

NECESSARY AND SUFFICIENT CONDITIONS FOR CONVERGENCE IN DISTRIBUTION OF QUANTILE AND P-P PROCESSES IN $L^1(0, 1)$

BRENDAN K. BEARE AND TETSUYA KAJI

ABSTRACT. We establish a necessary and sufficient condition for the quantile process based on iid sampling to converge in distribution in $L^1(0, 1)$. The condition is that the quantile function is locally absolutely continuous on the open unit interval and satisfies a slight strengthening of square integrability. We further establish a necessary and sufficient condition for the P-P process based on iid sampling from two populations to converge in distribution in $L^1(0, 1)$. The condition is that the P-P curve is locally absolutely continuous on the open unit interval. If either process converges in distribution then it may be approximated using the bootstrap.

1. INTRODUCTION

This article establishes necessary and sufficient conditions for quantile and procentile-procentile (P-P) processes constructed from independent and identically distributed (iid) samples to converge in distribution in $L^1(0, 1)$.¹ The more fundamental of the two processes is the quantile process. Given an empirical quantile function Q_n constructed from an iid sample of size n drawn from a distribution with quantile function Q , we will show that the quantile process $\sqrt{n}(Q_n - Q)$ converges in distribution in $L^1(0, 1)$ if and only if Q has the following property.

Property Q. The function $Q : (0, 1) \rightarrow \mathbb{R}$ is locally absolutely continuous and satisfies

$$\int_0^1 \sqrt{u(1-u)} dQ(u) < \infty. \quad (1.1)$$

Local absolute continuity of Q is defined to mean that the restriction of Q to each compact subinterval of $(0, 1)$ is absolutely continuous. The integral in (1.1) is defined in the Lebesgue-Stieltjes sense.² Requiring the integral to be finite is slightly more

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¹The notation $L^1(a, b)$ refers to the vector space of Borel measurable and integrable functions $h : (a, b) \rightarrow \mathbb{R}$, equipped with the norm $\|h\|_1 := \int_a^b |h(x)| dx$. Functions differing only on a null set are regarded to be equal as vectors in this space. Convergence in distribution in a normed space is defined in Section 2.1. Throughout this article, all vector spaces should be understood to be defined over the real field.

²See [Leoni \(2017\)](#), ch. 6 for a discussion of Lebesgue-Stieltjes measures and integrals. In particular, Theorem 6.7 therein associates every nondecreasing real function on an open interval with a unique Radon measure.

restrictive than requiring Q to be square integrable.³ We may rewrite (1.1) as

$$\int_{-\infty}^{\infty} \sqrt{F(x)(1-F(x))} dx < \infty, \quad (1.2)$$

where F is the cumulative distribution function (cdf) corresponding to Q . The integrals in (1.1) and (1.2) are equal by Lebesgue-Stieltjes substitution.

Let F_n be the empirical cdf corresponding to Q_n . Prior literature has established a necessary and sufficient condition for the empirical process $\sqrt{n}(F_n - F)$ to converge in distribution in $L^1(-\infty, \infty)$. Specifically, Theorem 2.1 in [del Barrio, Giné, and Matrán \(1999\)](#) establishes that $\sqrt{n}(F_n - F)$ converges in distribution in $L^1(-\infty, \infty)$ if and only if (1.2) is satisfied. Property **Q** is therefore sufficient, but not necessary, for said convergence in distribution. Our demonstration that Property **Q** is necessary and sufficient for $\sqrt{n}(Q_n - Q)$ to converge in distribution in $L^1(0, 1)$ makes it easy to construct examples in which $\sqrt{n}(F_n - F)$ converges in distribution in $L^1(-\infty, \infty)$ but $\sqrt{n}(Q_n - Q)$ does not converge in distribution in $L^1(0, 1)$. Any Q which is not locally absolutely continuous but satisfies (1.1) will do the job. A very simple example is obtained by choosing Q to be the quantile function for the Bernoulli distribution with success probability $p \in (0, 1)$. An example in which Q is continuous but not locally absolutely continuous is obtained by choosing Q to be the middle-thirds Cantor function or “Devil’s staircase”; see e.g. [Tao \(2011, p. 170\)](#). Either choice of Q is sufficiently irregular to ensure that $\sqrt{n}(Q_n - Q)$ does not converge in distribution in $L^1(0, 1)$. This may seem surprising since the sequence of real-valued random variables $\sqrt{n} \int_0^1 |Q_n(u) - Q(u)| du$ does converge in distribution for both of these choices of Q and, more generally, for any choice of Q satisfying (1.1). The latter convergence in distribution may be deduced from the fact that $\int_0^1 |Q_n(u) - Q(u)| du = \int_{-\infty}^{\infty} |F_n(x) - F(x)| dx$ for all n ; see Lemma 4.4 below.

Property **Q** is weaker than sufficient conditions for convergence in distribution of the quantile process in $L^1(0, 1)$ appearing in prior literature. Proposition 1.4 in [Kaji \(2018\)](#) establishes that the quantile process converges in distribution in $L^1(0, 1)$ if: (i) F has at most finitely many discontinuities and is elsewhere continuously differentiable with positive derivative, and (ii) $|Q|^{2+\delta}$ is integrable for some $\delta > 0$. Condition (i) implies, but is not implied by, local absolute continuity of Q , while condition (ii) implies, but is not implied by, (1.1). Lemma A.18 in the online appendix to [Weitkamp, Proksch, Tameling, and Munk \(2024\)](#) establishes that the quantile process converges in distribution in $L^1(0, 1)$ if: (a) F is supported on a compact interval $[a, b]$ with $-\infty < a < b < \infty$ and (b) F is continuously differentiable on $[a, b]$ with positive derivative on (a, b) . Condition (a) implies, but is not implied by, (1.1), while conditions (a) and (b) together imply, but are not implied by, local absolute continuity of Q .

Other literature has studied the convergence in distribution of the quantile process, or some special construction of the quantile process, in $L^2(0, 1)$. Consider the three special constructions defined for $u \in (0, 1)$ by

$$\sqrt{n}(Q_n(u) - Q(u)) \mathbb{1}(1/(n+1) < u < n/(n+1)), \quad (1.3)$$

$$\sqrt{n}(Q_n(u) - Q(\lfloor nu \rfloor + 1)/(n+1))) \quad \text{and} \quad (1.4)$$

³It is shown in [Hoeffding \(1973, pp. 64–5\)](#) that if Q satisfies (1.1) then $\int_0^1 Q(u)^2 du < \infty$, and that if there exists $\delta > 0$ such that $\int_0^1 Q(u)^2 [\log(1 + |Q(u)|)]^{1+\delta} du < \infty$ then Q satisfies (1.1).

$$\sqrt{n}(Q_n(u) - EQ_n(u)), \quad (1.5)$$

where $\lfloor \cdot \rfloor$ rounds down to the nearest integer. Theorems 1, 2, and 3 in [Mason \(1984\)](#) provide sufficient conditions for each of these special constructions of the quantile process to converge in distribution in $L^2(0, 1)$.⁴ The key requirement is that Q is locally absolutely continuous with density q satisfying

$$\int_0^1 u(1-u)q(u)^2 du < \infty. \quad (1.6)$$

This requirement is analogous to Property [Q](#), with the stronger norm on $L^2(0, 1)$ necessitating a stronger integrability condition. It is further assumed that F and q are continuous, and to obtain convergence in distribution of the processes in (1.4) and (1.5) it is assumed that there exist $\nu_1, \nu_2 \in \mathbb{R}$ and $M \in (0, \infty)$ such that $q(u) \leq Mu^{\nu_1}(1-u)^{\nu_2}$ for all $u \in (0, 1)$.

The task of establishing weak sufficient conditions for the quantile process $\sqrt{n}(Q_n - Q)$ to converge in distribution in $L^2(0, 1)$, without using any special construction to ameliorate troublesome behavior of Q near zero and one, is undertaken in [del Barrio, Giné, and Utzet \(2005\)](#). Theorem 4.6(i) therein establishes that the following conditions are collectively sufficient for $\sqrt{n}(Q_n - Q)$ to converge in distribution in $L^2(0, 1)$:

$$Q \text{ is locally absolutely continuous with density } q \text{ satisfying (1.6).} \quad (1.7)$$

$$F \text{ is twice differentiable on its open support } (a, b), \text{ with positive derivative } f. \quad (1.8)$$

$$\sup_{u \in (0, 1)} |u(1-u)q(u)^2 f'(Q(u))| < \infty. \quad (1.9)$$

$$\text{Either } a > -\infty \text{ or } \liminf_{u \downarrow 0} |uq(u)^2 f'(Q(u))| > 0. \quad (1.10)$$

$$\text{Either } b < \infty \text{ or } \liminf_{u \uparrow 1} |(1-u)q(u)^2 f'(Q(u))| > 0. \quad (1.11)$$

As mentioned above, (1.7) is analogous to Property [Q](#), our necessary and sufficient condition for convergence in distribution of the quantile process in $L^1(0, 1)$. No counterpart to (1.8)–(1.11) is required to obtain this convergence. Conditions (1.7)–(1.11) are not collectively necessary for the quantile process to converge in distribution in $L^2(0, 1)$. In particular, twice differentiability of F on its open support is not needed even to have convergence in distribution with respect to the uniform norm; see e.g. Lemma 21.4(ii) in [van der Vaart \(1998\)](#). The results of this article concern convergence in distribution in $L^1(0, 1)$, but future research might investigate whether our approach can be adapted to obtain necessary and sufficient conditions for convergence in distribution in $L^2(0, 1)$.

Our contribution is also closely related to earlier literature concerning the approximation of a special construction of the uniform quantile process under weighted L^p -norms. In particular, the following fact is established in [Csörgő, Horváth, and Shao \(1993\)](#). Let $U_{1,n} \leq U_{2,n} \leq \dots \leq U_{n,n}$ be the order statistics of n iid draws from the uniform

⁴The three processes in (1.3)–(1.5) are r_n^w , q_n^w , and p_n^w in the notation of [Mason \(1984\)](#), with the uniform weight function $w = 1$. [Mason \(1984\)](#) also studies weighted versions of the processes. Condition (1.6) here is Assumption (C) in [Mason \(1984\)](#) with $w = 1$.

distribution on $(0, 1)$, and let $U_n : (0, 1) \rightarrow \mathbb{R}$ be defined by

$$U_n(u) = U_{k,n}, \quad k/(n+2) < u \leq (k+1)/(n+2), \quad k = 0, \dots, n+1,$$

where $U_{0,n} = 0$ and $U_{n+1,n} = 1$. Let $p \in (0, \infty)$, and let $w : (0, 1) \rightarrow (0, \infty)$ be a Borel measurable function uniformly bounded on compact subintervals of $(0, 1)$. Then

$$\int_0^1 (u(1-u))^{p/2} w(u) du < \infty \quad (1.12)$$

if and only if we can define a sequence of Brownian bridges B_n such that

$$\int_0^1 |\sqrt{n}(u - U_n(u)) - B_n(u)|^p w(u) du = \text{op}(1). \quad (1.13)$$

This result is contained in Theorem 1.2 in [Csörgő, Horváth, and Shao \(1993\)](#); see also Theorem 1 in [Shorack and Wellner \(1986\)](#), p. 470). Note that U_n differs from the standard construction of an empirical quantile function, which is to take the generalized inverse of an empirical cdf, i.e. $Q_n(u) = \inf\{x : F_n(x) \geq u\}$. Instead, U_n has been defined in such a way that $U_n(u) = 0$ for all $u \in (0, 1/(n+2))$ and $U_n(u) = 1$ for all $u \in ((n+1)/(n+2), 1)$. If U_n were defined to be an empirical quantile function in the standard way then it would be uniformly bounded away from zero and one almost surely for all n , and consequently the integral in (1.13) would be almost surely infinite in cases where w is not integrable. The special construction of U_n is therefore critical to the equivalence of (1.12) and (1.13).

Our proof that the quantile process converges in distribution in $L^1(0, 1)$ if and only if Q has Property [Q](#) does not rely on any special construction of the quantile process. Nor does the uniform quantile process play an important role. To prove the sufficiency of Property [Q](#) we take as a starting point the convergence in distribution of $\sqrt{n}(F_n - F)$ in $L^1(-\infty, \infty)$ supplied by Theorem 2.1(a) in [del Barrio, Giné, and Matrán \(1999\)](#). From this we deduce the convergence in distribution of $\sqrt{n}(Q_n - Q)$ in $L^1(0, 1)$ by applying the delta method. The validity of a bootstrap approximation to $\sqrt{n}(Q_n - Q)$ in $L^1(0, 1)$ under Property [Q](#) is established as a byproduct of the delta method. We use a separate argument to prove the necessity of Property [Q](#), drawing on Theorem 2.1(b) in [del Barrio, Giné, and Matrán \(1999\)](#), which establishes that $\sqrt{n} \int_{-\infty}^{\infty} |F_n(x) - F(x)| dx$ is stochastically bounded if and only if (1.2) is satisfied.

Our application of the delta method is complicated by the fact that cdfs are not integrable over the real line and therefore do not belong to the space $L^1(-\infty, \infty)$. While $\sqrt{n}(F_n - F)$ takes values in $L^1(-\infty, \infty)$ if (1.2) is satisfied, the fact that neither F_n nor F belong to $L^1(-\infty, \infty)$ renders the standard definition of Hadamard differentiability—see e.g. [van der Vaart \(1998, ch. 20\)](#)—unsuitable for our purpose. To circumvent this difficulty we rely on a generalized version of the delta method introduced in [Beutner and Zähle \(2010\)](#) and developed further in [Krätschmer, Schied, and Zähle \(2015\)](#) and [Beutner and Zähle \(2016, 2018\)](#); see also [Volgushev and Shao \(2014\)](#). We provide a precise statement of this version of the delta method in Section 2.3.

The second main contribution of this article concerns the P-P process. Let F_n be the empirical cdf constructed from an iid sample of size n drawn from a distribution with cdf F , and let Q_n be the empirical quantile function constructed from an iid sample of size n drawn from a distribution with cdf G and quantile function Q , with the two samples assumed to be independent of one another. (We assume the two sample sizes

to be equal to simplify our discussion in this section but will later allow them to differ.) We call the map $R : (0, 1) \rightarrow [0, 1]$ defined by $R(u) = F(Q(u))$ the P-P curve, we call the map $R_n : (0, 1) \rightarrow [0, 1]$ defined by $R_n(u) = F_n(Q_n(u))$ the empirical P-P curve, and we call $\sqrt{n/2}(R_n - R)$ the P-P process. We will show that the P-P process converges in distribution in $L^1(0, 1)$ if and only if R is locally absolutely continuous.

Prior literature on the P-P process has focused on approximation with respect to a uniform or weighted uniform norm, rather than on convergence in distribution in $L^1(0, 1)$. The best available result for the uniform norm appears to be Theorem 3.1 in [Aly, Csörgő, and Horváth \(1987\)](#), which requires a Čibisov-O'Reilly condition to be satisfied. A Borel measurable function $w : (0, 1) \rightarrow (0, \infty)$ is said to be a Čibisov-O'Reilly weight function if it is uniformly bounded away from zero on compact subintervals of $(0, 1)$, nondecreasing in a neighborhood of zero and nonincreasing in a neighborhood of one, and satisfies

$$\int_0^1 (u(1-u))^{-1} \exp(-\delta(u(1-u))^{-1}w(u)^2) du < \infty \quad \text{for all } \delta > 0. \quad (1.14)$$

Theorem 3.1 in [Aly, Csörgő, and Horváth \(1987\)](#) establishes that if F and G are continuous and admit continuous positive densities f and g on their open supports, and if

$$\sup_{u \in (0,1)} \frac{f(Q(u))}{g(Q(u))} w(u) < \infty$$

for some Čibisov-O'Reilly weight function w , then we may define a sequence of pairs of independent Brownian bridges $(B_{1,n}, B_{2,n})$ such that

$$\sup_{u \in (0,1)} \left| \sqrt{n}(R_n(u) - R(u)) - B_{1,n}(R(u)) + \frac{f(Q(u))}{g(Q(u))} B_{2,n}(u) \right| = o_P(1). \quad (1.15)$$

Condition (1.14) is optimal in the sense that if

$$\liminf_{u \downarrow 0} \frac{f(Q(u))}{g(Q(u))} w(u) > 0 \quad \text{or} \quad \liminf_{u \uparrow 1} \frac{f(Q(u))}{g(Q(u))} w(u) > 0$$

for some Borel measurable $w : (0, 1) \rightarrow (0, \infty)$ that is bounded on compact subintervals of $(0, 1)$, nondecreasing in a neighborhood of zero and nonincreasing in a neighborhood of one, but not satisfying (1.14), then (1.15) cannot be satisfied. See also Theorem 2.2 in [Hsieh and Turnbull \(1996\)](#) for a closely related result in which the supremum in (1.15) is taken over a compact subinterval of $(0, 1)$.

Our demonstration that the P-P process converges in distribution in $L^1(0, 1)$ if and only if R is locally absolutely continuous contrasts with the result of [Aly, Csörgő, and Horváth \(1987\)](#) just described because there is no counterpart to the Čibisov-O'Reilly condition. The reason for this may be understood by considering the term $[f(Q(u))/g(Q(u))]B_{2,n}(u)$ appearing in (1.15). The role of the Čibisov-O'Reilly condition is to control the behavior of this term for u close to zero or one, which is necessary when seeking to approximate the P-P process under the uniform norm. Less regularity near zero and one is needed to establish convergence in distribution in $L^1(0, 1)$. The key requirement is integrability, which is automatically satisfied because the Brownian bridge has uniformly bounded sample paths and $f(Q)/g(Q)$ is a density for the uniformly bounded function R . In fact, we do not require densities for F and G to obtain

convergence in distribution in $L^1(0, 1)$, or even require F or G to be continuous; only a density for R is required.

Another result closely related to ours is Theorem 5.1(a) in [Pyke and Shorack \(1968\)](#). It pertains to a modified version of the P-P process based on the quantile functions \tilde{Q}_n and \tilde{Q} corresponding to the pooled cdfs $(F_n + G_n)/2$ and $(F + G)/2$. Theorem 5.1(a) establishes that if F and G are continuous then $\sqrt{n}(F_n(\tilde{Q}_n) - F(\tilde{Q}))$ converges in distribution in $L^1(0, 1)$.⁵ Note that $F(\tilde{Q})$ is guaranteed to be locally absolutely continuous because the Lebesgue-Stieltjes measure generated by F is absolutely continuous with respect to the one generated by $(F + G)/2$. The absence of any explicit requirement of local absolute continuity in the statement of Theorem 5.1(a) is therefore explained by the pooling of samples. Pooling also has the effect of ensuring that $F(\tilde{Q})$ has density no greater than 2. If, in addition to requiring that F and G are continuous, we require further that $F(\tilde{Q})$ is continuously differentiable on $(0, 1)$, with derivative having one-sided limits at zero and one, then Theorem 4.1(a) in [Pyke and Shorack \(1968\)](#) establishes that the convergence in distribution of $\sqrt{n}(F_n(\tilde{Q}_n) - F(\tilde{Q}))$ in $L^1(0, 1)$ continues to hold when the norm on $L^1(0, 1)$ is strengthened to the uniform norm. It is unnecessary to explicitly require a Čibisov-O'Reilly condition to be satisfied, or to specify that the one-sided limits of the derivative of $F(\tilde{Q})$ at zero and one must be finite, because the pooling of samples already guarantees a uniformly bounded density for $F(\tilde{Q})$. See also Theorem 3.3 in [Aly, Csörgő, and Horváth \(1987\)](#).

The preceding discussion of related literature has focused on a small number of contributions which we view to be the most closely connected to what is done in this article. There is an enormous volume of literature concerning the asymptotic behavior of the quantile process and other related processes. We make no attempt at a survey, but point to [Csörgő \(1983\)](#) and [Csörgő and Horváth \(1993\)](#) for detailed treatments of the quantile process with a particular emphasis on strong approximations with respect to a uniform or weighted uniform norm. We also point to [Zwingmann and Holzmann \(2020\)](#) for a recent treatment of the quantile process based on the concepts of epi- and hypoconvergence introduced in [Bücher, Segers, and Volgushev \(2014\)](#).

The remainder of this article is structured as follows. Section 2 contains necessary mathematical background, with Sections 2.1, 2.2, and 2.3 respectively concerning convergence in distribution in normed spaces, bootstrap approximation, and the delta method. Section 3 establishes conditions under which the generalized inverse mapping is quasi-Hadamard differentiable in the sense explained in Section 2.3. Much of the work needed to prove the sufficiency of our conditions for convergence in distribution in $L^1(0, 1)$ takes place in the proof of Theorem 3.1. Building on this result, Section 4 establishes that Property Q is necessary and sufficient for convergence in distribution of the quantile process in $L^1(0, 1)$, and Section 5 establishes that local absolute continuity of the P-P curve is necessary and sufficient for convergence in distribution of the P-P process in

⁵Theorem 5.1(a) in [Pyke and Shorack \(1968\)](#) is more general than described here because it permits different sample sizes and a weighted L^1 -norm. For the purposes of the present discussion we take, in the notation of [Pyke and Shorack \(1968\)](#), ν to be the Lebesgue measure, $q = 1$, and $m = n$. Consequently $\lambda_N = \lambda_0 = 1/2$ and the a.e. differentiability condition required in Theorem 5.1(a) is redundant. Moreover L'_N may be replaced with L_N because $\|L'_N - L_N\|_\nu = (1/N)|L_N(1/N)| \leq 2/\sqrt{N}$.

