Digital vs Physical Currency: A Difference That Makes a Difference[*]

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Abstract

This paper compares digital and physical currency, focusing on a single intrinsic difference: digital, unlike physical, currency allows to trace the monetary flows in and out of the accounts. We show that this technological advance in record-keeping can be used by the monetary authority to reward active balances relative to idle balances, helping achieve efficiency in a wide range of circumstances.

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1 Introduction

In academic and policy circles, there is an active debate about the implications of introducing sovereign digital currency, also known as central bank digital currency (CBDC), as payment instrument alongside or instead of physical currency. The literature has identified a number of benefits and costs of digital currency.[1] Benefits

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[1]See Chapman et al. (2023) for a literature review.

include lower transaction costs, increased competition in the banking sector, lower tax evasion. Costs include the breach of privacy, infringement of personal freedom, risk of disintermediation. Neither the benefits nor the costs are strictly connected with the monetary nature of the instrument, that is often conceived simply as an intangible version of cash.

In this paper, we identify a single intrinsic difference between digital and physical currency and show how it matters for optimal monetary policy. To make it transparent that this is the only driving force, we contrast two pure currency economies that are identical except for the type of currency that is used, either physical or digital, comparing their functioning and identifying the optimal monetary policy in each case. The only difference between the two instruments is that, with digital currency the monetary authority can keep track of flows of money balances in and out of the accounts, while with physical currency this is not possible.[2] The extra information collected through the digital technology can always be ignored, hence, any allocation attainable with physical currency can be reached with digital currency, but there are robust circumstances in which digital currency allows to improve strictly upon cash.

We consider a monetary search economy à la Lagos and Wright (2005), in which the velocity of circulation of currency is endogenous since the traders choose their search intensity, as in Rocheteau and Wright (2005). We characterize the equilibrium and identify the optimal intervention by the monetary authority with physical and digital currency, respectively. With physical currency, the optimal intervention is the Friedman rule, that, however, does not always achieve the efficient search intensity. Moreover, any deviation from the Friedman rule always reduces search intensity and the velocity of money.[3] With digital currency, instead, the efficient search intensity can always be achieved and in some cases the optimal policy is not the Friedman rule. The monetary authority achieves efficiency paying a positive nominal interest rate in accounts with a higher velocity of money and no interest in accounts with a lower velocity of money.

We show that the result holds in a robust set of circumstances, beyond the specific trading arrangement adopted for simplicity, i.e. the Kalai bargaining protocol. Following Gu and Wright (2016), we generalize the trading protocol to a mechanism

[2] This difference in the record-keeping possibilities with different types of payment instruments is in the spirit of Kocherlakota (1998).

[3] This occurs also in Liu, Wang and Wright (2011).

that subsumes among other schemes, both the Kalai and Nash protocol. Interestingly, we show that, under the Nash bargaining protocol, digital currency allows to improve the allocation relative to physical currency, even if the search intensity is exogenous. Although limited to the Nash protocol, this result shows that the key for the improvement is the presence of distortions, arising from extensive or intensive margins, that the Friedman rule alone cannot correct. We also show that the result survives when the taxation of balances is not feasible, as in Hu, Kennan and Wallace (2009). Finally, the result continues to hold even if the agents can try to manipulate the system at a cost, opening "shadow accounts" to obtain interest on balances that are not truly active.

A growing literature examines the impact of digital currencies, particularly CBDC. For example, recent papers that adopt the Lagos and Wright (2005) framework include Williamson (2022), Chiu, Davoodalhosseini, Jiang and Zhu (2023), and Keister and Sanches (2023). In these papers, interest can be paid with digital unlike physical currency, but *uniformly* on all balances. We show that, if the monetary authority can run the Friedman rule, restricting the scheme to the payment of interest uniformly on all balances, both active and idle, digital currency cannot enlarge the set of implementable allocations relative to physical currency, as the optimal allocation coincides with the one achieved under the Friedman rule in the corresponding physical currency economy. However, paying interest uniformly is sub-optimal, as it does not take into account the ability of the monetary authority to keep track of the monetary flows in and out of the accounts that is available with digital but not physical currency. This superior ability helps stimulate the traders' participation, through the payment of interest on active balances, allowing to achieve full efficiency on all margins, extensive and intensive. With endogenous participation, the ability to reward active balances is an improvement also relative to credit schemes that reward only idle balances, as in Berentsen, Camera and Waller (2007) or Ferraris and Watanabe (2008). Intuitively, this obtains because it is preferable to pay interest in accounts with higher rather than lower velocity to stimulate the traders' participation.

The paper proceeds as follows. Section 2 introduces the model. Section 3 presents the results and Section 4 compares them with the literature. Section 5 presents two extensions and Section 6 concludes. The Appendix contains the proofs that are not in the text and the technical details of two of the extensions.

2 Model

The model is a version of Lagos and Wright (2005) with an endogenous participation decision. In this section, we describe the fundamentals, the trading instruments and derive the efficient benchmark.

2.1 Fundamentals

Time is discrete and each period is divided into two sub-periods, called day and night. There is a continuum of infinitely lived agents who discount the future at rate $\beta \in (0, 1)$. During the day, the agents meet in a decentralized market to trade a commodity, whose non-negative quantity is denoted with q. Half of the agents are sellers of this commodity, that, to be produced, requires a cost represented by a twice differentiable strictly increasing and convex function, $c(q)$. The other half of the agents are potential consumers of this commodity. Consumption of the commodity gives utility represented by a twice differentiable strictly increasing and strictly concave function, $u(q)$. We assume $u(0) = c(0) = u'(\infty) = c'(0) = 0$, and $u'(0) = \infty$. The potential consumers choose search intensity $\alpha \in [0, 1]$ that represents the probability of meeting sellers in the day market, incurring a cost represented by a twice differentiable, strictly increasing and strictly convex function, $k(\alpha)$. We assume $k(0) = k'(0) = 0$. The buyers and sellers meet bilaterally, and the terms of trade within meetings are determined with the Kalai protocol, where $\theta \in (0, 1)$ denotes the bargaining power of the buyer.[4] The traders cannot commit to future actions and are anonymous during the day, their trades being not observable by outsiders. At night, all the agents meet in a centralized market where another commodity is traded that serves as the numeraire and can be produced and consumed by all agents, with linear payoffs.

2.2 Currency

Lack of monitoring and commitment prevents trade during the day in the absence of a trading instrument. We consider two alternative economies, one in which the trading instrument is physical currency, and another with digital currency. These economies are identical except for the use of the trading instrument. In particular,

[4]See Kalai (1977). Lagos and Wright (2005) use the Nash protocol. As discussed below, a similar logic would be at work with Nash bargaining but the algebra would be messier. We also show that our results hold in a larger class of trading protocols.

in both economies, currency is introduced through accounts open by the monetary authority for each agent at the beginning of the first period. We distinguish the trading instruments by the record-keeping technology embedded in the accounts. In the physical currency economy the monetary authority only keeps track of balance holdings in the accounts, while in the digital currency economy the monetary authority keeps track of balance holdings and balance flows in and out of the accounts.[5]

2.3 Efficiency

Due to the linearity of the payoffs, utility is transferable at night and has no impact on welfare. The efficient allocation consists of a participation rate α and a day quantity q that maximize $\alpha[u(q) - c(q)] - k(\alpha)$. The complementary slackness condition for the optimal participation is

$$(1-\alpha)[u(q) - c(q) - k'(\alpha)] = 0, \tag{1}$$

whereby the participation is full if the gains from trade are sufficiently large relative to the marginal cost of participating and partial otherwise. The optimum condition for the intensive margin is

$$u'(q) = c'(q). \tag{2}$$

Let the efficient allocation that satisfies (1) and (2) be denoted by (α^*, q^*) and define $\theta^* \equiv \frac{k'(1)}{u(q^*)-c(q^*)}$. The next Proposition characterizes the efficient allocation.

Proposition 1 *The efficient day quantity, q^*, equates marginal utility and cost; the efficient participation is full, $\alpha^* = 1$, if $\theta^* \leq 1$; and partial, $\alpha^* = k'^{-1}(u(q^*) - c(q^*)) \in (0, 1)$, if $\theta^* > 1$.*

3 Physical vs Digital Currency

In what follows, we first consider physical currency, and then digital currency. In both economies, we restrict attention to stationary symmetric equilibria with constant real balances. Throughout, we denote by ϕ the price of currency in the night market.

