Heterogenous Workers and Optimal Inflation

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Abstract

It is shown in this paper that interactions between the staggered price-setting of firms and skill ladder of workers can lead to the optimality of a positive inflation in the presence of nominal price rigidity. Specifically, a social benefit of inflation arises with the assignment of workers with low skill levels to price-adjusting firms and workers with high-skill levels to non-adjusting firms. It is also shown that the magnitude and sign of the optimal inflation rate depend on the nature of skill - either general or firm-specific or both as well as the pattern of price adjustment. Finally, theoretic models of this paper imply that the long-term U.S. optimal inflation rate is slightly above 2 percent under baseline parameter values, while numerical results can respond sensitively to changes in skill and productivity distributions.

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1 Introduction

A lot of academic and practical research works have been devoted to address the issue of what inflation rate is socially desirable while actual central banks usually set positive inflation targets. A recent trend in macroeconomics can be characterized by the emphasis on the importance of microeconomic heterogeneity for practical and normative macroeconomic policy prescriptions including central bank's inflation target. But little has been discussed on the potential role of interactions between the staggered price-setting of firms and skill ladder of workers in the determination of the optimal inflation rate. To fill such a gap, this paper analyzes the impact of assignment of heterogenous workers to firms on the optimal inflation rate in otherwise standard staggered price-setting models with nominal price rigidity.

It is shown in this paper that a positive inflation rate can be optimal when a social benefit of inflation is created by the assignment of workers with low skill levels to price-adjusting firms and workers with high-skill levels to non-adjusting firms. The mechanism behind this result is two-fold. First, a positive inflation affects relative output levels of individual firms whose prices are set in a non-synchronized fashion. Second, the attainment of efficient resource allocation depends on the assignment of workers to firms in the presence of output dispersion of individual firms. In particular, the aggregate output level is increased for given amounts of production inputs as workers with higher skill levels are assigned to firms with larger output levels. Hence inflation can be socially beneficial if workers are efficiently assigned to firms when a positive inflation rate leads to output dispersion of individual firms. In addition, it should be noted that such a mechanism does not exist without output dispersion where each firm produces the same amount of output. In this sense, a novel point of this paper is that interactions between the staggered price-setting of firms and skill ladder of workers play an important role in the optimality of a positive inflation in equilibrium models of nominal price rigidity.

It might be possible to explain this result heuristically by relying on the well-known optimality condition of welfare economics that the marginal rate of transformation should be equal to the marginal rate of substitution at the optimal allocation as follows. The steady-state ratio of relative prices between price-adjusting and non-price adjusting firms is equal to the aggregate gross inflation in staggered price-setting models of nominal price rigidity. The steady-state ratio of marginal products of labor between price-adjusting and non-price adjusting firms is the ratio of skill levels assigned to these firms. The steady-state optimal allocation thus can be attained when the aggregate gross inflation is equal to the ratio of skill levels between non-price adjusting and price-adjusting firms. It follows from this condition that when workers with low skill levels are assigned to price-adjusting firms and workers with high-skill levels to non-adjusting firms, the optimal gross inflation should be equal to the ratio of skill levels between these workers. To the extent which the ratio of skill levels is greater than one, it implies the optimality of a positive inflation rate in the presence of skill heterogeneity between workers.

The magnitude of the optimal inflation rate depends on the nature of skill - either general or firm-specific or both as well as the pattern of price adjustment. In the case of general skill that can be widely used across different individual firms, one should allow for the possibility of mismatch between supply and demand of workers with a skill level. For example, when the measure of price-adjusting firms is larger than that of workers with lowest skill level, the skill level of price-adjusting firms should be defined as a weighted average of different skill levels. It means that the ratio of skill levels between non-price adjusting and price-adjusting firms is affected by both skill distribution of workers and price distribution of firms. But this result does hold in the case of firm-specific skill where there is one-to-one correspondence between price and skill distributions. It is shown in this paper that the optimal inflation rate in the case of general skill tends to be higher than that in the case of firm-specific skill.

The equality between gross inflation and ratio of skill levels alone does not exclude the possibility that a negative inflation rate is optimal at the steady state. The reason for this argument is that when workers with high skill levels are assigned to price-adjusting firms and workers with low-skill levels to non-adjusting firms, the optimal gross inflation still should be equal to the ratio of skill levels between these workers. In this case, the equality condition for the optimal steady-state gross inflation implies the optimality of a negative steady-state inflation rate. For this reason, one might wonder if a negative inflation rate is optimal when the social planner is supposed to choose both the aggregate inflation and the assignment of workers to firms at the same time. It is shown in this paper that there are both a negative inflation rate and a positive inflation rate that attains the minimum relative price distortion in a symmetric two-period Taylor-type pricing model. But this result does not hold in a non-symmetric staggered price-setting model such as Calvo-type pricing model.

Finally, theoretic models of this paper imply the long-term U.S. optimal inflation rate is slightly above 2 percent under baseline parameter values. In this sense, the present paper would present a theoretical framework to help rationalize actual inflation targets on the basis of a standard staggered price-setting model in which the theoretical rationalization of a substantially positive optimal inflation rate has been deemed to be a challenging task. But a caveat is that the magnitude of the optimal inflation rate varies within a relatively wide range starting from less than 2 percent through about 8 percent depending on distributional assumptions of skill and price distributions. By and large, this result can be regarded as a product of two independent factors. The first factor is the fact that the magnitude of productivity dispersion statistics varies substantially across different related studies. Specifically, the interquartile range of firm-level productivity (defined as the log difference between measured values of productivity at three quarter and one quarter) ranges from 0.3 through 0.9. In particular, this feature results into a relatively wide range of possible values of each skill ladder. The second one is that the survival probability of prices is much smaller than the probability of skill expansion because the average life-span of individual prices is much shorter than the average employment tenure of individual workers.