$L^1(0, 1)$. Sections 4 and 5 also contain material concerning the bootstrap approximation of quantile and P-P processes in $L^1(0, 1)$. Appendix A addresses some technical details regarding measurability which arise in Sections 4 and 5. Appendix B states well-known properties of the generalized inverse to be applied in Sections 3, 4, and 5.

2. MATHEMATICAL BACKGROUND

In this section we introduce notation and terminology, and present necessary mathematical background for subsequent sections. Much of our discussion amounts to a summary of essential material from [Beutner and Zähle \(2016\)](#).

2.1. Convergence in distribution. Following [Beutner and Zähle \(2016\)](#), we adopt the general approach of [Pollard \(1984\)](#) toward convergence in distribution, wherein measurability with respect to the ball σ -algebra is used to handle technical complications associated with nonseparable spaces. We specialize the setting from that of a metric space to that of a normed space, which suffices for our purposes. While the prominent role played in this article by the L^1 -norm means that we often work with separable spaces, our treatment of the P-P process in Section 5 will require us also to work with nonseparable spaces.

Let \mathbf{E} be a normed space. We define the *Borel σ -algebra* on \mathbf{E} to be the σ -algebra generated by the open subsets of \mathbf{E} , and we define the *ball σ -algebra* on \mathbf{E} to be the σ -algebra generated by the open balls in \mathbf{E} . If \mathbf{E} is separable then its Borel and ball σ -algebras coincide, but in general the two may differ. We denote by \mathcal{B} and \mathcal{B}° the Borel and ball σ -algebras on \mathbf{E} , and assign the notation $\mathcal{B}(\mathbb{R})$ to the Borel σ -algebra on \mathbb{R} .

Definition 2.1 (Random variable). A map $\xi : \Omega \rightarrow \mathbf{E}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be an *\mathbf{E} -valued random variable* if it is $(\mathcal{F}, \mathcal{B}^\circ)$ -measurable; that is, if $\xi^{-1}B \in \mathcal{F}$ for all $B \in \mathcal{B}^\circ$. We use the notation $\text{law}(\xi)$ to refer to the probability measure $\mathbb{P}\xi^{-1}$ on $(\mathbf{E}, \mathcal{B}^\circ)$. If $\mathbf{E} = \mathbb{R}$ then we may simply say that ξ is a random variable.

Let BL_1° be the set of all $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurable functions $f : \mathbf{E} \rightarrow \mathbb{R}$ satisfying $|f(x)| \leq 1$ and $|f(x) - f(y)| \leq \|x - y\|_{\mathbf{E}}$ for all $x, y \in \mathbf{E}$. Let \mathcal{M}_1° be the set of all probability measures on $(\mathbf{E}, \mathcal{B}^\circ)$. Define the map $d_{\text{BL}}^\circ : \mathcal{M}_1^\circ \times \mathcal{M}_1^\circ \rightarrow [0, \infty)$ by

$$d_{\text{BL}}^\circ(\mu, \nu) := \sup_{f \in \text{BL}_1^\circ} \left| \int f d\mu - \int f d\nu \right|.$$

As discussed in [Beutner and Zähle \(2016, p. 1209\)](#), d_{BL}° is a pseudo-metric on \mathcal{M}_1° , or a metric if \mathbf{E} is separable. In the latter case d_{BL}° is called the bounded Lipschitz metric.

Definition 2.2 (Convergence in distribution). Let ξ be an \mathbf{E} -valued random variable, and (ξ_n) a sequence of \mathbf{E} -valued random variables. We say that ξ_n converges in distribution[◦] to ξ in \mathbf{E} if

$$\lim_{n \rightarrow \infty} d_{\text{BL}}^\circ(\text{law}(\xi_n), \text{law}(\xi)) = 0.$$

We write “ $\xi_n \rightsquigarrow^\circ \xi$ in \mathbf{E} ” as shorthand for “ ξ_n converges in distribution[◦] to ξ in \mathbf{E} ”. If \mathbf{E} is separable then we may write “converges in distribution” and “ \rightsquigarrow ” in place of “converges in distribution[◦]” and “ \rightsquigarrow° ”.

See [Beutner and Zähle \(2016\)](#), p. 1209) for a portmanteau theorem providing equivalent reformulations of convergence in distribution[◦].

2.2. Bootstrap versions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and \mathbf{V} a vector space, and let θ_n be a map from Ω into \mathbf{V} . We have in mind a statistical context in which θ_n is determined by a sample of size n and \mathbb{P} represents the uncertainty introduced by randomly drawing this sample from some population. In order to make precise the general idea of a bootstrap counterpart to θ_n we introduce a second probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and define a third probability space by

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}'), \quad (2.1)$$

where the symbol \otimes is used to define a product σ -algebra or product measure in the usual way. We think of the second probability measure \mathbb{P}' as representing the uncertainty introduced by some bootstrap procedure; for instance, by the random selection of multinomial weights for the Efron bootstrap. A bootstrap counterpart to θ_n ought to be influenced by both sampling uncertainty and the random variation inherent to bootstrapping, so it is natural to view it as a map $\theta_n^* : \bar{\Omega} \rightarrow \mathbf{V}$.

The following definition provides two ways to make precise the idea that the bootstrap “works”. It is an amalgamation of Definitions 2.1 and 2.2 in [Beutner and Zähle \(2016\)](#). The notation \mathbb{P}^{out} refers to the outer measure generated by \mathbb{P} .

Definition 2.3 (Bootstrap versions). Let \mathbf{V} be a vector space and \mathbf{E} a normed subspace of \mathbf{V} . Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ be probability spaces, and define $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ as in (2.1). Let (θ_n) be a sequence of maps from Ω into \mathbf{V} , and (θ_n^*) a sequence of maps from $\bar{\Omega}$ into \mathbf{V} . Let θ be a vector in \mathbf{V} . Let (a_n) be a sequence of positive real numbers tending to infinity. Require that $a_n(\theta_n - \theta)$ is an \mathbf{E} -valued random variable on Ω for each n , and that

$$a_n(\theta_n - \theta) \rightsquigarrow^\circ \xi \quad \text{in } \mathbf{E} \quad (2.2)$$

for some \mathbf{E} -valued random variable ξ . Require further that $a_n(\theta_n^* - \theta_n)$ is an \mathbf{E} -valued random variable on $\bar{\Omega}$ for each n . We say that (θ_n^*) is:

(i) *almost surely a bootstrap version of (θ_n) w.r.t. the convergence in (2.2)* if

$$a_n(\theta_n^*(\omega, \cdot) - \theta_n(\omega)) \rightsquigarrow^\circ \xi \quad \text{in } \mathbf{E} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

(ii) *a bootstrap version in outer probability of (θ_n) w.r.t. the convergence in (2.2)* if

$$\lim_{n \rightarrow \infty} \mathbb{P}^{\text{out}} \{ \omega \in \Omega : d_{\text{BL}}^\circ(\text{law}(a_n(\theta_n^*(\omega, \cdot) - \theta_n(\omega))), \text{law}(\xi)) \geq \epsilon \} = 0 \quad \text{for all } \epsilon > 0.$$

If \mathbf{E} is separable then in case (ii) we may replace \mathbb{P}^{out} with \mathbb{P} , and say instead that (θ_n^*) is a *bootstrap version in probability of (θ_n) w.r.t. the convergence in (2.2)*.

See [Beutner and Zähle \(2016\)](#) for an explanation of why we may replace \mathbb{P}^{out} with \mathbb{P} in case (ii) if \mathbf{E} is separable, and for further discussion of other technical issues related to Definition 2.3. In particular, Lemma D.2 in [Beutner and Zähle \(2016\)](#) shows that the mapping $\omega' \mapsto a_n(\theta_n^*(\omega, \omega') - \theta_n(\omega))$ is an \mathbf{E} -valued random variable on Ω' for every $\omega \in \Omega$.

2.3. The delta method. The version of the delta method we require for our treatment of quantile and P-P processes relies on a generalization of Hadamard differentiability called quasi-Hadamard differentiability. We define it as in Definition C.3 in [Beutner and Zähle \(2016\)](#).

Definition 2.4 (Quasi-Hadamard differentiability). Let \mathbf{V} and $\tilde{\mathbf{E}}$ be vector spaces, with $\tilde{\mathbf{E}}$ equipped with a norm $\|\cdot\|_{\tilde{\mathbf{E}}}$. Let \mathbf{E} be a subspace of \mathbf{V} equipped with a norm $\|\cdot\|_{\mathbf{E}}$. Let $f : \mathbf{V}_f \rightarrow \tilde{\mathbf{E}}$ be a map defined on a subset \mathbf{V}_f of \mathbf{V} . Given a vector $\theta \in \mathbf{V}_f$ and a set $\mathbf{E}_0 \subseteq \mathbf{E}$, we say that f is *quasi-Hadamard differentiable* at θ tangentially to $\mathbf{E}_0\langle \mathbf{E} \rangle$ if there is a continuous map $\dot{f}_\theta : \mathbf{E}_0 \rightarrow \tilde{\mathbf{E}}$ such that

$$\lim_{n \rightarrow \infty} \left\| \dot{f}_\theta(h) - \frac{f(\theta + t_n h_n) - f(\theta)}{t_n} \right\|_{\tilde{\mathbf{E}}} = 0$$

for each $h \in \mathbf{E}_0$, each vanishing sequence (t_n) of positive real numbers, and each sequence (h_n) of vectors in \mathbf{E} satisfying $\theta + t_n h_n \in \mathbf{V}_f$ for each n and $\|h_n - h\|_{\mathbf{E}} \rightarrow 0$. In this case the map \dot{f}_θ is called the *quasi-Hadamard derivative* of f at θ . If $\mathbf{E}_0 = \mathbf{E} = \mathbf{V}$ then we may omit “tangentially to $\mathbf{E}_0\langle \mathbf{E} \rangle$ ” and simply say that f is quasi-Hadamard differentiable at θ .

For our purposes, the critical distinction between quasi-Hadamard differentiability and the ordinary notion of Hadamard differentiability—as defined in, for instance, [van der Vaart \(1998, ch. 20\)](#)—is that quasi-Hadamard differentiability does not require the norm controlling the behavior of each sequence (h_n) to be defined at each point in the domain of f . See [Beutner and Zähle \(2010\)](#) for an extended discussion of the practical relevance of this distinction, focusing on applications involving weighted uniform norms. A second distinction between the two notions of differentiability is that the approximating map \dot{f}_θ is required to be linear in the definition of Hadamard differentiability, but not in the definition of quasi-Hadamard differentiability. The second distinction plays no role in the applications of the delta method in this article, as \dot{f}_θ will always be linear.

We now provide a precise statement of the version of the delta method we will use to establish the sufficiency of our conditions for convergence in distribution of the quantile and P-P processes in $L^1(0, 1)$. The following result is Theorem 3.1 in [Beutner and Zähle \(2016\)](#).

Theorem 2.1 (Delta method). *Let \mathbf{V} be a vector space, and \mathbf{E} a normed subspace of \mathbf{V} . Let $\tilde{\mathbf{E}}$ be a separable normed space. Let \mathcal{B}° be the ball σ -algebra on \mathbf{E} , and $\tilde{\mathcal{B}}$ the Borel σ -algebra on $\tilde{\mathbf{E}}$. Let \mathbf{E}_0 be a separable subspace of \mathbf{E} belonging to \mathcal{B}° . Let $f : \mathbf{V}_f \rightarrow \tilde{\mathbf{E}}$ be a map defined on a subset \mathbf{V}_f of \mathbf{V} . Let θ be a vector in \mathbf{V}_f . Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ be probability spaces, and define $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ as in (2.1). Let (θ_n) be a sequence of maps from Ω into \mathbf{V}_f , and (θ_n^*) a sequence of maps from $\bar{\Omega}$ into \mathbf{V}_f . Let (a_n) be a sequence of positive real numbers tending to infinity. Consider the following conditions:*

(a) $a_n(\theta_n - \theta)$ is an \mathbf{E} -valued random variable on Ω for each n , and

$$a_n(\theta_n - \theta) \rightsquigarrow^\circ \xi \quad \text{in } \mathbf{E} \tag{2.3}$$

for some \mathbf{E} -valued random variable ξ taking values only in \mathbf{E}_0 .

(b) $f(\theta_n)$ is an $\tilde{\mathbf{E}}$ -valued random variable on Ω .

- (c) f is quasi-Hadamard differentiable at θ tangentially to $\mathbf{E}_0\langle\mathbf{E}\rangle$.
- (d) The quasi-Hadamard derivative $\dot{f}_\theta : \mathbf{E}_0 \rightarrow \tilde{\mathbf{E}}$ can be extended from \mathbf{E}_0 to \mathbf{E} such that the extension $\dot{f}_\theta : \mathbf{E} \rightarrow \tilde{\mathbf{E}}$ is linear and $(\mathcal{B}^\circ, \tilde{\mathcal{B}})$ -measurable, and continuous at every point of \mathbf{E}_0 .
- (e) $f(\theta_n^*)$ is an $\tilde{\mathbf{E}}$ -valued random variable on $\bar{\Omega}$.
- (f) $a_n(\theta_n^* - \theta)$ and $a_n(\theta_n^* - \theta_n)$ are \mathbf{E} -valued random variables on $\bar{\Omega}$ for each n , and (θ_n^*) is almost surely a bootstrap version of (θ_n) w.r.t. the convergence in (2.3).
- (f') $a_n(\theta_n^* - \theta)$ and $a_n(\theta_n^* - \theta_n)$ are \mathbf{E} -valued random variables on $\bar{\Omega}$ for each n , and (θ_n^*) is a bootstrap version in outer probability of (θ_n) w.r.t. the convergence in (2.3).

The following assertions hold:

- (i) If conditions (a), (b), and (c) hold, then $a_n(f(\theta_n) - f(\theta))$ is an $\tilde{\mathbf{E}}$ -valued random variable on Ω for each n , $\dot{f}_\theta(\xi)$ is an $\tilde{\mathbf{E}}$ -valued random variable, and

$$a_n(f(\theta_n) - f(\theta)) \rightsquigarrow \dot{f}_\theta(\xi) \quad \text{in } \tilde{\mathbf{E}}. \quad (2.4)$$

- (ii) If conditions (a), (b), (c), (d), (e), and (f) hold, then $a_n(f(\theta_n^*) - f(\theta_n))$ is an $\tilde{\mathbf{E}}$ -valued random variable on $\bar{\Omega}$ for each n , and $(f(\theta_n^*))$ is a bootstrap version in probability of $(f(\theta_n))$ w.r.t. the convergence in (2.4).
- (iii) Assertion (ii) still holds when condition (f) is replaced by (f').

Conditions (a), (b), (c), (d), (e), (f), and (f') in Theorem 2.1 have the same meaning as the corresponding conditions in Theorem 3.1 in Beutner and Zähle (2016). Note that several measurability conditions appearing there are subsumed here into Definition 2.1.

3. QUASI-HADAMARD DIFFERENTIABILITY OF THE GENERALIZED INVERSE

In this section we establish that a mapping from cdfs to quantile functions is quasi-Hadamard differentiable under very mild conditions. The precise statement is given in Theorem 3.1 below. We say that the conditions for quasi-Hadamard differentiability are very mild because in Section 4 we will use Theorem 3.1 to establish that Property **Q** is sufficient for convergence in distribution of the quantile process in $L^1(0, 1)$, and then establish separately that Property **Q** is also necessary for such convergence.

Let $D(-\infty, \infty)$ be the vector space of real càdlàg functions on \mathbb{R} , and define the set

$$\mathbb{D}(-\infty, \infty) := \left\{ F \in D(-\infty, \infty) : F \text{ is nondecreasing, } \lim_{n \rightarrow \infty} F(-n) = 0, \lim_{n \rightarrow \infty} F(n) = 1 \right\}.$$

Thus $\mathbb{D}(-\infty, \infty)$ is the set of all cdfs on \mathbb{R} . For each $F \in \mathbb{D}(-\infty, \infty)$, we denote by F^{-1} the map from $(0, 1)$ into \mathbb{R} defined by

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\} \quad \text{for all } u \in (0, 1). \quad (3.1)$$

We call F^{-1} the *generalized inverse* of F , or sometimes the *quantile function* for F .

Define the set

$$\mathbb{D}_1 := \left\{ G \in \mathbb{D}(-\infty, \infty) : G^{-1} \in L^1(0, 1) \right\}.$$

Throughout the remainder of this section we fix a cdf $F \in \mathbb{D}_1$, and define $Q := F^{-1}$. Define

$$\alpha_F = \inf\{x \in \mathbb{R} : F(x) > 0\}, \quad \beta_F = \sup\{x \in \mathbb{R} : F(x) < 1\}. \quad (3.2)$$

and $\mathbb{D}_F := \{G \in \mathbb{D}_1 : G(x) = 0 \text{ for all } x < \alpha_F \text{ and } G(x) = 1 \text{ for all } x \geq \beta_F\}$.

Note that α_F and β_F need not be finite, and that $F \in \mathbb{D}_F$ by construction. Let $\phi : \mathbb{D}_F \rightarrow L^1(0, 1)$ be the map defined for $G \in \mathbb{D}_F$ by $\phi(G) = G^{-1}$.

Our objective in this section is to verify that if Q is locally absolutely continuous then ϕ is quasi-Hadamard differentiable at F . While Q must be integrable because $F \in \mathbb{D}_1$, we do not require Q to satisfy (1.1). Local absolute continuity implies that Q has a nonnegative and locally integrable density $q : (0, 1) \rightarrow \mathbb{R}$, uniquely defined up to null sets. The density q need not be integrable because Q need not be uniformly bounded.

In order to make precise the sense in which ϕ is quasi-Hadamard differentiable at F we must specify the vector spaces \mathbf{V} , \mathbf{E} , and $\tilde{\mathbf{E}}$ and the tangent set \mathbf{E}_0 appearing in Definition 2.4. We choose $\mathbf{V} = D(-\infty, \infty)$ and $\tilde{\mathbf{E}} = L^1(0, 1)$. Note that $D(-\infty, \infty)$ has not been equipped with a norm. Let $D^1(-\infty, \infty)$ be the vector space of integrable càdlàg functions $h : \mathbb{R} \rightarrow \mathbb{R}$, and equip $D^1(-\infty, \infty)$ with the norm $\|h\|_1 := \int_{-\infty}^{\infty} h(x)dx$. This makes $D^1(-\infty, \infty)$ a separable normed subspace of $D(-\infty, \infty)$. We choose $\mathbf{E} = D^1(-\infty, \infty)$. Let $C[0, 1]$ be the vector space of continuous functions $g : [0, 1] \rightarrow \mathbb{R}$ equipped with the uniform norm. We choose \mathbf{E}_0 to be the vector subspace of $D^1(-\infty, \infty)$ given by

$$T_F := \{h \in D^1(-\infty, \infty) : h = g(F) \text{ for some } g \in C[0, 1] \text{ such that } g(0) = g(1) = 0\}.$$

Theorem 3.1. *Let F be a function in \mathbb{D}_1 such that $Q := F^{-1}$ is locally absolutely continuous, and let q be a density for Q . Then $h(Q)q \in L^1(0, 1)$ for each $h \in D^1(-\infty, \infty)$. Moreover, $\phi : \mathbb{D}_F \rightarrow L^1(0, 1)$ is quasi-Hadamard differentiable at F tangentially to $T_F \langle D^1(-\infty, \infty) \rangle$, with quasi-Hadamard derivative $\dot{\phi}_F : T_F \rightarrow L^1(0, 1)$ satisfying $\dot{\phi}_F(h) = -h(Q)q$ for each $h \in T_F$. The last equality defines a continuous and linear extension of $\dot{\phi}_F$ to all of $D^1(-\infty, \infty)$.*

Our proof of Theorem 3.1 is somewhat long. This is largely due to the fact that we have placed no structure on the quantile function Q beyond integrability and local absolute continuity. A much shorter proof could be provided if we were to assume, say, that Q admits a density q that is uniformly bounded away from zero and infinity. Such an assumption must be avoided because it is not implied by Property Q. The purpose of Theorem 3.1 is to help us prove that Property Q is sufficient for convergence in distribution of the quantile process in $L^1(0, 1)$.

Proof of Theorem 3.1. Before establishing the quasi-Hadamard differentiability of ϕ we will verify that $h(Q)q \in L^1(0, 1)$ for every $h \in D^1(-\infty, \infty)$, and that $h \mapsto -h(Q)q$ is continuous and linear as a map from $D^1(-\infty, \infty)$ into $L^1(0, 1)$. By Lemma B.2 we have

$$\int_0^1 |h(Q(u))|dQ(u) = \int_{\alpha_F}^{\beta_F} |h(Q(F(x)))|dx \quad \text{for each } h \in D^1(-\infty, \infty).$$

We may replace $dQ(u)$ with $q(u)du$ in the first integral because Q is locally absolutely continuous with density q . Local absolute continuity of Q implies continuity of Q , and $0 < F(x) < 1$ for all $x \in (\alpha_F, \beta_F)$, so by Lemma B.1(iii) we may replace $Q(F(x))$ with

x in the second integral. Consequently we have

$$\int_0^1 |h(Q(u))|q(u)du = \int_{\alpha_F}^{\beta_F} |h(x)|dx \quad \text{for each } h \in D^1(-\infty, \infty). \quad (3.3)$$

Thus $h(Q)q \in L^1(0, 1)$ for each $h \in D^1(-\infty, \infty)$. Linearity of $h \mapsto -h(Q)q$ as a map from $D^1(-\infty, \infty)$ into $L^1(0, 1)$ is obvious. Continuity of this map follows from the fact that $\|h(Q)q\|_1 \leq \|h\|_1$ for each $h \in D^1(-\infty, \infty)$, by (3.3).

