Implementation in Stationary Markov-Perfect Equilibrium*

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Abstract

This paper studies the implementation of society's policy functions in stationary Markov-perfect equilibrium. We identify a monotonicity condition, a natural but nontrivial stationary recursive extension of Maskin monotonicity. We also show that our condition is sufficient under a mild regularity condition when there are three or more agents. We apply our result to a market solution for intertemporal trades and the recursive median voter solution for public capital accumulation.

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1. Introduction

Stationary Markov-perfect equilibrium (SMPE) is a game theoretical solution concept that captures our day-to-day economic life well because we often cannot commit to future actions. An SMPE is a refinement of the subgame perfect-equilibrium to extensive form games where a state variable encodes the whole payoff-relevant information and where strategies take the form of a time-independent function, called policy function, which maps each present state variable into an action (Fudenberg and Tirole (1991), Duffie et al. (1994), Maskin and Tirole (2001)).

While this refinement abstracts away complex dependence on histories, such as those considered in the repeated game literature, it allows us to describe intertemporal trades by responding to state variables. Moreover, it is economically appealing because it enables us to deal with capital accumulation and saving in a setting where the set of social outcomes/allocations may endogenously change over time via the evolution of state variables (see, for instance, Levhari and Mirman (1980), Ericson and Pakes (1995), Dockner and Sorger (1996) among many economic applications). In the price-taking approach, this recursive method is common in the dynamic general equilibrium theory (Prescott and Mehra (1980), Lucas Jr and Stokey (1984), Stokey and Lucas Jr (1989), Ljungqvist and Sargent (2018)).

We study the implementation of social choice objectives in SMPEs. From a mechanism design viewpoint, in a world in which agents cannot commit to future outcomes or future actions, a "good mechanism" or a "stable mechanism" would be such that people are fine to behave in an "automatic" way, period-by-period, without worrying about writing lengthy contracts that directly bind their future lives in a complicated manner. Thus, at each period, the planner is constrained in his elicitation process because he can elicit from agents only information about how society should advance in the next period and their current preferences.

We consider implementation problems in environments with complete information among agents about their permanent types. As is shown by Hayashi and Lombardi

(2019), *even when agents have complete information about their permanent types*, lack of commitment to future actions creates severe problems because agents reveal their information to the planner in a piece-by-piece manner, and this prevents the planner from understanding the real *intentions* of such information revelation. For instance, when an agent reports to the planner that she wants to save more, the planner cannot distinguish whether this is because she is purely patient or because she wants to manipulate equilibrium outcomes in the future by carrying over more wealth. Also, when an agent reports she prefers a high tax rate today, where the society cannot commit to tax rates in the future, the planner cannot distinguish whether this is because the agent purely prefers a high tax rate today or because she prefers the outcome in the future that follows as a consequence of a high tax rate today.

Hayashi and Lombardi (2019) show that the sequential or even recursive competitive equilibrium solution is not implementable (in subgame perfect-equilibria) because the planner cannot distinguish between the real intentions for saving decisions, as explained above. This negative result is also due to their setup, which does not allow internalizing pecuniary externalities due to saving decisions.[1] A saving decision has a pecuniary externality effect because it changes endowment/wealth allocations in the next period while there is no consumption/production externality. This issue is particularly relevant when we want to give a strategic foundation of the competitive equilibrium in a dynamic environment where we cannot commit to future actions because we need to define prices and allocation of wealth/assets in future periods *even after deviations.*

We develop a setup that allows us to internalize such pecuniary externalities by explicitly incorporating state variables for the next period as part of the object of choice for agents at each period. This lets us control saving decisions with pecuniary externalities through taxation or subsidies. Thus, given the current state of society,

[1] The definition of recursive competitive equilibrium due to Prescott and Mehra (1980) rules out this problem by assuming a continuum of traders.

we collect information about people's preferences over pairs, each specifying a current social outcome and a state variable for the next period.

The basic idea of Maskin monotonicity (Maskin (1999)) is that the social choice is invariant to any "unanimous lie." In static environments and in dynamic environments without the stationarity condition, there is no restriction on how such a "unanimous lie" is. Indeed, Hayashi and Lombardi (2019) study an abstract dynamic environment in which any backward induction argument works, and so there are no restrictions on the relation between how people "unanimously lie" after some history and how they "unanimously lie" after another history. In a stationary recursive environment, however, even such "unanimous lies" must be told in a stationary Markov manner while they do not need to be constant across histories.

We present a natural extension of Maskin monotonicity to our stationary recursive environment, which we call Stationary Recursive Monotonicity (SRM). We show that SRM is necessary for implementation in SMPEs. Moreover, under a natural regularity condition, we show that SRM is sufficient for implementation.

We provide two applications. One is the recursive Lindahl solution, a market-like solution that can fully internalize the pecuniary externalities of saving decisions and achieve intertemporal trades. It is worth emphasizing that the collective value function for a fully Pareto-efficient solution typically exhibits *non-stationary* time-discounting over sequences of consumption allocations when individuals' discount rates differ. So, it generally runs into time-inconsistency problems (Zuber (2011), Jackson and Yariv (2014),Jackson and Yariv (2015)). Nevertheless, we show that it is implementable in *stationary* Markov-perfect equilibrium when the next period asset allocation is "encoded" in the state variable, which restricts the society's decision in the next period. More precisely, we require that the society can commit to an allocation of such assets to be carried over to the next period but not to a sequence of consumption allocations for all future periods; that is, when the society can commit only period-by-period.

The other is the recursive median voter solution in the public capital accumulation problem due to Boylan and McKelvey (1995), Boylan et al. (1996), in which people vote on how much of a reproducible public good to save in each period.

Related Literature

This paper contributes to the literature on dynamic implementation, e.g., Moore and Repullo (1988), Lee and Sabourian (2011), and Mezzetti and Renou (2017).

Moore and Repullo (1988) also study the implementation of social choice processes in subgame perfect-equilibrium in a setting where agents' preferences over outcomes are fixed. However, they assume that society commits to the chosen alternative/plan when it reaches a decision. In our setup, a society selects a current outcome (e.g., consumption) and a society's state for the next period in each period. As a result, society cannot commit to plans or future actions in our setting.

Lee and Sabourian (2011) study infinitely repeated implementation problems in a complete information setting where agents' preferences change in each period according to an independent and identical probability distribution, the time horizon is infinite, and the discount factor is common across agents and is common knowledge among the agents and the planner. Lee and Sabourian (2011) show the importance of efficiency for repeated implementation and how any efficient social function can be repeatedly implemented in Nash equilibrium when the discount factor is high enough. In a similar setting, Mezzetti and Renou (2017) connect the work of Lee and Sabourian (2011) with Maskin's static implementation environment and identify a monotonicity-like condition, called dynamic monotonicity, that is necessary and, in some environments, also sufficient for repeated implementation in Nash equilibrium, regardless of the value of the discount factor and whether the horizon is finite or not.

Although the implementation models of Lee and Sabourian (2011) and Mezzetti and Renou (2017) naturally fit in many applications, their i.i.d. assumption rules out intertemporal trades, which are common in dynamic contexts where agents' preferences over sequences of outcomes are fixed. Moreover, since heterogeneity and information about time discounting are critical in intertemporal trades, agents may disagree on time discounting in our setup, and the planner needs to collect information about such disagreement and their per-period utilities. These modeling differences are crucial and motivated by the desire to give a strategic foundation to the dynamic general equilibrium environment where intertemporal trades are present.