[5]We follow Andolfatto (2010) and Wallace (2014) in assuming that the monetary authority can implement policies that condition on balances in the physical currency economy. The implementation of these policies naturally requires the observation of balances. We do not allow the authority to condition on the history of balances, which would be equivalent to conditioning on the flows.

3.1 Physical Currency

Consider a meeting between a buyer and a seller in the day market. Under Kalai bargaining, a buyer holding an amount m of currency chooses a payment $d \leq m$ in exchange for a quantity q to maximize $u(q) - \phi d$, subject to the constraint

$$(1-\theta)[u(q) - \phi d] = \theta[\phi d - c(q)]. \tag{3}$$

Note that the real balances of the seller do not affect the terms of trade. Moreover, since the seller has no use for real balances in the day market, in equilibrium it is never the case that the seller strictly prefers to bring a positive quantity of currency. We assume that the buyers spend all their balances during the day. Below, we will show that this is, in fact, the optimal choice of the agents. This implies $d = m$, and we can rewrite (3) as

$$\phi m = (1-\theta)u(q) + \theta c(q) \equiv g(q), \tag{4}$$

which determines the quantity produced by the seller as a function of the buyer's real balances.

Consider now the participation decision of a buyer with ϕm real balances at the beginning of the day. Choosing α, he incurs the cost $k(\alpha)$ and meets a seller with probability α, in which case he spends all his balances. If he does not meet a seller, he keeps his balances. Formally, the buyer chooses $\alpha \leq 1$ to maximize $\alpha u(q) + (1 - \alpha)\phi m - k(\alpha)$. Using (4), we obtain the following complementary slackness condition for the optimum,

$$(1-\alpha)\left\{\theta\left[u(q) - c(q)\right] - k'(\alpha)\right\} = 0, \tag{5}$$

where both terms in parentheses are non-negative. If the surplus appropriated by the buyer is sufficiently large, he chooses full participation; otherwise, the solution is interior and there is partial participation.

We now move to the night market, and consider the buyer's choice of the amount of balances to bring into the day market. He chooses m to maximize $-\phi m + \beta[-k(\alpha) + \alpha u(q) + (1-\alpha)\phi_{+1} m]$. Using equation (4), we obtain the inter-temporal condition for the optimum,

$$\phi = \beta\phi_{+1}\left[\alpha\frac{u'(q)}{g'(q)} + 1 - \alpha\right]. \tag{6}$$

In words, an extra unit of currency acquired presently can be spent next period on day consumption if the agent turns out to be a buyer or kept idle until the night market otherwise.

In the economy with physical currency, the monetary authority can inject or tax currency in a lump-sum fashion. If we let M denote the quantity of currency and τ denote the growth rate of currency change, we have $M_{+1} = (1+\tau)M$. Stationarity of real balances implies $\frac{\phi}{\phi_{+1}} = \frac{M_{+1}}{M} = 1+\tau$, and we can rewrite (6) as

$$\frac{u'(q)}{g'(q)} = \frac{1-\beta+\tau+\alpha\beta}{\alpha\beta}. \tag{7}$$

The existence of a monetary equilibrium requires $\tau \geq \beta - 1$ to prevent an infinite demand for money. This condition also ensures that it is never strictly optimal for the buyers to bring balances they do not plan to use, hence vindicating our initial assumption that the buyers spend all their balances. A stationary symmetric equilibrium is a pair (α, q) that satisfies (5) and (7). The next Proposition, whose proof appears in the Appendix, shows the existence of a unique equilibrium.

Proposition 2 *The equilibrium with physical currency exists and is unique.*

We now determine the optimal policy τ. First, observe that τ does not directly enter the participation condition (5). Also observe that, for any given α, it is optimal to set $q = q^*$ to maximize the surplus in a trade meeting. This can be achieved by implementing the Friedman rule, i.e., setting $\tau = \beta - 1$. At the Friedman rule, if $\theta \geq \theta^*$, buyers choose to fully participate in trade, and the first-best is achieved. If, instead, $\theta < \theta^*$, buyers choose $\alpha_c < \alpha^*$, where $k'(\alpha_c) = \theta\left[u(q^*) - c(q^*)\right]$. In this case, the Friedman rule is optimal but it does not achieve the first-best. The following Proposition summarizes this result.

Proposition 3 *The Friedman rule is optimal. It implements the first-best if and only if* $\theta^* \leq \theta < 1$.

The Friedman rule drives the intensive margin to efficiency rewarding balances at the rate of time preference to compensate for the elapse of time, but is unable, in general, to drive the participation decision towards full efficiency, as it lacks the tools to give the buyers the right incentive to participate.

3.2 Digital Currency

With digital currency, the record-keeping technology allows the monetary authority to track the flows of balances in and out of the accounts. In what follows, we use this information to separate balances at the end of the day market into two groups, labeled idle and active. A balance in an account at the end of the day market is idle if it was already in the account at the beginning of the day market; while a balance in an account at the end of the day market is active if it was transferred into the account during the day market. We consider interventions where the monetary authority treats these balances differently, offering two distinct nominal interest rates, i_p for the idle balances and i_a for the active balances. We proceed by first determining the equilibrium and then characterizing the optimal policy.

3.2.1 Equilibrium

Consider a meeting between a seller and a buyer with m units of digital currency in the day market. If the buyer transfers d balances to the seller and receives q units of goods, the surplus of the buyer is $S_b = u(q) - \phi(1 + i_p)d$, while the surplus of the seller is $S_s = -c(q) + \phi(1 + i_a)d$. Observe that the surplus of the buyer includes the interest lost in the currency that was transferred to the seller and, correspondingly, the surplus of the seller includes the interest gained in the process. The total surplus is $S = u(q) - c(q) + \phi(i_a - i_p)d$, namely the gain from trade and the net interest payment to the transferred balances.

With Kalai bargaining, a buyer holding an amount m of currency chooses $d \leq m$ in exchange for a quantity q to maximize $u(q) - \phi(1 + i_p)d$ subject to the constraint $(1 - \theta)S_b = \theta S_s$. As with physical currency, the real balances of the seller do not affect the terms of trade. However, the seller may want to bring real balances to the day market, depending on the nominal interest rate paid on idle balances. We will examine this incentive below, showing that it is never part of the optimal policy to give the sellers the incentive to bring balances. We will also show that, as in the case of physical currency, buyers spend all their balances during the day. This allows to rewrite the constraint $(1 - \theta)S_b = \theta S_s$ as

$$\phi m \equiv \frac{g(q)}{1 + \theta i_a + (1 - \theta)i_p}. \tag{8}$$

Consider now the participation decision of a buyer with balances m at the beginning of the day market. Choosing α, he incurs a cost $k(\alpha)$, meets a seller with probability α, and spends all his balances. If he does not meet a seller, he keeps his balances until the night market, and receives an interest rate payment i_p. Formally, the buyer chooses α to maximize $\alpha u(q) + (1-\alpha)(1+i_p)\phi m - k(\alpha)$. Using (8), we obtain the complementary slackness condition for the optimum,

$$(1-\alpha)\left[\theta\frac{(1+i_a)u(q)-(1+i_p)c(q)}{1+\theta i_a+(1-\theta)i_p} - k'(\alpha)\right] = 0, \tag{9}$$

where both terms in parenthesis are non-negative. If the surplus appropriated by the buyer is sufficiently large, he chooses full participation; otherwise, the solution is interior and participation is less than full. Observe that, for any given q, the incentive of the buyer to participate in trade increases with the interest rate paid on active balances and decreases with the interest paid on idle balances.

We now move to the night market. The buyer chooses m to maximize $-\phi m + \beta[\alpha u(q) + (1-\alpha)(1+i_p)\phi_{+1}m - k(\alpha)]$. Using (8), we obtain the inter-temporal condition for the optimum,

$$\phi = \beta\phi_{+1}\left\{\alpha\frac{u'(q)\left[1+\theta i_a+(1-\theta)i_p\right]}{g'(q)} + (1-\alpha)(1+i_p)\right\}. \tag{10}$$

In words, an extra unit of currency acquired presently can be spent next period on day consumption if the agent turns out to be a buyer or kept idle until the night market otherwise. In either case, the agent receives an interest payment for using actively the balances or keeping them idle.