The rest of this paper is organized as follows. The following section presents a simple model with non-degenerate price and skill distributions to discuss how workers with different skill levels are assigned to firms with different output levels and its consequences on aggregate production. The relative price distortion is also defined as the part of the aggregate output that is foregone because of price dispersion. Section 3 analyzes the impact of assignment of heterogenous workers to firms on the optimal inflation on the basis of the social planner's optimization to minimize relative price distortion given a pair of price and skill distributions. In section 4, theoretic results of previous sections are used to compute the optimal U.S. steady-state inflation rate under different sets of plausible parameter values. Section 5 shows how the optimal inflation rate can be implemented in a 'perpetual youth' overlapping-generations model of Blanchard (1985) and Yarri (1965) (hereafter Blanchard-Yarri model). Section 6 concludes.

2 Model

In the economy, there are heterogeneous workers with different levels of production skill and firms with different nominal prices of goods. The whole population of workers is partitioned into $(\bar{m}+1)$ different skill cohorts, $m = 0, \cdots, \bar{m}$. The whole set of firms is partitioned into $(\bar{k}+1)$ different price cohorts, $k = 0, \cdots, \bar{k}$. In addition, values of $\bar{m}$ and $\bar{k}$ are either finite or infinite. The skill distribution at period t of workers is represented by a list of skill levels and corresponding measures, $\{Z_{m,t}, \Gamma_m\}_{m=0}^{\bar{m}}$ with restrictions of $\sum_{m=0}^{\bar{m}} \Gamma_m = 1$ and $Z_{m,t} < Z_{m+1,t}$ for $m = 0, \cdots, \bar{m}-1$. The price distribution at period t of firms is represented by a list of nominal prices and corresponding measures, $\{P_{t-k}^{*}, \Phi_k\}_{k=0}^{\bar{k}}$ with a restriction of $\sum_{k=0}^{\bar{k}} \Phi_k = 1$, where P_{t-k}^{*} is the nominal price reset optimally at period $t-k$ reflecting the staggered price-setting of firms.

2.1 Goods Market

The distribution of individual demands is straightforwardly inherited from that of individual nominal prices when each firm's demand is determined by only two variables such as its own relative price and aggregate demand, following the model of Dixit and Stiglitz (1977). Let $D_{k,t}$ denote the demand at period t of a type k firm and D_t the aggregate demand at period t. The model of

Dixit and Stiglitz then implies that the demand curve facing a type k firm is given by the following equation:

$$D_{k,t} = (\frac{P^*_{t-k}}{P_t})^{-\epsilon} D_t \tag{2.1}$$

where ϵ is a positive constant greater than one and P_t is the aggregate price index at period t. The aggregate demand at period t is defined as

$$D_t = \sum_{k=0}^{\bar{k}} \Phi_k \frac{P^*_{t-k} D_{k,t}}{P_t} \tag{2.2}$$

The aggregate price index at period t is defined as

$$P_t = (\sum_{k=0}^{\bar{k}} \Phi_k (P^*_{t-k})^{1-\epsilon})^{1/(1-\epsilon)}. \tag{2.3}$$

In addition, it follows from equation (2.1) that the probability mass function of demands should be identical to that of nominal prices for $k = 0, \cdots, \bar{k}$.

Each period, production of consumption goods requires the assignment of heterogenous workers to firms reflecting the presence of skill cohorts of workers and price cohorts of firms. A skill cohort m consists of type m workers with their marginal product of labor $Z_{m,t}$ (> 0) at period t. A price cohort k consists of type k firms whose nominal price at period t is P^*_{t-k} optimally determined at period $t-k$. The output of consumption goods produced by a pair of a type m worker and a type k firm, $Y_t(m,k)$, is determined by the following linear function:

$$Y_t(m,k) = Z_{m,t} H_t(m,k) \tag{2.4}$$

where $H_t(m,k)$ is the amount of hours a type m worker works for a type k firm at period t.

2.2 Assignment of Workers to Firms

Let us begin with the definition of assignment function that records the measure of type m workers who are assigned to type k firms as follows.

Definition 2.1 (Assignment Function) An assignment function is a function $V_t(m,k) : \mathbb{N}_0 \times \mathbb{N}_0 \to [0,\ 1]$ satisfying the following conditions

$$\begin{aligned} \Phi_k &= \textstyle\sum_{m=0}^{\bar{m}} V_t(m,k) \\ \Gamma_m &= \textstyle\sum_{k=0}^{\bar{k}} V_t(m,k) \\ 1 &= \textstyle\sum_{m=0}^{\bar{m}} \sum_{k=0}^{\bar{k}} V_t(m,k) \end{aligned} \tag{2.5}$$

where $\mathbb{N}_0$ is the set of nonnegative integers given sets of $\{\Phi_k\}_{k=0}^{\bar{k}}$ and $\{\Gamma_m\}_{m=0}^{\bar{m}}$.

The first condition of equation (2.5) requires that the measure of type k firms should equal the sum of all types of workers who work for type k firms in the absence of inactive firms. The second condition requires that the measure of type m workers should equal the sum of all types of firms that hire type m workers in the absence of unemployment. The third condition is that the double summation of assignment function across all types of workers and firms is equal to one. The assignment function $V_t(m,k)$ thus can be interpreted as a joint probability mass function of individual households in terms of skill and workplace.[1]

It should be noted that if the magnitude of $D_{k,t}$ is regarded as the difficulty level of a type k firm's production task, the ratio of firm's demand level to worker's skill level, $F_t(m,k) = (D_{k,t}/Z_{m,t})$, can be interpreted as a measure of assignment function's performance because it records the length of time taken by a type m worker in the completion of a type k firm's production task. Given this interpretation, the assignment model of this section exhibits the absence of comparative advantage in light of Sattinger (1975, 1993) when $\log F_t(m,k)$ is additively separable between m and k.

It is emphasized in this paper that assignment function depends on government's inflation policy as well as the nature of skill. In order to show how government's inflation policy affects assignment function, let us assume that government can achieve its inflation target together with the following classification of its inflation policy.