Our contribution significantly differs from the dynamic mechanism design literature for the same reasons. Part of this literature studies the properties of profit-maximizing mechanisms in settings where agents' private information evolves stochastically over time. Earlier contributions are Baron and Besanko (1984), Besanko (1985), and Riordan and Sappington (1987). More recent contributions include, for instance, Courty and Li (2000), Battaglini (2005), Eso and Szentes (2007), and Kakade et al. (2013). Another body of work investigates how to implement Pareto efficient allocations in a dynamic setting where private information changes over time. Seminal works in this area are Bergemann and Valimaki (2010) and Atehy and Segal (2013). Pavan et al. (2014) extend the above contributions to a general dynamic setting. Their approach to the design of optimal mechanisms can be thought of as the dynamic analog of the approach proposed by Myerson (1981) for static settings.[2] Finally, another body of literature studies the design of efficient and profit-maximizing mechanisms in settings where agents' information is fixed but where agents or objects arrive stochastically over time. The contributions of this third body are summarized in Gershkov and Moldovanu (2014).

2. The Environment

Preferences, Types, and Society State: We consider environments with a discrete and infinite time horizon that begins at time 1, a society $I = \{1, \cdots, n\}$ that consists of a finite number n of infinitely-lived agents, a set of society types Θ describ-

[2]For a recent introduction to the literature on dynamic mechanism design, see Bergemann and Välimäki (2019).

ing agents' preferences over sequences of outcomes in X^∞, where X is the per-period set of outcomes and a society state space A. We assume complete information among players.[3]

Player i's preferences over X^∞ at $\theta \in \Theta$ are represented by an additively separable utility function

$$U_i(\mathbf{x}, \theta) = \sum_{t=1}^{\infty} \beta_i^{t-1}(\theta)\, u_i(x_t, \theta)$$

where $\beta_i(\theta)$ is agent i's discount factor at θ and $\mathbf{x}$ is a typical sequence in X^∞.

Each society state $a \in A$ specifies a set $Y(a) \subseteq X \times A$, which consists of pairs specifying a current outcome and a next-period society state. Let $Y = \bigcup_{a \in A} Y(a)$.

Example 1. In *a borrowing-lending economy without production,* a society state $a \in A$ specifies an allocation of net assets (credit/debt). Let $A = \left\{a \in \mathbb{R}_+^n : \sum_{i \in I} a_i = 0\right\}$ be the set of net asset allocations which add up to zero. Moreover, let us assume that each agent i earns e_i units of consumption good in each period, which is perishable. For each society state $a \in A$, the set $Y(a)$, defined by

$$Y(a) = \left\{x \in \mathbb{R}_+^n : \sum_{i \in I} x_i = \sum_{i \in I} e_i\right\} \times A,$$

describes a set of pairs, each of which specifies an allocation of current consumptions and an allocation of net assets for the next period.

Example 2. In an intertemporal production economy with physical capital, a society state specifies an allocation of capital, so that $A = \mathbb{R}_+^I$. Let $Q : \mathbb{R}_+^2 \to \mathbb{R}$ be a production function that satisfies the standard regularity properties and exhibits constant returns to scale. Then, for each society state $a \in A$, the set $Y(a)$ defined by

$$Y(a) = \left\{(x, b) \in \mathbb{R}_+^I \times \mathbb{R}_+^I : \sum_{i \in I} x_i + \sum_{i \in I} b_i = Q\left(\sum_{i \in I} a_i, \sum_{i \in I} l_i\right)\right\}$$

[3]Information is complete when agents' preferences and outcomes are common knowledge among all agents.

describes a set of pairs, each of which specifies a current consumption and an allocation of physical capital for the next period.

Example 3. There is one public good, which is reproducible over time. The social choice problem is how much to save at each period. We assume people have the same per-period utility function but disagree on time discounting.

Let $X = \mathbb{R}_+$ and $A = \mathbb{R}_+$. Let $Q : \mathbb{R}_+ \to \mathbb{R}$ be a production function which satisfies the standard regularity properties. Since saving choice determines the current consumption as well, the feasible set is

$$Y(a) = \{(x, h) \in \mathbb{R}^2_+ : x + b \leq Q(a)\}$$

for each $a \in A$.

Assume there is $u : X \to \mathbb{R}$ such that $u_i(z, \theta) = u(z)$ holds for all $\theta \in \Theta$ and all $i \in I$, where u satisfies the standard regularity properties. Hence, time discounting is the only source of heterogeneity among the agents. Let us assume that $\Theta = (0, 1)^I$ and that $\beta_i(\theta) = \theta_i$ for each $i \in I$.

Stationary Policy Function: A *stationary policy function* is a function $f : A \to Y$ such that $f(a) \in Y(a)$ for all $a \in A$. Since $Y(a) \subseteq X \times A$, a stationary policy function f can be written as $f = (z, \alpha)$ where $z : A \to X$ and $\alpha : A \to A$ are functions such that $f(a) = (z(a), \alpha(a))$ for all $a \in A$. For each society state $a \in A$, $f(a) = (z(a), \alpha(a))$ specifies the current outcome and the society state in the next period. Let $F(Y, A)$ denote the set of all stationary policy functions.

Fix any $i \in I$, any $f = (z, \alpha) \in F(A, X)$, any $\theta \in \Theta$ and any $a \in A$. Let $W_i(f, \theta, a)$ be the continuation lifetime utility for agent i at θ when society follows f after the social state a, which is defined by the following recursive formula

$$W_i(f, \theta, a) = u_i(z(a), \theta) + \beta_i(\theta) W_i(f, \theta, \alpha(a)).$$

Example 4. Let us consider the borrowing-lending economy without production described in Example 1. Assume that $u_i(\cdot, \theta) : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strongly increasing and strictly concave.

First, let us note that the recursive competitive equilibrium solution cannot be a well-defined concept when there are pecuniary externalities. For, let us formulate the recursive competitive equilibrium in our finite-agent setting. A stationary policy function f would be seen as recursive competitive equilibrium if there is a function $q : A \to \mathbb{R}_+$, which maps each state a into the price of asset $q(a)$ being measured by the consumption good in the given period, such that for each $i \in I$ it holds

$$(z_i(a), \alpha_i(a)) \in \arg\max_{(x_i, b_i)} u_i(x_i, \theta) + \beta_i(\theta) W_i(f, \theta, (b_i, \alpha_{-i}(a)))$$

subject to

$$x_i + q(a) b_i \leq a_i + e_i,$$

where (x_i, b_i) denotes individual i's potential deviation to choose a pair of current consumption and asset to carry over to the next period and $\alpha_{-i}(a)$ denotes the other individuals' asset holdings for the next period "in equilibrium."

But what is "$(b_i, \alpha_{-i}(a))$"? If i deviates to b_i, the asset/credit allocation $(b_i, \alpha_{-i}(a))$ is not feasible, even in a hypothetical sense, and we cannot define the continuation lifetime utility.