For the remainder of the proof we fix a function $h \in T_F$, a vanishing sequence (t_n) of positive real numbers, and a sequence (h_n) of functions in $D^1(-\infty, \infty)$ satisfying $F + t_n h_n \in \mathbb{D}_F$ for each n and $\|h_n - h\|_1 \rightarrow 0$. Our task is to show that

$$\|t_n^{-1}[(F + t_n h_n)^{-1} - Q] + h(Q)q\|_1 \rightarrow 0. \quad (3.4)$$

Define $F_n := F + t_n h_n$, $Q_n := F_n^{-1}$, and $\xi_n := t_n^{-1}(Q_n - Q) \in L^1(0, 1)$. Then we have

$$\xi_n(u) = t_n^{-1} \int_{\alpha_F}^{\beta_F} [\mathbb{1}(F(x) \geq u) - \mathbb{1}(F_n(x) \geq u)]dx \quad \text{for each } u \in (0, 1). \quad (3.5)$$

To understand why (3.5) is true, observe that for each $u \in (0, 1)$ we have

$$\int_{-\infty}^{Q(u) \vee Q_n(u)} \mathbb{1}(F(x) \geq u)dx = Q(u) \vee Q_n(u) - Q(u)$$

and

$$\int_{-\infty}^{Q(u) \vee Q_n(u)} \mathbb{1}(F_n(x) \geq u)dx = Q(u) \vee Q_n(u) - Q_n(u).$$

Taking the difference gives

$$\int_{-\infty}^{Q(u) \vee Q_n(u)} [\mathbb{1}(F(x) \geq u) - \mathbb{1}(F_n(x) \geq u)]dx = Q_n(u) - Q(u).$$

To verify (3.5) it suffices to show that the lower and upper limits of integration in the last integral may be replaced with α_F and β_F . Since $F, F_n \in \mathbb{D}_F$, we have $\mathbb{1}(F(x) \geq u) = \mathbb{1}(F_n(x) \geq u) = 0$ for all $x < \alpha_F$. Therefore we may replace the lower limit of integration with α_F . We also have $\mathbb{1}(F(x) \geq u) = \mathbb{1}(F_n(x) \geq u) = 1$ for all $x \geq Q(u) \vee Q_n(u)$ by Lemma B.1(i), and also for all $x \geq \beta_F$ because $F, F_n \in \mathbb{D}_F$. Therefore we may replace the upper limit of integration with β_F .

We now further simplify the expression for $\xi_n(u)$ given in (3.5). Define $g_n := h_n(Q)$, so that $F_n(Q) = F(Q) + t_n g_n$. For all $x \in (\alpha_F, \beta_F)$ we have $0 < F(x) < 1$ and thus, by Lemma B.1(iii), $Q(F(x)) = x$. Thus we may rewrite (3.5) as

$$\xi_n(u) = t_n^{-1} \int_{\alpha_F}^{\beta_F} [\mathbb{1}(F(x) \geq u) - \mathbb{1}(F(x) + t_n g_n(F(x)) \geq u)]dx \quad \text{for each } u \in (0, 1).$$

Now by applying Lemma B.2 we obtain

$$\xi_n(u) = t_n^{-1} \int_0^1 [\mathbb{1}(v \geq u) - \mathbb{1}(v + t_n g_n(v) \geq u)]dQ(v) \quad \text{for each } u \in (0, 1), \quad (3.6)$$

which will be more convenient to work with than (3.5).

Let \tilde{g} be a function in $C[0, 1]$ satisfying $\tilde{g}(F) = h$. Such a function exists because $h \in T_F$. Let g be the restriction of \tilde{g} to $(0, 1)$. The left-hand side of (3.4) may be

rewritten as $\|\xi_n + h(Q)q\|_1$, which we need to show converges to zero. We will show that it suffices to establish this convergence with $h(Q)$ replaced by g and with ξ_n replaced by the map $\zeta_n : (0, 1) \rightarrow \mathbb{R}$ defined by

$$\zeta_n(u) := t_n^{-1} \int_0^1 [\mathbb{1}(v \geq u) - \mathbb{1}(v + t_n g(v) \geq u)] dQ(v). \quad (3.7)$$

Observe first that

$$\|h(Q)q - gq\|_1 = \int_0^1 |h(Q(u)) - g(u)| dQ(u) = \int_{\alpha_F}^{\beta_F} |h(Q(F(x))) - h(x)| dx = 0,$$

where the second equality follows from Lemma B.2 and the third from Lemma B.1(iii). Observe next that

$$\begin{aligned} \|\xi_n - \zeta_n\|_1 &= t_n^{-1} \int_0^1 \left| \int_0^1 [\mathbb{1}(v + t_n g_n(v) \geq u) - \mathbb{1}(v + t_n g(v) \geq u)] dQ(v) \right| du \\ &\leq t_n^{-1} \int_0^1 \int_0^1 |\mathbb{1}(v + t_n g_n(v) \geq u) - \mathbb{1}(v + t_n g(v) \geq u)| du dQ(v) \\ &\leq t_n^{-1} \int_0^1 \int_{-\infty}^{\infty} |\mathbb{1}(v + t_n g_n(v) \geq u) - \mathbb{1}(v + t_n g(v) \geq u)| du dQ(v) \\ &= \int_0^1 |g_n(v) - g(v)| dQ(v), \end{aligned}$$

where the first equality follows from (3.6) and (3.7), and the first inequality is obtained by applying Fubini's theorem. Now we apply Lemma B.2 to the last integral to obtain

$$\begin{aligned} \|\xi_n - \zeta_n\|_1 &\leq \int_{\alpha_F}^{\beta_F} |g_n(F(x)) - g(F(x))| dx = \int_{\alpha_F}^{\beta_F} |h_n(Q(F(x))) - h(x)| dx \\ &= \int_{\alpha_F}^{\beta_F} |h_n(x) - h(x)| dx \leq \|h_n - h\|_1 \rightarrow 0, \end{aligned}$$

with the second equality following from Lemma B.1(iii), and the convergence to zero holding by construction. Since $\|\xi_n - \zeta_n\|_1 \rightarrow 0$ and $\|h(Q)q - gq\|_1 = 0$, if $\|\zeta_n + gq\|_1 \rightarrow 0$ then $\|\xi_n + h(Q)q\|_1 \rightarrow 0$. The proof is complete if we can show that $\|\zeta_n + gq\|_1 \rightarrow 0$.

Showing that $\|\zeta_n + gq\|_1 \rightarrow 0$ will take some work. We will need to introduce notation for several new functions. Define the function $\chi_n : (0, 1)^2 \rightarrow \{0, 1\}$ by

$$\chi_n(u, v) := \mathbb{1}(v + t_n g(v) \wedge 0 < u \leq v + t_n g(v) \vee 0).$$

For each u and v in $(0, 1)$ we have

$$\text{sgn}(g(v))\chi_n(u, v) = \mathbb{1}(v + t_n g(v) \geq u) - \mathbb{1}(v \geq u),$$

where sgn is the signum function. It therefore follows from (3.7) that

$$\zeta_n(u) = -t_n^{-1} \int_0^1 \chi_n(u, v) \text{sgn}(g(v)) dQ(v). \quad (3.8)$$

For each $\epsilon > 0$, define the function $\chi_{n,\epsilon} : (0, 1)^2 \rightarrow \mathbb{R}$ by

$$\chi_{n,\epsilon}(u, v) := \begin{cases} \chi_n(u, v) / \int_0^1 \chi_n(u, w) dw & \text{if } \int_0^1 \chi_n(u, w) dw \geq t_n \sqrt{\epsilon} \\ 0 & \text{otherwise,} \end{cases}$$

and the function $\zeta_{n,\epsilon} : (0, 1) \rightarrow \mathbb{R}$ by

$$\zeta_{n,\epsilon}(u) := - \int_0^1 \chi_{n,\epsilon}(u, v) g(v) dQ(v).$$

It will be useful to establish some inequalities involving the function χ_n . First observe that, for each $v \in (0, 1)$, $\chi_n(\cdot, v)$ is the indicator function of a set contained in an interval of length $t_n |g(v)|$. Thus

$$\int_0^1 \chi_n(u, v) du \leq t_n |g(v)| \quad \text{for all } n \in \mathbb{N}, v \in (0, 1). \quad (3.9)$$

For each $\epsilon > 0$, define

$$\delta_\epsilon := \sup\{\delta \in [0, 1) : |g(v) - g(u)| \leq \epsilon \text{ for all } u, v \in (0, 1) \text{ such that } |v - u| \leq \delta\}$$

and $N_{\epsilon,1} := \inf\{N \in \mathbb{N} : t_n |g(u)| < \delta_\epsilon \text{ for all } n \geq N, u \in (0, 1)\}$.

We have $\delta_\epsilon > 0$ for every $\epsilon > 0$ because g is uniformly continuous, and thus $N_{\epsilon,1} < \infty$ for every $\epsilon > 0$ because g is uniformly bounded and $t_n \rightarrow 0$. Observe that

$$|v - u| \leq t_n |g(v)| \quad \text{for all } n \in \mathbb{N} \text{ and } u, v \in (0, 1) \text{ such that } \chi_n(u, v) \neq 0. \quad (3.10)$$

Consequently,

$$|g(v) - g(u)| \leq \epsilon \quad \text{for all } \epsilon > 0, n \geq N_{\epsilon,1} \text{ and } u, v \in (0, 1) \text{ such that } \chi_n(u, v) \neq 0. \quad (3.11)$$

Let λ denote the Lebesgue measure. We deduce from (3.11) that for each $\epsilon > 0$, each $n \geq N_{\epsilon,1}$ and each $u \in (0, 1)$,

$$\begin{aligned} \int_0^1 \chi_n(u, v) dv &= \int_0^1 \mathbb{1}(|g(v) - g(u)| \leq \epsilon) \chi_n(u, v) dv \\ &= \lambda\{v \in (0, 1) : v + t_n g(v) \wedge 0 < u \leq v + t_n g(v) \vee 0, |g(v) - g(u)| \leq \epsilon\} \\ &\leq \lambda\{v \in (0, 1) : v + t_n(g(u) - \epsilon) \wedge 0 < u \leq v + t_n(g(u) + \epsilon) \vee 0\} \\ &= \lambda\{v \in (0, 1) : u - t_n(g(u) + \epsilon) \vee 0 \leq v < u - t_n(g(u) - \epsilon) \wedge 0\} \\ &\leq \lambda\{v \in (0, 1) : u - t_n g(u) \vee 0 - t_n \epsilon \leq v < u - t_n g(u) \wedge 0 + t_n \epsilon\}. \end{aligned} \quad (3.12)$$

Thus

$$\int_0^1 \chi_n(u, v) dv \leq t_n |g(u)| + 2t_n \epsilon \quad \text{for all } \epsilon > 0, n \geq N_{\epsilon,1}, u \in (0, 1). \quad (3.13)$$

Similar to (3.12), for each $\epsilon > 0$, each $n \geq N_{\epsilon,1}$ and each $u \in (0, 1)$, we have

$$\int_0^1 \chi_n(u, v) dv \geq \lambda\{v \in (0, 1) : u - t_n g(u) \vee 0 + t_n \epsilon \leq v < u - t_n g(u) \wedge 0 - t_n \epsilon\}. \quad (3.14)$$

For each $\epsilon > 0$, define

$$N_{\epsilon,2} := \inf\{N \in \mathbb{N} : t_n |g(u)| < \epsilon \text{ for all } n \geq N, u \in (0, 1)\}.$$

We have $N_{\epsilon,2} < \infty$ for each $\epsilon > 0$ because g is uniformly bounded and $t_n \rightarrow 0$. For each $\epsilon \in (0, 1/2)$, each $n \geq N_{\epsilon,2}$, and each $u \in (\epsilon, 1 - \epsilon)$ we have

$$u - t_n g(u) \vee 0 + t_n \epsilon > 0 \quad \text{and} \quad u - t_n g(u) \wedge 0 - t_n \epsilon < 1,$$

and thus, if also $n \geq N_{\epsilon,1}$, deduce from (3.14) that

$$\int_0^1 \chi_n(u, v) dv \geq \lambda \{v \in \mathbb{R} : u - t_n g(u) \vee 0 + t_n \epsilon \leq v < u - t_n g(u) \wedge 0 - t_n \epsilon\}.$$

Consequently, we have

$$\int_0^1 \chi_n(u, v) dv \geq t_n |g(u)| - 2t_n \epsilon \quad \text{for all } \epsilon \in (0, 1/2), n \geq N_{\epsilon,1} \vee N_{\epsilon,2}, u \in (\epsilon, 1 - \epsilon). \quad (3.15)$$

Observe further that, in view of (3.10), we have

$$\int_0^\epsilon \chi_n(u, v) du = 0 \quad \text{for all } \epsilon \in (0, 1/2), n \geq N_{\epsilon,2}, v \in (2\epsilon, 1), \quad (3.16)$$

and similarly

$$\int_{1-\epsilon}^1 \chi_n(u, v) du = 0 \quad \text{for all } \epsilon \in (0, 1/2), n \geq N_{\epsilon,2}, v \in (0, 1 - 2\epsilon). \quad (3.17)$$

We will also need to suitably control integrals involving the function $\chi_{n,\epsilon}$. From (3.9) and the definition of $\chi_{n,\epsilon}$ we deduce that

$$\int_0^1 \chi_{n,\epsilon}(u, v) du \leq \epsilon^{-1/2} |g(v)| \quad \text{for all } \epsilon > 0, n \in \mathbb{N}, v \in (0, 1). \quad (3.18)$$

For each $\epsilon \in (0, 1/2)$ define

$$A_\epsilon := \{u \in (\epsilon, 1 - \epsilon) : |g(u)| > 2\epsilon + \sqrt{\epsilon}\} \quad (3.19)$$

and define $A_\epsilon^c := (0, 1) \setminus A_\epsilon$. The bound (3.15) implies that

$$\int_0^1 \chi_n(u, v) dv \geq t_n \sqrt{\epsilon} \quad \text{for all } \epsilon \in (0, 1/2), n \geq N_{\epsilon,1} \vee N_{\epsilon,2}, u \in A_\epsilon. \quad (3.20)$$

We therefore deduce from the definition of $\chi_{n,\epsilon}$ that

$$\int_0^1 \chi_{n,\epsilon}(u, v) dv = 1 \quad \text{for all } \epsilon \in (0, 1/2), n \geq N_{\epsilon,1} \vee N_{\epsilon,2}, u \in A_\epsilon. \quad (3.21)$$

We now return to showing that $\|\zeta_n + gq\|_1 \rightarrow 0$. Let ϵ be a point in $(0, 1/2)$ and let n be an integer satisfying $n \geq N_{\epsilon,1} \vee N_{\epsilon,2}$. Observe that

$$\begin{aligned} \int_{A_\epsilon^c} |\zeta_n(u)| du &= \int_{A_\epsilon^c} \left| t_n^{-1} \int_0^1 \chi_n(u, v) \text{sgn}(g(v)) dQ(v) \right| du && \text{by (3.8)} \\ &\leq t_n^{-1} \int_0^1 \int_{A_\epsilon^c} \chi_n(u, v) du dQ(v) && \text{by Fubini's theorem} \\ &\leq t_n^{-1} \int_0^1 \int_0^1 \mathbb{1}(|g(u)| \leq 2\epsilon + \sqrt{\epsilon}) \chi_n(u, v) du dQ(v) \end{aligned}$$

$$\begin{aligned}
& + t_n^{-1} \int_0^1 \int_0^\epsilon \chi_n(u, v) du dQ(v) \\
& + t_n^{-1} \int_0^1 \int_{1-\epsilon}^1 \chi_n(u, v) du dQ(v) \quad \text{by (3.19).} \tag{3.22}
\end{aligned}$$

We bound the three terms in the upper bound in (3.22) in turn. For the first term, we have

$$\int_0^1 \mathbb{1}(|g(u)| \leq 2\epsilon + \sqrt{\epsilon}) \chi_n(u, v) du \leq \int_0^1 \mathbb{1}(|g(v)| \leq 3\epsilon + \sqrt{\epsilon}) \chi_n(u, v) du$$

by (3.11), and thus

$$\begin{aligned}
t_n^{-1} \int_0^1 \int_0^1 \mathbb{1}(|g(u)| \leq 2\epsilon + \sqrt{\epsilon}) \chi_n(u, v) du dQ(v) \\
\leq \int_0^1 \mathbb{1}(|g(v)| \leq 3\epsilon + \sqrt{\epsilon}) |g(v)| dQ(v) \tag{3.23}
\end{aligned}$$

by (3.9). For the second term, we have

$$\begin{aligned}
t_n^{-1} \int_0^1 \int_0^\epsilon \chi_n(u, v) du dQ(v) & = t_n^{-1} \int_0^{2\epsilon} \int_0^\epsilon \chi_n(u, v) du dQ(v) \quad \text{by (3.16)} \\
& \leq \int_0^{2\epsilon} |g(v)| dQ(v) \quad \text{by (3.9).} \tag{3.24}
\end{aligned}$$

For the third term, a similar argument using (3.17) in place of (3.16) shows that

$$t_n^{-1} \int_0^1 \int_{1-\epsilon}^1 \chi_n(u, v) du dQ(v) \leq \int_{1-2\epsilon}^1 |g(v)| dQ(v). \tag{3.25}$$

Next observe that, by (3.19),

$$\begin{aligned}
\int_{A_\epsilon^c} |g(u)| dQ(u) & \leq \int_0^1 \mathbb{1}(|g(u)| \leq 2\epsilon + \sqrt{\epsilon}) |g(u)| dQ(u) + \int_0^\epsilon |g(u)| dQ(u) \\
& \quad + \int_{1-\epsilon}^1 |g(u)| dQ(u). \tag{3.26}
\end{aligned}$$

Combining (3.22)–(3.26), we obtain

$$\begin{aligned}
\int_{A_\epsilon^c} |\zeta_n(u) + g(u)q(u)| du & \leq 2 \int_0^1 \mathbb{1}(|g(u)| \leq 3\epsilon + \sqrt{\epsilon}) |g(u)| dQ(u) \\
& \quad + 2 \int_0^{2\epsilon} |g(u)| dQ(u) + 2 \int_{1-2\epsilon}^1 |g(u)| dQ(u). \tag{3.27}
\end{aligned}$$

To bound the integral of $|\zeta_n + gq|$ over A_ϵ we bound separately the integrals of $|\zeta_n - \zeta_{n,\epsilon}|$ and $|\zeta_{n,\epsilon} + gq|$ over A_ϵ . First we bound the former integral. From (3.8) and the definition of $\zeta_{n,\epsilon}$ we have

$$\zeta_n(u) - \zeta_{n,\epsilon}(u) = - \int_0^1 [t_n^{-1} \chi_n(u, v) - \chi_{n,\epsilon}(u, v) |g(v)|] \text{sgn}(g(v)) dQ(v)$$

for all $u \in A_\epsilon$, and thus by Fubini's theorem

$$\int_{A_\epsilon} |\zeta_n(u) - \zeta_{n,\epsilon}(u)| du \leq \int_0^1 \int_{A_\epsilon} |t_n^{-1} \chi_n(u, v) - |g(v)| \chi_{n,\epsilon}(u, v)| du dQ(v). \quad (3.28)$$

In view of (3.20) and the definition of $\chi_{n,\epsilon}$, the inner integral on the right-hand side satisfies

$$\int_{A_\epsilon} |t_n^{-1} \chi_n(u, v) - |g(v)| \chi_{n,\epsilon}(u, v)| du = \int_{A_\epsilon} \chi_{n,\epsilon}(u, v) \left| t_n^{-1} \int_0^1 \chi_n(u, w) dw - |g(v)| \right| du.$$

Thus, by the triangle inequality,

$$\begin{aligned} \int_{A_\epsilon} |t_n^{-1} \chi_n(u, v) - |g(v)| \chi_{n,\epsilon}(u, v)| du &\leq \int_{A_\epsilon} \chi_{n,\epsilon}(u, v) \left| |g(u)| - |g(v)| \right| du \\ &\quad + \int_{A_\epsilon} \chi_{n,\epsilon}(u, v) \left| t_n^{-1} \int_0^1 \chi_n(u, w) dw - |g(u)| \right| du. \end{aligned} \quad (3.29)$$

The definition of $\chi_{n,\epsilon}$ implies that $\chi_{n,\epsilon}(u, v) = 0$ whenever $\chi_n(u, v) = 0$, so from (3.11) we have

$$\int_{A_\epsilon} \chi_{n,\epsilon}(u, v) \left| |g(u)| - |g(v)| \right| du \leq \epsilon \int_{A_\epsilon} \chi_{n,\epsilon}(u, v) du. \quad (3.30)$$