As in the economy with physical currency, the monetary authority can inject or tax currency in a lump-sum fashion. It can also inject or tax currency by using the nominal interest rates i_p and i_a. If we let τ denote the rate at which money is injected or taxed in the economy in a lump-sum way, we have $M_{+1} = [1+\tau+(1-\alpha)i_p+\alpha i_a]M$. Stationarity of real balances implies $\frac{\phi}{\phi_{+1}} = 1+\tau+(1-\alpha)i_p+\alpha i_a$, and we can rewrite (10) as

$$\frac{u'(q)}{g'(q)} = \frac{\tau+1-\beta+\alpha\beta+\alpha i_a+(1-\alpha)(1-\beta)i_p}{\alpha\beta\left[1+\theta i_a+(1-\theta)i_p\right]}. \tag{11}$$

The existence of a monetary equilibrium requires $\phi \geq \beta\phi_{+1}(1+i_p)$, otherwise an agent would have the incentive to demand an infinite amount of currency. We can rewrite this condition as

$$\frac{\tau + \alpha\,(i_a - i_p)}{1 + i_p} \geq \beta - 1. \tag{12}$$

A stationary symmetric equilibrium is a pair (α, q) that satisfies (9) and (11). The next Proposition, whose proof appears in the Appendix, shows the existence of a unique equilibrium.

Proposition 4 *The equilibrium with digital currency exists and is unique.*

As in the physical currency economy, an increase in τ has no direct impact on the participation of buyers; but it reduces the surplus, with a negative impact on the quantity produced in trade meetings. In contrast, changes in the nominal interest rates on active and idle balances impact both the extensive and the intensive margins.

We start with the intensive margin. To provide intuition, fix the extensive margin. An increase in interest rates leads to money creation, which negatively impacts the real rate of return on balances. This reduces the demand for real balances, and the quantity in trade meetings, as captured in the numerator on the right-hand side of (11). However, an increase in interest rates increases the quantity in trade meetings, for any given real balances ϕm. This is so because an increase in interest rates increases the relative surplus of the seller (an increase in i_a increases the surplus of the seller, while an increase in i_p reduces the surplus of the buyer) which must be balanced by an increase in the quantity produced in the meeting. This positive effect is captured in the denominator on the right-hand side of (11). The overall effect is ambiguous.

Consider now the extensive margin. To provide intuition, fix the intensive margin. An increase in the net interest payment $i_a - i_p$ has a positive impact in the participation of buyers. It does so either by increasing the region of parameters where full participation is optimal or by increasing the participation of buyers in the region of parameters with partial participation. This can be seen by observing that the term in brackets in (9) is strictly increasing in i_a and strictly decreasing in i_p. Intuitively, an increase in $i_a - i_p$ encourages participation due to its direct positive effect on the surplus in a trade meeting.

3.2.2 Optimal Policy

In the digital currency economy, in addition to impacting the overall return on currency through changes in τ, the monetary authority can use i_a and i_p to target how the return on currency will be distributed between agents holding idle balances and agents holding active balances. In what follows, we will show that the availability of these additional instruments allows the monetary authority to always implement the first-best.

We proceed using θ^* and θ to divide the parameter space into three regions. First, if $\theta^* \leq \theta < 1$, Proposition 2 shows that the first-best can be achieved in the physical currency economy. Since any policy in the physical currency economy can be replicated in the digital currency economy if $i_a = i_p = 0$, the first-best can also be achieved if the authority implements the Friedman rule. The next Lemma summarizes the result.

Lemma 1 *Let $\theta^* \leq \theta < 1$. The first-best involves full participation of the buyers and is implemented by the Friedman rule, i.e., $\tau = \beta - 1$ and $i_a = i_p = 0$.*

Next, consider $\theta < \theta^* \leq 1$. In this region, if an intervention implements the first-best, the efficient quantity is produced in trade meetings, and there is full participation of buyers. Thus, the right-hand-side of (11) evaluated at $\alpha = 1$ must be equal to one, which requires

$$\tau = (\beta - 1)(1 + i_p) - (1 - \beta\theta)(i_a - i_p). \tag{13}$$

Using (9), we also need

$$k'(1) \leq \theta \frac{(1 + i_a)u(q^*) - (1 + i_p)c(q^*)}{1 + \theta i_a + (1 - \theta)i_p}, \tag{14}$$

to ensure that buyers want to fully participate. Finally, under full participation, the existence of the monetary equilibrium requires $\tau + i_a - \beta i_p \geq \beta - 1$, which, using (13), can be rewritten as $i_a \geq i_p$.

A natural policy candidate for the implementation of the first-best involves setting $i_p = 0$. This is consistent with a zero lower bound on interest rates and it provides the buyers with the incentive to participate in trade while vindicating our initial claim that agents have no incentive to bring balances into the day market that they do not plan to use. Since $\theta^* \leq 1$ implies $k'(1) \leq u(q^*) - c(q^*)$, a sufficient condition for the buyers to fully participate is that $u(q^*) - c(q^*)$ equals the right hand side of (14), i.e.,

$$u(q^*) - c(q^*) = \theta \frac{(1+i_a)u(q^*) - (1+i_p)c(q^*)}{1+\theta i_a + (1-\theta)i_p},$$

which can be rewritten as

$$i_a^* = \frac{1-\theta}{\theta}\frac{u(q^*) - c(q^*)}{c(q^*)}.$$

The positive nominal interest rate on active balances compensates for the insufficient bargaining power of the buyer. In fact, note that i_a^* is strictly decreasing in θ, and it goes to zero when θ goes to one. Using (13), we also obtain

$$\tau^* = -\left[1 - \beta + (1-\beta\theta)\, i_a^*\right].$$

Note that $i_a^* > i_p = 0$ implies $\phi > \beta\phi_{+1}$ and the real rate of return on currency is lower than the discount rate. In this sense, the optimal intervention is away from the Friedman rule. In fact, there is money creation under the optimal policy if $\tau^* + i_a^* > 0$, which can be rewritten as

$$\beta > \frac{c(q^*)}{(1-\theta)\, u(q^*) + \theta c(q^*)}.$$

The next Lemma summarizes our result.

Lemma 2 *Let $\theta < \theta^* \leq 1$. The first-best involves full participation of the buyers and it can be implemented by the policy $\tau = \tau^*$, $i_a = i_a^*$, and $i_p = 0$, which deviates from the Friedman rule.*

We now turn to the region where $\theta^* > 1$. In this case, the first-best involves efficient production of q^* in trade meetings but partial participation of buyers, i.e., the efficient search intensity is given by $\alpha^* < 1$, where

$$k'(\alpha^*) = u(q^*) - c(q^*). \tag{15}$$

If a policy implements the efficient quantity, the right hand side of (11) evaluated at α^* must be equal to one, i.e.,

$$\tau = (\beta - 1)(1 + i_p) - (1-\beta\theta)\,\alpha^*\,(i_a - i_p). \tag{16}$$

In turn, in order to give buyers the incentive to search efficiently, the term in brackets in (9) must be satisfied at q^* and α^* with equality, i.e.,

$$k'(\alpha^*) = \frac{\theta[(1+i_a)u(q^*) - (1+i_p)c(q^*)]}{1+\theta i_a + (1-\theta)i_p}. \tag{17}$$

Finally, the existence of the monetary equilibrium requires (12) which, using (16), can be rewritten as $i_a \geq i_p$.

As in the region where $\theta^* \leq 1$, a natural policy candidate for the implementation of the first-best has $i_p = 0$. Using i_a to equate the right-hand side of (15) with the right-hand side of (17), we obtain $i_a = i_a^*$. In turn, using (16), we obtain

$$\tau^{**} = -\left[1 - \beta + \alpha^* \left(1 - \beta\theta\right) i_a^*\right].$$

As in the case where $\theta < \theta^* \leq 1$, $i_a^* > i_p = 0$ implies $\phi > \beta\phi_{+1}$, and the real rate of return on balances is lower than the discount rate, so the economy is away from the Friedman rule. The next Lemma summarizes our results.

Lemma 3 *Let $\theta^* > 1$. The first-best involves partial participation of the buyers and it can be implemented by the policy $\tau = \tau^{**}$, $i_a = i_a^*$, and $i_p = 0$, which deviates from the Friedman rule.*

The previous three Lemmas together imply the following Proposition.

Proposition 5 *If $\theta < \theta^*$, there exist policy schemes that deviate from the Friedman rule and implement the first-best rewarding active balances with positive interest and idle balances with zero interest.*

This is the main result of the paper. When there are distortions that would imply inefficiencies in both intensive and extensive margins, the optimal monetary policy in the physical currency economy, which is the Friedman rule, achieves efficiency along the intensive but not the extensive margin. In the corresponding digital currency economy, the optimal policy, which is not necessarily the Friedman rule, achieves efficiency also along the extensive margin stimulating the participation of the buyers in the process of trade through the payment of interest on active but not idle balances. Unlike in the physical currency economy, this is feasible in the digital currency economy, since the technology allows to observe the flows of balances in and out of the accounts, which allows to distinguish active and idle balances.