For that asset allocation to be feasible even after deviations, we must introduce a Lindahl-type solution. A stationary policy function f is as recursive Lindahl equilibrium if there are functions $q_i : A \to \mathbb{R}^I_+$, for each $i \in I$, where $q_{ij}(a)$ is the price of individual j's additional saving that is measured by individual i's consumption, such that $\sum_{i \in I} q_{ij}(a) = \overline{q}(a)$ for all $j \in I$ and it holds

$$(z_i(a), \alpha(a)) \in \arg\max_{(x_i, b_i)} u_i(x_i, \theta) + \beta_i(\theta) W_i(f, \theta, b)$$

subject to

$$x_i + q_i(a) \cdot b \leq a_i + e_i$$

for each $i \in I$.

Example 5. Let us consider the intertemporal production economy with the physical capital described in Example 2. Assume that $u_i(\cdot, \theta) : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strongly increasing and strictly concave. Assume that $Q : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strongly increasing and concave.

In this setup, a stationary policy function f would is recursive Lindahl equilibrium if there are functions $r : A \to \mathbb{R}_+$, $w : A \to \mathbb{R}_+$ and $q_i : A \to \mathbb{R}^I_+$, for each $i \in I$, such that $\sum_{i \in I} q_{ij}(a) = \overline{q}(a)$ for all $j \in I$ and such that it holds

$$(z_i(a), \alpha(a)) \in \arg\max_{(x_i, b_i)} u_i(x_i, \theta) + \beta_i(\theta) W_i(f, \theta, b)$$

subject to

$$x_i + q_i(a) \cdot b \leq r(a) a_i + w(a) l_i$$

for each $i \in I$, and

$$\left(\sum_{i \in I} a_i, \sum_{i \in I} l_i \right) \in \arg\max Q(K, L) - r(a) K - w(a) L.$$

Note that $r(a)$ denotes the price of input and $w(a)$ denotes the wage when the present state is a, which are measured by the consumption good in the given period.

Example 6. Let us consider the public capital accumulation problem described in Example 3. Given $f = (z, \alpha)$, one can define the value function by

$$W_i(f, \theta, a) = u(z(a)) + \beta_i(\theta) W_i(f, \theta, \alpha(a)).$$

We can also define the preference relation $R_i^f[\theta]$ by

$$(x,b)R_i^f[\theta](x',b') \quad \Leftrightarrow \quad u(x)+\beta_i(\theta)W_i(f,\theta,b) \geq u(x')+\beta_i(\theta)W_i(f,\theta,b')$$

In each period, given a society state $a \in A$, we run the majority rule over how much to save between 0 and $Q(a)$. f is a recursive Condorcet winner if $f(a)$ is the Condorcet winner for the preference profile $(R_i^f[\theta])_{i\in I}$ in such voting for each $a \in A$.

One can show that the recursive Condorcet winner must be equal to the optimal policy function for the individual with the median discount factor; hence, it is unique. It has been shown by Boylan and McKelvey (1995) and Boylan et al. (1996) in the finite-period setting, and an analogous result holds in the stationary infinite-horizon case.

The recursive Condorcet winner solution is "dictatorial" in the sense that it is always optimal for the individual with median discount factor (Zuber (2011), Jackson and Yariv (2014, 2015)), but it is a fair rule in the sense that it satisfies anonymity (Hayashi and Lombardi (2021)).

Social Choice Processes: A *social choice process* is a function $\phi : \Theta \to F(Y, A)$ specifying a stationary policy function for each $\theta \in \Theta$. That is, for each $\theta \in \Theta$, ϕ specifies a stationary policy function $\phi[\theta] = (z[\theta], \alpha[\theta]) \in F(Y, A)$ such that $\phi[\theta](a) = (z[\theta](a), \alpha[\theta](a)) \in Y(a)$ for all $a \in A$.

Let $\phi[\Theta] = \{\phi[\theta] : \theta \in \Theta\}$ be the set of stationary policy functions that are consistent with some type under ϕ.

Mechanisms and Stationary Markov Strategies: Let B be any set. A mechanism Γ is a tuple $\Gamma = \left((M_i)_{i\in I}, (g,h)\right)$, where for each $i \in I$, $M_i : A \to 2^B$ is a nonempty function such that $M_i(a) \subseteq B$ is the set of messages available to agent i at a, and $(g,h) : D \to Y$ is an outcome function, where $D = \{(a,m) \in A \times M : m \in M(a)\}$, which assigns to each society state a and a feasible profile of messages at a, $m \in M(a)$, a current outcome and a society state for the next period; that is, $(g(a,m), h(a,m)) \in Y(a)$ for all $(a,m) \in D$.

Fix any mechanism Γ. A (stationary Markov) strategy for agent i is a function $\sigma_i : A \to B$ such that $\sigma_i(a) \in M_i(a) \subseteq B$ for all $a \in A$. A typical profile of strategies is denoted by $\sigma = (\sigma_i)_{i \in I}$. A strategy profile σ is often written as (σ_i, σ_{-i}).

Stationary Markow-Perfect Equilibrium (SMPE): Fix any mechanism Γ. Fix any strategy profile σ, any $\theta \in \Theta$ and any $a \in A$. Let $W_i^{\Gamma}(\sigma, \theta, a)$ be the continuation lifetime utility for agent i at θ when society follows σ after the society state a, which is defined by the following recursive formula

$$W_i^{\Gamma}(\sigma, \theta, a) = u_i(g(a, \sigma(a)), \theta) + \beta_i(\theta) W_i^{\Gamma}(\sigma, \theta, h(a, \sigma(a))),$$

which is uniquely determined by its recursive formulation.

A strategy profile σ is a Stationary Markov-Perfect Equilibrium (SMPE) of (Γ, θ) if for all $i \in I$,

$$\begin{aligned} & u_i(g(a, \sigma(a)), \theta) + \beta_i(\theta) W_i^{\Gamma}(\sigma, \theta, h(a, \sigma(a))) \\ \geq \ & u_i(g(a, (m_i, \sigma_{-i}(a))), \theta) + \beta_i(\theta) W_i^{\Gamma}(\sigma, \theta, h(a, (m_i, \sigma_{-i}(a)))) \end{aligned}$$

for all $a \in A$ and all $m_i \in M_i(a)$. The set of stationary Markov perfect equilibrium of (Γ, θ) is denoted by SMPE(Γ, θ)

Implementation in SMPEs: A mechanism Γ implements $\phi : \Theta \to F(A, X)$ in SMPEs if for all $\theta \in \Theta$, SMPE$(\Gamma, \theta) \neq \emptyset$ and all $\sigma \in$SMPE(Γ, θ),

$$\phi[\theta](a) = (g(a, \sigma(a)), h(a, \sigma(a)))$$

for all $a \in A$. When such a mechanism exists, ϕ is implementable in SMPEs.

3. A Necessary Condition: Stationary Recursive Monotonicity

We show that only stationary recursive monotonic social choice processes are implementable in SMPEs. To introduce this condition, we need additional notation.