From (3.13) and (3.15) we have

$$\left| t_n^{-1} \int_0^1 \chi_n(u, w) dw - |g(u)| \right| \leq 2\epsilon \quad \text{for all } u \in A_\epsilon,$$

and thus

$$\int_{A_\epsilon} \chi_{n,\epsilon}(u, v) \left| t_n^{-1} \int_0^1 \chi_n(u, w) dw - |g(u)| \right| du \leq 2\epsilon \int_{A_\epsilon} \chi_{n,\epsilon}(u, v) du. \quad (3.31)$$

Consequently,

$$\begin{aligned} \int_{A_\epsilon} |t_n^{-1} \chi_n(u, v) - |g(v)| \chi_{n,\epsilon}(u, v)| du &\leq 3\epsilon \int_{A_\epsilon} \chi_{n,\epsilon}(u, v) du \quad \text{by (3.29)–(3.31)} \\ &\leq 3\sqrt{\epsilon} |g(v)| \quad \text{by (3.18).} \end{aligned} \quad (3.32)$$

Combining (3.28) and (3.32), we obtain

$$\int_{A_\epsilon} |\zeta_n(u) - \zeta_{n,\epsilon}(u)| du \leq 3\sqrt{\epsilon} \int_0^1 |g(u)| dQ(u). \quad (3.33)$$

Next we bound the integral of $|\zeta_{n,\epsilon} + gq|$ over A_ϵ . From (3.21) and the definition of $\zeta_{n,\epsilon}$ we have

$$\zeta_{n,\epsilon}(u) + g(u)q(u) = - \int_0^1 \chi_{n,\epsilon}(u, v) [g(v)q(v) - g(u)q(u)] dv$$

for all $u \in A_\epsilon$, and thus

$$\int_{A_\epsilon} |\zeta_{n,\epsilon}(u) + g(u)q(u)| du \leq \int_{A_\epsilon} \int_0^1 \chi_{n,\epsilon}(u, v) |g(v)q(v) - g(u)q(u)| dv du. \quad (3.34)$$

Define the function $f : (0, 1) \rightarrow \mathbb{R}$ by $f(u) := g(u)q(u)$. Since $f \in L^1(0, 1)$, there exists (see e.g. Theorem 1.3.20 in Tao, 2011) a uniformly continuous function $f_\epsilon : (0, 1) \rightarrow \mathbb{R}$

such that $\|f_\epsilon - f\|_1 \leq \epsilon$. Using the triangle inequality and Fubini's theorem, we deduce from (3.34) that

$$\begin{aligned} \int_{A_\epsilon} |\zeta_{n,\epsilon}(u) + g(u)q(u)| du &\leq \int_0^1 \int_{A_\epsilon} \chi_{n,\epsilon}(u,v) |f_\epsilon(v) - f(v)| du dv \\ &\quad + \int_{A_\epsilon} \int_0^1 \chi_{n,\epsilon}(u,v) |f_\epsilon(u) - f(u)| dv du \\ &\quad + \int_{A_\epsilon} \int_0^1 \chi_{n,\epsilon}(u,v) |f_\epsilon(v) - f_\epsilon(u)| dv du. \end{aligned}$$

From (3.18) we have

$$\int_0^1 \int_{A_\epsilon} \chi_{n,\epsilon}(u,v) |f_\epsilon(v) - f(v)| du dv \leq \epsilon^{-1/2} \int_0^1 |g(v)| |f_\epsilon(v) - f(v)| dv \leq \sqrt{\epsilon} \sup_{v \in (0,1)} |g(v)|,$$

and from (3.21) we have

$$\int_{A_\epsilon} \int_0^1 \chi_{n,\epsilon}(u,v) |f_\epsilon(u) - f(u)| dv du = \int_{A_\epsilon} |f_\epsilon(v) - f(v)| dv \leq \epsilon.$$

Thus

$$\begin{aligned} \int_{A_\epsilon} |\zeta_{n,\epsilon}(u) + g(u)q(u)| du &\leq \sqrt{\epsilon} \sup_{u \in (0,1)} |g(u)| + \epsilon \\ &\quad + \int_{A_\epsilon} \int_0^1 \chi_{n,\epsilon}(u,v) |f_\epsilon(v) - f_\epsilon(u)| dv du. \end{aligned} \quad (3.35)$$

The inequalities (3.27), (3.33), and (3.35) collectively establish the following fact: For every $\epsilon \in (0, 1/2)$, there exists an integer N_ϵ and a uniformly continuous function $f_\epsilon : (0, 1) \rightarrow \mathbb{R}$ such that, for every $n \geq N_\epsilon$, we have

$$\begin{aligned} \|\zeta_n + gq\|_1 &\leq 2 \int_0^1 \mathbb{1}(|g(u)| \leq 3\epsilon + \sqrt{\epsilon}) |g(u)| dQ(u) + 2 \int_0^{2\epsilon} |g(u)| dQ(u) \\ &\quad + 2 \int_{1-2\epsilon}^1 |g(u)| dQ(u) + 3\sqrt{\epsilon} \int_0^1 |g(u)| dQ(u) + \sqrt{\epsilon} \sup_{u \in (0,1)} |g(u)| + \epsilon \\ &\quad + \int_{A_\epsilon} \int_0^1 \chi_{n,\epsilon}(u,v) |f_\epsilon(v) - f_\epsilon(u)| dv du. \end{aligned}$$

The first six terms on the right-hand side of this inequality do not depend on n . Since $h(Q)q \in L^1(0, 1)$ (established at the beginning of the proof) we know that $\int_0^1 |g(u)| dQ(u) < \infty$. Thus, by the monotone convergence theorem, the first four terms can be made arbitrarily small by choosing ϵ sufficiently small. Since g is uniformly bounded, the fifth term can also be made arbitrarily small by choosing ϵ sufficiently small. The sixth term is simply ϵ . If we can show that the seventh and final term converges to zero as $n \rightarrow \infty$ for every $\epsilon \in (0, 1/2)$ then we are done.

Fix arbitrary real numbers $\eta > 0$ and $\epsilon \in (0, 1/2)$. Since f_ϵ is uniformly continuous, there exists a positive real number $\kappa_{\epsilon,\eta}$ such that

$$|f_\epsilon(v) - f_\epsilon(u)| \leq \eta \quad \text{for all } u, v \in (0, 1) \text{ such that } |v - u| \leq \kappa_{\epsilon,\eta}. \quad (3.36)$$

Define

$$M_{\epsilon,\eta} := \inf\{N \in \mathbb{N} : t_n|g(u)| \leq \kappa_{\epsilon,\eta} \text{ for all } n \geq N, u \in (0, 1)\}.$$

We have $M_{\epsilon,\eta} < \infty$ because g is uniformly bounded and $t_n \rightarrow 0$. We deduce from (3.10) that

$$|v - u| \leq \kappa_{\epsilon,\eta} \quad \text{for all } n \geq M_{\epsilon,\eta} \text{ and } u, v \in (0, 1) \text{ such that } \chi_n(u, v) \neq 0. \quad (3.37)$$

The definition of $\chi_{n,\epsilon}$ implies that $\chi_{n,\epsilon}(u, v) = 0$ whenever $\chi_n(u, v) = 0$. Therefore, for all $n \geq M_{\epsilon,\eta}$, we have

$$\begin{aligned} \int_{A_\epsilon} \int_0^1 \chi_{n,\epsilon}(u, v) |f_\epsilon(v) - f_\epsilon(u)| dv du &\leq \eta \int_{A_\epsilon} \int_0^1 \chi_{n,\epsilon}(u, v) dv du \quad \text{by (3.36) and (3.37)} \\ &= \eta \int_{A_\epsilon} 1 du \leq \eta \quad \text{by (3.21).} \end{aligned}$$

Since η may be chosen arbitrarily small, we are done. \square

4. THE QUANTILE PROCESS

Following some preliminary discussion in Section 4.1, we will establish in Section 4.2 that Property **Q** is sufficient for convergence in distribution of the quantile process in $L^1(0, 1)$ and justifies the use of the bootstrap. The necessity of Property **Q** is established in Section 4.3.

4.1. Preliminary discussion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and (X_n) a sequence of random variables $X_n : \Omega \rightarrow \mathbb{R}$. Assume the sequence (X_n) to be iid with cdf $F \in \mathbb{D}(-\infty, \infty)$, and define $Q := F^{-1}$. For each $n \in \mathbb{N}$ let $F_n : \Omega \rightarrow D(-\infty, \infty)$ be the map defined by

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(X_i \leq x) \quad \text{for all } x \in \mathbb{R}.$$

We follow the notational convention of writing $F_n(x)$ with $x \in \mathbb{R}$, rather than $F_n(\omega)$ with $\omega \in \Omega$, while simultaneously regarding F_n to be a map from Ω into $D(-\infty, \infty)$. For each $n \in \mathbb{N}$ the generalized inverse $Q_n := F_n^{-1}$ may be viewed as a map from Ω into $L^1(0, 1)$. Again as a matter of notational convention, we will generally write $Q_n(u)$ with $u \in (0, 1)$ rather than $Q_n(\omega)$ with $\omega \in \Omega$. The maps F_n and Q_n are the usual empirical cdf and empirical quantile function for the random variables X_1, \dots, X_n , viewed as maps from Ω into $D(-\infty, \infty)$ and $L^1(0, 1)$ respectively.

To define bootstrap counterparts to F_n and Q_n we introduce a second probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ on which bootstrap weights will be defined, and define the product probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ as in (2.1). For simplicity, and because resampling procedures are not the primary focus of this article, we confine attention to the Efron bootstrap. For each $n \in \mathbb{N}$, let $W_{n,1}, \dots, W_{n,n}$ be random variables on Ω' which take values only in $\{0, 1, \dots, n\}$ and whose sum is equal to n . Each n -tuple $(W_{n,1}, \dots, W_{n,n})$ is assumed to have the multinomial distribution based on n draws from the categories $1, \dots, n$, with

equal probability assigned to each category. For each $n \in \mathbb{N}$ let $F_n^* : \bar{\Omega} = \Omega \times \Omega' \rightarrow D(-\infty, \infty)$ be the map defined by

$$F_n^*(x) = \frac{1}{n} \sum_{i=1}^n W_{n,i} \mathbb{1}(X_i \leq x) \quad \text{for all } x \in \mathbb{R}.$$

We regard the generalized inverse $Q_n^* := F_n^{*-1}$ to be a map from $\bar{\Omega}$ into $L^1(0, 1)$. The maps F_n^* and Q_n^* may be understood to be Efron bootstrap counterparts to F_n and Q_n .

The maps Q_n and Q_n^* are $L^1(0, 1)$ -valued random variables in the sense of Definition 2.1. We provide further discussion of this detail in Appendix A. As a general rule, the resolution of questions of measurability arising in this section or in Section 5 is deferred to Appendix A.

We refer to $\sqrt{n}(Q_n - Q)$ as the *quantile process*. If $Q \in L^1(0, 1)$, which must be the case if Q has Property Q, then the quantile process is an $L^1(0, 1)$ -valued random variable. In Section 4.2 we will establish that if Q has Property Q then the quantile process converges in distribution to an $L^1(0, 1)$ -valued random variable \mathcal{Q} constructed from a Q -integrable Brownian bridge.

Definition 4.1 (Brownian bridge and Q -integrable Brownian bridge). We say that a map $B : \Omega \times [0, 1] \rightarrow \mathbb{R}$ is a *Brownian bridge* if:

- (a) $B(\cdot, u)$ is a random variable on Ω for each $u \in [0, 1]$.
- (b) $B(\omega, \cdot)$ is continuous on $[0, 1]$ for each $\omega \in \Omega$.
- (c) $B(\omega, 0) = B(\omega, 1) = 0$ for each $\omega \in \Omega$.
- (d) For each $d \in \mathbb{N}$ and each d -tuple (u_1, \dots, u_d) of distinct points in $[0, 1]$, the \mathbb{R}^d -valued random variable $(B(\cdot, u_1), \dots, B(\cdot, u_d))$ on Ω is multivariate normal with zero mean and covariances given by

$$\int_{\Omega} B(\omega, u_i) B(\omega, u_j) d\mathbb{P}(\omega) = u_i(1 - u_j), \quad i, j \in \{1, \dots, d\}.$$

If a Brownian bridge satisfies the additional property

- (e) $\int_0^1 |B(\omega, u)| dQ(u) < \infty$ for each $\omega \in \Omega$,

then we say that it is a *Q -integrable Brownian bridge*.

The following result, established in Csörgő, Horváth, and Shao (1993), identifies a necessary and sufficient condition on Q such that any Brownian bridge may be made Q -integrable by modifying it on a set of probability zero.⁶

Lemma 4.1. *Let $B : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be a Brownian bridge. Then*

$$\int_0^1 |B(\omega, u)| dQ(u) < \infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega$$

if and only if Q satisfies (1.1).

In what follows we will write $B(u)$ rather than $B(\cdot, u)$, suppressing the first argument.

⁶Lemma 4.1 is obtained by applying Theorem 2.1 in Csörgő, Horváth, and Shao (1993) with ξ a Brownian bridge and μ the Lebesgue-Stieltjes measure generated by Q . Condition (2.14) of Theorem 2.1 is satisfied with $r = 2$ and $C = 2/\pi$ because the square of the first absolute moment of a normal random variable with mean zero and variance σ^2 is $2\sigma^2/\pi$.

4.2. Sufficiency of Property Q . In this section we establish the following result.

Theorem 4.1. *Assume that Q has Property Q , and let q be a density for Q . Let $B : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be a Q -integrable Brownian bridge, and let $\mathcal{Q} : \Omega \rightarrow L^1(0, 1)$ be defined by*

$$\mathcal{Q}(u) = -q(u)B(u) \quad \text{for all } u \in (0, 1).$$

Then $\sqrt{n}(Q_n - Q)$ is an $L^1(0, 1)$ -valued random variable on Ω for each n , \mathcal{Q} is an $L^1(0, 1)$ -valued random variable on Ω , and

$$\sqrt{n}(Q_n - Q) \rightsquigarrow \mathcal{Q} \quad \text{in } L^1(0, 1). \quad (4.1)$$

Moreover, $\sqrt{n}(Q_n^ - Q_n)$ is an $L^1(0, 1)$ -valued random variable on $\bar{\Omega}$ for each n , and (Q_n^*) is a bootstrap version in probability of (Q_n) w.r.t. the convergence in (4.1).*

We will prove Theorem 4.1 by applying Theorem 2.1, i.e. the delta method. Any application of the delta method requires a probabilistic ingredient and an analytic ingredient. Our analytic ingredient is the quasi-Hadamard differentiability of the generalized inverse map supplied by Theorem 3.1. Our probabilistic ingredient is supplied by Lemma 4.2 below. The proof consists mostly of references to prior work. We refer to arguments given in [Baillo, Cárcamo, and Mora-Corral \(2024\)](#) for the part of Lemma 4.2 concerning the bootstrap, but the result seems to be long known; see e.g. Remark 2.5 in [Giné and Zinn \(1990\)](#). The part of Lemma 4.2 not concerning the bootstrap is essentially contained in Theorem 2.1(a) in [del Barrio, Giné, and Matrán \(1999\)](#).

Lemma 4.2. *Assume that Q satisfies (1.1), and let $B : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be a Q -integrable Brownian bridge. Let $\mathcal{F} : \Omega \rightarrow D(-\infty, \infty)$ be defined by*

$$\mathcal{F}(x) = B(F(x)) \quad \text{for all } x \in \mathbb{R}.$$

Then $\sqrt{n}(F_n - F)$ is a $D^1(-\infty, \infty)$ -valued random variable on Ω for each n , \mathcal{F} is a $D^1(-\infty, \infty)$ -valued random variable on Ω , and

$$\sqrt{n}(F_n - F) \rightsquigarrow \mathcal{F} \quad \text{in } D^1(-\infty, \infty). \quad (4.2)$$

Moreover, $\sqrt{n}(F_n^ - F_n)$ is a $D^1(-\infty, \infty)$ -valued random variable on $\bar{\Omega}$ for each n , and (F_n^*) is almost surely a bootstrap version of (F_n) w.r.t. the convergence in (4.2).*

Proof. For each $n \in \mathbb{N}$ we have $F_n^*(x) = F_n(x) = 0$ for all sufficiently large negative x and $F_n^*(x) = F_n(x) = 1$ for all sufficiently large positive x . Thus $\sqrt{n}(F_n^* - F_n)$ takes values only in $D^1(-\infty, \infty)$. Since $Q \in L^1(0, 1)$, $\sqrt{n}(F_n - F)$ also takes values only in $D^1(-\infty, \infty)$. And since B is Q -integrable, \mathcal{F} also takes values only in $D^1(-\infty, \infty)$. The measurability requirements for $\sqrt{n}(F_n - F)$, $\sqrt{n}(F_n^* - F_n)$, and \mathcal{F} to be $D^1(-\infty, \infty)$ -valued random variables are satisfied; we defer discussion of this matter to Appendix A.

Theorem 2.1(a) in [del Barrio, Giné, and Matrán \(1999\)](#) establishes that

$$\sqrt{n}(F_n - F) \rightsquigarrow \mathcal{F} \quad \text{in } L^1(-\infty, \infty). \quad (4.3)$$

Therefore, by Definition 2.2, for every bounded and Lipschitz continuous function $f : L^1(-\infty, \infty) \rightarrow \mathbb{R}$ we have

$$\mathbb{E}f(\sqrt{n}(F_n - F)) \rightarrow \mathbb{E}f(\mathcal{F}), \quad (4.4)$$

where we write \mathbb{E} for the expected value under \mathbb{P} . Now choose any bounded and Lipschitz continuous function $g : D^1(-\infty, \infty) \rightarrow \mathbb{R}$. By the theorem in Section M9 in [Billingsley \(1999, pp. 242–3\)](#), there exists a bounded and Lipschitz continuous function $f : L^1(-\infty, \infty) \rightarrow \mathbb{R}$ which coincides with g on $D^1(-\infty, \infty)$. Thus (4.4) holds with f replaced by g . Consequently, we may appeal to the equivalence of (f) and (g) in Theorem A.3 in [Beutner and Zähle \(2016\)](#), a version of the portmanteau theorem, to deduce that (4.2) is satisfied.

It remains to show that (F_n^*) is almost surely a bootstrap version of (F_n) w.r.t. the convergence in (4.2). It is shown on pp. 5–8 of the online appendix to [Baillo, Cárcamo, and Mora-Corral \(2024\)](#) that (F_n^*) is almost surely a bootstrap version of (F_n) w.r.t. the convergence in (4.3). That the latter property of (F_n^*) implies the former may be shown by applying the portmanteau theorem as above. \square

Proof of Theorem 4.1. We assume that the random variables X_1, X_2, \dots take values only in $[\alpha_F, \beta_F]$, where α_F and β_F are defined as in (3.2). This assumption is made without loss of generality because $\mathbb{P}\{\omega \in \Omega : \alpha_F \leq X_i(\omega) \leq \beta_F \text{ for all } i \in \mathbb{N}\} = 1$.

We will apply Theorem 2.1 with $\mathbf{V} = D(-\infty, \infty)$, $\mathbf{E} = D^1(-\infty, \infty)$, $\tilde{\mathbf{E}} = L^1(0, 1)$, $\mathbf{E}_0 = T_F$, $f = \phi$, $\mathbf{V}_f = \mathbb{D}_F$, $\theta = F$, $\theta_n = F_n$, $\theta_n^* = F_n^*$, $\xi = \mathcal{F}$, and $a_n = \sqrt{n}$, where T_F , ϕ , and \mathbb{D}_F are defined in Section 3. Thus the requirement in Theorem 2.1 that $\tilde{\mathbf{E}}$ and \mathbf{E}_0 are separable is met, and \mathbf{E} is also separable. The requirement that \mathbf{E}_0 belongs to the ball σ -algebra on \mathbf{E} is verified in Lemma A.1. The requirement that $\theta \in \mathbf{V}_f$ is met because $Q \in L^1(0, 1)$. The requirement that θ_n and θ_n^* take values only in \mathbf{V}_f is met because Q_n and Q_n^* take values only in $L^1(0, 1)$ and because of the assumption in the previous paragraph.

Conditions (a), (b), (c), (d), (e), and (f) of Theorem 2.1 are met for the following reasons. Conditions (a) and (f) follow from Lemma 4.2. Conditions (c) and (d) follow from Theorem 3.1. Conditions (b) and (e) are satisfied because Q_n and Q_n^* are $L^1(0, 1)$ -valued random variables, as discussed in Appendix A.

Theorem 2.1 delivers the assertions of Theorem 4.1. Note that $\dot{\phi}_F(\mathcal{F}) = \mathcal{Q}$. \square

4.3. Necessity of Property **Q**.

In this section we establish the following result.

Theorem 4.2. *Assume that $Q \in L^1(0, 1)$, and that there exists an $L^1(0, 1)$ -valued random variable \mathcal{Z} such that $\sqrt{n}(Q_n - Q) \rightsquigarrow \mathcal{Z}$ in $L^1(0, 1)$. Then Q has Property **Q**.*

To prove Theorem 4.2 we will apply the following lemma, which collects together two results proved in [del Barrio, Giné, and Matrán \(1999\)](#).

Lemma 4.3. *Assume that $Q \in L^1(0, 1)$, and for each $n \in \mathbb{N}$ define*

$$\zeta_n := \sqrt{n} \int_{-\infty}^{\infty} |F_n(x) - F(x)| dx.$$

Let $B : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be a Brownian bridge. Then:

- (i) (ζ_n) is stochastically bounded if and only if Q satisfies (1.1).