3.3 Velocity of circulation

It is well known from the work of Levine (1991), Kehoe, Levine and Woodford (1992) and Wallace (2014) that extensive margins play a key role in monetary economies. In our framework, the extensive margin is captured by the endogenous participation decision of the buyers. The ability to observe the flows of money balances allows to implement intervention schemes that improve welfare by giving extra incentives to the buyers to participate in trade, thus, increasing the velocity of money. To see this, compare the velocity of circulation of currency across economies under the optimal intervention. In both economies, there is a proportion α of traders who end up in meetings in which currency changes hands for sure, with unit velocity of circulation, and a fraction $1-\alpha$ of traders who keep their currency in the account, in which case the velocity is nil. Therefore, the velocity of circulation of either physical or digital currency is given by $v = \alpha \times 1 + (1-\alpha) \times 0 = \alpha$. If $\theta^* \leq \theta < 1$, both economies implement the first-best and velocity is the same. If, instead, $\theta < \theta^*$, the velocity is higher in the digital currency economy because participation is inefficiently low in the physical currency economy. Thus, the key message of our paper can be interpreted as follows. The observation of balance holdings, which is feasible in the physical currency economy, impacts the intensive margin, as it allows to set a positive real rate of return on currency, thus reducing the cost of carrying balances across periods. In turn, the observation of balance flows in and out of the accounts, which is feasible in the digital currency economy, allows to set a spread between interest rates on active and idle balances that impacts the extensive margin, giving the buyers the incentive to participate in trade and, as a consequence, increasing the velocity of money.

4 Constrained Intervention

In this section, we compare our results with the literature by exploring constrained intervention schemes. First, we examine interventions that pay the same interest rate on idle and active balances. Second, we consider interventions that pay interest only on idle balances. Finally, we examine interventions that do not involve the taxation of balances.

4.1 Uniform Interest on Balances

The potential impact of the introduction of CBDC on financial intermediation has spurred a large literature. The idea is that CBDC may compete with banks on the liability side, thus affecting their ability to fund investments, as, for instance, in Andolfatto (2021), Williamson (2022), Keister and Sanches (2023), and Chiu et al. (2023). For the most part, this literature distinguishes CBDC and cash assuming that they are imperfect substitutes as means of payment. In particular, CBDC is a closer substitute to bank issued debt than cash. Apart from this difference, CBDC is treated as a digital form of cash, and the only notable difference between them is the fact that interest can be paid on CBDC.

In our model, treating digital currency as physical currency in digital form means assuming that the former embeds the same record-keeping technology as the latter, which implies that the monetary authority can no longer distinguish between idle and active balances, and must set $i_a = i_p$. Letting $i_a = i_p = i$ in (11), we obtain

$$\frac{u'(q)}{g'(q)} = \frac{1 - \beta + \frac{\tau}{1+i} + \alpha\beta}{\alpha\beta}. \tag{18}$$

Note that the only difference between (18) and (7) is in the first-term in the numerator of these equations. In fact, while in the physical currency economy, the rate τ at which money is injected or taxed in the economy in a lump-sum way is the only policy instrument; in the digital currency economy with uniform interest the policy instrument is given by τ divided by the gross nominal interest rate.

Consider now the extensive margin of the digital currency economy. If $i_a = i_p = i$ in (9), we obtain

$$(1 - \alpha)\left\{\theta\left[u(q) - c(q)\right] - k'(\alpha)\right\} = 0,$$

which coincides with (5), the extensive margin in the physical currency economy. We have immediately the following proposition.

Proposition 6 *Let τ_c (resp. τ_d) be the rate at which money is injected or taxed in a lump sum way in the physical currency (resp. digital currency) economy, and let i be the uniform interest payment on balances in the digital currency economy. An outcome (α, q) is an equilibrium in the physical currency economy if and only if it is also an equilibrium in the digital currency economy.*

This result highlights that, if one restricts attention to intrinsic properties, a key fundamental difference between physical and digital currency is the fact that the latter embeds a technology which allows to keep track of flows and holdings across accounts. In turn, this allows the monetary authority to implement interventions that condition on this information. In other words, if one assumes that digital currency is simply a digital form of physical currency, differences between these instruments must be related to extrinsic features of the economy, i.e., frictions that make physical and digital currencies imperfect substitutes as means of payments.

4.2 Rewarding Idle Balances

The literature has examined the role of banks and liquidity markets to reallocate idle balances in a model based on Lagos and Wright (2005), as in Berentsen, Camera and Waller (2007), Ferraris and Watanabe (2008) and Geromichalos and Herrenbrueck (2017). These models usually feature an exogenous intensive margin and competitive markets, so the Friedman rule implements the first-best. Away from the Friedman rule, it is shown that banks and liquidity markets can improve the allocation by transferring balances from the agents without to those with a consumption opportunity. This transfer is beneficial as it allows the payment of positive interest to agents without a consumption opportunity, who would otherwise keep their balances idle. Intuitively, paying interest on idle balances is beneficial because it insures agents against the risk of not participating in trade.

To replicate the role of banks and liquidity markets in reallocating balances with digital currency, consider an intervention by the monetary authority that pays positive interest on idle balances but no interest on active balances. To make the insurance effect of paying interest on idle balances relevant, we restrict attention to the region of parameters where $\theta^* > 1$ and full participation is not efficient.

Let $i_a = 0$. We can rewrite (11) as

$$\frac{u'(q)}{g'(q)} = \frac{1-\beta+\alpha\beta+\tau+(1-\alpha)(1-\beta)\, i_p}{\alpha\beta\left[1+(1-\theta)i_p\right]}, \tag{19}$$

and we can rewrite (9) as

$$k'(\alpha) = \theta\frac{u(q)-(1+i_p)c(q)}{1+(1-\theta)i_p}. \tag{20}$$

The existence of the monetary equilibrium requires $\phi \geq \beta\phi_{+1}(1 + i_p)$, which can be rewritten as $\tau \geq (\beta - 1)(1 + i_p) + \alpha i_p \equiv \tau(i_p)$. It follows immediately that the optimal policy if $i_a = 0$ is the Friedman rule, given by $\tau = \beta - 1$ and $i_p = 0$. This policy achieves the efficient quantity and it encourages participation by offering zero interest on idle balances. As shown in the previous section, this policy is dominated by an intervention that sets positive interest on active balances and zero interest on idle balances.

Now, let $i_a = 0$ and consider the scenario where $\phi > \beta\phi_{+1}(1+i_p)$ and the economy is away from the Friedman rule. In particular, fix $\tau = \tau(i_p) + \delta$. We can rewrite (19) as

$$\frac{u'(q)}{g'(q)} = \frac{1 + i_p + \frac{\delta}{\alpha\beta}}{1 + (1 - \theta)i_p}.$$

and an increase in i_p leads to an increase in the quantity produced by the seller if and only if $\delta > \frac{\alpha\beta\theta}{1-\theta}$, and the economy is sufficiently away from the Friedman rule. Intuitively, an increase in i_p insures buyers against the risk of not meeting a seller. However, unlike banks, the insurance provided by the monetary authority requires the injection of currency into the economy, which reduces its real rate of return. The overall effect on the intensive margin is thus ambiguous. If the economy is not too far away from the Friedman rule, the insurance benefit is less relevant and a higher interest rate on idle balances reduces the quantity in trade meetings. Since the increase in i_p also reduces participation in trade, welfare decreases. However, if the economy is farther away from the Friedman rule, the insurance effect dominates and an increase in i_p increases the quantity in trade meetings. However, since participation goes down, the overall welfare effect is ambiguous.

Summarizing, digital currency may replicate the insurance role of banks in improving the intensive margin by paying interest on idle balances, but it does so at the cost of creating currency to fund the payment of the interest. Allowing for the payment of interest on active balances, instead, digital currency can improve both the intensive and the extensive margins of trade.

4.3 No taxation of Balances

The taxation of balances may not be feasible in pure currency economies, as pointed out, among others, by Hu, Kennan and Wallace (2009), Andolfatto (2010), Wallace (2014), and Bajaj et al. (2017). The idea is that the same frictions on commitment and monitoring that render money essential prevent also the working of lump-sum taxation schemes. In our case, this translates into a restriction to policies with $\tau \geq 0$. In Appendix B, we show that, even if balances cannot be taxed, there still exists an open region of parameters that replicates the results obtained in the previous section, i.e., the observation of balance flows still allows digital currency to dominate physical currency and implement the first-best. The intuition is the following.

