type bidder is more likely to be chosen as the winner in the SS auction, the expected quality is higher for the SS auction than the FS auction.

The comparisons with respect to the expected score (the buyer's expected payoff) and final price are also analogous to the comparative statics of the score and price on β. For an auction format $k = I, II$, the expected score S^k is expressed as

$$S^k = E\left[\sum_{i\in\{1,2\}} A_i^k(v_1, v_2)\phi(v_i)\right] - 2\underline{\Pi}^k,$$

where

$$\phi(v) \equiv v - \frac{1 - G(v)}{g(v)}$$

is the *virtual valuation function*. By the monotone virtual valuation, the more efficient allocation induces the higher expected score. Similarly, the expected final price P^k is expressed as

$$P^k = E\left[\sum_{i \in \{1,2\}} A_i^k(v_1, v_2)\psi(v_i)\right] + 2\underline{\Pi}^k.$$

When the virtual cost function ψ is decreasing, the more efficient allocation induces the lower expected price.

However, the worst-type payoff $\underline{\Pi}^k$ differs across the auction formats, which requires an additional condition to obtain a clear ranking between the FS and SS auctions. In the second main theorem below, we suppose

$$\beta(v(\underline{\theta}) - v(\bar{v})) \leq \frac{1}{\sqrt{2}} \tag{11}$$

to guarantee that the worst-type payoff in the FS auction is higher than that in the SS auction.[13]

Theorem 6 *Suppose that all assumptions stated above hold with (11). Then, the expected score is higher in the SS auction than in the FS auction. In addition, if C_q/C_θ is non-decreasing in q, the expected price in the SS auction is lower than that in the FS auction.*

Overall, when uncertainty in quality evaluation is inevitable, the SS auction achieves more efficient allocation than the FS auction, and the SS auction performs better than the FS auction in terms of buyer's expected payoff, quality, and price.

5 General Noise Structure

Much of the analysis in the previous section relies on the specific noise structure, so it is natural to question whether our results are sustained under a more general noise

[13] Appendix B provides other sufficient conditions under which the expected score and price are ranked.

structure and a general number of bidders. In this section, we discuss how the results in the previous section are robust to a general noise structure.

First, it should be emphasized that the dominant strategy property of the SS auction is robust to noise structure and the number of bidders. It is weakly dominant for bidders to bid their true pseudo-valuation for any noise structure A and any number of rival bidders. As the precision of the quality evaluation increases and an efficient bidder becomes more likely to win, both social surplus and expected quality improve. Furthermore, if the worst-type payoff is non-increasing in the evaluation precision, the winning score (i.e., the buyer's payoff) increases and the expected price decreases, given the monotonicity of the virtual cost.

As for the FS auction, the existence of an equilibrium can be guaranteed from the standard argument. The expected payoff of the FS auction at bidder i's pseudo-value v_i and intended score s_i is expressed by (5). If A is smooth and strictly increasing (in the domain played in equilibrium), then $E[A'(s_i - s(v_j))] > 0$, and the expected payoff π^I clearly satisfies the single crossing property of Athey (2001). Thus, a monotone Bayesian Nash equilibrium exists.

However, it is difficult to obtain an explicit equilibrium strategy of the FS auction for a general noise structure. This is because, unlike the standard auction model, the winning probability in equilibrium is not reduced to the probability distribution of order statistics of types. Thus, the equilibrium strategy cannot be characterized even in the form of a differential equation in s. Indeed, suppose that there exists a symmetric monotone equilibrium s^I and let

$$A^*(s_i) \equiv E_{v_j}[A(s_i - s^I(v_j))] \tag{12}$$

denote the winning probability when submitting an intended score s_i. Then, the first order condition of expected payoff maximization yields

$$(A^*)'(s^I(v_i))(v_i - s^I(v_i)) - A^*(s^I(v_i)) = 0$$

and thus, we have

$$s^I(v_i) = v_i - \frac{A^*(s^I(v_i))}{(A^*)'(s^I(v_i))}. \tag{13}$$

It is difficult to analyze the equilibrium score bidding strategy further.

Nevertheless, it is still possible to discuss some properties of the FS auction under a general noise structure. First, as stated in Lemma 1, that the optimal quality bid is independent of noise and intended score and depends only on the bidder's type. The noise structure will affect only the price bidding.

Second, our comparison of FS and SS auctions could be sustained for a more general A. Our results are derived by the equilibrium property of the FS auction that the score-bid difference is smaller than the pseudo-value difference. Suppose that the FS auction has a symmetric monotone equilibrium s^I for some distribution A. If the equilibrium satisfies $ds^I/dv \leq 1$, the SS auction chooses the efficient bidder with a higher probability than the FS auction. Thus, we can conclude that the SS auction yields a higher expected social welfare and quality than the FS auction. In addition, the expected score is higher and the expected price is lower in the SS auction if $\underline{\Pi}^I \geq \underline{\Pi}^{II}$ and the monotonicity of C^* hold. Equation (13) implies that $ds^I/dv \leq 1$ holds if $(A^*)'(x)/A^*(x)$ is non-increasing in x. Thus, if A^* possesses the non-increasing *reverse hazard rate* property, then the SS auction performs better in equilibrium than the FS auction.[14] It is difficult to obtain a primitive condition for A^* to have a non-increasing reverse hazard rate, but we explicitly state this property as follows.