Definition 2.2 (Inflation Policy) A zero inflation policy corresponds to $\bar{\Pi}_{t-k,t} = 1$ for all k in each period $t = 0, \cdots, \infty$. A nonzero inflation policy corresponds to $\bar{\Pi}_{t-k,t} \neq 1$ for all k in each period $t = 0, \cdots, \infty$ where $\bar{\Pi}_{t-k,t}$ represents a geometric mean of gross inflations between periods $(t-k)$ and t: $\bar{\Pi}_{t-k,t} = (\prod_{i=1}^{k} \Pi_{t-k+i})^{1/k}$. A positive inflation policy corresponds to $\bar{\Pi}_{t-k,t} > 1$ and a negative inflation policy corresponds to $\bar{\Pi}_{t-k,t} < 1$ for all k in each period $t = 0, \cdots, \infty$.

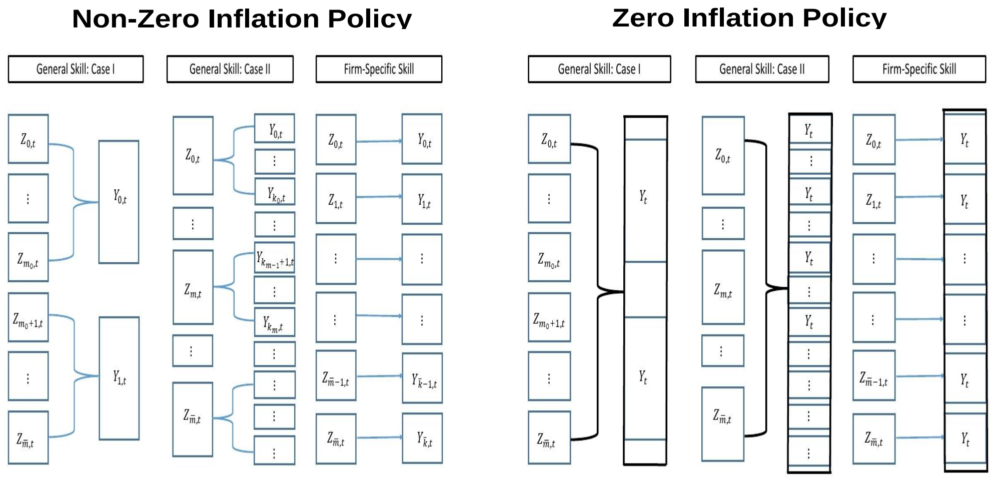
Equation (2.1) is then used to show that relative output levels of individual firms are affected by government's inflation policy as follows.

$$\frac{Y_{k,t}}{Y_t} = \bar{\Pi}^{\epsilon k}_{t-k,t}(p^*_{t-k})^{-\epsilon} \tag{2.6}$$

for $k = 0, \cdots,$ and where $Y_{k,t} = D_{k,t}$ are obtained from market clearing conditions for individual goods and p^*_{t-k} $(= P^*_{t-k}/P_{t-k})$ is the ratio of the nominal price determined at period $t-k$ to the aggregate price index at period $t-k$. It follows from equation (2.6) that the equality of $Y_{k,t} = Y_t$ holds under a zero inflation policy when individual firms set prices $P^*_{t-k} = P_{t-k}$ for all k. It means that output levels of all individual firms are the same even with the staggered price-setting of firms. To the extent which variations of p^*_{t-k} are not large enough, it follows from equation (2.6)

[1] In this case, the first and second conditions require that marginal probability mass functions of skill and workplace implied by a given assignment function correspond to probability mass functions of skill and firm cohorts respectively.

Figure 1: Inflation Policy and Assignment of Workers to Firms



that a positive inflation policy leads each firm's output to increase with the time length of fixing price from the last price change, whereas a negative inflation policy leads each firm's output to decreases with the time length of fixing price from the last price change. Specifically, a positive inflation policy leads to $Y_{k,t} < Y_{i,t}$ if $k < i$ and a negative inflation policy leads to $Y_{k,t} > Y_{i,t}$ if $k < i$ for all non-negative integers i and k not greater than $\bar{k}$.

The reason why assignment function is affected by the nature of skill is that a firm-specific skill is productive for only one firm whereas a general skill is useful for all firms, while such a distinction between firm-specific and general skills can be found in the related literature such as Acemoglu and Pischke (1999) and Lazear (2009). In the case of general skill, there are two distinct possibilities of either a single skill cohort being assigned to multiple skill cohorts or multiple skill cohorts working at a single firm cohort. But a single skill cohort should be assigned to a single firm cohort in the case of firm-specific skill with the assumption that skill levels required for production can be achieved only with continuation of employment.

<Figure 1> illustrates how the assignment of workers to firms is affected by inflation policy and the nature of worker's skill. The right panel corresponds to the case of a zero inflation policy and the left panel corresponds to a non-zero inflation policy. In each column of the right panel, a black box is included to indicate the impact of a zero inflation policy on the assignment of workers to firms where output levels of individual firms are the same across different firms.

In order to distinguish between firm-specific and general skills, each panel contains three different columns. The first two columns correspond to general skill and the third one is firm-specific skill.

In addition, a single type firms can hire different types of workers in the first column while a single type workers can spread across different types of firms in the second column. The third column shows that distributions of skill levels and price levels are the same when skills of households are purely firm-specific.