In the physical currency economy, we consider a scheme where the monetary authority makes a transfer of real balances to each account that conditions on the real balance in the account. As in Bajaj et al. (2017), the optimal scheme compensates for the inflation tax accounts with real balances larger than or equal to the socially optimal amount, i.e., the amount of real balances that induces the sellers to produce the efficient quantity. This ensures that the efficient quantity is produced. However, since this transfer scheme cannot separate between idle and active balances, it fails to directly give the buyers the incentive to participate. In the digital currency economy, instead, since flows in an out of accounts are also observed by the authority, the optimal scheme compensates for the inflation tax only active balances. This ensures not only an efficient production in trade meetings but also an efficient participation of buyers.

Finally, the only difference between these schemes and the optimal interventions considered in the case where real balances can be taxed, is that agents must be sufficiently patient, both in the physical currency and in the digital currency economy. This is so because, unlike the scenario where real balances can be taxed and the optimal intervention can condition on the agent's discount factor, here the injection of real balances must be sufficiently large to convert the decision of a buyer on how much real balances to bring, into a binary choice between bringing the socially optimal amount and bringing zero balances. Patience is required for the former option to dominate.

5 Robustness

In this section, we extend the result to a larger class of bargaining procedures than just Kalai bargaining and show that the scheme with the payment of interest on active balances is robust to manipulation attempts that involve the opening of shadow accounts.

5.1 Bargaining

In the search and matching monetary literature, since trade is not mediated by the Walrasian auctioneer, the pricing mechanisms are key. In our setting, trade meetings are bilateral and we have adopted the Kalai bargaining procedure for simplicity, as it makes computations easier. Other bargaining schemes could be adopted without altering the main results. In Appendix C, we adapt to our setting the monetary mechanisms proposed by Gu and Wright (2016) and show that our results remain valid. Particular cases of their mechanism include Kalai bargaining, Nash bargaining, and competitive price taking. In what follows, we consider Nash bargaining in more detail, since it unveils an additional channel through which the observation of balance flows allows the implementation of policies that improve welfare.

It is well known from Aruoba, Rocheteau and Waller (2007) that, even without the endogenous participation decision, i.e. when the extensive margin plays no role, the Friedman rule implements the efficient quantity within trade meetings under Kalai but not Nash bargaining, unless the buyers have full bargaining power. This occurs because the traders' payoffs are non-monotonic in real balances under Nash bargaining.[6] Hence, under Nash bargaining, there may be room for digital currency to allow the implementation of the efficient allocation even without the endogenous participation decision.

Consider the digital currency economy with an exogenous extensive margin, so a buyer meets a seller in the day market with an exogenous probability α. It is straightforward to show that, under Nash bargaining, if a buyer with an amount m of currency meets a seller, terms of trade are given by

$$\phi m = \frac{(1-\theta)u(q)c'(q) + \theta c(q)u'(q)}{\theta(1+i_a)u'(q) + (1+i_p)(1-\theta)c'(q)} \equiv N(q, i_a, i_p). \tag{21}$$

In turn, in the previous night market, the buyer chooses m to maximize $-\phi m + \beta[\alpha u(q) + (1-\alpha)\phi_{+1}(1+i_p)m]$. Using the derivative of (21) with respect to q, we obtain the intertemporal condition for the optimum,

$$\frac{u'(q)}{N_q(q, i_a, i_p)} = \frac{1-\beta+\alpha\beta+\tau+\alpha i_a+(1-\alpha)(1-\beta)i_p}{\alpha\beta}. \tag{22}$$

[6]See Hu and Rocheteau (2020) for a unified strategic foundation of the Kalai and Nash protocols that shed further light on their differences.

We can replicate the allocation in the economy with physical currency by letting $i_a = i_p = 0$, which delivers

$$\frac{u'(q)}{N_q(q,0,0)} = \frac{1-\beta+\alpha\beta+\tau}{\alpha\beta}. \tag{23}$$

It is straightforward to show that the solution q_c to (23) is strictly lower than q^*, even if $\tau = \beta - 1$ and the economy is at the Friedman rule. Moreover, q_c converges to q^* as θ goes to one. These are well known results in the literature. Notice that (21) implies that an increase in the interest rates allows the buyer to consume the same quantity with lower real balances, since an increase in i_a increases the surplus in trade meetings, while an increase in i_p increases the outside option of the buyer. This suggests that the buyer may be encouraged to bring more real balances to the day market if he expects positive interest rates. However, positive interest rates require currency creation, which reduces the real rate of return on currency and lowers the incentive to bring real balances. The overall effect is ambiguous. Should the monetary authority set $i_a = i_p = i > 0$, (22) would collapse into (23), leading to the same allocation as in the physical currency economy. Thus, under uniform interest rates, it is still the case that the surplus of the buyer may decrease while the total surplus increases, which is at the root of the inefficiency under Nash bargaining. In the following Proposition, whose proof is in the Appendix, we show that setting different interest rates on active and passive balances allows the monetary authority to achieve the efficient quantity.

Proposition 7 *Fix α and assume the Nash bargaining protocol, with $\theta < 1$. Setting $\tau = \beta - 1$ and $i_p = 0$, there exists a unique $i_a > 0$ that achieves the first best, $q = q^*$.*

Summarizing, the observation of balance flows offers an additional instrument to the monetary authority, i.e., the spread between the interest rate on active and idle balances. If the extensive margin is endogenous, this additional instrument can be used to improve welfare by increasing the participation of buyers and the velocity of money. If the extensive margin is exogenous and terms of trade are determined by Nash bargaining, this additional instrument can be used to improve welfare by aligning the surplus of the buyer and the total surplus in trade meetings.

5.2 Manipulation

We would like to find out whether our results are robust to the possibility of manipulation, whereby agents try to obtain interest on idle balances equal to the interest paid on active balances. For example, if an agent is allowed to open more than one account with the monetary authority, she can transfer unused balances across accounts and be paid an interest i_a on those balances. We prevent this possibility by assuming that each agent has only one account with the monetary authority.

In what follows, we introduce the possibility of manipulation by assuming that agents can open a shadow account that the authority cannot associate with the owner of any existing account. However, in order to do so, they incur a cost $\gamma > 0$ during the night market. This account can then be used in the next day market to transfer balances in order to receive the interest i_a.

Consider the incentive of a single buyer to deviate and incur the cost γ. To make it transparent the benefits of incurring this cost, we consider the region of parameters where $\theta^* > 1$ and full participation is not efficient. In this case, Lemma 3 implies that the optimal policy is given by $i_p = 0$, $i_a = i_a^*$ and $\tau = \tau^{**}$. If the deviant buyer holds an amount $\tilde{m}$ of currency, he chooses $\tilde{q}$ and $\tilde{d} \leq \tilde{m}$ to maximize $u(\tilde{q}) - \phi(1+i_a^*)\tilde{d}$ subject to the constraint $(1-\theta)\tilde{S}_b = \theta\tilde{S}_s$, where $\tilde{S}_b = u(\tilde{q}) - \phi(1+i_a^*)\tilde{d}$ and $\tilde{S}_s = -c(\tilde{q}) + \phi(1+i_a^*)\tilde{d}$. Note that the buyer receives interest i_a^* on his passive balances. This allows to rewrite the constraint $(1-\theta)\tilde{S}_b = \theta\tilde{S}_s$ as $\phi\tilde{m} = \frac{g(\tilde{q})}{1+i_a^*}$. Consider now the participation decision. Optimization leads to the complementary slackness condition for the participation rate $\{\theta[u(\tilde{q}) - c(\tilde{q})] - k'(\tilde{\alpha})\}(1-\tilde{\alpha}) = 0$, where both terms in parenthesis are non-negative. Finally, moving to the night market, the intertemporal condition for the optimum can be written as

$$\frac{u'(\tilde{q})}{g'(\tilde{q})} = \frac{1 + \tau + \alpha^* i_a^* - \beta(1+i_a^*)(1-\tilde{\alpha})}{\beta\tilde{\alpha}(1+i_a^*)}. \tag{24}$$

If the buyer chooses not to incur the cost, his participation is given by (9) and the intertemporal condition is given by (11), both evaluated at the optimal policy. Since the optimal policy implements the first-best, the net benefit of not incurring the cost is

$$\beta\left[\alpha u(q^*) - k(\alpha^*)\right] - \frac{1 + \tau + \alpha^* i_a^* - \beta(1-\alpha^*)}{1+\theta i_a^*} g(q^*), \tag{25}$$

while the net benefit if the buyer chooses to incur the cost is

$$-\gamma + \beta\left[\tilde{\alpha}u(\tilde{q}) - k(\tilde{\alpha})\right] - \frac{1+\tau+\alpha^{*}i_a^{*} - \beta(1-\tilde{\alpha})(1+i_a^{*})}{1+i_a^{*}}g(\tilde{q}). \tag{26}$$

Provided (25) is not smaller than (26), the buyer has no incentive to incur the cost. The next Proposition, whose proof appears in the appendix, shows that there exists an open region of parameters under which the buyer has no incentive to deviate and create a shadow account.