Proposition 1 *Suppose that Assumptions 1 and 2 hold. If a score-bid strategy* s^I *satisfies (13), in which* A^* *is defined by (12), and if* $(A^*)'(x)/A^*(x)$ *is non-increasing, then* s^I *is the symmetric equilibrium of the FS auction. The expected social surplus and quality are higher in the SS auction than in the FS auction. In addition, if* $\underline{\Pi}^I \geq \underline{\Pi}^{II}$*, the SS auction yields a higher expected score than the FS auction. Also, if* $\underline{\Pi}^I \geq \underline{\Pi}^{II}$ *and* C_q/C_θ *is non-decreasing in* q*, the SS auction yields a lower expected price than the FS auction.*

[14]The term "reverse hazard rate" is used here informally because A^* is not a probability distribution.

6 Concluding Remarks

This study examines the effect of uncertainty in the evaluation of quality in scoring auctions. In scoring auctions for public-works procurement, bids include various non-monetary attributes, and there is much room for reviewers to use their discretion to evaluate these quality bids. This makes it difficult for bidders to anticipate perfectly the evaluation scores of their quality bids at the time of bidding. Similar phenomena are prevalent in other markets.

We have shown that in an FS auction, the noisy evaluation of quality creates an element of chance in determining winning bidders, which decreases the incentive for bidders to make aggressive bids and thus weakens bidder competition. In an SS auction, by contrast, the truthful weakly dominant strategy exists, so evaluation uncertainty does not affect the bidding incentive. Further, as the precision of the quality evaluation improves, the expected social surplus, score (the buyer's payoff), and quality improves. In addition, under certain conditions, the expected price decreases through the effect of competition. As the SS auction chooses the winner more efficiently, the SS auction achieves a higher social surplus, score, and quality at a lower price than the FS auction. Our results imply that when evaluation uncertainty is inevitable, the SS auction is preferable to the FS auction for both the buyer and social welfare.

The comparative statics presented here have a practical implication for public-works procurement, particularly for large-scale projects such as bridge construction in which superior quality is perceived as highly desirable, causing scoring auctions to often be used to procure such projects. Uncertainty in quality evaluation adversely affects governments through not only the selection of the winning bidder but also the endogenous responses of bidders. This uncertainty affects bidding behavior and efficiency, which in turn can influence the attributes of the final products.

There are a number of potential extensions for further research. One important extension would be a theoretical consideration of scoring rules other than the quasilinear scoring rule. Following Che (1993), this study analyzed a quasilinear scoring rule, but real-world procurement auctions often use different scoring rules such as the price-per-quality-ratio rule (Takahashi, 2018; Hanazono, Nakabayashi, Sano and Tsuruoka, 2024). Such analyses of alternative models with such non-linear scoring rules are left for future research.

Acknowledgements

The authors thank Michihiro Kandori, Fuhito Kojima, Nozomu Muto, Rieko Ishii, Takashi Ui, Philip C. MacLellan, Takeshi Nishimura, participants at the 2025 Game Theory Workshop, EARIE 2025, and 2025 Japanese Economic Association Autumn Meeting, and members of the construction industry who have been granted anonymity for their helpful comments. We acknowledge financial support from the Japan Society for the Promotion of Science (KAKENHI 21K01401, 23H00051, and 25K05073).

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A Proofs

A.1 Proof of Theorem 1

Suppose that bidder 2 submits an intended score $s_2 = q_2 - p_2$ and that the subjective quality evaluation is realized as $\varepsilon = \varepsilon_2 - \varepsilon_1$. Bidder 1 wins the auction if their intended score $s_1 > s_2 + \varepsilon$. When bidder 1 wins, they sign a contract $(q_1 - s_2 - \varepsilon, q_1)$, and the associated payoff is

$$u^{II} = q_1 - C(q_1, \theta_1) - (s_2 + \varepsilon).$$

Given that bidder 1 submits a quality bid $q^*(\theta_1)$, they earn a nonnegative winning profit if and only if $v(\theta_1) \geq s_2 + \varepsilon$. Thus, by submitting an intended score $s_1 = v(\theta_1)$, bidder 1 wins the auction whenever the winning payoff is nonnegative. Thus, similar to a standard second-price auction, it is always optimal for bidder 1 to submit $s_1 = v(\theta_1)$ for any (s_2, ε). □