Fix any $f = (z, \alpha) \in F(A, Y)$ and any $\theta \in \Theta$. Agent i's marginal preferences induced by the stationary policy function f at θ over Y, denoted by $R_i^f[\theta]$, is defined by

$$(x, b) R_i^f[\theta](x', b')$$

if and only if

$$u_i(x, \theta) + \beta_i(\theta) W_i(f, \theta, b) \geq u_i(x', \theta) + \beta_i(\theta) W_i(f, \theta, b')$$

for all $(x, b), (x', b') \in X \times A$. That is, when society follows f at θ, agent i ranks pairs specifying a current outcome and a society state in the next period according to $R_i^f[\theta]$. A preference profile of preference orderings induced by f at θ is a list of agents' preference orderings induced by the stationary policy function f at θ over Y and it is denoted by $R^f[\theta]$.

Agent i's lower contour set at (f, θ, x, b), denoted by $\mathcal{L}_i^f((x, b), \theta)$, is defined by

$$\mathcal{L}_i^f((x, b), \theta) = \left\{(x', b') \in Y : (x, b) R_i^f[\theta](x', b')\right\}.$$

To understand our necessary condition, let us call any map $\pi : \Theta \times A \to \Theta$ as a (Markovian) society deception; that is, agents coordinates their reports to $\pi(\theta, a)$, though their preferences are described by θ and the society state is a. In the case that agents' reports are not coordinated, the planner detects a lie and can punish the agents.

Given a social choice process ϕ and a society deception π, they induce a social choice process, denoted by $\phi \circ \pi$, which is defined by

$$\phi \circ \pi (\theta, a') = \phi [\pi (\theta, a')] (a')$$

for all $(\theta, a') \in \Theta \times A$. Since $\phi \circ \pi(\theta, \cdot) = (z \circ \pi(\theta, \cdot), \alpha \circ \pi(\theta, \cdot)) \in F(Y, A)$ for each θ, $\phi \circ \pi$ can be equivalently defined by

$$\begin{aligned} z \circ \pi (\theta, a') &= z [\pi (\theta, a')] (a') \\ \alpha \circ \pi (\theta, a') &= \alpha [\pi (\theta, a')] (a') \end{aligned}$$

for all $(\theta, a') \in \Theta \times A$. When ϕ is the social choice process and the society follows the deception π, the actual social choice process is $\phi \circ \pi$. This should not be confused with social choice process at $\pi (\theta, a)$, that is, $\phi [\pi (\theta, a)]$, which appears below. Indeed, $\phi [\pi (\theta, a)]$ is the stationary policy function followed by the society when the society always pretends that $\pi (\theta, a)$ describes their preferences; that is, $\pi (\theta, a)$ is fixed.

Definition 1. $\phi : \Theta \to F(A, Y)$ satisfies Stationary Recursive Monotonicity (SRM) provided that for all $\theta \in \Theta$ and all $\pi \in \Theta^{\Theta \times A}$, if for all $i \in I$,

$$\mathcal{L}_i^{\phi[\pi(\theta,a)]} (\phi \circ \pi (\theta, a), \pi (\theta, a)) \cap Y(a) \subseteq \mathcal{L}_i^{\phi \circ \pi(\theta,\cdot)} (\phi \circ \pi (\theta, a), \theta)$$

for all $a \in A$, then

$$\phi [\theta] = \phi \circ \pi(\theta, \cdot).$$

Let us introduce SRM by considering the simple case where $A = \{a_1, a_2\}$ and $\Theta = \{\theta, \theta'\}$. Let us suppose that θ is the actual society type and that agents coordinate on a society deception π such that $\pi(\theta, a_1) = \theta$ and $\pi(\theta, a_2) = \theta'$. In other words, agents truthful report the society type in state a_1, whereas they lie in state a_2. Since θ is the society type, the socially optimal stationary policy function is $\phi[\theta]$.

Suppose that the social choice process ϕ assigns in state a_1 the pair $\phi \circ \pi (\theta, a_1)$ when agents report $\pi (\theta, a_1)$ and commit to continuation policy function $\phi [\pi (\theta, a_1)]$

and that it assigns in state a_2 the pair $\phi \circ \pi(\theta, a_2)$ when agents report $\pi(\theta, a_2)$ and commit to $\phi[\pi(\theta, a_2)]$.

Moreover, suppose that:

- the marginal preferences of each agent i change from $R_i^{\phi[\pi(\theta,a_1)]}[\pi(\theta, a_1)]$ to $R_i^{\phi\circ\pi(\theta,\cdot)}[\theta]$ in a monotonic way around $\phi\circ\pi(\theta, a_1)$ (that is, whenever $(\phi \circ \pi(\theta, a_1))$ $R_i^{\phi[\pi(\theta,a_1)]}[\pi(\theta, a_1)](x', a')$, it holds that $(\phi \circ \pi(\theta, a_1)) R_i^{\phi\circ\pi(\theta,\cdot)}[\theta](x', a')$); and

- the marginal preferences of each agent i change from $R_i^{\phi[\pi(\theta,a_2)]}[\pi(\theta, a_2)]$ to $R_i^{\phi\circ\pi(\theta,\cdot)}[\theta]$ in a monotonic way around $\phi \circ \pi(\theta, a_2)$.

Then, SRM demands that the stationary policy function $\phi\circ\pi(\theta, \cdot)$ *without commitment* is the socially optimal stationary policy function at θ (that is, $\phi\circ\pi(\theta, \cdot) = \phi[\theta]$).

Our first main result is that only social choice processes satisfying SRM are implementable in SMPEs.

Theorem 1. If $\phi : \Theta \to F(A, Y)$ is implementable in SMPEs, then it satisfies SRM.

Proof. Suppose that Γ implements $\phi : \Theta \to F(A, Y)$ in SMPEs. Fix any $\theta \in \Theta$ and any $\pi \in \Theta^{\Theta\times A}$. Suppose that for all $i \in I$,

$$\mathcal{L}_i^{\phi[\pi(\theta,a)]}(\phi \circ \pi(\theta, a), \pi(\theta, a)) \cap Y(a) \subseteq \mathcal{L}_i^{\phi\circ\pi(\theta,\cdot)}(\phi \circ \pi(\theta, a), \theta) \tag{1}$$

for all $a \in A$. We show that $\phi[\theta] = \phi \circ \pi(\theta, \cdot)$.

For each $a' \in A$, fix any $\sigma[\pi(\theta, a')] \in$SMPE$(\Gamma, \pi(\theta, a'))$. Let $\sigma \circ \pi(\theta, \cdot)$ be defined by

$$\sigma \circ \pi(\theta, a) \equiv \sigma[\pi(\theta, a)](a)$$

for all $a \in A$. Since Γ implements ϕ in SMPEs, it follows that

$$\phi[\pi(\theta, a)](a) = (g(a, \sigma[\pi(\theta, a)](a)), h(a, \sigma[\pi(\theta, a)](a))) \tag{2}$$

for all $a \in A$, and so

$$\begin{aligned} z\left[\pi\left(\theta,a\right)\right]\left(a\right) &= g\left(a,\sigma\left[\pi\left(\theta,a\right)\right]\left(a\right)\right) \\ \alpha\left[\pi\left(\theta,a\right)\right]\left(a\right) &= h\left(a,\sigma\left[\pi\left(\theta,a\right)\right]\left(a\right)\right) \end{aligned}$$