(ii) If Q satisfies (1.1) then

$$\lim_{n \rightarrow \infty} \mathbb{E} \zeta_n^2 = \mathbb{E} \left(\int_0^1 |B(u)| dQ(u) \right)^2 < \infty,$$

which implies that (ζ_n) is uniformly integrable.

Proof. Part (i) is Theorem 2.1(b) in [del Barrio, Giné, and Matrán \(1999\)](#). The equality in part (ii) is Theorem 2.4(a) in [del Barrio, Giné, and Matrán \(1999\)](#) with $p = r = 2$. Finiteness of the limit follows from the fact that

$$\begin{aligned} \mathbb{E} \left(\int_0^1 |B(u)| dQ(u) \right)^2 &= \int_0^1 \int_0^1 \mathbb{E} |B(u)B(v)| dQ(u) dQ(v) \\ &\leq \int_0^1 \int_0^1 \sqrt{(\mathbb{E} B(u)^2)(\mathbb{E} B(v)^2)} dQ(u) dQ(v) \\ &= \left(\int_0^1 \sqrt{u(1-u)} dQ(u) \right)^2 < \infty. \end{aligned}$$

Since the limit is finite, uniform integrability of (ζ_n) follows from the usual argument with Liapounov's inequality. \square

We also use the following lemma in our proof of Theorem 4.2. It may be understood heuristically by visualizing a graph of F and G and mentally rotating it by 90 degrees.

Lemma 4.4. *Let F and G be cdfs, and a and b real numbers with $0 < a < b < 1$. Then*

$$\begin{aligned} \int_a^b |F^{-1}(u) - G^{-1}(u)| du &\geq \int_{F^{-1}(a) \vee G^{-1}(a)}^{F^{-1}(b) \wedge G^{-1}(b)} |F(x) - G(x)| dx, \\ \text{and } \int_0^1 |F^{-1}(u) - G^{-1}(u)| du &= \int_{-\infty}^{\infty} |F(x) - G(x)| dx. \end{aligned}$$

Proof. First observe that for each $x \in \mathbb{R}$,

$$|F(x) - G(x)| = \int_0^1 |\mathbb{1}(F(x) \geq u) - \mathbb{1}(G(x) \geq u)| du, \quad (4.5)$$

and for each $u \in (0, 1)$,

$$|F^{-1}(u) - G^{-1}(u)| = \int_{-\infty}^{\infty} |\mathbb{1}(x \geq F^{-1}(u)) - \mathbb{1}(x \geq G^{-1}(u))| dx. \quad (4.6)$$

Further observe that

$$\begin{aligned} &\int_{F^{-1}(a) \vee G^{-1}(a)}^{F^{-1}(b) \wedge G^{-1}(b)} |F(x) - G(x)| dx \\ &= \int_{-\infty}^{\infty} \mathbb{1}(F^{-1}(a) \leq x < F^{-1}(b)) \mathbb{1}(G^{-1}(a) \leq x < G^{-1}(b)) |F(x) - G(x)| dx \\ &= \int_{-\infty}^{\infty} \mathbb{1}(a \leq F(x) < b) \mathbb{1}(a \leq G(x) < b) |F(x) - G(x)| dx, \end{aligned}$$

using Lemma B.1(i) to obtain the last equality. Now we apply (4.5) to obtain

$$\begin{aligned} & \int_{F^{-1}(a) \vee G^{-1}(a)}^{F^{-1}(b) \wedge G^{-1}(b)} |F(x) - G(x)| dx \\ &= \int_{-\infty}^{\infty} \int_0^1 \mathbb{1}(a \leq F(x) < b) \mathbb{1}(a \leq G(x) < b) |\mathbb{1}(F(x) \geq u) - \mathbb{1}(G(x) \geq u)| du dx. \end{aligned}$$

For each $x \in \mathbb{R}$ and $u \in (0, 1)$ such that $\mathbb{1}(G(x) \geq u) \neq \mathbb{1}(F(x) \geq u)$, one of the two values $F(x)$ and $G(x)$ is no less than u , and the other less than u . Thus the inequalities $a \leq F(x) < b$ and $a \leq G(x) < b$ together imply that $a \leq u < b$. Consequently,

$$\int_{F^{-1}(a) \vee G^{-1}(a)}^{F^{-1}(b) \wedge G^{-1}(b)} |F(x) - G(x)| dx \leq \int_{-\infty}^{\infty} \int_a^b |\mathbb{1}(F(x) \geq u) - \mathbb{1}(G(x) \geq u)| du dx.$$

The first assertion of Lemma 4.4 follows from the previous inequality by applying Fubini's theorem and (4.6). The second assertion follows from an obvious modification to the preceding argument. \square

We require one further lemma for our proof of Theorem 4.2. It will also be used in Section 5 to prove Theorem 5.2.

Lemma 4.5. *Assume that Q satisfies (1.1), and let $B : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be a Q -integrable Brownian bridge. Let a and b be continuity points of Q with $0 < a < b < 1$. Then*

$$\liminf_{n \rightarrow \infty} \mathbb{E} \sqrt{n} \int_a^b |Q_n(u) - Q(u)| du \geq \mathbb{E} \int_{[a,b)} |B(u)| dQ(u).$$

Proof. By applying the first assertion of Lemma 4.4 we find that, for each $n \in \mathbb{N}$,

$$\begin{aligned} \int_a^b |Q_n(u) - Q(u)| du &\geq \int_{Q_n(a) \vee Q(a)}^{Q_n(b) \wedge Q(b)} |F_n(x) - F(x)| dx \\ &\geq \int_{Q(a)}^{Q(b)} |F_n(x) - F(x)| dx - \int_{Q_n(a) \wedge Q(a)}^{Q_n(b) \vee Q(b)} |F_n(x) - F(x)| dx. \end{aligned} \tag{4.7}$$

By Lemma 4.2 and the continuous mapping theorem, and by applying Lemma B.2,

$$\sqrt{n} \int_{Q(a)}^{Q(b)} |F_n(x) - F(x)| dx \rightsquigarrow \int_{[a,b)} |B(u)| dQ(u) \quad \text{in } \mathbb{R}. \tag{4.8}$$

For each $n \in \mathbb{N}$ and each $c \in \{a, b\}$ we have

$$\sqrt{n} \int_{Q_n(c) \wedge Q(c)}^{Q_n(c) \vee Q(c)} |F_n(x) - F(x)| dx \leq |Q_n(c) - Q(c)| \cdot \sup_{x \in \mathbb{R}} \sqrt{n} |F_n(x) - F(x)|.$$

The sequence of random variables $\sup_{x \in \mathbb{R}} \sqrt{n} |F_n(x) - F(x)|$, $n \in \mathbb{N}$, converges in distribution in \mathbb{R} by Donsker's theorem. Moreover, since a and b are continuity points of Q ,

Lemma 21.2 in [van der Vaart \(1998\)](#) and the Glivenko-Cantelli theorem together imply that $Q_n(c) \rightarrow Q(c)$ a.s. Thus, by the Slutsky theorem, for each $c \in \{a, b\}$ we have

$$\sqrt{n} \int_{Q_n(c) \wedge Q(c)}^{Q_n(c) \vee Q(c)} |F_n(x) - F(x)| dx \rightsquigarrow 0 \quad \text{in } \mathbb{R}. \quad (4.9)$$

In view of the uniform integrability condition established in Lemma [4.3\(ii\)](#), the statements of convergence in distribution in [\(4.8\)](#) and [\(4.9\)](#) imply statements of convergence in mean:

$$\mathbb{E} \sqrt{n} \int_{Q(a)}^{Q(b)} |F_n(x) - F(x)| dx \rightarrow \mathbb{E} \int_{[a,b]} |B(u)| dQ(u) \quad (4.10)$$

$$\text{and } \mathbb{E} \sqrt{n} \int_{Q_n(c) \wedge Q(c)}^{Q_n(c) \vee Q(c)} |F_n(x) - F(x)| dx \rightarrow 0 \quad \text{for each } c \in \{a, b\}. \quad (4.11)$$

By combining [\(4.7\)](#), [\(4.10\)](#), and [\(4.11\)](#), we obtain the assertion of Lemma [4.5](#). \square

Proof of Theorem 4.2. The assumed convergence in distribution in $L^1(0, 1)$ implies, via the continuous mapping theorem, that the sequence $\sqrt{n} \int_0^1 |Q_n(u) - Q(u)| du$, $n \in \mathbb{N}$, is stochastically bounded. In view of the second assertion of Lemma [4.4](#), this is precisely the sequence (ζ_n) defined in Lemma [4.3](#). Thus Lemma [4.3\(i\)](#) establishes that Q satisfies [\(1.1\)](#).

Let $B : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be a Q -integrable Brownian bridge, noting that, by Lemma [4.1](#) and the fact that Q satisfies [\(1.1\)](#), any Brownian bridge may be made Q -integrable by modifying it on a set of probability zero. We assume without loss of generality that the $L^1(0, 1)$ -valued random variable \mathcal{Z} appearing in the statement of Theorem [4.2](#) has domain Ω . Our assumption that $\sqrt{n}(Q_n - Q) \rightsquigarrow \mathcal{Z}$ in $L^1(0, 1)$ implies, via the continuous mapping theorem, that for every $a, b \in [0, 1]$ with $a < b$ we have

$$\sqrt{n} \int_a^b |Q_n(u) - Q(u)| du \rightsquigarrow \int_a^b |\mathcal{Z}(u)| du \quad \text{in } \mathbb{R}.$$

Lemma [4.3\(ii\)](#) and the second assertion of Lemma [4.4](#) together imply that the sequence of random variables $\sqrt{n} \int_a^b |Q_n(u) - Q(u)| du$, $n \in \mathbb{N}$, is uniformly integrable. Thus we deduce from the last statement of convergence in distribution that

$$\mathbb{E} \sqrt{n} \int_a^b |Q_n(u) - Q(u)| du \rightarrow \mathbb{E} \int_a^b |\mathcal{Z}(u)| du < \infty \quad \text{for all } a, b \in [0, 1], a < b. \quad (4.12)$$

We next establish that

$$\mathbb{E} \int_a^b |\mathcal{Z}(u)| du \geq \mathbb{E} \int_{[a,b]} |B(u)| dQ(u) \quad \text{for all } a, b \in (0, 1), a < b. \quad (4.13)$$

Fix $a, b \in (0, 1)$ with $a < b$. Since Q is monotone, it has at most countably many discontinuities. Thus we may choose an increasing sequence (a_k) of continuity points of Q converging to a , and a decreasing sequence (b_k) of continuity points of Q converging to b . We have $\mathbb{E} \int_0^1 |\mathcal{Z}(u)| du < \infty$ as a consequence of [\(4.12\)](#). This justifies using the

dominated convergence theorem to write

$$\mathbb{E} \int_a^b |\mathcal{Z}(u)| du = \lim_{k \rightarrow \infty} \mathbb{E} \int_{a_k}^{b_k} |\mathcal{Z}(u)| du. \quad (4.14)$$

We have $\mathbb{E} \int_0^1 |B(u)| dQ(u) < \infty$ by Lemma 4.3(ii) because Q satisfies (1.1). This justifies using the dominated convergence theorem to write

$$\mathbb{E} \int_{[a,b]} |B(u)| dQ(u) = \lim_{k \rightarrow \infty} \mathbb{E} \int_{[a_k, b_k]} |B(u)| dQ(u). \quad (4.15)$$

Since each a_k and b_k is a continuity point of Q , Lemma 4.5 and (4.12) together imply that

$$\lim_{k \rightarrow \infty} \mathbb{E} \int_{a_k}^{b_k} |\mathcal{Z}(u)| du \geq \lim_{k \rightarrow \infty} \mathbb{E} \int_{[a_k, b_k]} |B(u)| dQ(u). \quad (4.16)$$

Taken together, (4.14), (4.15), and (4.16) establish that (4.13) is satisfied.

Finally we use (4.13) to show that Q is locally absolutely continuous. Let λ be the Lebesgue measure on $(0, 1)$, and let μ_Q be the Lebesgue-Stieltjes measure on $(0, 1)$ generated by Q . Fix $\eta \in (0, 1/2)$, and let \mathcal{I}_η be the collection of all finite unions of closed intervals with endpoints in $[\eta, 1 - \eta]$. It is enough to show that the restriction of Q to $[\eta, 1 - \eta]$ is absolutely continuous. By definition, such absolute continuity is satisfied if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu_Q(A) \leq \epsilon$ for every $A \in \mathcal{I}_\eta$ such that $\lambda(A) \leq \delta$.

Fix $\epsilon > 0$, and let A be a set in \mathcal{I}_η . Since $A \subseteq [\eta, 1 - \eta]$, and since $u(1 - u) \geq \eta(1 - \eta)$ for each $u \in [\eta, 1 - \eta]$, we have

$$\mu_Q(A) = \int_A 1 dQ(u) \leq \frac{1}{\sqrt{\eta(1 - \eta)}} \int_A \sqrt{u(1 - u)} dQ(u). \quad (4.17)$$

The square of the first absolute moment of a normal random variable with mean zero and variance σ^2 is $2\sigma^2/\pi$. Thus, by Fubini's theorem,

$$\int_A \sqrt{u(1 - u)} dQ(u) = \sqrt{\frac{\pi}{2}} \mathbb{E} \int_A |B(u)| dQ(u). \quad (4.18)$$

Since A is a finite union of closed intervals, by combining (4.13), (4.17), and (4.18) we obtain

$$\mu_Q(A) \leq \sqrt{\frac{\pi}{2\eta(1 - \eta)}} \mathbb{E} \int_A |\mathcal{Z}(u)| du.$$

Observe that, for each $N \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E} \int_A |\mathcal{Z}(u)| du &= \mathbb{E} \int_A \mathbb{1}(|\mathcal{Z}(u)| \leq N) |\mathcal{Z}(u)| du + \mathbb{E} \int_A \mathbb{1}(|\mathcal{Z}(u)| > N) |\mathcal{Z}(u)| du \\ &\leq N \lambda(A) + \mathbb{E} \int_0^1 \mathbb{1}(|\mathcal{Z}(u)| > N) |\mathcal{Z}(u)| du. \end{aligned}$$

By the dominated convergence theorem,

$$\lim_{N \rightarrow \infty} \mathbb{E} \int_0^1 \mathbb{1}(|\mathcal{Z}(u)| > N) |\mathcal{Z}(u)| du = 0.$$

Thus there exists $N_0 \in \mathbb{N}$, not depending on A , such that

$$\mu_Q(A) \leq \sqrt{\frac{\pi}{2\eta(1-\eta)}} N_0 \lambda(A) + \epsilon/2.$$

Now if we set

$$\delta = \sqrt{\frac{\eta(1-\eta)}{2\pi N_0^2}} \epsilon$$

then we have $\mu_Q(A) \leq \epsilon$ if $\lambda(A) \leq \delta$. This shows that the restriction of Q to $[\eta, 1-\eta]$ is absolutely continuous. \square

5. THE P-P PROCESS

Following some preliminary discussion in Section 5.1, we will establish in Section 5.2 that local absolute continuity of the P-P curve is sufficient for convergence in distribution of the P-P process in $L^1(0, 1)$ and justifies the use of the bootstrap. The necessity of local absolute continuity is established in Section 5.3.

5.1. Preliminary discussion. In this section we deal with a sampling framework in which samples are drawn from two populations. Let (X_n) and (Y_n) be sequences of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume the two sequences to be independent of one another, assume that (X_n) is iid with cdf F and quantile function F^{-1} , and assume that (Y_n) is iid with cdf G and quantile function $Q := G^{-1}$. Note that, unlike Sections 3 and 4, Q does not denote the quantile function for F . We call the composition $R := F(Q)$ the *P-P curve*.

Until now we have regarded the cdf of a random variable to be a map from \mathbb{R} into $[0, 1]$, and the corresponding quantile function to be a map from $(0, 1)$ into \mathbb{R} . We will maintain this convention for G and Q ; i.e. we have $G : \mathbb{R} \rightarrow [0, 1]$ and $Q : (0, 1) \rightarrow \mathbb{R}$. It will be more convenient to adopt a different convention for F and F^{-1} . We will regard the cdf F to be a map from $[-\infty, \infty]$ into $[0, 1]$, and define F^{-1} to be the map from $[0, 1]$ into $[-\infty, \infty]$ given by $F^{-1}(u) = \inf\{y \in \mathbb{R} : F(y) \geq u\}$ for $u \in [0, 1]$. Consistent with our convention that Q has domain $(0, 1)$, we regard the P-P curve R to be a map from $(0, 1)$ into $[0, 1]$. Thus $R : (0, 1) \rightarrow [0, 1]$ is defined by $R(u) = F(Q(u))$ for $u \in (0, 1)$.

In Section 3 we defined $D(-\infty, \infty)$ to be the vector space of all càdlàg functions from \mathbb{R} into \mathbb{R} . We similarly define $D[-\infty, \infty]$ to be the vector space of all càdlàg functions from $[-\infty, \infty]$ into \mathbb{R} . Thus we have $F \in D[-\infty, \infty]$ and $G \in D(-\infty, \infty)$.

The P-P curve R will be estimated from two samples of possibly different sizes. We introduce two nondecreasing maps $n_x : \mathbb{N} \rightarrow \mathbb{N}$ and $n_y : \mathbb{N} \rightarrow \mathbb{N}$, to be interpreted as sequences of sample sizes. For each $n \in \mathbb{N}$ define

$$a_n := \sqrt{\frac{n_x(n)n_y(n)}{n_x(n) + n_y(n)}}.$$

We assume throughout this section that as $n \rightarrow \infty$ we have $a_n \rightarrow \infty$ and

$$\frac{n_y(n)}{n_x(n) + n_y(n)} \rightarrow \rho \quad \text{for some } \rho \in [0, 1]. \quad (5.1)$$

Note that our assumption implies that $n_x(n) \rightarrow \infty$ and $n_y(n) \rightarrow \infty$ as $n \rightarrow \infty$.

For each $n \in \mathbb{N}$ let $F_n : \Omega \rightarrow D[-\infty, \infty]$ and $G_n : \Omega \rightarrow D(-\infty, \infty)$ be defined by

$$F_n(z) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(X_i \leq z) \quad \text{for all } z \in [-\infty, \infty]$$

and $G_n(z) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(Y_i \leq z) \quad \text{for all } z \in \mathbb{R}.$

We regard the generalized inverse $Q_n := G_n^{-1}$ to be a map from Ω into $L^1(0, 1)$. Thus, for each $n \in \mathbb{N}$, F_n is the empirical cdf for X_1, \dots, X_n , and G_n and Q_n are the empirical cdf and quantile function for Y_1, \dots, Y_n . For each $n \in \mathbb{N}$ we define $R_n : \Omega \rightarrow L^1(0, 1)$ by

$$R_n(u) = F_{n_x(n)}(Q_{n_y(n)}(u)) \quad \text{for all } u \in (0, 1).$$

Thus R_n provides an estimate of R based on the samples $X_1, \dots, X_{n_x(n)}$ and $Y_1, \dots, Y_{n_y(n)}$. We call R_n the *empirical P-P curve*, and we call $a_n(R_n - R)$ the *P-P process*. In what follows we write n_x and n_y rather than $n_x(n)$ and $n_y(n)$, suppressing the dependence on n .

Next we introduce a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ on which bootstrap weights will be defined. For each $n \in \mathbb{N}$, let $W_{n,1}, \dots, W_{n,n}$ and $V_{n,1}, \dots, V_{n,n}$ be random variables on Ω' taking values only in $\{0, 1, \dots, n\}$ and satisfying $\sum_{i=1}^n W_{n,i} = 1$ and $\sum_{i=1}^n V_{n,i} = 1$. Each n -tuple $(W_{n,1}, \dots, W_{n,n})$ and each n -tuple $(V_{n,1}, \dots, V_{n,n})$ is assumed to have the multinomial distribution based on n draws from the categories $1, \dots, n$, with equal probability assigned to each category. Moreover, we assume that $(W_{n,1}, \dots, W_{n,n})$ and $(V_{n,1}, \dots, V_{n,n})$ are independent for each $n \in \mathbb{N}$.

Define the product probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$. For each $n \in \mathbb{N}$ let $F_n^* : \bar{\Omega} \rightarrow D[-\infty, \infty]$ and $G_n^* : \bar{\Omega} \rightarrow D(-\infty, \infty)$ be the maps defined by

$$F_n^*(z) = \frac{1}{n} \sum_{i=1}^n W_{n,i} \mathbb{1}(X_i \leq z) \quad \text{for all } z \in [-\infty, \infty]$$

and $G_n^*(z) = \frac{1}{n} \sum_{i=1}^n V_{n,i} \mathbb{1}(Y_i \leq z) \quad \text{for all } z \in \mathbb{R}.$

We regard the generalized inverse $Q_n^* := G_n^{*-1}$ to be a map from $\bar{\Omega}$ into $L^1(0, 1)$. The maps F_n^* , G_n^* , and Q_n^* may be understood to be Efron bootstrap counterparts to F_n , G_n , and Q_n . For each $n \in \mathbb{N}$ we define $R_n^* : \bar{\Omega} \rightarrow L^1(0, 1)$ by

$$R_n^*(u) = F_{n_x}^*(Q_{n_y}^*(u)) \quad \text{for all } u \in (0, 1).$$

The map R_n^* may be understood to be an Efron bootstrap counterpart to R_n . Both R_n and R_n^* are $L^1(0, 1)$ -valued random variables; see Appendix A for further discussion of this detail.