A.2 Proof of Theorem 2

Suppose that there exists a symmetric Bayesian Nash equilibrium and that the equilibrium score bid function is $s^I : \Theta \to \mathbb{R}$. Suppose that bidder 2 takes the equilibrium score-bid strategy s^I. Suppose that bidder 1 has a pseudo-valuation $v(\theta_1)$ and submits a score bid s_1, where

$$|s_1 - s^I(\theta_2)| \leq \frac{1}{2\beta} \tag{14}$$

for all θ_2. Then, the associated expected payoff is given by

$$\begin{aligned} \pi^I(s_1, \theta_1) &= E_{\theta_2}[A(s_1 - s^I(\theta_2))](v(\theta_1) - s_1) \\ &= \left(\frac{1}{2} + \beta s_1 - \beta E[s^I(\theta_2)]\right)(v(\theta_1) - s_1). \end{aligned} \tag{15}$$

Taking the first order condition with respect to s_1, we have

$$\beta(v(\theta_1) - s_1) - \frac{1}{2} - \beta s_1 + \beta E[s^I(\theta_2)] = 0. \tag{16}$$

By substituting symmetric strategy $s_1 = s^I(\theta_1)$ into (16), we have

$$s^I(\theta_1) = \frac{v(\theta_1)}{2} - \frac{1}{4\beta} + \frac{E[s^I(\theta_2)]}{2}.$$

Because $E[s^I(\theta_1)] = E[s^I(\theta_2)]$, we have

$$E[s^I(\theta_i)] = \mu - \frac{1}{2\beta}.$$

Thus, we have a candidate for equilibrium score bid function

$$s^I(\theta_i) = \frac{1}{2}\left(v(\theta_i) + \mu - \frac{1}{\beta}\right). \tag{17}$$

The condition (14) holds for all $s_1 \in [s^I(\bar{\theta}), s^I(\underline{\theta})]$, and the expected payoff is described as (15) if $s^I(\underline{\theta}) - s^I(\bar{\theta}) \leq 1/2\beta$, which holds under (7).

In what follows, we show that any off-path score bid is not a best response. Suppose that (7) holds, and that bidder 2 takes strategy (17). Note that (15) is a quadratic function which takes the optimum in $[s^I(\bar{\theta}), s^I(\underline{\theta})]$. We abuse notations and treat pseudo-valuations $v_i \in [\underline{v}, \bar{v}]$ as type. Suppose that bidder 1 submits $s_1 > s^I(\bar{v})$ and $s_1 > s^I(v_2) + \frac{1}{2\beta}$ with positive probability. Then, the winning probability $E[A(s_1 - s^I(v_2))] \leq \frac{1}{2} + \beta(s_1 - E[s^I(v_2)])$, so that bidder 1's expected payoff is bounded from above by (15). Thus, any score bid $s_1 > s^I(\bar{v})$ is not a best response for any v_1.

Now suppose that bidder 1 submits $s_1 < s^I(\underline{v})$. Define $\hat{s} \leq s^I(\underline{v})$ by

$$\hat{s} \equiv \frac{\bar{v} + \mu}{2} - \frac{1}{\beta}.$$

Thus, we have $s^I(\bar{v}) = \hat{s} + \frac{1}{2\beta}$. Then, the expected payoff π^I is given by (15) for $s_1 \geq \hat{s}$. Hence, suppose $s_1 < \hat{s}$. Bidder 1 loses with probability 1 if $s^I(v_2) > s_1 + \frac{1}{2\beta}$, and the expected payoff is given by[15]

$$\pi^I(s_1, v_1) = (v_1 - s_1) \int_{\underline{v}}^{s^{-1}(s_1 + \frac{1}{2\beta})} \left(\frac{1}{2} + \beta(s_1 - s^I(v_2)) \right) dG(v_2). \tag{18}$$

By differentiating with respect to s_1, we have

$$\begin{aligned} \frac{\partial \pi^I}{\partial s_1} &= \int_{\underline{v}}^{s^{-1}(s_1 + \frac{1}{2\beta})} \left(\beta(v_1 - s_1) - \frac{1}{2} - \beta(s_1 - s^I(v_2)) \right) dG(v_2) \\ &= \beta G\left(s^{-1}\left(s_1 + \frac{1}{2\beta} \right) \right) \left(v_1 + \frac{\mu + E^*[v_2 \mid s_1]}{2} - \frac{1}{\beta} - 2s_1 \right), \end{aligned} \tag{19}$$

where

$$E^*[v_2 \mid s_1] \equiv E\left[v_2 \,\middle|\, v_2 < s^{-1}\left(s_1 + \frac{1}{2\beta} \right) \right].$$

We can conclude that $s^I(v_1)$ is indeed a best response if $\partial \pi^I / \partial s_1 \geq 0$ for $s_1 \leq \hat{s}$.