Proposition 8 *Suppose the system can be manipulated at a cost, $\gamma \geq 0$. For any $\gamma > 0$, there exists an open interval of values of θ in which the optimal policy with digital currency identified above is incentive compatible.*

The optimal policy that reproduces the Friedman rule and achieves the best allocation attainable with physical currency is always available. Hence, for any positive cost of manipulation, the optimal policy with digital currency can always at least replicate the outcome with physical currency and, in an open set of economies, strictly improve upon it. Intuitively, this is achieved when the interest payments that induce efficiency can be kept small. If the digital system can be manipulated at no cost, obviously, the digital technology cannot constitute an effective improvement over traditional currency. If it is even slightly costly to game the system, however, there are robust circumstances in which the extra information provided by the digital technology can be exploited to achieve the fully efficient allocation, while discouraging the manipulation.

In principle, the agents could try to manipulate the system forming partnerships with other agents in order to transfer balances between the accounts and get undue interest payments. However, in our economy, by assumption, the agents meet randomly and bilaterally and cannot commit to future actions. These frictions, that are key to generate an essential role for currency as a medium of exchange, prevent the formation of partnerships of this type as well as market trade and group interactions in general. Hence, the only feasible manipulation schemes are individual.

6 Conclusion

We have argued that the key difference between digital currency and cash consists in the technological ability to trace monetary flows in and out of the accounts. Since the extra information collected with this superior technology can be ignored, any allocation achieved with cash can be attained also with digital currency. However, if the extra information is used to stimulate the velocity of circulation of money, there are robust circumstances in which digital currency can help attain the efficient allocation that would be unattainable with cash. The availability of such information does indeed make the difference for the design of optimal policy.[7]

The payment of interest on active balances outperforms other reward schemes that have been identified as welfare improving in physical currency economies, such as the payment of interest on idle balances, obtained through monetary loans. This occurs because the positive interest differential between active and idle balances stimulates the velocity of circulation through its effect on the extensive margin, while rewarding idle balances affects positively the intensive but not the extensive margin.

Interestingly, in some cases, notably when the trading protocol is Nash bargaining, the interest payment on active balances in the digital economy can be used to correct intensive margin distortions, that optimal policy would be unable to redress in the corresponding physical currency economy. Hence, this type of intervention can play a role in addressing inefficiencies that arise from both extensive and intensive margins. This is ultimately due to the availability of a further policy instrument that can be brought into play thanks to the extra information obtained through the technology that underlies the digital payment system relative to the information made available when the payment system is based on cash.

The banking system plays no special role in our environment. Indeed, we show that the efficient allocation can be achieved without banks. However, this does not mean that the adoption of CBDC would necessarily lead to disintermediation. On the contrary, banks could play a key role in the actual implementation of the system, as the digital payments could continue to work exactly as they currently do with the interest on active balances being paid through the banking system.

[7] In systems theory and cybernetics à la Wiener and Shannon, information is defined as neg-entropy or ”the difference that makes a difference”, e.g. Bateson (1972), a definition we alluded to playfully in the title.

7 Appendix

In Appendix A, we provide the proofs omitted in the main text. In Appendix B, absent taxation of balances, we adapt the transfer scheme of Bajaj et al.(2017) to our setting. In Appendix C, we show that our results are valid in a more general class of trading mechanisms than the Kalai bargaining procedure adopted in the text.

7.1 Appendix A: Omitted Proofs

Proof of Proposition 1. Suppose $\alpha^* > 0$. Define $\Phi(q) \equiv u'(q) - c'(q)$, a continuous function with $\Phi(0) = \infty$, $\Phi(\infty) < 0$ and $\Phi'(q) < 0$ by the properties of the fundamentals. Hence, a $q^* \in (0, \infty)$ that satisfies $\Phi(q) = 0$ exists and is unique. The function $k'(\alpha)$ is monotonic in α, hence, invertible. By equation (1), if $\theta^* \leq 1$, $\alpha^* = 1$, otherwise $\alpha^* = k'^{-1}(u(q^*) - c(q^*)) \in (0, 1)$. In both cases, $\alpha^* > 0$. QED

Proof of Proposition 2. Solve (7) for the quantity $q = f^{-1}(\alpha, \tau)$ and plug it into (5). Define $\hat{\theta}(\tau)$ as the unique value of θ that satisfies the first term at equality in equation (5) with $\alpha = 1$. Such a cutoff that exists and is unique by the properties of the fundamentals is at least as large as θ^*. There are two possible cases. Suppose, first, $\theta \geq \hat{\theta}(\tau)$, then, $\alpha = 1$ and $q = f^{-1}(1, \tau)$. The existence and uniqueness of the solution of (7) for any $\tau \geq \beta - 1$ in this case follows immediately from the continuity of the function $f(\cdot)$, with $f(\infty) = 0$, $f(0) = \infty$ and $f'(q) < 0$. Second, suppose $\theta < \hat{\theta}(\tau)$, then $\alpha < 1$ solves $\Delta(\alpha, \tau) \equiv \theta[u(f^{-1}(\alpha, \tau)) - c(f^{-1}(\alpha, \tau))] - k'(\alpha) = 0$ and $q = f^{-1}(\alpha, \tau)$. The solution exists for any $\tau \geq \beta - 1$ since $\Delta(\cdot, \tau)$ is continuous in α, with $\Delta(0, \tau) = 0$, $\Delta(1, \tau) < 0$ and $\Delta_\alpha(0, \tau) = \infty$. Uniqueness can be guaranteed for τ not too large. Once α is determined, equation (7) gives uniquely q. Let $\alpha(\tau)$ and $q(\tau)$ denote the equilibrium variables as function of policy, τ. Implicit differentiation shows that $q'(\tau) < 0$ and $\alpha'(\tau) < 0$ when $\alpha < 1$. QED

Proof of Proposition 4. Define $\iota \equiv (\tau, i_a, i_p)$. Solve (11) for the quantity $q = f^{-1}(\alpha, \iota)$. and plug it into (9). Define $\underline{\theta}(\iota)$ as the unique value of θ that satisfies the first term at equality in equation (9) with $\alpha = 1$. Such a cutoff that exists and is unique by the properties of the fundamentals is at least as large as θ^*. There are two possible cases. Suppose, first, $\theta \geq \underline{\theta}(\iota)$, then, $q = f^{-1}(1, \iota)$ and $\alpha = 1$. The existence and uniqueness of the solution of (11) for any τ that satisfies (12) in this case follows immediately from the continuity of the function $f(\cdot)$, with $f(\infty) = 0$, $f(0) = \infty$ and $f'(q) < 0$. Second, suppose $\theta < \underline{\theta}(i)$, then $\alpha < 1$ solves $\Gamma(\alpha, \iota) \equiv$

$\theta[u(f^{-1}(\alpha,\iota)) - c(f^{-1}(\alpha,\iota))] - k'(\alpha) = 0$ and $q = f^{-1}(\alpha,\iota)$. The solution exists for any τ that satisfies (12) since $\Gamma(\cdot,\iota)$ is continuous in α, with $\Gamma(0,\iota) = 0$, $\Gamma(1,\iota) < 0$ and $\Gamma_\alpha(0,\iota) = \infty$. Uniqueness can be guaranteed for τ not too large. Once α is determined, equation (11) gives uniquely q. Let $\alpha(\iota)$ and $q(\iota)$ denote the equilibrium variables as function of policy. Implicit differentiation shows that $q_\tau(\iota) < 0$ and $\alpha_\tau(\iota) < 0$ when $\alpha < 1$; $q_{i_a}(\iota) > 0$ and $\alpha_{i_a}(\iota) > 0$; the signs of the derivatives of $q(\iota)$ and $\alpha(\iota)$ wrt i_p are ambiguous. QED