The vector spaces $D(-\infty, \infty)$ and $D[-\infty, \infty]$ have not been equipped with any norm. In what follows we will regard $D[-\infty, \infty]$ to be equipped with the uniform norm $\|h\|_\infty := \sup_{z \in [-\infty, \infty]} |h(z)|$. This makes $D[-\infty, \infty]$ a nonseparable normed space. Note that since $[-\infty, \infty]$ is compact, the requirement that each $h \in D[-\infty, \infty]$ is càdlàg implies that each $h \in D[-\infty, \infty]$ is uniformly bounded.

Our treatment of the P-P process will require us to work with products of normed spaces. Given vector spaces \mathbf{V} and \mathbf{W} with norms $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|_{\mathbf{W}}$, we define $\mathbf{V} \otimes \mathbf{W}$ to be the normed space formed by equipping the Cartesian product $\mathbf{V} \times \mathbf{W}$ with the obvious notions of addition and scalar multiplication and the norm $\|(v, w)\|_{\mathbf{V} \otimes \mathbf{W}} := \max\{\|v\|_{\mathbf{V}}, \|w\|_{\mathbf{W}}\}$.

5.2. Sufficiency of local absolute continuity. We will prove the following result.

Theorem 5.1. *Assume that R is locally absolutely continuous, and let r be a density for R . Let $B_1 : \Omega \times [0, 1] \rightarrow \mathbb{R}$ and $B_2 : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be independent Brownian bridges. Let $\mathcal{R} : \Omega \rightarrow L^1(0, 1)$ be defined by*

$$\mathcal{R}(u) = \rho^{1/2} B_1(R(u)) - (1 - \rho)^{1/2} r(u) B_2(u) \quad \text{for all } u \in (0, 1).$$

Then $a_n(R_n - R)$ is an $L^1(0, 1)$ -valued random variable on Ω for each n , \mathcal{R} is an $L^1(0, 1)$ -valued random variable on Ω , and

$$a_n(R_n - R) \rightsquigarrow \mathcal{R} \quad \text{in } L^1(0, 1). \quad (5.2)$$

Moreover, $a_n(R_n^ - R_n)$ is an $L^1(0, 1)$ -valued random variable on $\bar{\Omega}$ for each n , and (R_n^*) is a bootstrap version in probability of (R_n) w.r.t. the convergence in (5.2).*

We will prove Theorem 5.1 by applying the delta method with a composition mapping. Quasi-Hadamard differentiability of this mapping is established in Lemma 5.3 below. The probabilistic ingredient for our application of the delta method is supplied by Lemma 5.1. It establishes that local absolute continuity of R is sufficient for the convergence in distribution^o of $a_n(F_{n_x} - F, F(Q_{n_y}) - R)$ in $D[-\infty, \infty] \otimes L^1(0, 1)$ and justifies a corresponding bootstrap approximation.

Lemma 5.1. *Assume that R is locally absolutely continuous, and let r be a density for R . Let $B_1 : \Omega \times [0, 1] \rightarrow \mathbb{R}$ and $B_2 : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be independent Brownian bridges. Further introduce the following notation:*

- (a) θ is the vector (F, R) in $D[-\infty, \infty] \otimes L^1(0, 1)$.
- (b) $\theta_n : \Omega \rightarrow D[-\infty, \infty] \otimes L^1(0, 1)$ is defined by $\theta_n = (F_{n_x}, F(Q_{n_y}))$.
- (c) $\theta_n^* : \bar{\Omega} \rightarrow D[-\infty, \infty] \otimes L^1(0, 1)$ is defined by $\theta_n^* = (F_{n_x}^*, F(Q_{n_y}^*))$.
- (d) $\xi_1 : \Omega \rightarrow D[-\infty, \infty]$ is defined by $\xi_1(z) = \rho^{1/2} B_1(F(z))$ for all $z \in [-\infty, \infty]$.
- (e) $\xi_2 : \Omega \rightarrow L^1(0, 1)$ is defined by $\xi_2(u) = -(1 - \rho)^{1/2} r(u) B_2(u)$ for all $u \in (0, 1)$.

Then $a_n(\theta_n - \theta)$ is a $D[-\infty, \infty] \otimes L^1(0, 1)$ -valued random variable on Ω for each n , $\xi := (\xi_1, \xi_2)$ is a $D[-\infty, \infty] \otimes L^1(0, 1)$ -valued random variable on Ω , and

$$a_n(\theta_n - \theta) \rightsquigarrow^o \xi \quad \text{in } D[-\infty, \infty] \otimes L^1(0, 1). \quad (5.3)$$

Moreover, $a_n(\theta_n^ - \theta_n)$ is a $D[-\infty, \infty] \otimes L^1(0, 1)$ -valued random variable on $\bar{\Omega}$ for each n , and (θ_n^*) is a bootstrap version in outer probability of (θ_n) w.r.t. the convergence in (5.3).*

To prove Lemma 5.1, and also Theorem 5.2 below, we apply the following simple lemma.

Lemma 5.2. *Let $\tilde{R} : (0, 1) \rightarrow [0, 1]$ be the left-continuous version of R defined by*

$$\tilde{R}(u) = \sup_{v \in (0, u)} R(v) \quad \text{for each } u \in (0, 1).$$

For each $n \in \mathbb{N}$, let $\tilde{Y}_n = F(Y_n)$. Then (\tilde{Y}_n) is iid with quantile function \tilde{R} .

Proof. Plainly (\tilde{Y}_n) is iid, so it suffices to show that \tilde{Y}_n has quantile function \tilde{R} . Let U be a random variable uniformly distributed on $(0, 1)$. To show that \tilde{Y}_n has quantile function \tilde{R} , it suffices to show that \tilde{Y}_n and $\tilde{R}(U)$ are equal in law. The functions R and \tilde{R} differ only at points at which R is not left-continuous. Thus $R(u) = \tilde{R}(u)$ for a.e. $u \in (0, 1)$. Consequently $R(U)$ and $\tilde{R}(U)$ are equal in law. But $R(U) = F(Q(U))$, and $Q(U)$ is equal in law to Y_n , so it must be the case that $\tilde{R}(U)$ is equal in law to $F(Y_n)$, which is \tilde{Y}_n . \square

Proof of Lemma 5.1. In this proof we will use the notation $\zeta_n := F_{n_x}$, $\zeta_n^* := F_{n_x}^*$, $\eta_n := F(Q_{n_y})$ and $\eta_n^* = F(Q_{n_y}^*)$. Thus $\theta_n = (\zeta_n, \eta_n)$ and $\theta_n^* = (\zeta_n^*, \eta_n^*)$. We will separately establish the following two facts:

(i) $a_n(\zeta_n - F)$ is a $D[-\infty, \infty]$ -valued random variable on Ω for each n , ξ_1 is a $D[-\infty, \infty]$ -valued random variable on Ω , and

$$a_n(\zeta_n - F) \rightsquigarrow \xi_1 \quad \text{in } D[-\infty, \infty]. \quad (5.4)$$

Moreover, $a_n(\zeta_n^* - \zeta_n)$ is a $D[-\infty, \infty]$ -valued random variable on $\bar{\Omega}$ for each n , and (ζ_n^*) is a bootstrap version in outer probability of (ζ_n) w.r.t the convergence in (5.4).

(ii) $a_n(\eta_n - R)$ is an $L^1(0, 1)$ -valued random variable on Ω for each n , ξ_2 is an $L^1(0, 1)$ -valued random variable on Ω , and

$$a_n(\eta_n - R) \rightsquigarrow \xi_2 \quad \text{in } L^1(0, 1). \quad (5.5)$$

Moreover, $a_n(\eta_n^* - \eta_n)$ is an $L^1(0, 1)$ -valued random variable on $\bar{\Omega}$ for each n , and (η_n^*) is a bootstrap version in probability of (η_n) w.r.t the convergence in (5.5).

Proving (i) and (ii) suffices to prove Lemma 5.1 due to the independence of $a_n(\zeta_n - F)$ and $a_n(\eta_n - R)$ for each n , and the independence of ξ_1 and ξ_2 .

First we prove (i). This is almost immediate from classical results. The maps ζ_n , ζ_n^* , and ξ_1 are $D[-\infty, \infty]$ -valued random variables⁷ for reasons discussed on pp. 89–90 in Pollard (1984); see also our discussion in Appendix A. Theorem 11 on p. 97 in Pollard (1984), a version of Donsker’s theorem, establishes that

$$\sqrt{n_x}(\zeta_n - F) \rightsquigarrow B_1(F) \quad \text{in } D[-\infty, \infty]. \quad (5.6)$$

Since $a_n/\sqrt{n_x} \rightarrow \rho^{1/2}$ and $\xi_1 = \rho^{1/2}B_1(F)$, we now obtain (5.4) by an application of Slutsky’s theorem; see e.g. Beutner and Zähle (2018, pp. 15–6).

Theorem 3.7.1 in van der Vaart and Wellner (2023) establishes that (ζ_n^*) is a bootstrap version in outer probability of (ζ_n) w.r.t the convergence in (5.6). To show that this implies the same to be true w.r.t. the convergence in (5.4) we argue as follows. Choose

⁷Note that under Definition 2.1 a $D[-\infty, \infty]$ -valued random variable is required to be ball measurable but need not be Borel measurable.

any $f \in \text{BL}_1^\circ$, where BL_1° is defined as in Section 2.1 with $\mathbf{E} = D[-\infty, \infty]$. For each $n \in \mathbb{N}$ we have $a_n/\sqrt{n_x} \in (0, 1)$, and may therefore define $f_n \in \text{BL}_1^\circ$ by

$$f_n(h) = f\left(\frac{a_n}{\sqrt{n_x}}h\right), \quad h \in D[-\infty, \infty].$$

Then, writing \mathbb{E} and \mathbb{E}' for the expected values under \mathbb{P} and \mathbb{P}' respectively, for each $n \in \mathbb{N}$ we have

$$\mathbb{E}'f(a_n(\zeta_n^* - \zeta_n)) - \mathbb{E}f(a_n(\zeta_n - F)) = \mathbb{E}'f_n(\sqrt{n_x}(\zeta_n^* - \zeta_n)) - \mathbb{E}f_n(\sqrt{n_x}(\zeta_n - F))$$

everywhere on Ω . Consequently

$$\begin{aligned} \sup_{f \in \text{BL}_1^\circ} |\mathbb{E}'f(a_n(\zeta_n^* - \zeta_n)) - \mathbb{E}f(a_n(\zeta_n - F))| \\ \leq \sup_{f \in \text{BL}_1^\circ} |\mathbb{E}'f(\sqrt{n_x}(\zeta_n^* - \zeta_n)) - \mathbb{E}f(\sqrt{n_x}(\zeta_n - F))| \end{aligned}$$

everywhere on Ω . Viewed as a map from Ω into \mathbb{R} , the right-hand side of the last inequality converges in outer probability to zero because (ζ_n^*) is a bootstrap version in outer probability of (ζ_n) w.r.t the convergence in (5.6). Therefore, (ζ_n^*) is also a bootstrap version in outer probability of (ζ_n) w.r.t the convergence in (5.4). This proves (i).

Next we prove (ii). Define the sequence of random variables (\tilde{Y}_n) as in Lemma 5.2. Since R is continuous, Lemma 5.2 establishes that (\tilde{Y}_n) is an iid sequence of random variables with quantile function R . For each $n \in \mathbb{N}$, define \tilde{Q}_n and \tilde{Q}_n^* just as we defined Q_n and Q_n^* in Section 5.1, but with the random variables (Y_1, \dots, Y_n) replaced by $(\tilde{Y}_1, \dots, \tilde{Y}_n)$. Let $\tilde{\mathcal{Q}} : \Omega \rightarrow L^1(0, 1)$ be the map defined by $\tilde{\mathcal{Q}}(u) = -r(u)B_2(u)$ for all $u \in (0, 1)$. Note that rB_2 takes values in $L^1(0, 1)$ because B_2 has uniformly bounded sample paths and because r is integrable due to the uniform boundedness of R . Theorem 4.1 establishes that \tilde{Q}_n , \tilde{Q}_n^* , and $\tilde{\mathcal{Q}}$ are $L^1(0, 1)$ -valued random variables, that

$$\sqrt{n}(\tilde{Q}_n - R) \rightsquigarrow \tilde{\mathcal{Q}} \quad \text{in } L^1(0, 1), \quad (5.7)$$

and that (\tilde{Q}_n^*) is a bootstrap version in probability of (\tilde{Q}_n) w.r.t. the convergence in (5.7).

Now observe that $\eta_n = \tilde{Q}_{n_y}$ and $\eta_n^* = \tilde{Q}_{n_y}^*$. We deduce immediately that η_n and η_n^* are $L^1(0, 1)$ -valued random variables, that

$$\sqrt{n_y}(\eta_n - R) \rightsquigarrow \tilde{\mathcal{Q}} \quad \text{in } L^1(0, 1), \quad (5.8)$$

and that (η_n^*) is a bootstrap version in probability of (η_n) w.r.t. the convergence in (5.8). Since $a_n/\sqrt{n_y} \rightarrow (1 - \rho)^{1/2}$ and $\xi_2 = (1 - \rho)^{1/2}\tilde{\mathcal{Q}}$, we obtain (5.5) from (5.8) by an application of Slutsky's theorem. We may show that (η_n^*) is a bootstrap version in probability of (η_n) w.r.t. the convergence in (5.5) using a simple argument with bounded Lipschitz functions, as we did in our proof of (i). Thus (ii) is also proved. \square

Our next task is to establish quasi-Hadamard differentiability of a suitable composition mapping. Care is needed to deal with the fact that distinct Borel measurable and integrable functions on $(0, 1)$ are equal as vectors in $L^1(0, 1)$ if they differ only on a null set. Given a Borel set $A \subseteq \mathbb{R}$ and a Borel measurable and uniformly bounded function $g : A \rightarrow \mathbb{R}$, and given a Borel measurable function $f : (0, 1) \rightarrow \mathbb{R}$ such that $f(u) \in A$ for

a.e. $u \in (0, 1)$, we will denote by $g \circ f$ any Borel measurable function from $(0, 1)$ into \mathbb{R} satisfying

$$[g \circ f](u) = g(f(u)) \quad \text{for a.e. } u \in f^{-1}A.$$

Such a function always exists; for instance we may set $g \circ f = g(f)$ on $f^{-1}A$ and $g \circ f = 0$ elsewhere. Moreover, any such function is essentially bounded, and any two such functions differ only on a null set. Thus we may understand $g \circ f$ to denote a unique vector in $L^1(0, 1)$, with this vector being unaffected by modifications to the function f on a null set.

Let $\text{ran}(F)$ be the range of F , and define the sets

$$\mathbb{L}_F := \{h \in L^1(0, 1) : h(u) \in \text{ran}(F) \text{ for a.e. } u \in (0, 1)\}$$

$$\text{and } S_F := \{h \in D[-\infty, \infty] : h = g(F) \text{ for some } g \in C[0, 1]\},$$

with the latter set constituting a subspace of $D[-\infty, \infty]$. Let $\psi : D[-\infty, \infty] \times \mathbb{L}_F \rightarrow L^1(0, 1)$ be the map defined by $\psi(g, h) = g(F^{-1}) \circ h$. Note that $g(F^{-1}) \circ h$ validly defines a unique vector in $L^1(0, 1)$ because $g(F^{-1})$ is a Borel measurable and uniformly bounded function on $[0, 1]$ and $h(u) \in [0, 1]$ for a.e. $u \in (0, 1)$.

Lemma 5.3. *Let $\theta_2 \in \mathbb{L}_F$. Then ψ is quasi-Hadamard differentiable at $\theta := (F, \theta_2)$ tangentially to $(S_F \otimes L^1(0, 1)) \langle D[-\infty, \infty] \otimes L^1(0, 1) \rangle$. Its quasi-Hadamard derivative $\psi_\theta : S_F \otimes L^1(0, 1) \rightarrow L^1(0, 1)$ is given by $\psi_\theta(g, h) = g(F^{-1}) \circ \theta_2 + h$. The last equality defines a continuous, linear, and ball measurable extension of ψ_θ to all of $D[-\infty, \infty] \otimes L^1(0, 1)$.*

Proof. The map $(g, h) \mapsto g(F^{-1}) \circ \theta_2 + h$ from $D[-\infty, \infty] \otimes L^1(0, 1)$ into $L^1(0, 1)$ is plainly linear, and is continuous because

$$\|g(F^{-1}) \circ \theta_2 + h\|_1 \leq \|g(F^{-1}) \circ \theta_2\|_1 + \|h\|_1 \leq \|g\|_\infty + \|h\|_1$$

for all $(g, h) \in D[-\infty, \infty] \otimes L^1(0, 1)$. Thus this map defines a continuous and linear extension of our posited quasi-Hadamard derivative to all of $D[-\infty, \infty] \otimes L^1(0, 1)$. Ball measurability of the extension follows from Lemma A.2.

For the remainder of the proof we fix vectors $g \in S_F$ and $h \in L^1(0, 1)$, a vanishing sequence (t_n) of positive real numbers, a sequence (g_n) of vectors in $D[-\infty, \infty]$ such that $\|g_n - g\|_\infty \rightarrow 0$, and a sequence (h_n) of vectors in $L^1(0, 1)$ such that $\theta_2 + t_n h_n \in \mathbb{L}_F$ for each n and $\|h_n - h\|_1 \rightarrow 0$. Our task is to show that

$$\left\| \frac{\psi(F + t_n g_n, \theta_2 + t_n h_n) - \psi(F, \theta_2)}{t_n} - g(F^{-1}) \circ \theta_2 - h \right\|_1 \rightarrow 0. \quad (5.9)$$

From the definition of ψ we have

$$\begin{aligned} \psi(F + t_n g_n, \theta_2 + t_n h_n) &= F(F^{-1}) \circ (\theta_2 + t_n h_n) + t_n g_n(F^{-1}) \circ (\theta_2 + t_n h_n) \\ \text{and } \psi(F, \theta_2) &= F(F^{-1}) \circ \theta_2. \end{aligned}$$

Since $\theta_2(u) \in \text{ran}(F)$ and $\theta_2(u) + t_n h_n(u) \in \text{ran}(F)$ for a.e. $u \in (0, 1)$, Lemma B.1(ii) implies that

$$F(F^{-1}) \circ (\theta_2 + t_n h_n) = \theta_2 + t_n h_n \quad \text{and} \quad F(F^{-1}) \circ \theta_2 = \theta_2. \quad (5.10)$$

The norm in (5.9) is therefore equal to

$$\|h_n + g_n(F^{-1}) \circ (\theta_2 + t_n h_n) - g(F^{-1}) \circ \theta_2 - h\|_1,$$

which is easily seen to be bounded by

$$\begin{aligned} \|h_n - h\|_1 + & \|(g_n(F^{-1}) - g(F^{-1})) \circ (\theta_2 + t_n h_n)\|_1 \\ & + \|g(F^{-1}) \circ (\theta_2 + t_n h_n) - g(F^{-1}) \circ \theta_2\|_1. \end{aligned} \quad (5.11)$$

We seek to show that the three terms in (5.11) converge to zero as $n \rightarrow \infty$. For the first term, this is true by construction. For the second term, it is true since $\|g_n - g\|_\infty \rightarrow 0$. For the third term, observe that since $g \in S_F$ there exists $f \in C[0, 1]$ such that $g = f(F)$. Consequently, using (5.10) again, the third term in (5.11) satisfies

$$\|g(F^{-1}) \circ (\theta_2 + t_n h_n) - g(F^{-1}) \circ \theta_2\|_1 = \|f \circ (\theta_2 + t_n h_n) - f \circ \theta_2\|_1.$$

Thus it remains only to show that

$$\|f \circ (\theta_2 + t_n h_n) - f \circ \theta_2\|_1 \rightarrow 0. \quad (5.12)$$

Fix $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(u) - f(v)| \leq \epsilon/2$ for all $u, v \in [0, 1]$ satisfying $|u - v| \leq \delta$. Consequently

$$\int_0^1 \mathbb{1}(|t_n h_n(u)| \leq \delta) |f(\theta_2(u) + t_n h_n(u)) - f(\theta_2(u))| du \leq \epsilon/2. \quad (5.13)$$

Moreover, we have

$$\begin{aligned} \int_0^1 \mathbb{1}(|t_n h_n(u)| > \delta) |f(\theta_2(u) + t_n h_n(u)) - f(\theta_2(u))| du \\ \leq \delta^{-1} t_n \int_0^1 |h_n(u)| |f(\theta_2(u) + t_n h_n(u)) - f(\theta_2(u))| du \end{aligned}$$

by Chebyshev's inequality. The upper bound converges to zero since f is uniformly bounded, $\|h_n - h\|_1 \rightarrow 0$, $h \in L^1(0, 1)$, and $t_n \rightarrow 0$. Thus

$$\int_0^1 \mathbb{1}(|t_n h_n(u)| > \delta) |f(\theta_2(u) + t_n h_n(u)) - f(\theta_2(u))| du \leq \epsilon/2 \quad (5.14)$$

for all sufficiently large n . Combining (5.13) and (5.14) and noting that ϵ may be chosen arbitrarily small verifies (5.12). \square

We note in passing that the proof of Lemma 5.3 establishes that ψ is Hadamard differentiable (in the sense of van der Vaart, 1998, ch. 20) at θ tangentially to $S_F \otimes L^1(0, 1)$. The additional generality afforded by quasi-Hadamard differentiability is not needed here, but was critical to our treatment of the generalized inverse mapping in Section 3.