Fix an arbitrary v_1 and let

$$h(x) \equiv v_1 + \frac{\mu + E^*[v_2 \mid x]}{2} - \frac{1}{\beta} - 2x. \tag{20}$$

Note that because $E^*[v_2 \mid \hat{s}] = \mu$, we have

$$h(\hat{s}) = v_1 + \mu - \frac{1}{\beta} - 2\hat{s} = v_1 - \bar{v} + \frac{1}{\beta} \geq 0$$

The second line follows from (7). In addition,

$$\begin{aligned} h'(x) &= \frac{1}{2G(s^{-1}(x + \frac{1}{2\beta}))^2} \underbrace{(s^{-1})'}_{=2} \cdot \left[s^{-1}(\cdot) g(s^{-1}(\cdot)) G(s^{-1}(\cdot)) - g(s^{-1}(\cdot)) \int_{\underline{v}}^{s^{-1}(\cdot)} v_2 dG(v_2) \right] - 2 \\ &= \frac{g(s^{-1}(x + \frac{1}{2\beta}))}{G(s^{-1}(x + \frac{1}{2\beta}))} \left[s^{-1}(x + \frac{1}{2\beta}) - E^*[v_2 \mid x] \right] - 2 \\ &= \frac{d}{dy} E[v_i \mid v_i < y] \Big|_{y = s^{-1}(x + \frac{1}{2\beta})} - 2. \end{aligned}$$

Hence, we have $h'(x) \leq 0$ under Assumption 4. Then, we have $h(s_1) \geq 0$ and $\partial \pi^I / \partial s_1 \geq 0$ for all $s_1 \leq \hat{s}$. □

[15] We assume $s_1 \geq s^I(\underline{v}) - \frac{1}{2\beta}$ so that bidder 1 wins with positive probability.

A.3 Proof of Theorem 3

Consider the FS auction. By abusing notations, the optimal quality for type $\theta = v^{-1}(v)$ is denoted by $q^*(v)$. Because the optimal quality is non-increasing in type θ, $q^*(v)$ is non-decreasing in v.

Bidders' pseudo-valuations are independently and identically distributed with G. Let H^I be the cumulative distribution of the winner's pseudo-valuation in the FS auction. Then, we have

$$\begin{aligned} H^I(x) &= G(x)^2 + 2\int_{\underline{v}}^{x}\int_{x}^{\bar{v}} \left(1 - A\left(\frac{v_1}{2} - \frac{v_2}{2}\right)\right) dG(v_1)dG(v_2) \\ &= G(x)^2 + \int_{\underline{v}}^{x}\int_{x}^{\bar{v}} (1 + \beta(v_2 - v_1))\, dG(v_1)dG(v_2). \end{aligned}$$

Thus, $H^I(x)$ is clearly decreasing in β because $v_2 < v_1$ in the integrand. Hence, it is clear that the expected social surplus increases with β by stochastic ordering. The expected score also increases with β because of the monotonicity of the score bid s^I in both v and β and the stochastic ordering. The expected quality is $E_{H^I}[q^*(v)]$, so that it increases with β by the stochastic ordering.

Consider the SS auction. The proof for the expected social surplus and expected quality is analogous to that of the FS auction, so that it is omitted. Given an arbitrary pseudo-valuation profile (v_1, v_2), bidder 1 wins with probability

$$A_1^{II}(v_1, v_2) \equiv A(v_1 - v_2) = \begin{cases} 0 & \text{if } v_1 - v_2 < -\frac{1}{2\beta} \\ \frac{1}{2} + \beta(v_1 - v_2) & \text{if } |v_1 - v_2| \leq \frac{1}{2\beta} \\ 1 & \text{if } v_1 - v_2 > \frac{1}{2\beta} \end{cases}. \tag{21}$$

Let

$$\phi(v) \equiv v - \frac{1 - G(v)}{g(v)} \tag{22}$$

be the virtual valuation function, which is strictly increasing by Assumption 2. By the standard calculation, the expected score in the SS auction, denoted by S^{II}, is expressed as

$$S^{II} = E\left[\sum_{i\in\{1,2\}} A_i^{II}(v_i, v_j)\phi(v_i)\right] - 2\underline{\Pi}^{II}, \tag{23}$$

where $\underline{\Pi}^{II}$ denotes the interim expected payoff of a bidder with pseudo-value $\underline{v}$ in equilibrium. The first term in (23) is clearly increasing in β.

In the rest of the proof, we show $\underline{\Pi}^{II}$ is decreasing in β. To show this, we first suppose

$$\beta \leq \frac{1}{2(\bar{v}-\underline{v})}. \tag{24}$$

Then, bidder 1 of pseudo-value $\underline{v}$ wins with probability $E[\frac{1}{2}+\beta(\underline{v}-v_2)] = \frac{1}{2}+\beta(\underline{v}-\mu)$. The expected "score payment" is:

$$\int_{\underline{v}}^{\bar{v}} \int_{-1/2\beta}^{\underline{v}-v_2} (\varepsilon + v_2)\beta g(v_2) d\varepsilon dv_2 = \frac{\beta}{2}\left(\underline{v}^2 - E[v^2]\right) - \frac{1}{8\beta} + \frac{\mu}{2}.$$