for all $a \in A$. Since implementability implies that

$$W_i\left(\phi\left[\pi\left(\theta,a\right)\right],\pi\left(\theta,a\right),\alpha\left[\pi\left(\theta,a\right)\right]\left(a\right)\right) = W_i^{\Gamma}\left(\sigma\left[\pi\left(\theta,a\right)\right],\pi\left(\theta,a\right),h\left(a,\sigma\left[\pi\left(\theta,a\right)\right]\left(a\right)\right)\right)$$

for all $i \in I$ and all $a \in A$, it holds that

$$\left(g\left(a,\left(m_i,\sigma_{-i}\left[\pi\left(\theta,a\right)\right]\left(a\right)\right)\right),h\left(a,\left(m_i,\sigma_{-i}\left[\pi\left(\theta,a\right)\right]\left(a\right)\right)\right)\right) \in \mathcal{L}_i^{\phi\left[\pi\left(\theta,a\right)\right]}\left(\phi\circ\pi\left(\theta,a\right),\pi\left(\theta,a\right)\right)\cap Y\left(a\right) \tag{3}$$

for all $i \in I$, all $a \in A$ and all $m_i \in M_i\left(a\right)$. For all $i \in I$ and all $a \in A$, it follows from (7) and (3) that

$$\left(g\left(a,\left(m_i,\sigma_{-i}\left[\pi\left(\theta,a\right)\right]\left(a\right)\right)\right),h\left(a,\left(m_i,\sigma_{-i}\left[\pi\left(\theta,a\right)\right]\left(a\right)\right)\right)\right) \in \mathcal{L}_i^{\phi\circ\pi\left(\theta,\cdot\right)}\left(\phi\circ\pi\left(\theta,a\right),\theta\right) \tag{4}$$

for all $m_i \in M_i\left(a\right)$. Since (8) holds for all $i \in I$, all $a \in A$ and all $m_i \in M_i\left(a\right)$ and since

$$W_i\left(\phi\circ\pi(\theta,\cdot),\theta,\alpha\left[\theta\right]\left(a\right)\right) = W_i^{\Gamma}\left(\sigma\circ\pi(\theta,\cdot),\theta,h\left(a,\sigma\left[\pi\left(\theta,a\right)\right]\left(a\right)\right)\right)$$

for all $i \in I$ and all $a \in A$, it follows that $\sigma\circ\pi(\theta,\cdot) \in SMPE\left(\Gamma,\theta\right)$. Since Γ implements ϕ in SMPE, we have that

$$\phi\left[\theta\right]\left(a\right) = \left(g\left(a,\sigma\left[\pi\left(\theta,a\right)\right]\left(a\right)\right),h\left(a,\sigma\left[\pi\left(\theta,a\right)\right]\left(a\right)\right)\right) \tag{5}$$

for all $a \in A$. (2) and (5) imply that $\phi\left[\theta\right]\left(a\right) = \phi\left[\pi\left(\theta,a\right)\right]\left(a\right)$ for all $a \in A$. Thus, ϕ satisfies SRM. ■

4. A Full Characterization for Economic Environments

To proceed with our construction of the mechanism, let $R^{\phi[\theta]}[\theta]$ denote the profile of agents' preferences induced by the stationary policy function $\phi[\theta]$ at θ over Y, that is,

$$R^{\phi[\theta]}[\theta] = \left(R_i^{\phi[\theta]}[\theta]\right)_{i\in I}$$

for all $\theta \in \Theta$. Moreover, let

$$\mathcal{R}(\Theta) = \left\{R^{\phi[\theta]}[\theta] : \theta \in \Theta\right\}$$

be the domain of preferences induced by the social choice process ϕ over Θ.

Although SRM is a necessary condition for implementation in SMPEs, it is not sufficient. Since our interest in this notion of implementation is driven mainly by its economic applications, we tackle sufficiency in economic environments.

Definition 2 (Economic environment)**.** The environment is economic if there is no state $\theta \in \Theta$, stationary policy function $f \in F(A, Y)$ and society state $a \in A$ such that

$$\left|\left\{i \in I : (x, b) \in \max_{R_i^f[\theta]} Y(a)\right\}\right| \geq n - 1$$

for some $(x, b) \in Y(a)$.

Theorem 2. Assume $n \geq 3$ and an economic environment. If $\phi : \Theta \to F(A, Y)$ satisfies SRM, then it is implementable in SMPEs.

The proof of Theorem 2 is based on the construction of a canonical mechanism that is similar to the existing mechanisms in the literature of Nash implementation, though it is modified appropriately to deal with the problem at hands.

Proof of Theorem 2. Assume $n \geq 3$ and an economic environment. Suppose that $\phi : \Theta \to F(A, Y)$ satisfies SRM.

For all $i \in I$ and all $a \in A$, agent i's message space is defined by

$$M_i(a) = \mathcal{R}(\Theta) \times Y(a) \times \mathbb{Z}_+,$$

where $\mathbb{Z}_+$ is the set of non-negative integers, with $(R^i, (x^i, b^i), k^i)$ as typical agent i's message. Let us define the outcome function (g, h) according the following three rules. For all $a \in A$ and all $m \in M(a)$,

RULE 1: If there exists $\left(R^{\phi[\bar{\theta}]}[\bar{\theta}], (x, b)\right) \in \mathcal{R}(\Theta) \times Y(a)$ such that $m_i = (R^i, (x^i, b^i), k^i) = \left(R^{\phi[\bar{\theta}]}[\bar{\theta}], (x, b), k^i\right)$ for all $i \in I$ and $(x, b) = \phi\left[\bar{\theta}\right](a)$, then $g(a, m) = x$ and $h(a, m) = b$.

RULE 2: For all $i \in I$, if there exists $\left(R^{\phi[\bar{\theta}]}[\bar{\theta}], (x, b)\right) \in \mathcal{R}(\Theta) \times Y(a)$ such that $m_j = \left(R^{\phi[\bar{\theta}]}[\bar{\theta}], (x, b), k^j\right)$ for all $j \in I \backslash \{i\}$ and $(x, b) = \phi\left[\bar{\theta}\right](a)$ and $(R^i, (x^i, b^i)) \neq \left(R^{\phi[\bar{\theta}]}[\bar{\theta}], (x, b)\right)$, then:

1. If $(x, b)\, R_i^{\phi[\bar{\theta}]}[\bar{\theta}]\, (x^i, b^i)$, then $g(a, m) = x^i$ and $h(a, m) = b^i$.

2. Otherwise, $g(a, m) = x$ and $h(a, m) = b$.

RULE 3: Otherwise, an integer game is played: identify the agent who plays the highest integer (if there is a tie at the top, pick the agent with the lowest index among them.) This agent is declared the winner of the game, and the alternative implemented is the one she selects.

Suppose that $\theta \in \Theta$ is the true state. Let us show that $\Gamma = \left((M_i)_{i \in I}, (g, h)\right)$ implements ϕ in SMPEs. To show this, let us define the strategy profile σ by

$$\sigma_i(a) = \left(R^{\phi[\theta]}[\theta], \phi[\theta](a), k^i\right) \in M_i(a)$$

for all $a \in A$ and all $i \in I$. It is clear that the above strategy profile is an SMPE for (Γ, θ); that is, $\sigma \in$SMPE(Γ, θ).