2.3 Aggregation and Relative Price Distortion

The aggregate production function can be obtained from the linear aggregation of individual firms' outputs given an assignment function. Specifically, the aggregate output is expressed as a function of the aggregate labor input, the aggregate labor productivity, and relative price distortion. In addition, relative price distortion can be regarded as a part of the aggregate output that is foregone because of a non-degenerate distribution of nominal prices set by individual firms. Specifically, let Δ_t represent relative price distortion at period t that is defined as a weighted average of relative outputs of individual firms:

$$\Delta_t = \sum_{k=0}^{\bar{k}} \sum_{m=0}^{\bar{m}} V_t(m,k) \frac{Z_t}{Z_{m,t}} (\frac{P^*_{t-k}}{P_t})^{-\epsilon} \tag{2.7}$$

The aggregate labor input can be derived from the linear aggregation of working hours of all workers:

$$H_t = \sum_{k=0}^{\bar{k}} \sum_{m=0}^{\bar{m}} V_t(m,k) H_t(m,k) \tag{2.8}$$

where H_t is the aggregate hours worked at period t. The aggregate productivity of workers is defined as a weighted harmonic mean of skill levels of individual workers:

$$Z_t = (\sum_{k=0}^{\bar{k}} \sum_{m=0}^{\bar{m}} V_t(m,k) Z^{-1}_{m,t})^{-1} \tag{2.9}$$

where Z_t is the aggregate labor-augmenting productivity at period t. In particular, such a definition of the aggregate labor productivity is derived from the requirement that relative price distortion should a weighted average of relative output levels of individual firms as can be seen in equation (2.7).The substitution of equation (2.4) into equation (2.8) together with the definition of relative price distortion leads to the aggregate production function as follows.

$$Y_t = \Delta_t^{-1} Z_t H_t \tag{2.10}$$

where Y_t is the aggregate real output at period t.

It should be also emphasized that the impact of aggregate inflation on relative price distortion arises through its impact on the distribution of relative output levels as discussed above. In line

Table 3.1: Social Planner's Optimization Problem

Description	Equation
Minimization of Objective Function	$\min_{\{\Pi_t, p_t^*, \{\{V_t(m,k)\}_{m=0}^{\bar{m}}\}_{k=0}^{\bar{k}}\}} \sum_{m=0}^{\bar{m}} \sum_{k=0}^{\bar{k}} V_t(m,k) \frac{Z_t}{Z_{m,t}} (\frac{\bar{\Pi}_{t-k,t}^{k}}{p_{t-k}^*})^{\epsilon}$ subject to
Constraint 1	$p_t^* = (\Phi_0^{-1}(1 - \Pi_t^{\epsilon-1} \sum_{k=1}^{\bar{k}} \Phi_k (\frac{\bar{\Pi}_{t-k,t-1}^{k-1}}{p_{t-k}^*})^{\epsilon-1}))^{-\frac{1}{\epsilon-1}}$
Constraint 2	$\Phi_k = \sum_{m=0}^{\bar{m}} V_t(m,k)$ for $k = 0, 1, \cdots, \bar{k}$
Constraint 3	$\Gamma_m = \sum_{k=0}^{\bar{k}} V_t(m,k)$ for $m = 0, 1, \cdots, \bar{m}$
Constraint 4	$1 = \sum_{m=0}^{\bar{m}} \sum_{k=0}^{\bar{k}} V_t(m,k)$
Constraint 5	$\bar{\Pi}_{t-k,t} = (\prod_{i=1}^{k} \Pi_{t-k+i})^{1/k}$ for $k = 1, \cdots, \bar{k}$

Note: Each period, the social planner takes as given sets of predetermined variables, exogenous variables, and probability mass functions such as $\{p_{t-k}^*\}_{k=1}^{\bar{k}}$, $\{\Pi_{t-\bar{k}+i}\}_{i=1}^{\bar{k}-1}$, $\{Z_{m,t}\}_{m=0}^{\bar{m}}$, $\{\Phi_k\}_{k=0}^{\bar{k}}$, and $\{\Gamma_m\}_{m=0}^{\bar{m}}$. In addition, $\bar{\Pi}_{t,t} = \bar{\Pi}_{t-1,t-1} = 1$.

with this feature, equation (2.7) implies that there is an explicit relationship between aggregate inflation and relative price distortion as can be seen below.

$$\Delta_t = \sum_{k=0}^{\bar{k}} \sum_{m=0}^{\bar{m}} V_t(m,k) \frac{Z_t}{Z_{m,t}} \bar{\Pi}_{t-k,t}^{\epsilon k} (p_{t-k}^*)^{-\epsilon} \tag{2.11}$$

It then follows from equation (2.11) that $\Delta_t = 1$ under a zero inflation policy in light of definition 2.2 when individual firms set prices $P_{t-k}^* = P_{t-k}$ for all k. As a result, relative price distortion disappears under a zero inflation policy in light of definition 2.2, whereas it deviates from one under a nonzero inflation policy.

3 Social Planner's Problem

<Table 3.1> summarizes the social planner's optimization problem formulated on the basis of the model presented in the previous section. The social planner is supposed to choose an assignment function and aggregate gross inflation by minimizing relative price distortion given skill and price distributions of workers and firms. The social planner's objective function in this table is relative price distortion specified in equation (2.11). The first constraint is derived from the definition equation of the aggregate price index (2.3). The second constraint through the fourth constraint reflect three restrictions on assignment function specified in (2.5). The fifth constraint includes the geometric mean of gross inflations starting from the reset time period of type k firms through the current period. The social planner is also supposed to take as given probability mass functions of individual goods prices and predetermined goods prices as well as skill levels and their probability mass functions.

But this result is no longer true when the government deviates from a zero inflation policy. In particular, simple examples of <Figure 1> can be used to show that a positive inflation policy makes it optimal to assign more efficient workers to firms with larger outputs. In order to show that this argument is correct, one needs to show that the assignment of more efficient workers to firms with larger outputs under a positive inflation policy is consistent with the solution of the optimization problem specified in <Table 3.1>.

3.1 Optimal Assignment of Workers to Firms

The aim of this subsection is to address the issue of how workers are efficiently assigned to firms given a set of relative output levels $\{Y_{k,t}/Y_t\}_{k=0}^{\bar{k}}$. In order to address this issue, let us illustrate simple examples with simplifying assumptions specified below. In this light, let us use simple examples of <Figure 1>. In addition, the following two sets of conditions are imposed on the two cases of general skill.