Thus, the expected payoff of pseudo-value $\underline{v}$ is:

$$\begin{aligned}\underline{\Pi}^{II} &= \left(\frac{1}{2}+\beta(\underline{v}-\mu)\right)\underline{v} - \frac{\beta}{2}\left(\underline{v}^2 - E[v^2]\right) + \frac{1}{8\beta} - \frac{\mu}{2} \\ &= \frac{\beta}{2}E[(v-\underline{v})^2] - \frac{1}{2}(\mu-\underline{v}) + \frac{1}{8\beta}.\end{aligned} \tag{25}$$

By differentiation, we have

$$\frac{d\underline{\Pi}^{II}}{d\beta} = \frac{1}{2}E[(v-\underline{v})^2] - \frac{1}{8\beta^2} < \frac{1}{2}(\bar{v}-\underline{v})^2 - \frac{1}{8\beta^2} \leq 0. \tag{26}$$

The last inequality follows from (24).

Next, we suppose

$$\frac{1}{2(\bar{v}-\underline{v})} < \beta \leq \frac{1}{\bar{v}-\underline{v}} \tag{27}$$

and let

$$\hat{v} \equiv \underline{v} + \frac{1}{2\beta}.$$

By $\varepsilon \geq -1/2\beta$, bidder 1 with pseudo-value $\underline{v}$ never wins if $v_2 \geq \hat{v}$. Thus, the expected payoff of the worst pseudo-value $\underline{v}$ is given by:

$$\begin{aligned}\underline{\Pi}^{II} &= \frac{\beta}{2}\int_{\underline{v}}^{\hat{v}} (v-\underline{v})^2 g(v)dv - \frac{1}{2}\int_{\underline{v}}^{\hat{v}} (v-\underline{v})g(v)dv + \frac{G(\hat{v})}{8\beta} \\ &= \frac{\beta}{2}\int_{\underline{v}}^{\hat{v}} (v-\hat{v})^2 g(v)dv.\end{aligned} \tag{28}$$

By differentiation, noting that $\hat{v} = \underline{v} + 1/2\beta$, we have

$$\begin{aligned}\frac{d\underline{\Pi}^{II}}{d\beta} &= \frac{1}{2}\int_{\underline{v}}^{\hat{v}} (v-\hat{v})^2 g(v)dv + \frac{\beta}{2}\cdot\frac{1}{2\beta^2}\int_{\underline{v}}^{\hat{v}} 2(v-\hat{v})g(v)dv \\ &= \frac{1}{2\beta}\int_{\underline{v}}^{\hat{v}} (v-\hat{v})\left(\beta(v-\hat{v})+1\right)g(v)dv \\ &< 0.\end{aligned}$$

The inequality follows because $\beta(v-\hat{v})+1 \geq \frac{1}{2}$ for all $v \leq \hat{v}$, which implies that the integrand is negative.

Therefore, the expected score S^{II} increases with β. $\square$

A.4 Proof of Theorem 4

Consider the FS auction. By the standard envelope argument, the equilibrium interim expected payoff of a bidder with pseudo-value v satisfies

$$\Pi^I(v) = \underline{\Pi}^I + \int_{\underline{v}}^{v} A^I(x)dx.$$

Thus, by the standard calculations, we have

$$\begin{aligned}E[P^I(v)] &= E\left[\Pi^I(v) + A^I(v)C^*(v)\right] \\ &= \underline{\Pi}^I + E\left[A^I(v)\left(C^*(v) + \frac{1-G(v)}{g(v)}\right)\right]\end{aligned}$$

and the expected price for the buyer in the FS auction is expressed by (9):

$$P^I = 2\underline{\Pi}^I + E\left[\sum_{i\in\{1,2\}} A_i^I(v_1,v_2)\psi(v_i)\right].$$

Suppose C_q/C_θ is non-decreasing in q; thus, it holds that

$$C_{qq}C_\theta - C_qC_{q\theta} \geq 0.$$

By differentiation, we have

$$\begin{aligned}\frac{d}{d\theta}C(q^*(\theta),\theta) &= (q^*)'(\theta)C_q(q^*(\theta),\theta) + C_\theta(q^*(\theta),\theta) \\ &= -\frac{C_{q\theta}(q^*,\theta)}{C_{qq}(q^*,\theta)}C_q(q^*,\theta) + C_\theta(q^*,\theta) \\ &\geq 0,\end{aligned}$$

where the second line follows from the implicit function theorem regarding the first order condition for the optimal quality (3). Hence, $C(q^*(\theta), \theta)$ is non-decreasing in θ. Therefore, because $v(\cdot)$ is decreasing in θ, C^* is non-increasing. By Assumption 2, the virtual cost function (10) is non-increasing, and the latter term of (9) weakly decreases with β.