To complete the proof, fix any $\sigma \in$SMPE(Γ, θ). Given that we are in an economic environment, it follows that there is no $a \in A$ such that $\sigma(a)$ falls either in **Rule 2** or **Rule 3**. Thus, for all $a \in A$, $\sigma(a)$ falls into **Rule 1**. Then, there exists a deception $\pi \in \Theta^{\Theta \times A}$ such that for all $a \in A$ and all $i \in I$,

$$\sigma_i(a) = \left(R^{\phi[\pi(\theta,a)]}[\pi(\theta, a)], \phi \circ \pi(\theta, a), k^i \right).$$

Rule 1 implies

$$\phi[\pi(\theta, a)](a) = (g(a, \sigma(a)), h(a, \sigma(a)))$$

for all $a \in A$, and so

$$\begin{aligned} z[\pi(\theta, a)](a) &= g(a, \sigma(a)) \\ \alpha[\pi(\theta, a)](a) &= h(a, \sigma(a)) \end{aligned}$$

for all $a \in A$.

Fix any society state $a \in A$. Moreover, fix any $i \in I$ and any

$$\left(x^i, b^i\right) \in \mathcal{L}_i^{\phi[\pi(\theta,a)]}(\phi \circ \pi(\theta, a), \pi(\theta, a)) \cap Y(a).$$

We show that

$$\left(x^i, b^i\right) \in \mathcal{L}_i^{\phi \circ \pi(\theta, \cdot)}(\phi \circ \pi(\theta, a), \theta). \tag{6}$$

Fix any $m_i = (R^i, (x^i, b^i), k^i) \in M_i(a)$ such that $\left(R^i, (x^i, b^i), \bar{k}^i\right) \neq \left(R^{\phi[\pi(\theta,a)]}, \phi \circ \pi(\theta, a), k^i\right)$. Suppose that $(R^i, (x^i, b^i)) \neq \left(R^{\phi[\pi(\theta,a)]}, \phi \circ \pi(\theta, a)\right)$. If

$$\phi[\pi(\theta, a)](a) \, R_i^{\phi[\pi(\theta,a)]}[\pi(\theta, a)] \left(x^i, b^i\right),$$

then **Rule 2** implies that $g(a, (m_i, \sigma_{-i}(a))) = x^i$ and $h(a, (m_i, \sigma_{-i}(a))) = b^i$. Otherwise, **Rule 2** implies that

$$\begin{aligned} g\left(a,\left(m_i,\sigma_{-i}\left(a\right)\right)\right) &= z\left[\pi\left(\theta,a\right)\right]\left(a\right) \\ h\left(a,\left(m_i,\sigma_{-i}\left(a\right)\right)\right) &= \alpha\left[\pi\left(\theta,a\right)\right]\left(a\right). \end{aligned}$$

Suppose that $(R^i,(x^i,b^i)) = \left(R^{\phi[\pi(\theta,a)]},\phi\circ\pi\left(\theta,a\right)\right)$. Then, **Rule 1** implies that

$$\begin{aligned} g\left(a,\left(m_i,\sigma_{-i}\left(a\right)\right)\right) &= z\left[\pi\left(\theta,a\right)\right]\left(a\right) \\ h\left(a,\left(m_i,\sigma_{-i}\left(a\right)\right)\right) &= \alpha\left[\pi\left(\theta,a\right)\right]\left(a\right). \end{aligned}$$

Since $\sigma \in$SMPE(Γ,θ), it follows that

$$\phi\left[\pi\left(\theta,a\right)\right]\left(a\right) R_i^{\phi\circ\pi(\theta,\cdot)}\left[\theta\right]\left(x^i,b^i\right),$$

and so (6) follows.

Since society state a, agent i, pair (x^i,b^i) and i's message m_i were arbitrarily chosen, it follows that

$$\mathcal{L}_i^{\phi[\pi(\theta,a)]}\left(\phi\circ\pi\left(\theta,a\right),\pi\left(\theta,a\right)\right)\cap Y\left(a\right)\subseteq\mathcal{L}_i^{\phi\circ\pi(\theta,\cdot)}\left(\phi\circ\pi\left(\theta,a\right),\theta\right)$$

for all $i\in I$ and all $a\in A$. SRM implies that $\phi\left[\theta\right]=\phi\circ\pi(\theta,\cdot)$. Thus, Γ implements ϕ in SMPEs. ■

5. Applications

In what follows, we provide two relevant economic applications.

5.1. Internalizing pecuniary externality of saving

This subsection shows that the recursive Lindahl solution, which allows internalization of pecuniary externalities of saving decisions, is implementable in SMPE. We show this result under the simplifying assumption of no production. To this end, in what follows, let us bear in mind the borrowing/lending economy with no production described in Example 1 and Example 4 and the regularity conditions spelled out in those examples. We have the following positive result.

Proposition 1. Assume $n \geq 3$. The recursive Lindahl solution is implementable in SMPEs provided that it is unique for all $\theta \in \Theta$.

Proof. Assume $n \geq 3$. In light of Theorem 2, we need only to check that the Economic Environment assumption is satisfied and that the recursive Lindahl solution satisfies SRM.

Since consumptions are private, the Economic Environment assumption is automatically satisfied.

Finally, to show that the recursive Lindahl solution satisfies SRM, pick any $a \in A$ and any $\theta \in \Theta$ and suppose that the premises of SRM are satisfied. Since $\phi[\pi(\theta, a)](a)$ is the recursive Lindahl solution under type $\pi(\theta, a)$, it holds

$$B_i(a) \subseteq \mathcal{L}_i^{\phi[\pi(\theta,a)]} \left(\phi \circ \pi\left(\theta, a\right), \pi\left(\theta, a\right)\right) \cap Y\left(a\right)$$

for all $i \in I$, where $B_i(a) = \{(x, b) \in Y(a) : x_i + q_i(a) \cdot b = a_i + e_i\}$ denotes the corresponding budget set that is constrained by the feasible set.

Fix any $i \in I$. By the presumption of SRM, it holds

$$B_i(a) \subseteq \mathcal{L}_i^{\phi \circ \pi(\theta,\cdot)} \left(\phi \circ \pi\left(\theta, a\right), \theta\right).$$

This shows that $\phi \circ \pi(\theta, a)$ is an optimal choice in $B_i(a)$ for i with type θ, provided that $\phi \circ \pi(\theta, \cdot)$ is the continuation policy function. Since i and a were arbitrary, $\phi \circ \pi(\theta, \cdot)$ is a recursive Lindahl solution for θ. Under uniqueness of the Lindahl solution, we obtain $\phi[\theta] = \phi \circ \pi(\theta, \cdot)$. Thus, the recursive Lindahl solution satisfies SRM. ■

How much does the solution internalize pecuniary externality? Here is an efficiency property that is met by the recursive Lindahl solution in a direct logical manner, like in the first welfare theorem.

Definition 3. A policy function f is recursively efficient for type θ if for all $a \in A$ there is no $(x, b) \in Y(a)$ such that

$$u_i(x_i, \theta) + \beta_i(\theta) W_i(f, \theta, b) \geq u_i(z(a), \theta) + \beta_i(\theta) W_i(f, \theta, \alpha(a))$$

for all $i \in I$ and

$$u_i(x_i, \theta) + \beta_i(\theta) W_i(f, \theta, b) > u_i(z(a), \theta) + \beta_i(\theta) W_i(f, \theta, \alpha(a))$$

for some $i \in I$.