One further lemma is required for our proof of Theorem 5.1.

Lemma 5.4. *For each $n \in \mathbb{N}$ we have:*

- (i) $F_n(F^{-1}(F(z))) = F_n(z)$ for all $z \in [-\infty, \infty]$ a.s.
- (ii) $F_n^*(F^{-1}(F(z))) = F_n^*(z)$ for all $z \in [-\infty, \infty]$ a.s.

Proof. Let x_1, \dots, x_n be real numbers satisfying $F^{-1}(F(x_i)) = x_i$ for all i . Then

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}(x_i \leq F^{-1}(F(z))) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(F^{-1}(F(x_i)) \leq F^{-1}(F(z))) \quad \text{for all } z \in [-\infty, \infty].$$

By Lemma B.1(i), $F^{-1}(F(x_i)) \leq F^{-1}(F(z))$ if and only if $F(x_i) \leq F(F^{-1}(F(z)))$. And by Lemma B.1(ii), $F(F^{-1}(F(z))) = F(z)$. Thus we obtain

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}(x_i \leq F^{-1}(F(z))) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(F(x_i) \leq F(z)) \quad \text{for all } z \in [-\infty, \infty].$$

By Lemma B.1(i), $F(x_i) \leq F(z)$ if and only if $F^{-1}(F(x_i)) \leq z$. Thus, using the equality $F^{-1}(F(x_i)) = x_i$, we obtain

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}(x_i \leq F^{-1}(F(z))) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(x_i \leq z) \quad \text{for all } z \in [-\infty, \infty].$$

If each x_i is replaced with X_i then the last equality becomes $F_n(F^{-1}(F(z))) = F_n(z)$. Thus claim (i) follows from the fact that $F^{-1}(F(X_i)) = X_i$ for all $i \in \mathbb{N}$ a.s., a consequence of Lemma B.1(iv). An obvious modification to the preceding argument establishes claim (ii). \square

Proof of Theorem 5.1. We assume that $F_n(F^{-1}(F)) = F_n$ everywhere on Ω and that $F_n^*(F^{-1}(F)) = F_n^*$ everywhere on $\bar{\Omega}$ for all $n \in \mathbb{N}$. This assumption is made without loss of generality because, by Lemma 5.4, it amounts to excluding a zero probability subset of $\bar{\Omega}$.

We will apply Theorem 2.1 with $\mathbf{V} = \mathbf{E} = D[-\infty, \infty] \otimes L^1(0, 1)$, $\mathbf{E}_0 = S_F \otimes L^1(0, 1)$, $\tilde{\mathbf{E}} = L^1(0, 1)$, $f = \psi$, and $\mathbf{V}_f = D[-\infty, \infty] \times \mathbb{L}_F$, and with $\theta, \theta_n, \theta_n^*$, and ξ defined as in Lemma 5.1. The requirement that $\tilde{\mathbf{E}}$ and \mathbf{E}_0 are separable is thus met. The requirement that \mathbf{E}_0 belongs to the ball σ -algebra on \mathbf{E} is verified in Lemma A.3. The requirement that $\theta \in \mathbf{V}_f$ is met because $R(u) \in \text{ran}(F)$ for all $u \in (0, 1)$. The requirement that θ_n and θ_n^* take values only in \mathbf{V}_f is met because $F(Q_{n_y}(u))$ and $F(Q_{n_y}^*(u))$ take values only in $\text{ran}(F)$ for all $u \in (0, 1)$.

Observe that

$$f(\theta_n) = F_{n_x}(F^{-1}) \circ F(Q_{n_y}) \quad \text{and} \quad f(\theta_n^*) = F_{n_x}^*(F^{-1}) \circ F(Q_{n_y}^*).$$

The assumption made without loss of generality at the beginning of the proof therefore implies that $f(\theta_n) = R_n$ and $f(\theta_n^*) = R_n^*$. We also have $f(\theta) = R$ by Lemma B.1(ii).

Conditions (a), (b), (c), (d), (e), and (f') of Theorem 2.1 are met for the following reasons. Conditions (a) and (f') follow from Lemma 5.1. Conditions (c) and (d) follow from Lemma 5.3. Conditions (b) and (e) are satisfied because R_n and R_n^* are $L^1(0, 1)$ -valued random variables, as discussed in Appendix A.

Theorem 2.1 delivers the assertions of Theorem 5.1. Note that the distributional limit in (5.2) is as claimed because

$$\begin{aligned} \dot{f}_\theta(\xi) &= \dot{\psi}_{(F, R)} \left(\rho^{1/2} B_1(F), -(1 - \rho)^{1/2} r B_2 \right) \\ &= \rho^{1/2} B_1(F(F^{-1})) \circ R - (1 - \rho)^{1/2} r B_2 = \rho^{1/2} B_1(R) - (1 - \rho)^{1/2} r B_2 = \mathcal{R}, \end{aligned}$$

where the first equality follows from the definitions of θ and ξ given in Lemma 5.1, the second equality follows from the expression for $\dot{\psi}_\theta$ given in Lemma 5.3, and the third equality follows from the fact that $F(F^{-1}(R)) = R$ by Lemma B.1(ii). \square

5.3. Necessity of local absolute continuity. We will prove the following result.

Theorem 5.2. *Assume that $\rho \in [0, 1]$, and that there exists an $L^1(0, 1)$ -valued random variable \mathcal{Z} such that $a_n(R_n - R) \rightsquigarrow \mathcal{Z}$ in $L^1(0, 1)$. Then R is locally absolutely continuous.*

We make use of the following lemma in the proof of Theorem 5.2.

Lemma 5.5. *The sequence of random variables $a_n\|R_n - R\|_1$, $n \in \mathbb{N}$, is uniformly integrable.*

Proof. Fix $n \in \mathbb{N}$, and observe that

$$a_n\|R_n - R\|_1 \leq \sqrt{n_x}\|F_{n_x}(Q_{n_y}) - F(Q_{n_y})\|_1 + \sqrt{n_y}\|F(Q_{n_y}) - R\|_1, \quad (5.15)$$

where we have used the fact that $a_n \leq \sqrt{n_x}$ and $a_n \leq \sqrt{n_y}$. Let $H : \mathbb{R} \rightarrow [0, 1]$ and $H_{n_y} : \mathbb{R} \rightarrow [0, 1]$ be the cdf and empirical cdf for the iid sample $F(Y_1), \dots, F(Y_{n_y})$. The empirical quantile function for this sample is $F(Q_{n_y})$. Moreover, Lemma 5.2 establishes that $H^{-1} = \tilde{R}$, where $\tilde{R}(u) := \sup_{v \in (0, u)} R(v)$ for all $u \in (0, 1)$. The second term on the right-hand side of (5.15) satisfies

$$\sqrt{n_y}\|F(Q_{n_y}) - R\|_1 = \sqrt{n_y}\|F(Q_{n_y}) - \tilde{R}\|_1 = \sqrt{n_y}\|H_{n_y} - H\|_1 \leq \sqrt{n_y}\|H_{n_y} - H\|_\infty,$$

where the first equality follows from the fact that $\tilde{R} = R$ a.e., the second equality follows from the second assertion of Lemma 4.4, and the inequality follows from the fact that $H_{n_y} - H$ is zero outside of the interval $[0, 1]$. The first term on the right-hand side of (5.15) is bounded by $\sqrt{n_x}\|F_{n_x} - F\|_\infty$, so we now have

$$a_n\|R_n - R\|_1 \leq \sqrt{n_x}\|F_{n_x} - F\|_\infty + \sqrt{n_y}\|H_{n_y} - H\|_\infty.$$

Thus we are done if we can show that the sequences of random variables $\sqrt{n}\|F_n - F\|_\infty$ and $\sqrt{n}\|H_n - H\|_\infty$, $n \in \mathbb{N}$, are uniformly integrable. But this is well-known: by applying the Dvoretzky-Kiefer-Wolfowitz inequality (see e.g. van der Vaart, 1998, p. 268) we obtain

$$\mathbb{E}(\sqrt{n}\|F_n - F\|_\infty)^2 = \int_0^\infty 2t\mathbb{P}(\sqrt{n}\|F_n - F\|_\infty > t)dt \leq 4 \int_0^\infty te^{-2t^2}dt < \infty \quad (5.16)$$

and may similarly bound the second moment of $\sqrt{n}\|H_n - H\|_\infty$. Uniform integrability of both sequences then follows from the usual argument with Liapounov's inequality. \square

Proof of Theorem 5.2. Let $B : \Omega \times [0, 1] \rightarrow \mathbb{R}$ be a Brownian bridge, and assume without loss of generality that the $L^1(0, 1)$ -valued random variable \mathcal{Z} appearing in the statement of Theorem 5.2 has domain Ω . Our assumption that $\sqrt{n}(R_n - R) \rightsquigarrow \mathcal{Z}$ in $L^1(0, 1)$ implies, via the continuous mapping theorem, that for every $a, b \in [0, 1]$ with $a < b$ we have

$$\sqrt{n} \int_a^b |R_n(u) - R(u)|du \rightsquigarrow \int_a^b |\mathcal{Z}(u)|du \quad \text{in } \mathbb{R}.$$

Lemma 5.5 establishes that the sequence of random variables $\sqrt{n} \int_a^b |R_n(u) - R(u)|du$, $n \in \mathbb{N}$, is uniformly integrable. Thus we deduce from the last statement of convergence in distribution that

$$\mathbb{E}\sqrt{n} \int_a^b |R_n(u) - R(u)|du \rightarrow \mathbb{E} \int_a^b |\mathcal{Z}(u)|du < \infty \quad \text{for all } a, b \in [0, 1], a < b. \quad (5.17)$$

We next establish that

$$\mathbb{E} \int_a^b |\mathcal{Z}(u)| du + \sqrt{\frac{\pi\rho}{2}}(b-a) \geq \sqrt{1-\rho} \mathbb{E} \int_{[a,b)} |B(u)| dR(u) \quad (5.18)$$

for all continuity points a, b of R such that $0 < a < b < 1$.

Let a and b be continuity points of R with $0 < a < b < 1$. For each $n \in \mathbb{N}$ define $\gamma_n := R_n - F(Q_{n_y})$. Then

$$\mathbb{E} \int_a^b |F(Q_{n_y}(u)) - R(u)| du \leq \mathbb{E} \int_a^b |R_n(u) - R(u)| du + \mathbb{E} \int_a^b |\gamma_n(u)| du.$$

Consequently, to establish (5.18), it suffices to verify that

$$\limsup_{n \rightarrow \infty} \mathbb{E} \sqrt{n_y} \int_a^b |R_n(u) - R(u)| du \leq \sqrt{\frac{1}{1-\rho}} \mathbb{E} \int_a^b |\mathcal{Z}(u)| du, \quad (5.19)$$

$$\limsup_{n \rightarrow \infty} \mathbb{E} \sqrt{n_y} \int_a^b |\gamma_n(u)| du \leq \sqrt{\frac{\pi\rho}{2(1-\rho)}}(b-a), \quad (5.20)$$

$$\text{and } \liminf_{n \rightarrow \infty} \mathbb{E} \sqrt{n_y} \int_a^b |F(Q_{n_y}(u)) - R(u)| du \geq \mathbb{E} \int_{[a,b)} |B(u)| dR(u). \quad (5.21)$$

Since $\sqrt{n_y}/a_n \rightarrow 1/\sqrt{1-\rho}$, we obtain (5.19) as an immediate consequence of (5.17).

Next we verify (5.20). From the definition of γ_n we have $\gamma_n = F_{n_x}(Q_{n_y}) - F(Q_{n_y})$, so

$$\int_a^b |\gamma_n(u)| du \leq \|F_{n_x} - F\|_\infty(b-a).$$

By applying the Dvoretzky-Kiefer-Wolfowitz inequality we obtain, for each $n \in \mathbb{N}$,

$$\mathbb{E} \sqrt{n} \|F_n - F\|_\infty = \int_0^\infty \mathbb{P}(\sqrt{n} \|F_n - F\|_\infty > t) dt \leq 2 \int_0^\infty e^{-2t^2} dt = \sqrt{\frac{\pi}{2}}.$$

Thus, for each $n \in \mathbb{N}$,

$$\mathbb{E} \sqrt{n_y} \int_a^b |\gamma_n(u)| du \leq \sqrt{\frac{\pi n_y}{2n_x}}(b-a).$$

Since $n_y/n_x \rightarrow \rho/(1-\rho)$, we conclude that (5.20) is satisfied.

Next we verify (5.21). Lemma 5.2 establishes that $F(Y_1)$ has quantile function \tilde{R} satisfying $\tilde{R}(u) = \sup_{v \in (0,u)} R(v)$ for all $u \in (0,1)$. Moreover, for each $n \in \mathbb{N}$, $F(Q_n)$ is the empirical quantile function for the iid sample $F(Y_1), \dots, F(Y_n)$. Therefore, since a and b are continuity points of R , and thus of \tilde{R} , Lemma 4.5 establishes that

$$\liminf_{n \rightarrow \infty} \mathbb{E} \sqrt{n} \int_a^b |F(Q_n(u)) - \tilde{R}(u)| du \geq \mathbb{E} \int_{[a,b)} |B(u)| d\tilde{R}(u).$$

The limit inferior of a sequence is no greater than the limit inferior of any subsequence. Thus

$$\liminf_{n \rightarrow \infty} \mathbb{E} \sqrt{n_y} \int_a^b |F(Q_{n_y}(u)) - \tilde{R}(u)| du \geq \mathbb{E} \int_{[a,b)} |B(u)| d\tilde{R}(u).$$

The nondecreasing functions R and \tilde{R} are equal a.e. and therefore generate the same Lebesgue-Stieltjes measure on $(0, 1)$. Thus (5.21) is satisfied. Having verified (5.19), (5.20), and (5.21), we conclude that (5.18) is satisfied.

Next we strengthen (5.18) by showing that

$$\mathbb{E} \int_a^b |\mathcal{Z}(u)| du + \sqrt{\frac{\pi\rho}{2}}(b-a) \geq \sqrt{1-\rho} \mathbb{E} \int_{[a,b]} |B(u)| dR(u) \quad (5.22)$$

for all $a, b \in (0, 1)$ such that $a < b$,

which may be compared to (4.13) in the proof of Theorem 4.2. Fix $a, b \in (0, 1)$ with $a < b$. Since R is monotone, it has at most countably many discontinuities. Thus we may choose an increasing sequence (a_k) of continuity points of R converging to a , and a decreasing sequence (b_k) of continuity points of R converging to b . We have $\mathbb{E} \int_0^1 |\mathcal{Z}(u)| du < \infty$ as a consequence of (5.17). This justifies using the dominated convergence theorem to write

$$\mathbb{E} \int_a^b |\mathcal{Z}(u)| du = \lim_{k \rightarrow \infty} \mathbb{E} \int_{a_k}^{b_k} |\mathcal{Z}(u)| du. \quad (5.23)$$

We have $\mathbb{E} \int_0^1 |B(u)| dR(u) < \infty$ by Lemma 4.3(ii) because R is uniformly bounded. This justifies using the dominated convergence theorem to write

$$\mathbb{E} \int_{[a,b]} |B(u)| dR(u) = \lim_{k \rightarrow \infty} \mathbb{E} \int_{[a_k, b_k]} |B(u)| dR(u). \quad (5.24)$$

Plainly, we also have

$$\sqrt{\frac{\pi\rho}{2}}(b-a) = \lim_{k \rightarrow \infty} \sqrt{\frac{\pi\rho}{2}}(b_k - a_k). \quad (5.25)$$

Since each a_k and b_k is a continuity point of R , it follows from (5.18) that

$$\mathbb{E} \int_{a_k}^{b_k} |\mathcal{Z}(u)| du + \sqrt{\frac{\pi\rho}{2}}(b_k - a_k) \geq \sqrt{1-\rho} \mathbb{E} \int_{[a_k, b_k]} |B(u)| dR(u) \quad \text{for each } k \in \mathbb{N}. \quad (5.26)$$

Taken together, (5.23), (5.24), (5.25), and (5.26) establish that (5.22) is satisfied.

Finally we use (5.22) to show that R is locally absolutely continuous. Let λ be the Lebesgue measure on $(0, 1)$, and let μ_R be the Lebesgue-Stieltjes measure on $(0, 1)$ generated by R . Fix $\eta \in (0, 1/2)$, and let \mathcal{I}_η be the collection of all finite unions of closed intervals with endpoints in $[\eta, 1 - \eta]$. It is enough to show that the restriction of R to $[\eta, 1 - \eta]$ is absolutely continuous. By definition, such absolute continuity is satisfied if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu_R(A) \leq \epsilon$ for every $A \in \mathcal{I}_\eta$ such that $\lambda(A) \leq \delta$.

Fix $\epsilon > 0$, and let A be a set in \mathcal{I}_η . The arguments used to establish (4.17) and (4.18) in the proof of Theorem 4.2 also serve to establish that

$$\mu_R(A) \leq \sqrt{\frac{\pi}{2\eta(1-\eta)}} \mathbb{E} \int_A |B(u)| dR(u).$$

Since A is a finite union of closed intervals, by applying (5.22) we obtain

$$\mu_R(A) \leq \sqrt{\frac{\pi}{2\eta(1-\eta)(1-\rho)}} \left(\mathbb{E} \int_A |\mathcal{Z}(u)| du + \sqrt{\frac{\pi\rho}{2}} \lambda(A) \right)$$

Again arguing as in the proof of Theorem 4.2, we have

$$\begin{aligned} \mathbb{E} \int_A |\mathcal{Z}(u)| du &\leq N\lambda(A) + \mathbb{E} \int_0^1 \mathbb{1}(|\mathcal{Z}(u)| > N) |\mathcal{Z}(u)| du \text{ for each } N \in \mathbb{N}, \\ \text{and } \lim_{N \rightarrow \infty} \mathbb{E} \int_0^1 \mathbb{1}(|\mathcal{Z}(u)| > N) |\mathcal{Z}(u)| du &= 0. \end{aligned}$$

Thus there exists $N_0 \in \mathbb{N}$, not depending on A , such that

$$\mu_R(A) \leq \sqrt{\frac{\pi}{2\eta(1-\eta)(1-\rho)}} \left(N_0\lambda(A) + \sqrt{\frac{\pi\rho}{2}}\lambda(A) \right) + \epsilon/2.$$

Now if we set

$$\delta = \sqrt{\frac{\eta(1-\eta)(1-\rho)}{2\pi(N_0 + \sqrt{\pi\rho/2})^2}} \epsilon$$

then we have $\mu_R(A) \leq \epsilon$ if $\lambda(A) \leq \delta$. Since η can be chosen arbitrarily small, we conclude that R is locally absolutely continuous. \square

APPENDIX A. MEASURABILITY

It remains to tie up some loose ends in Sections 4 and 5 related to measurability. Later in this appendix we establish three lemmas appealed to in the proofs of Theorems 4.1 and 5.1 and Lemma 5.3. We begin by providing a brief account of standard arguments used to establish suitable measurability of the maps F_n , Q_n , and R_n , and their bootstrap counterparts.

First consider the empirical quantile function Q_n defined in Section 4. Denote by $X_{1,n}, \dots, X_{n,n}$ the ascending order statistics for X_1, \dots, X_n . The map $Q_n : \Omega \rightarrow L^1(0, 1)$ sends each $\omega \in \Omega$ to the function in $L^1(0, 1)$ that sends each $u \in (0, 1)$ to

$$\sum_{i=1}^n X_{i,n}(\omega) \mathbb{1}((i-1)/n < u \leq i/n).$$

The last expression defines an $(\mathcal{F} \otimes \mathcal{B}(0, 1), \mathcal{B}(\mathbb{R}))$ -measurable map from $\Omega \times (0, 1)$ into \mathbb{R} because it is a sum of products of measurable maps $\omega \mapsto X_{i,n}(\omega)$ and $u \mapsto \mathbb{1}((i-1)/n < u \leq i/n)$. Thus, for any $h \in L^1(0, 1)$, Fubini's theorem implies that $\|Q_n - h\|_1$, viewed as a map from Ω into \mathbb{R} , is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable. The preimage under Q_n of any open ball in $L^1(0, 1)$ therefore belongs to \mathcal{F} . This shows that Q_n is an $L^1(0, 1)$ -valued random variable. Essentially the same argument shows that $Q_n^* : \bar{\Omega} \rightarrow L^1(0, 1)$ is an $L^1(0, 1)$ -valued random variable; one need merely replace the order statistics with their bootstrap counterparts.