By Theorem 2, we have

$$\begin{aligned}\underline{\Pi}^I &= \frac{1}{2}(1+\beta(\underline{v}-\mu))\left(\underline{v}-\frac{1}{2}\left(\underline{v}+\mu-\frac{1}{\beta}\right)\right)\\ &= \frac{\beta}{4}\left(\mu-\underline{v}-\frac{1}{\beta}\right)^2. \end{aligned} \tag{29}$$

By differentiation, we have

$$\frac{d\underline{\Pi}^I}{d\beta} = \frac{1}{4}\left((\mu-\underline{v})^2-\frac{1}{\beta^2}\right) < \frac{1}{4\beta^2}\left(\beta^2(\bar{v}-\underline{v})^2-1\right) \leq 0.$$

The last inequality follows from (7). Therefore, the expected price (9) strictly decreases with β.

The proof for the SS auction is analogous to that for the FS auction, so it is omitted.

□

A.5 Proof of Theorem 5

Given an arbitrary pseudo-valuation profile (v_1, v_2), bidder 1's winning probability A_1^{II} in the SS auction is given by (21) in the proof of Theorem 3. In the FS auction, bidder 1 wins with probability

$$A_1^I(v_1, v_2) = \frac{1}{2}\left(1+\beta(v_1-v_2)\right),$$

thus, we have

$$A_1^{II}(v_1,v_2) - A_1^I(v_1,v_2) = \begin{cases} \frac{\beta}{2}(v_1-v_2) & \text{if } |v_1-v_2| \leq \frac{1}{2\beta} \\ \frac{1}{2}-\frac{\beta}{2}(v_1-v_2) & \text{if } v_1-v_2 > \frac{1}{2\beta} \\ -\frac{1}{2}-\frac{\beta}{2}(v_1-v_2) & \text{if } v_1-v_2 < -\frac{1}{2\beta} \end{cases}, \tag{30}$$

and

$$A_1^{II}(v_1, v_2) - A_1^I(v_1, v_2) \begin{cases} \geq 0 & \text{if } v_1 \geq v_2 \\ \leq 0 & \text{if } v_1 \leq v_2 \end{cases} \tag{31}$$

by $\beta(\bar{v} - \underline{v}) \leq 1$.

Because the equilibrium social surplus is equal to $v(\theta)$ for a winner's type θ, it is clear that the SS auction selects the winner more efficiently and achieves a higher social surplus. Also, the equilibrium quality is determined solely by the winner's type or pseudo-valuation, and $q^*(v)$ is non-decreasing in pseudo-valuation v. Hence, the expected quality in the SS auction is greater than that in the FS auction. □

A.6 Proof of Theorem 6

The expected score in the SS auction is expressed by (23) in the proof of Theorem 3. Similarly, the expected score in the FS auction, denoted by S^I, is given by

$$S^I = E\left[\sum_{i \in \{1,2\}} A_i^I(v_1, v_2)\phi(v_i)\right] - 2\underline{\Pi}^I. \tag{32}$$

Hence, we have

$$S^{II} - S^I = E\left[\left(A_1^{II}(v_1, v_2) - A_1^I(v_1, v_2)\right)(\phi(v_1) - \phi(v_2))\right] + 2(\underline{\Pi}^I - \underline{\Pi}^{II}). \tag{33}$$

The first term is positive by the monotonicity of ϕ and (31). Hence, the expected score is higher for the SS auction than for the FS auction if $\underline{\Pi}^I \geq \underline{\Pi}^{II}$ holds.

As for the expected price, the expected final price in auction $k \in \{I, II\}$ is expressed by (9):

$$P^k = E\left[\sum_{i \in \{1,2\}} A_i^k(v_1, v_2)\psi(v_i)\right] + 2\underline{\Pi}^k.$$

Hence, we have

$$P^I - P^{II} = E\left[\left(A_1^I(v_1, v_2) - A_1^{II}(v_1.v_2)\right)(\psi(v_1) - \psi(v_2))\right] + 2(\underline{\Pi}^I - \underline{\Pi}^{II}). \tag{34}$$

Given that C_q/C_θ is non-decreasing and by Assumption 2, ψ is non-increasing and the former term of (34) is positive. Hence, the expected price is lower in the SS auction than in the FS auction if $\underline{\Pi}^I \geq \underline{\Pi}^{II}$.

In the rest of the proof, we show $\underline{\Pi}^I \geq \underline{\Pi}^{II}$. We first suppose

$$\beta \leq \frac{1}{2(\bar{v} - \underline{v})}.$$

Then, as shown in the proof of Theorem 3, the expected payoff of the worst pseudo-value for the SS auction is given by (25). For the FS auction, the expected payoff of the worst pseudo-value is given by (29). Hence,

$$\begin{aligned}
\underline{\Pi}^I - \underline{\Pi}^{II} &= \frac{\beta}{4}(\mu - \underline{v})^2 - \frac{1}{2}(\mu - \underline{v}) + \frac{1}{4\beta} - \frac{\beta}{2}E[(v - \underline{v})^2] + \frac{1}{2}(\mu - \underline{v}) - \frac{1}{8\beta} \\
&= \frac{1}{8\beta} - \frac{\beta}{4}\left((\mu - \underline{v})^2 + 2\sigma^2\right) \\
&> \frac{1}{8\beta} - \frac{\beta}{4}\left(\frac{1}{4\beta^2} + \frac{1}{8\beta^2}\right) \\
&> 0,
\end{aligned}$$

where σ^2 denotes the variance of pseudo-valuation distribution G. The inequality in the forth line follows from $(\mu - \underline{v})^2 < (\bar{v} - \underline{v})^2 \leq 1/4\beta^2$ and a fact $\sigma^2 \leq \frac{1}{16\beta^2}$.[16]