Proof of Proposition 7. Consider the equation (22) evaluated at q^*, $\tau = \beta - 1$ and $i_p = 0$. Manipulating such a condition, we obtain a quadratic equation of the interest on active balances, i_a, $\Phi(i_a) = 0$. Since $\Phi(0) < 0$, $\Phi(\infty) > 0$, $\Phi'(i_a) > 0$ for all $i_a \geq 0$, we conclude that there exists a unique $i_a > 0$ that induces the efficient allocation. QED

Proof of Proposition 8. Consider the case $\theta^* > 1$. Let the authority adopt the optimal policy that implements the first-best. Use (25) to define the function $\Psi(\theta)$ and use (24) and (26) to define $\tilde{\Psi}(\theta)$. The function $\Psi(\theta) - \tilde{\Psi}(\theta)$ is continuous in θ and $\Psi(1) - \tilde{\Psi}(1) = \gamma$. Hence, for $\gamma > 0$, by continuity there is an interval of values of $\theta \approx 1$ so that $\Psi(\theta) - \tilde{\Psi}(\theta) \geq 0$. Finally, to guarantee that the agents do not create shadow accounts even if they do not want to participate in trade, it has to be that $\frac{(1+\tau)(1+\gamma)}{\beta} \geq 1 + i_a$, that is satisfied for $\gamma > 0$ if $\theta \approx 1$. Consider the case $\theta^* \leq 1$ with $\theta < \theta^*$. Let the authority adopt the optimal policy that implements the first-best with $\tau = \beta - 1$, $i_a > 0$ and $i_p < 0$. Taking $\theta \to \theta^*$ from below, an analogous continuity argument applies for $\theta \approx \theta^*$. QED

7.2 Appendix B: No Taxation

Suppose taxation of balances is not feasible. For simplicity, we assume that $\theta < \theta^* \leq 1$, implying that the first-best involves full participation of buyers, and it cannot be achieved in the physical currency economy. We begin with physical currency. The transfer scheme we use is adapted from Bajaj et al. (2017). Assume that the monetary authority transfers $\tau_c(\phi m) \geq 0$ real balances to each account with ϕm real balances at the end of the day market, for all $m \geq 0$.[8] Thus, if the buyer gives ϕm real balances

[8]This transfer scheme is feasible because balances can be observed in a physical currency economy. In Bajaj et al. (2017) transfers are given to agents at the beginning of the day market, while we are considering a slightly modified version, where agents receive their transfers at the end of the day market. As it will become clear, this change is immaterial in the physical currency economy but it matters in the digital currency economy.

to a seller with zero balances during the day market, the seller anticipates that she will receive a transfer of $\tau_c(\phi m)$ real balances at the end of the day market. Under Kalai bargaining, the quantity produced in the meeting satisfies

$$\phi m + \tau_c(\phi m) = (1-\theta)u\left(q\right) + \theta c(q) \equiv g(q). \tag{27}$$

Bajaj et al. (2017) propose a transfer scheme that compensates for the inflation tax the buyers who bring at least the socially optimal amount of real balances, i.e., the amount of real balances that induces sellers to produce the efficient quantity. In our setting, this translates into the following transfer scheme

$$\tau_c\left(\phi m\right) = \begin{cases} 0 \text{ if } \phi m < \frac{g(q^*)}{1+\tau} \\ \frac{\tau g(q^*)}{1+\tau}, \text{ if } \phi m \geq \frac{g(q^*)}{1+\tau} \end{cases}, \tag{28}$$

where $M_{+1} = (1+\tau)M$. Note that (27) implies that this transfer scheme gives the seller the incentive to produce the efficient quantity if the buyer brings $\frac{g(q^*)}{1+\tau}$ real balances, since $\frac{g(q^*)}{1+\tau} + \tau_c\left(\frac{g(q^*)}{1+\tau}\right) = g(q^*)$. Moreover, since $\tau_c\left(\phi m\right) > 0$ is a step function at $\frac{g(q^*)}{1+\tau}$, carrying balances above $\frac{g(q^*)}{1+\tau}$ is costly and the seller does not receive additional transfers if she receives more than $\frac{g(q^*)}{1+\tau}$ from the buyer. As a result, the buyer has no incentive to bring more than $\frac{g(q^*)}{1+\tau}$ real balances into the day market. In the day market, the buyer with ϕm real balances makes his participation decision, solving

$$\max_{\alpha}\left\{\alpha u(q) + (1-\alpha)\left[\phi m + \tau_c(\phi m)\right] - k(\alpha)\right\}.$$

Using (27) and (28), the optimum satisfies

$$k'(\alpha) = \theta\left[u(q) - c(q)\right], \tag{29}$$

so that the participation decision of the buyer only depends on the surplus in the trade meeting. In the night market, the buyer chooses the money balances. There are two scenarios. If the buyer brings $\frac{g(q^*)}{1+\tau}$ real balances, his payoff is[9]

$$U_c^+ = -g(q^*) + \beta\left[\alpha_c u(q^*) + (1-\alpha_c)g(q^*) - k(\alpha_c)\right],$$

where α_c solves (29) evaluated at $q = q^*$. Since the efficient quantity is produced in trade meetings, and $\alpha_c < 1$ solves (5), we obtain the same allocation achieved under the Friedman rule when the balances can be taxed. If, instead, the buyer brings $\phi m < \frac{g(q^*)}{1+\tau}$, his payoff is

$$U^- = -\phi_{-1}m + \beta\left[\alpha u(q) + (1-\alpha)g(q) - k(\alpha)\right],$$

where α solves (29) evaluated at q. Note that, since $\phi \to 0$ when $\tau \to \infty$, (27) implies that $q \to 0$ when $\tau \to \infty$. Thus, there exists $\overline{\tau}$ such that $U^- \leq 0$ for all $\tau \geq \overline{\tau}$. As a result, if $\tau \geq \overline{\tau}$, a sufficient condition for the buyer to bring $\frac{g(q^*)}{1+\tau}$ real balances is that $U^+ \geq 0$, which can be rewritten as $\beta \geq \beta(\alpha_c) \equiv \frac{g(q^*)}{g(q^*)+\alpha_c k'(\alpha_c)-k(\alpha_c)}$. Therefore, if $\theta < \theta^* \leq 1$. If $\tau \geq \overline{\tau}$ and $\beta \geq \beta(\alpha_c)$, there exists a transfer scheme that replicates the allocation achieved under the optimal policy in the physical currency economy with taxation of balances.

Consider now the digital currency economy. The analysis is similar to the one in the physical currency economy, but with a modified transfer scheme,

$$\tau_d(\phi m) = \begin{cases} 0 \text{ if } \phi m < \frac{g(q^*)}{1+\tau} \\ 0, \text{ if } \phi m \geq \frac{g(q^*)}{1+\tau} \text{ and the balance is passive} \\ \frac{\tau}{1+\tau}g(q^*), \text{ if } \phi m \geq \frac{g(q^*)}{1+\tau} \text{ and the balance is active} \end{cases}.$$

This scheme only gives transfers to active balances that are at least equal to the socially optimal amount. This is the case of the balances held by the seller that participated in a trade meeting where the buyer brought the socially optimal amount of real balances. As in the physical currency economy, the sellers reciprocate by producing the efficient quantity in exchange for these balances. Moreover, since $\tau_d(\phi m) > 0$ is a step function at $\frac{g(q^*)}{1+\tau}$, the buyer has no incentive to bring more than $\frac{g(q^*)}{1+\tau}$ real balances into the day market. Consider now the participation decision of the buyer. If

[9] In more detail, if $\phi m = \frac{g(q^*)}{1+\tau}$, then $m = \frac{g(q^*)}{\phi(1+\tau)}$ and the cost of acquiring m units of currency in the previous night market is $\phi_{-1}m = \frac{\phi_{-1}g(q^*)}{\phi(1+\tau)} = g(q^*)$, since $\frac{\phi_{-1}}{\phi} = \frac{M}{M_{-1}} = 1+\tau$. In turn, since the buyer receives a transfer $\frac{\tau g(q^*)}{1+\tau}$ if he holds $\frac{g(q^*)}{1+\tau}$ real balances at the end of the day market, his total balances if he does not participate in a trade meeting is given by $g(q^*)$.