Condition 3.1 A: If $\bar{k} = 1$ and $\bar{m} > 1$, then there exist positive integers m_0 and $\hat{m}_0$ satisfying $\sum_{m=0}^{m_0} \Gamma_m = \Phi_0$ and $\sum_{m=0}^{\hat{m}_0} \Gamma_m = \Phi_1$ respectively (General Skill, Case I).
B: If $\bar{k} > 1$ and $\bar{m} = 1$, then there exist positive integers k_0 and $\hat{k}_0$ satisfying $\sum_{k=0}^{k_0} \Phi_k = \Gamma_0$ and $\sum_{k=0}^{\hat{k}_0} \Phi_k = \Gamma_1$ respectively (General Skill, Case II).

Lemma 3.1 Optimal Assignment Function If condition 3.1.A holds, then the solution to the social planner's optimization problem specified in <Table 3.1> leads to the following assignment function:

$$\begin{array}{llll} & Y_{0,t} < Y_{1,t} & & Y_{0,t} > Y_{1,t} \\ V_t(m,0) & = \Gamma_m \ \text{ if } \ m \in [0, m_0] & V_t(m,0) & = \Gamma_m \ \text{ if } \ m \in [0, \hat{m}_0] \\ V_t(m,1) & = \Gamma_m \ \text{ if } \ m \in [m_0+1, \bar{m}] & V_t(m,1) & = \Gamma_m \ \text{ if } \ m \in [\hat{m}_0+1, \bar{m}] \end{array} \tag{3.1}$$

If condition 3.1.B holds, then the solution to the social planner's optimization problem specified in <Table 3.1> leads to the following assignment function:

$$\begin{array}{llll} & Y_{0,t} < Y_{1,t} \cdots < Y_{\bar{k},t} & & Y_{0,t} > Y_{1,t} \cdots > Y_{\bar{k},t} \\ V_t(0,k) & = \Phi_k \ \text{ if } \ k \in [0, k_0] & V_t(0,k) & = \Phi_k \ \text{ if } \ k \in [0, \hat{k}_0] \\ V_t(1,k) & = \Phi_k \ \text{ if } \ k \in [k_0+1, \bar{k}] & V_t(1,k) & = \Phi_k \ \text{ if } \ k \in [\hat{k}_0+1, \bar{k}] \end{array} \tag{3.2}$$

Lemma 3.1 summarizes how the optimal assignment function is determined in four different cases: two different orderings of output levels for condition 3.1.A and condition 3.1.B respectively. The key point of lemma 3.1 is that workers with higher skill levels should be assigned to firms with larger output levels. For this reason, the ordering of relative output levels of firms plays an important role in the determination of the optimal assignment function.

Table 3.2: Determination of Optimal Assignment Function: Simple Cases

General Skill: Case I	General Skill: Case II
$\Delta_t^\circ = \Phi_0 \frac{Y_{0,t}}{Y_t}(\frac{\hat{Z}_{1,t}}{Z_t})^{-1} + \Phi_1 \frac{Y_{1,t}}{Y_t}(\frac{\hat{Z}_{0,t}}{Z_t})^{-1}$	$\Delta_t^\circ = \Gamma_1 \frac{\hat{Y}_{0,t}}{Y_t}(\frac{Z_{1,t}}{Z_t})^{-1} + \Gamma_0 \frac{\hat{Y}_{1,t}}{Y_t}(\frac{Z_{0,t}}{Z_t})^{-1}$
$\Delta_t^\star = \Phi_0 \frac{Y_{0,t}}{Y_t}(\frac{\bar{Z}_{0,t}}{Z_t})^{-1} + \Phi_1 \frac{Y_{1,t}}{Y_t}(\frac{\bar{Z}_{1,t}}{Z_t})^{-1}$	$\Delta_t^\star = \Gamma_0 \frac{\bar{Y}_{0,t}}{Y_t}(\frac{Z_{0,t}}{Z_t})^{-1} + \Gamma_1 \frac{\bar{Y}_{1,t}}{Y_t}(\frac{Z_{1,t}}{Z_t})^{-1}$
$\hat{Z}_{0,t} = (\sum_{m=0}^{\hat{m}_0} \Phi_1^{-1}\Gamma_m Z_{m,t}^{-1})^{-1}$	$\hat{Y}_{0,t} = \sum_{\underline{k}=0}^{\hat{k}_0} \Gamma_1^{-1}\Phi_k Y_{k,t}$
$\hat{Z}_{1,t} = (\sum_{m=\hat{m}_0+1}^{\bar{m}} \Phi_0^{-1}\Gamma_m Z_{m,t}^{-1})^{-1}$	$\hat{Y}_{1,t} = \sum_{k=\hat{k}_0+1}^{\bar{k}} \Gamma_0^{-1}\Phi_k Y_{k,t}$
$\bar{Z}_{0,t} = (\sum_{m=0}^{m_0} \Phi_0^{-1}\Gamma_m Z_{m,t}^{-1})^{-1}$	$\bar{Y}_{0,t} = \sum_{\underline{k}=0}^{k_0} \Gamma_0^{-1}\Phi_k Y_{k,t}$
$\bar{Z}_{1,t} = (\sum_{m=m_0+1}^{\bar{m}} \Phi_1^{-1}\Gamma_m Z_{m,t}^{-1})^{-1}$	$\bar{Y}_{1,t} = \sum_{k=k_0+1}^{\bar{k}} \Gamma_1^{-1}\Phi_m Y_{m,t}$
$\Phi_1 = \sum_{m=0}^{\hat{m}_0} \Gamma_k$	$\Gamma_1 = \sum_{k=0}^{\hat{k}_0} \Phi_k$
$\Phi_0 = \sum_{m=0}^{m_0} \Gamma_k$	$\Gamma_0 = \sum_{k=0}^{k_0} \Phi_k$
$1 = \Phi_0(\frac{\hat{Z}_{0,t}}{Z_t})^{-1} + \Phi_1(\frac{\hat{Z}_{1,t}}{Z_t})^{-1}$	$1 = \Gamma_1 \frac{\hat{Y}_{0,t}}{Y_t} + \Gamma_0 \frac{\hat{Y}_{1,t}}{Y_t}$
$1 = \Phi_0(\frac{\bar{Z}_{0,t}}{Z_t})^{-1} + \Phi_1(\frac{\bar{Z}_{1,t}}{Z_t})^{-1}$	$1 = \Gamma_0 \frac{\bar{Y}_{0,t}}{Y_t} + \Gamma_1 \frac{\bar{Y}_{1,t}}{Y_t}$