Next, we suppose

$$\frac{1}{2(\bar{v} - \underline{v})} < \beta \leq \frac{1}{\sqrt{2}(\bar{v} - \underline{v})}$$

and let

$$\hat{v} \equiv \underline{v} + \frac{1}{2\beta}.$$

By $\varepsilon \geq -1/2\beta$, bidder 1 with pseudo-value $\underline{v}$ never wins if $v_2 \geq \hat{v}$. The expected payoff of the worst pseudo-value $\underline{v}$ for the SS auction is given by (28). Hence,

$$\begin{aligned}
\underline{\Pi}^I - \underline{\Pi}^{II} &= \frac{\beta}{4}\left(\mu - \underline{v} - \frac{1}{\beta}\right)^2 - \frac{\beta}{2}\int_{\underline{v}}^{\hat{v}}\left(v - \underline{v} - \frac{1}{2\beta}\right)^2 dG(v) \\
&= \frac{\beta}{4}\int\left(\frac{1}{2\beta^2} - \mu^2 + 2v\underline{v} - \underline{v}^2 - 2(v^2 - \mu^2)\right) dG(v) + \frac{\beta}{2}\int_{\hat{v}}^{\bar{v}}(v - \hat{v})^2 dG(v) \\
&= \frac{\beta}{4}\left(\frac{1}{2\beta^2} + 2\int_{\hat{v}}^{\bar{v}}(v - \hat{v})^2 dG(v) - \left((\mu - \underline{v})^2 + 2\sigma^2\right)\right).
\end{aligned}$$

Note that the variance σ^2 is bounded by

$$\sigma^2 \leq (\bar{v} - \mu)(\mu - \underline{v}),$$

[16]For an arbitrary distribution over $[a, b]$, its variance σ^2 is bounded by $\sigma^2 \leq \frac{(b-a)^2}{4}$, which is known as the Popoviciu's inequality on variance.

which is known as the Bhatia-Davis inequality.[17] Hence,

$$\begin{aligned}(\mu-\underline{v})^2+2\sigma^2 &\le (\mu-\underline{v})^2+2(\bar{v}-\mu)(\mu-\underline{v})\\ &=(\bar{v}-\underline{v})^2-(\bar{v}-\mu)^2\\ &<(\bar{v}-\underline{v})^2.\end{aligned}$$

By $\beta(\bar{v}-\underline{v})\le 1/\sqrt{2}$, we have

$$\begin{aligned}\underline{\Pi}^I-\underline{\Pi}^{II} &= \frac{\beta}{4}\left(\frac{1}{2\beta^2}+2\int_{\hat{v}}^{\bar{v}}(v-\hat{v})^2dG(v)-\left((\mu-\underline{v})^2+2\sigma^2\right)\right)\\ &>\frac{\beta}{4}\left(\frac{1}{2\beta^2}-(\bar{v}-\underline{v})^2\right)\\ &\ge 0,\end{aligned}$$

which completes the proof. □

A.7 Proof of Proposition 1

Suppose that the other bidder takes the strategy s^I. Then, by (5) and differentiation, we have

$$\begin{aligned}\frac{\partial \pi^I(s_i,v_i)}{\partial s_i} &= (A^*)'(s_i)(v_i-s_i)-A^*(s_i)\\ &=(A^*)'(s_i)\left(v_i-s_i-\frac{A^*(s_i)}{(A^*)'(s_i)}\right).\end{aligned}$$

Because $(A^*)'(x)/A^*(x)$ is non-increasing in x,

$$s_i+\frac{A^*(s_i)}{(A^*)'(s_i)}$$

is strictly increasing in s_i. Hence, the strategy satisfying the first order condition (13) is the best response, and s^I is the symmetric equilibrium strategy.

[17] Note that

$$0\le E[(\bar{v}-v)(v-\underline{v})]=-E[v^2]+(\bar{v}+\underline{v})\mu-\bar{v}\underline{v}.$$

Hence, we have

$$\sigma^2=E[v^2]-\mu^2\le-\mu^2+(\bar{v}+\underline{v})\mu-\bar{v}\underline{v}=(\bar{v}-\mu)(\mu-\underline{v}).$$

It is straightforward to verify $0 < (s^I)'(v_i) \leq 1$ by differentiating (13). The comparison with the SS auction can be made in a manner analogous to Theorems 5 and 6. $\square$

B Other Conditions for Score and Price Ranking

In this appendix, we provide other conditions for the expected score and price being ranked. In what follows, we provide two sufficient conditions under which $\underline{\Pi}^I \geq \underline{\Pi}^{II}$ holds.