A social choice process ϕ is said to be recursively efficient if $\phi[\theta]$ is recursively efficient for every θ.

To see that the recursive Lindahl solution satisfies recursive efficiency, suppose that there is a pair (x', b') that improves over $f(a)$ provided that the society follows f from the next period onwards. Then, it must be that

$$x_i' + q_i(a) \cdot b' \geq a_i + e_i$$

for all $i \in I$ and

$$x_i' + q_i(a) \cdot b' > a_i + e_i$$

for some $i \in I$. By summing up the above inequalities, we obtain that

$$\left(\sum_{i \in I} q_i(a)\right) \cdot b' = \overline{q}(a) \sum_{i \in I} b_i' > 0,$$

which is a contradiction.

Recursive efficiency is a weaker requirement than full efficiency of intertemporal consumption allocations, however.

Definition 4. A policy function f is Pareto-efficient for type θ if for all $a \in A$ there is no $\mathbf{x} \in X^\infty(a)$ such that

$$U_i(\mathbf{x}_i, \theta) \geq W_i(f, \theta, a)$$

for all $i \in I$ and

$$U_i(\mathbf{x}_i, \theta) > W_i(f, \theta, a)$$

for some $i \in I$, where $X^\infty(a)$ denotes the set of sequences of consumption allocations which are feasible given a.

A social choice process ϕ is said to be Pareto-efficient if $\phi[\theta]$ is Pareto-efficient for every θ.

In the next example, we show that recursive efficiency does not imply Pareto efficiency. Such efficiency loss can arise because of a lack of ability to commit to future allocations, but the below is the most extreme one.

Example 7. Fix any per-period consumption allocation x^*. Let $f = (z, \alpha)$ be a policy function such that $z(a) = x^*$ for all $a \in A$ and α is arbitrary. The continuation lifetime utility of each i, that is, $W_i(f, \theta, b)$, is constant in b. Therefore, everybody is indifferent in the allocation of assets for the next period and cares only about current consumption. Under the assumption that there is just one physical good at each period, $(x^*, \alpha(a))$ is recursively efficient, provided that f is the policy function used from the next period onwards. Thus, f is recursively efficient but it is obviously Pareto-inefficient.

We argue, however, that Pareto-efficient allocation *can be* obtained as a recursive Lindahl equilibrium. Hence, if recursive Lindahl equilibrium is unique it fully internalizes pecuniary externality of saving decisions.

To see this, assume the standard regularity conditions on preferences and earnings, and let $\mathbf{x}(b)$ denote the interior Pareto-efficient allocation such that

$$b_i = \sum_{t=1}^{\infty} p_t(b)(x_{it}(b) - e_i)$$

holds for each i, where $p(b) = (p_1(b), p_2(b), p_3(b), \cdots)$ denotes the sequence of supporting prices with $p_1(b)$ being normalized to 1.

Now consider the policy function $f = (z, \alpha)$ given by

$$\begin{aligned} z_i(a) &= x_{i1}(a) \\ \alpha_i(b) &= \frac{b_i - x_{i1}(b) + e_i}{p_2(b)} \end{aligned}$$

for each i. Then, the continuation lifetime utility $W_i(f, \theta, b)$ is concave in b, increasing in b_i, and decreasing in b_j for all $j \neq i$, following the argument analogous to Lucas Jr and Stokey (1984).

Consider the preference induced over pairs of current consumption and asset allocation for the next period, which is represented in the form

$$u_i(x_i, \theta) + \beta_i(\theta) W_i(f, \theta, b).$$

Now, for each i and j, set $q_{ij}(a)$ be equal to the marginal rate of substitution of quantity b_j for quantity x_i at $(z_i(a), \alpha(a))$. Then $(z_i(a), \alpha(a))$ is individual i's optimal choice under the budget constraint

$$x_i + q_i(a) \cdot b \leq a_i + e_i.$$

5.2. Heterogeneous discounting and recursive median voter equilibrium in public capital accumulation

Let us consider the public capital accumulation problem described in Example 3 and in Example 6 where the regularity conditions spelled out in those examples hold. We show below that the recursive Condorcet winner is implementable in SMPEs when agents always have distinct discount factors.

Proposition 2. Assume $n \geq 3$. Assume that for each $\theta \in \Theta$, every agent i's discount factor differs from others. Then, the recursive Condorcet winner ϕ is implementable in SMPEs.

Proof. Assume $n \geq 3$. In light of Theorem 2, we need only to check that the Economic Environment assumption is satisfied and that the recursive Condorcet winner satisfies SRM.

The Economic Environment assumption is met when, for each $\theta \in \Theta$, every agent i's discount factor differs from those of others.

Finally, to show that the recursive Condorcet winner satisfies SRM, pick any $a \in A$ and any $\theta \in \Theta$. For each i, her upper contour set above the Condorcet winner $\phi \circ \pi(\theta, a)$ under type $\pi(\theta, a)$ when the continuation policy function is $\phi[\pi(\theta, a)]$ is left/right to $\phi \circ \pi(\theta, a)$ whenever her discount factor is smaller/larger than the median, which follows from Lemmata in the appendix.

Then, by the presumption of SRM, her upper contour set above $\phi \circ \pi(\theta, a)$ under type θ when the continuation policy function is $\phi \circ \pi(\theta, a)$ is left/right to $\phi \circ \pi(\theta, a)$. Hence, $\phi \circ \pi(\theta, a)$ remains the Condorcet winner for type θ when the continuation policy function is $\phi \circ \pi(\theta, \cdot)$. Since the choice of $a \in A$ was arbitrary, $\phi \circ \pi(\theta, \cdot)$ is a recursive Condorcet winner for θ. By the uniqueness of the recursive Condorcet winner, we have $\phi[\theta] = \phi \circ \pi(\theta, \cdot)$.

Since the choice of $\theta \in \Theta$ was arbitrary, we conclude that the recursive Condorcet winner satisfies SRM.

■

6. Conclusions

A standard assumption in dynamic implementation theory is that the agents (and so the planner) can commit to future plans/actions once the social decision is made (see, for instance, Moore and Repullo (1988)). Although this assumption has led to important insights, it is often made for technical convenience and is violated in many situations. Another critical assumption is that agents' preferences uncertainly change over time in a transitory manner, and the planner's objective is to repeatedly implement the same social objective upon realization of such uncertainty each time (see, for instance, Lee and Sabourian (2011) and Mezzetti and Renou (2017)). Although this assumption naturally fits in many applications, it rules out intertemporal trades, which are common in dynamic contexts where agents' preferences are permanent. Moreover, most existing studies on dynamic mechanism design assume that time discount is common and known, while heterogeneity and information about time discounting are critical in intertemporal trades. The classical dynamic general equilibrium model involves intertemporal trades.

In this paper, we drop the above three assumptions and study the implementation of (single-valued) social choice processes in stationary Markov-perfect equilibrium (SMPE). This game theoretical solution concept captures our day-to-day economic life well because we often cannot commit to future actions.