Note: $\hat{Z}_{0,t}$ and $\hat{Z}_{1,t}$ corresponds to averages skill levels of workers who work for type 1 and type 0 firms respectively, while $\bar{Z}_{0,t}$ and $\bar{Z}_{1,t}$ corresponds to averages skill levels of workers who work for type 0 and type 1 firms respectively. In addition, $\hat{Y}_{0,t}$ and $\hat{Y}_{1,t}$ denote average output levels of firms that hire type 1 and type 0 workers respectively, while $\bar{Y}_{0,t}$ and $\bar{Y}_{1,t}$ denote average output levels of firms that hire type 0 and type 1 workers.

<Table 3.2> summarizes how to compute values of relative price distortion under condition 3.1.A and condition 3.1.B respectively. In two columns of this table, Δ_t° is the value of relative price distortion under sub-optimal assignment and $\Delta_t^\star$ is that of relative price distortion under optimal assignment. In the left column of <Table 3.2>, the difference between two values of relative price distortion is

$$\begin{aligned} \Delta_t^\circ - \Delta_t^\star &= \frac{Y_{0,t}}{Y_t}(\Phi_0(\frac{\hat{Z}_{1,t}}{Z_t})^{-1} - \Phi_0(\frac{\bar{Z}_{0,t}}{Z_t})^{-1}) + \frac{Y_{1,t}}{Y_t}(\Phi_1(\frac{\hat{Z}_{0,t}}{Z_t})^{-1} - \Phi_1(\frac{\bar{Z}_{1,t}}{Z_t})^{-1}) \\ &= \Phi_0(\frac{Y_{0,t}}{Y_t} - \frac{Y_{1,t}}{Y_t})((\frac{\hat{Z}_{1,t}}{Z_t})^{-1} - (\frac{\bar{Z}_{0,t}}{Z_t})^{-1}) \ > \ 0 \end{aligned} \tag{3.3}$$

where the second equality reflects final two lines in the left column of <Table 3.2> and strict inequality holds with the assignment function specified in equation (3.1).

In the right column of <Table 3.2>, the difference between two values of relative price distortion is

$$\begin{aligned} \Delta_t^\circ - \Delta_t^\star &= (\frac{Z_{0,t}}{Z_t})^{-1}(\Gamma_0 \frac{\hat{Y}_{1,t}}{Y_t} - \Gamma_0 \frac{\bar{Y}_{0,t}}{Y_t}) + (\frac{Z_{1,t}}{Z_t})^{-1}(\Gamma_1 \frac{\hat{Y}_{0,t}}{Y_t} - \Gamma_1 \frac{\bar{Y}_{1,t}}{Y_t}) \\ &= \Gamma_0(\frac{\hat{Y}_{0,t}}{Y_t} - \frac{\bar{Y}_{1,t}}{Y_t})((\frac{Z_{1,t}}{Z_t})^{-1} - (\frac{Z_{0,t}}{Z_t})^{-1}) \ > \ 0 \end{aligned} \tag{3.4}$$

where the second equality reflects final two lines in the right column of Table 3.2 and strict inequality holds with the assignment function specified in equation (3.2). In sum, equations (3.3) and (3.4) imply that statements of lemma 3.1 are correct.[2]

[2] The proof of lemma 3.1 relies on the assumption that relative output levels of individual firms are independent of how to assign workers to firms. In relation to this assumption, one can find from equation (2.1) that relative output levels are determined by relative prices.

Table 3.3: Social Planner's Optimization Problem: Simple Cases

General Skill: Case I	General Skill: Case II
Price Distribution $\Phi_0 = \Phi_1 = 1/2$	Price Distribution $\Phi_k = (1-\alpha)\alpha^k, \quad k = 0, \cdots$
Optimization Problem Condition 3.1.A with $Y_0 \leq Y_1$ $\min_{\Pi} (\frac{2}{1+\Pi^{\epsilon-1}})^{\frac{\epsilon}{\epsilon-1}} (\frac{\gamma+\Pi^{\epsilon}}{\gamma+1})$ subject to $\Pi \geq 1$	Optimization Problem Condition 3.1.B with $Y_0 \leq \cdots \leq Y_{\bar{k}}$ $\min_{\Pi} \frac{1-\alpha(1-\gamma)\Pi^{\epsilon}}{1-\alpha(1-\gamma)} \Delta_p$ subject to $\Delta_p = (\frac{1-\alpha}{1-\alpha\Pi^{\epsilon}})(\frac{1-\alpha\Pi^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}}$ $\Pi \geq 1$
Optimization Problem Condition 3.1.A with $Y_0 > Y_1$ $\min_{\Pi} (\frac{2}{1+\Pi^{\epsilon-1}})^{\frac{\epsilon}{\epsilon-1}} (\frac{\gamma^{-1}+\Pi^{\epsilon}}{\gamma^{-1}+1})$ subject to $\Pi < 1$	Optimization Problem Condition 3.1.B with $Y_0 > \cdots > Y_{\bar{k}}$ $\min_{\Pi} \frac{1-\alpha(1-\gamma^{-1})\Pi^{\epsilon}}{1-\alpha(1-\gamma^{-1})} \Delta_p$ subject to $\Delta_p = (\frac{1-\alpha}{1-\alpha\Pi^{\epsilon}})(\frac{1-\alpha\Pi^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}}$ $\Pi < 1$

Note: Δ_p is the steady-state relative price distortion in models of homogenous workers and γ is the skill ratio between skilled and unskilled workers: $\gamma = Z_1/Z_0$.