Suppose

$$\frac{1}{2(\bar{v} - \underline{v})} < \beta \leq \frac{1}{\bar{v} - \underline{v}} \tag{35}$$

and let

$$\hat{v} = \underline{v} + \frac{1}{2\beta}.$$

Then, as shown in the proof of Theorem 6,

$$\underline{\Pi}^I - \underline{\Pi}^{II} = \frac{\beta}{4}\left(\frac{1}{2\beta^2} + 2\int_{\hat{v}}^{\bar{v}} (v - \hat{v})^2 dG(v) - \left((\mu - \underline{v})^2 + 2\sigma^2\right)\right).$$

Let

$$h(\beta) \equiv 2\int_{\hat{v}}^{\bar{v}} (v - \hat{v})^2 dG(v) + \frac{1}{2\beta^2} \tag{36}$$

be a function of β. By differentiation, we have

$$\begin{aligned} h'(\beta) = \frac{2}{\beta^2}\left(\int_{\hat{v}}^{\bar{v}} (v - \hat{v}) dG(v) - \frac{1}{2\beta}\right) &\leq \frac{2}{\beta^2}\left((1 - G(\hat{v}))\,(\bar{v} - \hat{v}) - \frac{1}{2\beta}\right) \\ &\leq -\frac{G(\hat{v})}{\beta^3} \\ &< 0. \end{aligned}$$

The third line follows from (35), which induces $\bar{v} - \hat{v} \leq 1/2\beta$. Hence, $h(\beta)$ is decreasing in β. Because

$$\underline{\Pi}^I - \underline{\Pi}^{II} = \frac{\beta}{4}(h(\beta) - (\mu - \underline{v})^2 - 2\sigma^2),$$

we have $\underline{\Pi}^I \geq \underline{\Pi}^{II}$ for all β if it holds for $\beta = 1/(\bar{v} - \underline{v})$.

Condition 1 By the fact known as the Bhatia-Davis inequality, the variance σ^2 is bounded by $\sigma^2 \leq (\bar{v} - \mu)(\mu - \underline{v})$, so that

$$\begin{aligned}(\mu - \underline{v})^2 + 2\sigma^2 &\leq (\mu - \underline{v})^2 + 2(\bar{v} - \mu)(\mu - \underline{v}) \\ &= (\bar{v} - \underline{v})^2 - (\bar{v} - \mu)^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}h\left(\frac{1}{\bar{v} - \underline{v}}\right) - (\mu - \underline{v})^2 - 2\sigma^2 &\geq \frac{(\bar{v} - \underline{v})^2}{2} - (\bar{v} - \underline{v})^2 + (\bar{v} - \mu)^2 \\ &\geq (\bar{v} - \mu)^2 - \frac{(\bar{v} - \underline{v})^2}{2}.\end{aligned}$$

We have

$$(\bar{v} - \mu)^2 \geq \frac{(\bar{v} - \underline{v})^2}{2}$$

if

$$\mu \leq \bar{v} - \frac{\bar{v} - \underline{v}}{\sqrt{2}} = \frac{2 - \sqrt{2}}{2}\bar{v} + \frac{\sqrt{2}}{2}\underline{v}.$$

Condition 2 Suppose that the pseudo valuation v is symmetrically distributed: that is, $\mu = \frac{\bar{v} + \underline{v}}{2}$ and $g(\mu + x) = g(\mu - x)$ holds for all x. Then, we have

$$2\int_{\hat{v}}^{\bar{v}} (v - \hat{v})^2 dG(v) = 2\int_{\mu}^{\bar{v}} (v - \mu)^2 dG(v) = \sigma^2.$$

Hence, we have

$$\begin{aligned}h\left(\frac{1}{\bar{v} - \underline{v}}\right) - (\mu - \underline{v})^2 - 2\sigma^2 &\geq \frac{(\bar{v} - \underline{v})^2}{2} - (\frac{\bar{v} + \underline{v}}{2} - \underline{v})^2 - (\bar{v} - \frac{\bar{v} + \underline{v}}{2})(\frac{\bar{v} + \underline{v}}{2} - \underline{v}) \\ &= 0\end{aligned}$$

Therefore, we have shown the following result.

Proposition 2 *Suppose that all the assumptions hold with $\beta(\bar{v} - \underline{v}) \leq 1$. Then, the worst-type payoffs of the FS and SS auctions satisfy $\underline{\Pi}^I \geq \underline{\Pi}^{II}$ if either one of the following conditions holds:*

1. *$\mu \leq \frac{2-\sqrt{2}}{2}\bar{v} + \frac{\sqrt{2}}{2}\underline{v}$ holds, or*

2. *G is symmetrically distributed.*