We develop a framework where agents' preferences are permanent but cannot commit to future actions. Consequently, the planner can elicit agents' preferences only piece-by-piece, period-by-period. More precisely, in our framework, the current state of the society is described by a state variable, and the objective of the planner is to collect information about agents' preferences over pairs, each specifying a current social outcome and a state variable for the next period.

In this framework, we show that only stationary Markov monotonic social choice processes are implementable in SMPEs. Moreover, we show that this condition is also sufficient when a regularity condition is satisfied. This characterization result allowed us to implement the recursive Lindhal solution and the recursive median voter solution in the public capital accumulation problem in SMPEs.

Appendices

Let $W(f, a, \beta, u)$ denote the continuation lifetime utility of the agent with per-period utility u, discount factor β, where the current capital amount is a, and the continuation policy function is f. The lemma below shows that the W satisfies the increasing difference property in (a, β) for *arbitrary* increasing function f.

This property implies that the incremental gain for choosing a higher a (i.e., $\tilde{a}$ rather than a) is greater when β is higher. That is, $W(f, \tilde{a}, \beta, u) - W(f, a, \beta, u)$ is increasing in β. Equivalently, if $\tilde{\beta} > \beta$, then $W\left(f, \tilde{a}, \tilde{\beta}, u\right) - W\left(f, a, \tilde{\beta}, u\right)$ is increasing in a. In other words, saving and discount factors are complements of each other. This result can be stated as follows.

Lemma 1. *Assume that $u_i(\cdot, \theta) = u$ for every voter $i = 1, ..., n$ and every $\theta \in \Theta$, where u is a continuous, strictly increasing, and strictly concave utility function. Assume that the policy function f is continuous and strictly increasing and assume that it is such that difference $Q(\cdot) - f(\cdot)$ is strictly increasing in the public good holding. Let $\tilde{\beta} > \beta$ and $\tilde{a} > a$. Then:*

$$W\left(f, \tilde{a}, \tilde{\beta}, u\right) - W\left(f, a, \tilde{\beta}, u\right) > W(f, \tilde{a}, \beta, u) - W(f, a, \beta, u)$$

and

$$\tilde{\beta}\left[W\left(f, \tilde{a}, \tilde{\beta}, u\right) - W\left(f, a, \tilde{\beta}, u\right)\right] > \beta\left[W(f, \tilde{a}, \beta, u) - W(f, a, \beta, u)\right].$$

Proof. Let the premises hold. Then, inductively, $f^t(\cdot)$ is continuous and strictly increasing. Moreover, since $Q(f^{t-1}(\cdot)) - f^t(\cdot) = Q(f^{t-1}(\cdot)) - f(f^{t-1}(\cdot))$ and $f^{t-1}(\cdot)$ is strictly increasing, it follows that $Q(f^{t-1}(\cdot)) - f^t(\cdot)$ is strictly increasing. Thus, by our initial suppositions that $\tilde{\beta} > \beta$ and $\tilde{a} > a$, we have that

$$\begin{aligned}
& W\left(f, \tilde{a}, \tilde{\beta}, u\right) - W\left(f, a, \tilde{\beta}, u\right) \\
= & \sum_{t=1}^{\infty} \tilde{\beta}^{t-1} \left[\ u\left(Q\left(f^{t-1}\left(\tilde{a}\right)\right) - f^{t}\left(\tilde{a}\right)\right) - u\left(Q\left(f^{t-1}(a)\right) - f^{t}(a)\right) \ \right] \\
> & \sum_{t=1}^{\infty} \beta^{t-1} \left[\ u\left(Q\left(f^{t-1}\left(\tilde{a}\right)\right) - f^{t}\left(\tilde{a}\right)\right) - u\left(Q\left(f^{t-1}(a)\right) - f^{t}(a)\right) \ \right] \\
= & W\left(f, \tilde{a}, \beta, u\right) - W\left(f, a, \beta, u\right),
\end{aligned}$$

and that

$$\begin{aligned}
\tilde{\beta}\left[W\left(f, \tilde{a}, \tilde{\beta}, u\right) - W\left(f, a, \tilde{\beta}, u\right)\right] & > \tilde{\beta}\left[W\left(f, \tilde{a}, \beta, u\right) - W\left(f, a, \beta, u\right)\right] \\
& > \beta\left[W\left(f, \tilde{a}, \beta, u\right) - W\left(f, a, \beta, u\right)\right].
\end{aligned}$$

■

The strict increasing differences in (a, β) of voters' continuation values allow us to show the following result. Fix the saving peak and the discount factor of voter i. Every other voter less patient than voter i dislikes savings more than voter i's saving peak. Every other voter more patient than voter i dislikes savings less than voter i's saving peak. This result can be stated as follows.

Lemma 2. *Assume that $u_i(\cdot, \theta) = u$ for every voter $i = 1, ..., n$ and every $\theta \in \Theta$, where u is a continuous, strictly increasing, and strictly concave utility function. Assume that the policy function f is continuous and strictly increasing and assume that it is such that difference $Q(\cdot) - f(\cdot)$ is strictly increasing in the public good holding. Fix any $\beta_\ell \in \{\beta_1, ..., \beta_n\}$ and any*

$$a'_\ell \in \arg \max_{a' \in [0, Q(a)]} \left(\ u(Q(a) - a') + \beta_\ell W\left(f, a', \beta_\ell, u\right) \ \right). \tag{7}$$

Then, for all $a' > a'_\ell$ and all $\beta < \beta_\ell$, it holds

$$u(Q(a) - a'_\ell) + \beta W\left(a'_\ell, \phi, \beta, u\right) > u(Q(a) - a') + \beta W\left(f, a', \beta, u\right), \tag{8}$$

and for all $a' < a'_\ell$ and all $\beta > \beta_\ell$, it holds

$$u(Q(a) - a'_\ell) + \beta W\left(f, a'_\ell, \beta, u\right) > u(Q(a) - a') + \beta W\left(f, a', \beta, u\right). \tag{9}$$

Proof. Let the premises hold. Take any $a' > a'_\ell$ and any $\beta < \beta_\ell$. Thus, by Lemma 1 and by the fact that a'_ℓ is an element of (7), it follows that

$$\begin{aligned}
& \left[\ u(Q(a) - a') + \beta W\left(f, a', \beta, u\right)\ \right] - \left[\ u(Q(a) - a'_\ell) + \beta W\left(f, a'_\ell, \beta, u\right)\ \right] \\
= & \left[\ u(Q(a) - a') - u(Q(a) - a'_\ell)\ \right] + \beta \left[\ W\left(f, a', \beta, u\right) - W\left(f, a'_\ell, \beta, u\right)\ \right] \\
< & \left[\ u(Q(a) - a') - u(Q(a) - a'_\ell)\ \right] + \beta_\ell \left[\ W\left(f, a', \beta_\ell, u\right) - W\left(f, a'_\ell, \beta_\ell, u\right)\ \right] \\
= & \left[\ u(Q(a) - a') + \beta_\ell W\left(f, a', \beta_\ell, u\right)\ \right] - \left[\ u(Q(a) - a_\ell) + \beta_\ell W\left(f, a'_\ell, \beta_\ell, u\right)\ \right] \\
\leqq & \ 0,
\end{aligned}$$

which establishes (8).

The case of $a' < a'_\ell$ and $\beta > \beta_\ell$ can be proved similarly. ■

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