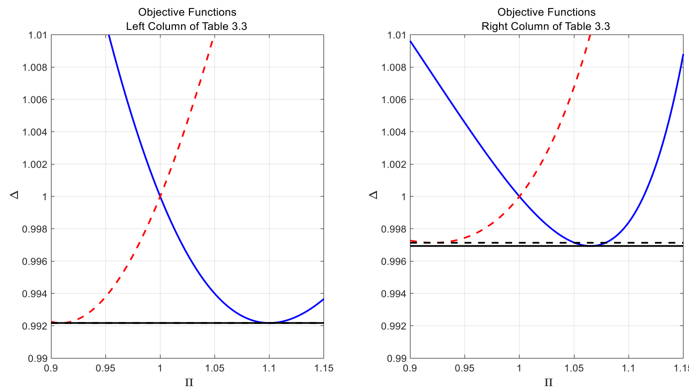
3.2 Optimal Inflation

The analysis of this subsection is focused on the steady-state relation between inflation and assignment function because it permits closed-form solutions of social planner's optimization problems. But a potential caveat is that the resulting solution to the social planner's optimization problem can be different from that of the corresponding dynamic model.

Let us move onto the discussion of the steady-state optimal inflation. The steady-state optimal inflation is defined as the aggregate inflation that minimizes the steady-state value of relative price distortion. The determination of the steady-state optimal inflation rate is summarized in <Table 3.3>. Tow columns of this table include two different optimization problems facing the social planner, each of which corresponds to case I and case II of general skill respectively. In particular, these two optimization problems are obtained by substituting the optimal assignment function presented in lemma 3.1 into the steady-state version of the social planner's optimization problem presented in <Table 3.1>.

It should be noted that cases I and II of general skill in <Table 3.3> correspond to different staggered price-setting models. In the left column of this table, price distribution is set to be $\Phi_0 = \Phi_1 = 1/2$, which means that individual firms reset prices every two periods as is done in a symmetric Taylor-type staggered price-setting model. In the right column of this table, price distribution is set to be $\Phi_k = (1-\alpha)\alpha^k$ (with $0 < \alpha < 1$) for $k = 0, \cdots$ as is done in a Calvo-type staggered

Figure 2: Objective Functions of Optimization Problems in Table 3.3



Note: In each graph, the x-axis is the steady-state aggregate gross inflation and the y-axis is the steady-state relative price distortion.

price-setting model. In this case, only a fraction $(1-\alpha)$ of firms set a new price while the other fraction of firms do not change their prices in each period $t = 0, \cdots$.

Comparing the second line with the third line in each column, one can find two different optimization problems facing the social planner respectively. The second-line optimization problem satisfies condition 3.1.A that leads to an ascending order of individual firms' output levels such as $Y_0 \leq Y_1 \leq \cdots$. The third-line optimization problem satisfies condition 3.1.B that leads to an descending order of individual firms' output levels such as $Y_0 \geq Y_1 \geq \cdots$. It should be noted that this difference is reflected in their constraints. Specifically, the steady-state optimal inflation rate should take a positive value in the second-line optimization problem, whereas it should take a negative value in the third-line optimization problem.

The reason why this distinction should be made can be explained as follows. The first point is that the sign of the aggregate inflation rate determines the ordering of individual firms' output levels. The second one is that the ordering of individual firms' output levels affects the assignment of workers to firms. The third one is that the assignment of workers to firms affects steady-state relative price distortion.

In order to check if the social planner's optimization problems are well-defined optimization problems, one might want to present graphs of objective functions drawn under a set of plausible parameter values rather than going through mathematical equations.[3] <Figure 2> illustrates

[3]The price elasticity of demand for individual firms is set to be $\epsilon = 7$ following a set of papers such as Coibion, Gorodinichenko, and Wieland (2012) and Nakamura, Steinsson, Sun, and Villar (2018). The parameter of nominal price rigidity is chosen to be $\alpha = (0.593)^4$, which is consistent with estimation results of Ascari and Sbordone (2014) but slightly higher than the numerical value set by Coibion, Gorodinichenko, and Wieland (2012) and Adam and

graphs of objective functions that correspond to optimization problems in <Table 3.3>. The two graphs in the left panel are objective functions of the left column in <Table 3.3> and the two graphs in the right panel are objective functions of the right column in <Table 3.3>. The blue horizontal lines in the two panels mark minimum values in a range of $\Pi \geq 1$ that correspond to second-line optimization problems. The red dotted lines in the two panels mark minimum values in a range of $\Pi < 1$ that correspond to third-line optimization problems. As a result, one can conclude that the social planner's optimization problems in <Table 3.3> are all well-defined optimization problems.

The implication of <Figure 2> for the steady-state optimal inflation can be summarized as follows. The black dotted (horizontal) lines are minimum values of red dotted lines and the black straight (horizontal) lines are minimum values of blue straight lines. In this figure, one can find that both black dotted and straight lines coincide in the left panel, whereas straight line stays below dotted line in the right panel. The left panel implies the existence of two different values of the steady-state optimal inflation rate in a symmetric two-period Taylor-type staggered price-setting model: One is positive and the other is negative. The right panel implies the existence of a unique positive steady-state optimal inflation rate in a Calvo-type staggered price-setting model. In sum, the following lemma summarize results of the social planner's optimization problem specified in <Table 3.3>.