he brings less than $\frac{g(q^*)}{1+\tau}$ real balances, his participation solves (29), as in the physical currency economy. Things are different if a buyer brings $\frac{g(q^*)}{1+\tau}$. In this case, he solves

$$\max_{\alpha}\left[\alpha u(q^*)+(1-\alpha)\frac{g(q^*)}{1+\tau}-k(\alpha)\right],$$

and the optimum satisfies

$$(1-\alpha)\left[u(q^*)-\frac{g(q^*)}{1+\tau}-k'(\alpha)\right]=0,$$

where both terms in parenthesis are non-negative. Full participation requires

$$k'(1)<u(q^*)-\frac{g(q^*)}{1+\tau}.$$

Since $\theta^*\leq 1$ implies $k'(1)\leq u(q^*)-c(q^*)$, a sufficient condition for buyers to fully participate is

$$\tau\geq\frac{(1-\theta)\left[u\left(q^*\right)-c(q^*)\right]}{c(q^*)}.$$

A higher rate of money growth gives the incentive for full participation by imposing an inflation tax on idle balances, even if the balances are equal to the socially optimal amount. In the night market, the buyer chooses his balances. There are two scenarios. If the buyer brings $\phi m<\frac{g(q^*)}{1+\tau}$, his payoff is the same as in the physical currency economy, given by U^-. Thus, since $\phi\to 0$ when $\tau\to\infty$, (27) implies that $q\to 0$ when $\tau\to\infty$ and there exists $\overline{\tau}$ such that $U^-\leq 0$ for all $\tau\geq\overline{\tau}$. If, instead, the buyer brings $\frac{g(q^*)}{1+\tau}$ real balances, his payoff is

$$U_d^+=-g(q^*)+\beta\left[u(q^*)-k(1)\right].$$

Thus, if $\tau\geq\overline{\tau}$, a sufficient condition for the buyer to bring $\frac{g(q^*)}{1+\tau}$ real balances is that $U_d^+\geq 0$, which can be rewritten as

$$\beta(1)\geq\frac{g(q^*)}{g(q^*)+k'(1)-k(1)}.$$

Therefore, in the case $\theta<\theta^*\leq 1$, if $\tau\geq\overline{\tau}$ and $\beta\geq\beta\left(1\right)$, there exists a transfer scheme that replicates the allocation achieved under the optimal policy in the digital currency economy with the taxation of balances.

7.3 Appendix C: General Mechanism

Gu and Wright (2016) propose a trading mechanism that satisfies a few desirable axioms, namely feasibility, individual rationality, monotonicity and bilateral efficiency, and subsumes, as particular cases, both Kalai and Nash bargaining, and competitive price taking, among other solution concepts. In what follows, we first provide a general form of their mechanism that is consistent with the record keeping technology under digital currency. We then show that our main result holds, i.e., the observation of balance flows allows to achieve the first best. Consider the following trading mechanism $\Gamma = (\Gamma_p(z), \Gamma_q(z))$, where $\Gamma_p(z)$ sets the real balances transferred from the buyer to the seller, and is given by

$$\Gamma_p(z) = \begin{cases} z \text{ if } z < p^* \\ p^* \text{ if } z \geq p^* \end{cases},$$

where p^* is defined as the minimum payment required for a buyer to receive q^* units of goods from the seller. In turn, $\Gamma_q(z)$ sets the quantity produced by the seller and is given by

$$\Gamma_q(z) = \begin{cases} v^{-1}((1+i_a)z) \text{ if } z < \frac{p^*}{1+i_a} \\ q^* \text{ if } z \geq \frac{p^*}{1+i_a} \end{cases},$$

where $v(\cdot)$ is a strictly increasing, twice continuously differentiable function, with $v(0) = 0$ and $v(q^*) = \frac{p^*}{1+i_a}$. This is the same trading mechanism as in Gu and Wright (2016) if $i_a = 0$. In the digital currency economy, we need to take into account that, if the buyer transfers z real balances to the seller in the day market, at the beginning of the night market, the seller will have $(1+i_a)z$ real balances. Note that the terms of trade do not depend on the real balances of the seller, and the efficient quantity is produced if $(1+i_a)z \geq p^*$. In turn, without loss in generality insofar as optimal allocations are concerned, henceforth, we restrict attention to interventions that pay zero interest on idle balances. This implies that, in equilibrium, the sellers do not bring balances and the buyers do not bring balances they are not planning to use. We start with the participation decision of a buyer with real balances z at the beginning of the day market. He chooses α to maximize $\alpha u(q) + (1-\alpha)z - k(\alpha)$. The complementary slackness condition for the optimum is

$$(1-\alpha)\left[u(q) - \frac{v(q)}{1+i_a} - k'(\alpha)\right] = 0,$$

where both terms in parenthesis are non-negative. Note that, given q, an increase in i_a increases the incentive of the buyer to participate in trade. Consider now the night market. The buyer chooses z to maximize $-\phi_{-1}z/\phi + \beta[\alpha u(q) + (1-\alpha)z - k(\alpha)]$. The intertemporal condition for the optimum is

$$\frac{u'(q)}{v'(q)} = \frac{1-\beta+\alpha\beta+\tau+\alpha i_a}{\alpha\beta(1+i_a)}.$$

where we used $M_{+1} = (1+\tau+\alpha i_a)M$ and the stationarity of real balances. We now determine the optimal intervention. First, let $\theta^* > 1$, so the first-best involves $q = q^*$ and $\alpha^* < 1$, where

$$u(q^*) - c(q^*) = k'(\alpha^*). \tag{30}$$

If a policy implements the first-best, we must have

$$\frac{u'(q^*)}{v'(q^*)} = \frac{1-\beta+\alpha^*\beta+\tau+\alpha^* i_a}{\alpha^*\beta(1+i_a)}, \tag{31}$$

and

$$u\left(q^*\right) - \frac{v(q^*)}{1+i_a} = k'(\alpha^*). \tag{32}$$

If $i_a = 0$, efficient participation requires $v(q^*) = c(q^*)$, which is not generically satisfied. For example, if $v(q)$ is determined by Kalai bargaining or Nash bargaining, this is only possible if $\theta = 1$, and the buyer has full bargaining power. Thus, the first-best cannot be implemented if the authority cannot distinguish between active and idle balances. If, instead, $i_a > 0$, we can equate the left hand side of (30) and (32) and achieve efficient participation by setting

$$i_a^* = \frac{v(q^*) - c(q^*)}{c(q^*)}.$$

Using (31), we determine the optimal τ, which is given by

$$\tau^* = \beta - 1 + \alpha^*\beta\frac{u'(q^*) - v'(q^*)}{v'(q^*)} + \alpha^*\frac{\beta u'(q^*) - v'(q^*)}{v'(q^*)} i_a^*.$$

The existence of the monetary equilibrium requires $\tau^* \geq \beta - 1 - \alpha^* i_a^*$. Thus, the first-best can be implemented by the policy (τ^*, i_a^*) if and only if

$$(1 + i_a^*) \frac{u'(q^*)}{v'(q^*)} \geq 1. \tag{33}$$

Consider Kalai bargaining, as in the text. In this case, we have

$$v(q) = \frac{(1 + i_a^*) \left[(1 - \theta)u(q) + \theta c(q)\right]}{1 + \theta i_a^*},$$

which implies $i_a^* = \frac{1-\theta}{\theta} \frac{u(q^*)-c(q^*)}{c(q^*)}$, i.e. the optimal policy considered in the text. In this case, we have proved that the first-best can be implemented. Given the regularity of the solution, a transversality argument shows that the same occurs for an open and dense set of mechanisms that are close to the Kalai protocol. Consider now the Nash bargaining protocol. We have

$$v(q) = \frac{(1 + i_a^*) \left[(1 - \theta)u(q)c'(q) + \theta c(q)u'(q)\right]}{\theta(1 + i_a^*)u'(q) + (1 - \theta)c'(q)},$$

which implies $i_a^* = \frac{1-\theta}{\theta} \frac{u(q^*)-c(q^*)}{c(q^*)}$, as in the case of Kalai. Computing $v'(q^*)$ and inserting it into (33) together with $i_a^* = \frac{1-\theta}{\theta} \frac{u(q^*)-c(q^*)}{c(q^*)}$, we obtain the condition $\Gamma(\theta) \geq 0$. Since $\Gamma(1) > 0$, by continuity there exists a non-empty interval of values of θ such that (33) is satisfied. Finally, under competitive pricing, we have $v(q) = pq$, where p is taken as given but is equal to $c'(q)$ in equilibrium. In this case, (33) becomes

$$\frac{c'(q^*)q^*}{c(q^*)} \geq 1,$$

which is always satisfied under our assumptions on the fundamentals.

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