Next consider the empirical P-P curve R_n defined in Section 5. Let $Y_{1,n_y}, \dots, Y_{n_y,n_y}$ be the ascending order statistics for Y_1, \dots, Y_{n_y} . The map $R_n : \Omega \rightarrow L^1(0, 1)$ sends each $\omega \in \Omega$ to the function in $L^1(0, 1)$ that sends each $u \in (0, 1)$ to

$$\frac{1}{n_x} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \mathbb{1}(X_i(\omega) \leq Y_{j,n_y}(\omega)) \mathbb{1}((j-1)/n_y < u \leq j/n_y).$$

The last expression defines an $(\mathcal{F} \otimes \mathcal{B}(0, 1), \mathcal{B}(\mathbb{R}))$ -measurable map from $\Omega \times (0, 1)$ into \mathbb{R} because it is a scalar multiple of a sum of products of measurable maps $\omega \mapsto \mathbb{1}(X_i(\omega) \leq Y_{j, n_y}(\omega))$ and $u \mapsto \mathbb{1}((j-1)/n_y < u \leq j/n_y)$. By arguing as in the previous paragraph with Fubini's theorem we may deduce that R_n is an $L^1(0, 1)$ -valued random variable. Essentially the same argument shows that $R_n^* : \bar{\Omega} \rightarrow L^1(0, 1)$ is an $L^1(0, 1)$ -valued random variable

Next consider the empirical cdf F_n as defined in Section 4. For each $i \in \{1, \dots, n\}$ the map $(\omega, x) \mapsto \mathbb{1}(X_i(\omega) \leq x)$ from $\Omega \times \mathbb{R}$ into \mathbb{R} is $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$ -measurable. Thus

$$(\omega, x) \mapsto \frac{1}{n} \sum_{i=1}^n \mathbb{1}(X_i(\omega) \leq x) - F(x) \quad (\text{A.1})$$

is $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$ -measurable. If $F(x)\mathbb{1}(x < 0)$ and $(1 - F(x))\mathbb{1}(x > 0)$ are integrable functions of x —which is the case if and only if Q is integrable—then the map in (A.1) is integrable as a function of x for each fixed ω . Thus by applying Fubini's theorem as above we deduce that $F_n - F$ is a $D^1(-\infty, \infty)$ -valued random variable, as asserted in the proof of Lemma 4.2. Essentially the same argument verifies the assertion made there that $F_n^* - F_n$ and \mathcal{F} are $D^1(-\infty, \infty)$ -valued random variables. Moreover, $F_n - F$, $F_n^* - F_n$, and \mathcal{F} may also be regarded to be $L^1(-\infty, \infty)$ -valued random variables due to the fact that the embedding $h \mapsto h$ from $D^1(-\infty, \infty)$ into $L^1(-\infty, \infty)$ is (trivially) continuous.

In Section 5 we instead defined F_n to be a map from Ω into $D[-\infty, \infty]$, and claimed that F_n is a $D[-\infty, \infty]$ -valued random variable. To see why, let $h \in D[-\infty, \infty]$ and let $x \in \mathbb{Q} \cup \{\infty\}$, and consider the $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable map

$$\omega \mapsto n^{-1} \sum_{i=1}^n \mathbb{1}(X_i(\omega) \leq x) - h(x).$$

Now take the pointwise supremum over all $x \in \mathbb{Q} \cup \{\infty\}$ to obtain another $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable map. Since h is right-continuous on $[-\infty, \infty)$ we have

$$\sup_{x \in \mathbb{Q} \cup \{\infty\}} \left| n^{-1} \sum_{i=1}^n \mathbb{1}(X_i \leq x) - h(x) \right| = \|F_n - h\|_\infty.$$

Thus $\|F_n - h\|_\infty$, viewed as a map from Ω into \mathbb{R} , is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable. The preimage under F_n of any open ball in $D[-\infty, \infty]$ therefore belongs to \mathcal{F} . This shows that F_n is a $D[-\infty, \infty]$ -valued random variable. Essentially the same argument shows that $F_n^* : \bar{\Omega} \rightarrow D[-\infty, \infty]$ is a $D[-\infty, \infty]$ -valued random variable. Note that random variables are not required to be Borel measurable under Definition 2.1, only ball measurable.

The facts established in the preceding discussion of measurability are well-known. Next we establish three lemmas used in Section 4 and 5. The first of these, Lemma A.1, was used in the proof of Theorem 4.1 to verify the tangent space measurability required by Theorem 2.1. It concerns the set T_F defined immediately prior to the statement of Theorem 3.1.

Lemma A.1. *T_F is a Borel subset of $D^1(-\infty, \infty)$.*

To prove Lemma A.1 we introduce another vector space. Let $L_{\text{loc}}^1(-\infty, \infty)$ be the vector space of Borel measurable and locally integrable functions $h : \mathbb{R} \rightarrow \mathbb{R}$. Functions differing only on a null set are regarded to be equal as vectors in this space. Equip $L_{\text{loc}}^1(-\infty, \infty)$ with the metric

$$d_{\text{loc}}^1(g, h) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\int_{-n}^n |g(x) - h(x)| dx}{1 + \int_{-n}^n |g(x) - h(x)| dx}.$$

Lemma 5.17 in [Meise and Vogt \(1997\)](#) establishes that the last equality defines a valid metric, and that under this metric $L_{\text{loc}}^1(-\infty, \infty)$ is complete and separable.

Proof of Lemma A.1. We will apply Theorem 15.1 in [Kechris \(1995\)](#), which states that if $f : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is a continuous map between Polish spaces \mathcal{P}_1 and \mathcal{P}_2 , and A is a Borel subset of \mathcal{P}_1 on which f is injective, then $f(A)$ is a Borel subset of \mathcal{P}_2 . We choose $\mathcal{P}_1 = C[0, 1]$ and $\mathcal{P}_2 = L_{\text{loc}}^1(-\infty, \infty)$, and define $f : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ by $f(g) = g(F)$. Note that $g(F)$ is locally integrable because g is uniformly bounded. Also note that f is continuous because if $g, h \in C[0, 1]$ satisfy $\|g - h\|_{\infty} \leq \epsilon$ then

$$d_{\text{loc}}^1(f(g), f(h)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\int_{-n}^n |g(F(x)) - h(F(x))| dx}{1 + \int_{-n}^n |g(F(x)) - h(F(x))| dx} \leq \epsilon \sum_{n=1}^{\infty} \frac{n}{2^{n-1}}.$$

Define the set

$$A_1 := \left\{ g \in C[0, 1] : \int_{-\infty}^{\infty} |g(F(x))| dx < \infty \text{ and } g(0) = g(1) = 0 \right\},$$

and observe that $f(A_1) = T_F$. We now show that A_1 is a Borel subset of $C[0, 1]$. The set $\{g \in C[0, 1] : g(0) = g(1) = 0\}$ is Borel because it is closed, so it suffices to show that $g \mapsto \int_{-\infty}^{\infty} |g(F(x))| dx$ is a Borel measurable map from $C[0, 1]$ into $\mathbb{R} \cup \{\infty\}$. Such measurability is implied by the fact that $g \mapsto \int_{-\infty}^{\infty} |g(F(x))| dx$ is the pointwise supremum over $n \in \mathbb{N}$ of the countable collection of measurable maps $g \mapsto \int_{-n}^n |g(F(x))| dx$ from $C[0, 1]$ into \mathbb{R} . Thus A_1 is a Borel subset of $C[0, 1]$.

The function f need not be injective on A_1 . Specifically, if two functions $g_1, g_2 \in A_1$ differ only on a subset of $(0, 1)$ not belonging to the range of F , then $f(g_1) = f(g_2)$. Let $J \subset \mathbb{R}$ be the set of discontinuities of F , a finite or countable set because F is nondecreasing. Define

$$A_2 := \{g \in C[0, 1] : g \text{ is linear on } [F(j-), F(j)] \text{ for each } j \in J\}.$$

The set A_2 is closed, thus $A_1 \cap A_2$ is Borel. We have $f(A_1 \cap A_2) = T_F$, and f is injective on $A_1 \cap A_2$, so Theorem 15.1 in [Kechris \(1995\)](#) implies that T_F is a Borel subset of $L_{\text{loc}}^1(-\infty, \infty)$.

We are done if we can show that the embedding $h \mapsto h$ from $D^1(-\infty, \infty)$ into $L_{\text{loc}}^1(-\infty, \infty)$ is continuous. But this is obvious, because if $g, h \in D^1(-\infty, \infty)$ satisfy $\|g - h\|_1 \leq \epsilon$ then

$$d_{\text{loc}}^1(g, h) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\int_{-n}^n |g(x) - h(x)| dx}{1 + \int_{-n}^n |g(x) - h(x)| dx} \leq \epsilon \sum_{n=1}^{\infty} \frac{n}{2^{n-1}}. \quad \square$$

The next lemma was used in the proof of Lemma 5.3. The notation \mathbb{L}_F and \circ was defined immediately prior to the statement of Lemma 5.3.

Lemma A.2. *For each $F \in \mathbb{D}[-\infty, \infty]$ and each $\theta \in \mathbb{L}_F$, the map $g \mapsto g(F^{-1}) \circ \theta$ from $D[-\infty, \infty]$ into $L^1(0, 1)$ is ball measurable.*

Proof. Let \mathcal{B}° denote the ball σ -algebra on $D[-\infty, \infty]$. The projection σ -algebra on $D[-\infty, \infty]$ is equal to \mathcal{B}° (van der Vaart and Wellner, 2023, p. 46). Thus, for every $x \in [-\infty, \infty]$, the coordinate projection $g \mapsto g(x)$ is $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurable.

Let $n \in \mathbb{N}$, and consider the map from $D[-\infty, \infty] \times [-\infty, \infty]$ into \mathbb{R} given by

$$(g, x) \mapsto \sum_{k=-n^{2^n}}^{n^{2^n}} g(k2^{-n}) \mathbb{1}((k-1)2^{-n} \leq x < k2^{-n}) + g(-\infty) \mathbb{1}(x = -\infty) + g(\infty) \mathbb{1}(x = \infty). \quad (\text{A.2})$$

This map is $(\mathcal{B}^\circ \otimes \mathcal{B}[-\infty, \infty], \mathcal{B}(\mathbb{R}))$ -measurable due to the $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurability of the coordinate projections $g \mapsto g(k2^{-n})$, $g \mapsto g(-\infty)$, and $g \mapsto g(\infty)$. Moreover, since each $g \in D[-\infty, \infty]$ is càdlàg, the map in (A.2) converges pointwise as $n \rightarrow \infty$ to the map $(g, x) \mapsto g(x)$. Thus the latter map is also $(\mathcal{B}^\circ \otimes \mathcal{B}[-\infty, \infty], \mathcal{B}(\mathbb{R}))$ -measurable.

For the remainder of the proof we assume that the function $\theta : (0, 1) \rightarrow \mathbb{R}$ takes values only in $[0, 1]$. This assumption is made without loss of generality because $\theta(u) \in \text{ran}(F)$ for a.e. $u \in (0, 1)$ by the definition of \mathbb{L}_F , and because the vector $g(F^{-1}) \circ \theta$ is unaffected by modifications to θ on a null set. We write $g(F^{-1}(\theta))$ rather than $g(F^{-1}) \circ \theta$ in what follows.

Fix a function $h \in L^1(0, 1)$. Since $F^{-1} : [0, 1] \rightarrow [-\infty, \infty]$ and $\theta : (0, 1) \rightarrow [0, 1]$ are Borel measurable, so is the composition $F^{-1}(\theta) : (0, 1) \rightarrow [-\infty, \infty]$. The Borel measurability of $F^{-1}(\theta)$ and h and the $(\mathcal{B}^\circ \otimes \mathcal{B}[-\infty, \infty], \mathcal{B}(\mathbb{R}))$ -measurability of $(g, x) \mapsto g(x)$ together imply that the map from $D[-\infty, \infty] \times (0, 1)$ into \mathbb{R} given by

$$(g, u) \mapsto g(F^{-1}(\theta(u))) - h(u)$$

is $(\mathcal{B}^\circ \otimes \mathcal{B}(0, 1), \mathcal{B}(\mathbb{R}))$ -measurable. This map is integrable as a function of u for each fixed g due to the uniform boundedness of g and the integrability of h . Thus, by Fubini's theorem, the map $g \mapsto \|g(F^{-1}(\theta)) - h\|_1$ from $D[-\infty, \infty]$ into \mathbb{R} is $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurable. The preimage under $g \mapsto g(F^{-1}(\theta))$ of every open ball in $L^1(0, 1)$ therefore belongs to \mathcal{B}° . This shows that $g \mapsto g(F^{-1}(\theta))$ is ball measurable. \square

The final lemma to be established was used in the proof of Theorem 5.1 to verify the tangent space measurability required by Theorem 2.1. It concerns the set S_F defined immediately prior to the statement of Lemma 5.3.

Lemma A.3. *S_F belongs to the ball σ -algebra on $D[-\infty, \infty]$.*

Proof. It suffices to show that S_F is closed and separable (van der Vaart and Wellner, 2023, p. 48). Define A_2 as in the proof of Lemma A.1, and let $f : A_2 \rightarrow S_F$ be defined by $f(g) = g(F)$. Then f is a bijective isometry. Since A_2 is closed and separable, it follows that S_F is closed and separable. \square

APPENDIX B. GENERALIZED INVERSES

Here we collect together some well-known properties of generalized inverses to which we have appealed repeatedly throughout this article. In Section 3 we defined the generalized inverse of a cdf $F : \mathbb{R} \rightarrow [0, 1]$ to be the left-continuous and nondecreasing map $F^{-1} : (0, 1) \rightarrow \mathbb{R}$ defined by

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\} \quad (\text{B.1})$$

for each $u \in (0, 1)$. In Section 5 we similarly defined the generalized inverse of a cdf $F : [-\infty, \infty] \rightarrow [0, 1]$ to be the left-continuous and nondecreasing map $F^{-1} : [0, 1] \rightarrow [-\infty, \infty]$ defined by (B.1) for each $u \in [0, 1]$. In both cases it should be understood that F is the cdf of a real-valued random variable, not of a random variable possibly taking the values $\pm\infty$.

Useful discussions of the properties of generalized inverses may be found in [Embrechts and Hofert \(2013\)](#) and on pp. 3–8 in [Shorack and Wellner \(1986\)](#). We have referred to the properties enumerated in the following lemma in arguments given throughout this article.

Lemma B.1. *Let F be a cdf defined on \mathbb{R} or on $[-\infty, \infty]$. Then:*

- (i) *If x and u belong to the domains of F and F^{-1} then $F(x) \geq u$ if and only if $x \geq F^{-1}(u)$.*
- (ii) *$F(F^{-1}(u)) = u$ for all u in the range of F and in the domain of F^{-1} .*
- (iii) *If F^{-1} is continuous on $(0, 1)$ then $F^{-1}(F(x)) = x$ for all x such that $0 < F(x) < 1$.*

Moreover, if F is defined on $[-\infty, \infty]$, then:

- (iv) *If X is a random variable with cdf F then $F^{-1}(F(X)) = X$ a.s.*

The next lemma, applied several times throughout this article, states a well-known substitution rule for Lebesgue-Stieltjes integrals involving the generalized inverse.

Lemma B.2. *Let F be a function belonging to $\mathbb{D}(-\infty, \infty)$, and let $Q = F^{-1}$. Let*

$$\alpha_F = \inf\{x \in \mathbb{R} : F(x) > 0\} \quad \text{and} \quad \beta_F = \sup\{x \in \mathbb{R} : F(x) < 1\}.$$

Let $h : (0, 1) \rightarrow \mathbb{R}$ be a Borel measurable function that satisfies $\int_0^1 |h(u)| dQ(u) < \infty$ or is nonnegative. Then

$$\int_{[a,b)} h(u) dQ(u) = \int_{Q(a)}^{Q(b)} h(F(x)) dx \quad \text{for all } a, b \in (0, 1) \text{ satisfying } a < b,$$

and

$$\int_0^1 h(u) dQ(u) = \int_{\alpha_F}^{\beta_F} h(F(x)) dx.$$

It is straightforward to prove Lemma B.2 using Theorem 3.6.1 in [Bogachev \(2007\)](#) and Lemma B.1(i). We omit the details. See also Theorem 6.14 in [Leoni \(2017\)](#).

REFERENCES

Aly, E.A.A., Csörgő, M., Horváth, L. (1987). P-P plots, rank processes, and Chernoff-Savage theorems. In Puri, M.L., Vilaplana, J.P., Wertz, W. (Eds.), *New Perspectives in Theoretical and Applied Statistics*, 135–56. Wiley. [\[URL\]](#)

Baíllo, A., Cárcamo, J., Mora-Corral, C. (2024). Tests for almost stochastic dominance. *Journal of Business and Economic Statistics*, in press. [\[DOI\]](#)

del Barrio, E., Giné, E., Matrán, C. (1999). Central limit theorems for the Wasserstein distance between the empirical and the true distributions. *Annals of Probability*, 27(2):1009–71. [\[DOI\]](#)

del Barrio, E., Giné, E., Utzet, F. (2005). Asymptotics for L_2 functionals of the empirical quantile process, with applications to tests of fit based on weighted Wasserstein distances. *Bernoulli*, 11(1):131–89. [\[DOI\]](#)

Beutner, E., Zähle, H. (2010). A modified functional delta method and its application to the estimation of risk functionals. *Journal of Multivariate Analysis*, 101(10):2452–63. [\[DOI\]](#)

Beutner, E., Zähle, H. (2016). Functional delta-method for the bootstrap of quasi-Hadamard differentiable functionals. *Electronic Journal of Statistics*, 10(1):1181–222. [\[DOI\]](#)

Beutner, E., Zähle, H. (2018). Bootstrapping average value at risk of single and collective risks. *Risks*, 6(3):96. [\[DOI\]](#)

Billingsley, P. (1999). *Convergence of Probability Measures*, second ed. Wiley. [\[DOI\]](#)

Bogachev, V.I. (2007). *Measure Theory*. Springer. [\[DOI\]](#)

Bücher, A., Segers, J., Volgushev, S. (2014). When uniform weak convergence fails: Empirical processes for dependence functions and residuals via epi- and hypographs. *Annals of Statistics*, 42(4):1598–634. [\[DOI\]](#)

Csörgő, M. (1983). *Quantile Processes with Statistical Applications*. Society for Industrial and Applied Mathematics. [\[DOI\]](#)

Csörgő, M., Horváth, L. (1993). *Weighted Approximations in Probability and Statistics*. Wiley. [\[URL\]](#)

Csörgő, M., Horváth, L., Shao, Q.-M. (1993). Convergence of integrals of uniform empirical and quantile processes. *Stochastic Processes and their Applications*, 45(2):283–94. [\[DOI\]](#)

Embrechts, P., Hofert, M. (2013). A note on generalized inverses. *Mathematical Methods of Operations Research*, 77(3):423–32. [\[DOI\]](#)

Giné, E., Zinn, J. (1990). Bootstrapping general empirical measures. *Annals of Probability*, 18(2):851–69. [\[DOI\]](#)

Hoeffding, W. (1973). On the centering of a simple linear rank statistic. *Annals of Statistics*, 1(1):54–66. [\[DOI\]](#)

Hsieh, F., Turnbull, B.W. (1996). Nonparametric and semiparametric estimation of the receiver operating characteristic curve. *Annals of Statistics*, 24(1):24–40. [\[DOI\]](#)

Kaji, T. (2018). Essays on asymptotic methods in econometrics. Doctoral thesis, Massachusetts Institute of Technology. [\[DOI\]](#)

Kechris, A.S. (1995). *Classical Descriptive Set Theory*. Springer. [\[DOI\]](#)

Krätschmer, V., Schied, A., Zähle, H. (2015). Quasi-Hadamard differentiability of general risk functionals and its application. *Statistics & Risk Modeling*, 32(1):25–47. [\[DOI\]](#)

Leoni, G. (2017). *A First Course in Sobolev Spaces*, second ed. American Mathematical Society. [\[DOI\]](#)

Mason, D.M. (1984). Weak convergence of the weighted empirical quantile process in $L^2(0, 1)$. *Annals of Probability*, 12(1):243–55. [\[DOI\]](#)

Meise, R., Vogt, D. (1997). *Introduction to Functional Analysis*. Oxford University Press. [\[DOI\]](#)

Pollard, D. (1984). *Convergence of Stochastic Processes*. Springer. [\[DOI\]](#)

Pyke, R., Shorack, G.R. (1968). Weak convergence of a two-sample empirical process and a new approach to Chernoff-Savage theorems. *Annals of Mathematical Statistics*, 39(3):755–71. [\[DOI\]](#)

Shorack, G.R., Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics*. Wiley. [\[DOI\]](#)

Tao, T. (2011). *An Introduction to Measure Theory*. American Mathematical Society. [\[DOI\]](#)

van der Vaart, A.W. (1998). *Asymptotic Statistics*. Cambridge University Press. [\[DOI\]](#)

van der Vaart, A.W., Wellner, J.A. (2023). *Weak Convergence and Empirical Processes*, second ed. Springer. [\[DOI\]](#)

Volgushev, S., Shao, X. (2014). A general approach to the joint asymptotic analysis of statistics from sub-samples. *Electronic Journal of Statistics*, 8(1):390–431. [\[DOI\]](#)

Weitkamp, C.A., Proksch, K., Tameling, C., Munk, A. (2024). Distribution of distances based object matching: Asymptotic inference. *Journal of the American Statistical Association*, 119(545):538–51. [\[DOI\]](#)

Zwingmann, T., Holzmann, H. (2020). Weak convergence of quantile and expectile processes under general assumptions. *Bernoulli*, 26(1):323–51. [\[DOI\]](#)

SCHOOL OF ECONOMICS, UNIVERSITY OF SYDNEY, CITY ROAD. DARLINGTON, NSW 2006, AUSTRALIA.

Email address: `brendan.beare@sydney.edu.au`

BOOTH SCHOOL OF BUSINESS, UNIVERSITY OF CHICAGO, 5807 S. WOODLAWN AVE. CHICAGO, IL 60637, USA.

Email address: `tkaji@chicagobooth.edu`