Lemma 3.2 Suppose that $\gamma\ (= Z_1/Z_0) \geq 1$. If **condition 3.1.A** holds with $\Phi_0 = \Phi_1 = 1/2$, then two different values of the steady-state optimal inflation $\Pi = \gamma$ and $\Pi = \gamma^{-1}$ attains the minimized value of relative price distortion:

$$\Delta = \frac{2}{\gamma+1}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{1-\epsilon}} \tag{3.5}$$

If **condition 3.1.B** holds with $k_0 = 0$ and $\Phi_k = (1-\alpha)\alpha^k$, then there is a unique steady-state optimal inflation (greater than one) satisfying the following condition

$$(1-\phi(1-\alpha))(\Pi-1) + (1-\phi)(1-\alpha\Pi^{\epsilon})^2 = 0 \tag{3.6}$$

The proof of lemma 3.2 is provided in appendix. The implication of this lemma for the steady-state optimal inflation can be summarized as follows. The first point is that the steady-state optimal gross inflation is equal to one when all workers are homogenous, which is consistent with results of the related literature such as Woodford (2003) and Yun (2005). The second point is that the steady-state optimal gross inflation is indeterminate in a symmetric two-period Taylor-type

Weber (2019). In addition, the skill ratio between skilled and unskilled workers is set to be $\gamma = 1.1$, which means that the amount of hours worked by a skilled worker is less than that of unskilled workers by 10 percent in order to complete the same production task.

staggered price-setting model. In this case, the steady-state optimal inflation rate can be either positive ($\Pi = \gamma - 1$) or negative ($\Pi = \gamma^{-1} - 1$) that gives the same value of relative price distortion. The third point is that the steady-state optimal gross inflation is greater than one in a Calvo-style staggered price-setting model with $k_0 = 0$.

4 The Optimal Inflation Rate in the United States

In this section, a theoretic model of the previous section is used to analyze the optimal inflation rate in the United States. The first part of this section is to show that a sufficient condition for the optimality of a positive steady-state inflation in the case of general production skill (of workers) is the presence of a constraint to ensure that higher types of firms are supposed to employ higher skill levels of workers. The second part is to analyze the implication of this theoretic result for the steady-state U.S. optimal inflation rate on the basis of plausible parameter values.

4.1 Assortative Assignment Constraint and Social Planner's Problem

In this section, assortative assignment constraint acts as a constraint facing the social planner who is supposed to minimize relative price distortion. The reason for the inclusion of assortative assignment constraint is to guarantee the optimality of a positive inflation rate, reflecting results of lemma 3.2. In particular, the introduction of assortative assignment constraint into models with staggered price-setting of firms helps eliminate the possibility of a negative steady-state optimal inflation rate. The analysis of this section is therefore focused on finding a positive optimal inflation rate under the requirement of assortative assignment.

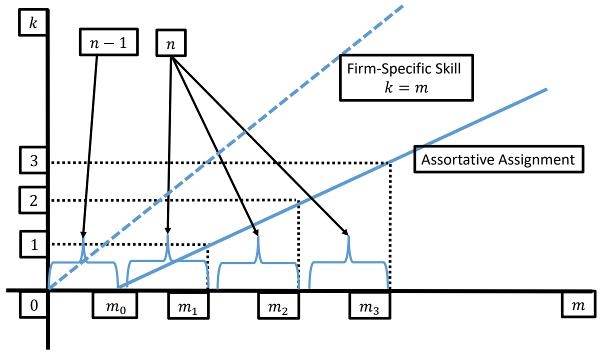
Definition 4.1 (Assortative Assignment Constraint) Let m_k denote an upper bound on skill levels of workers assigned to a type k firm. An assortative assignment of workers to firms is defined as an assignment scheme under which m_k is an increasing function of k. The corresponding assignment function is defined as follows.

$$V_t(m, k) = \Gamma_m \quad \text{if} \quad m \in [m_{k-1} + 1, m_k] \tag{4.1}$$

for $k = 0, 1, \cdots, \infty$ and where $m_{-1} = -1$. In addition, a random selection device is used to implement a random assignment of type $(m_{k-1} + 1)$ workers through type m_k workers to type k firms.

In light of definition 4.1, firm-specific skill can be regarded as a subset of the assortative assignment constraint. The case of firm-specific skill means that a type k firm must hire a type k worker, which in turn implies that $m_k = k$. Hence the case of firm-specific skill satisfies the requirement

Figure 3: Assortative Assignment



that m_k should be an increasing function of k. But it is possible to show that there are cases of $m_k \neq k$ among which m_k is an increasing function of k. The following set of conditions helps ensure the existence of such cases. It also can be interpreted as a subset of sufficient conditions to guarantee the existence of a closed-form solution to the social planner's optimization problem summarized in <Table 3.1> as will be seen.

Condition 4.1 A: An infinite set of integers $\{m_k : \sum_{i=0}^{k} \Phi_i = \sum_{m=0}^{m_k} \Gamma_m, k = 0, \cdots, \infty\}$ exists. **B:** $\Gamma_m = (1-\omega)\omega^m$ for $m = 0, \cdots, \infty$ and $\Phi_k = (1-\alpha)\alpha^k$ for $k = 0, \cdots, \infty$ where $0 < \alpha < \omega < 1$.

Condition 4.1.A can be viewed as an integer constraint for m_k that helps obtain a closed-form characterization of how relative price distortion evolves over time under the assortative assignment constraint in light of definition 4.1. Condition 4.1.B is that both skill and price distributions follow geometric distributions. Specifically, Φ_k can be interpreted as a probability of k failures of having a new price after the last price change and Γ_m as a probability of m updates of skill after being employed for the first time. In addition, a strict inequality of $\alpha < \omega$ in condition 4.1.B means that the survival probability of existing prices is less than the skill update probability of currently employed workers, which will be shown later to be consistent with actual data.

It will be shown that m_k is a linear function of k under conditions 4.1.A and 4.1.B. Specifically, the substitution of condition 4.1.B into condition 4.1.A leads to a linear representation of m_k:

$$m_k = nk + n - 1 \tag{4.2}$$

for $k = 0, 1, \cdots, \infty$ and where n $(= \log\alpha/\log\omega)$ is a positive integer greater than one. It follows from equation (4.2) that $n = 1$ in the case of $\alpha = \omega$, which corresponds to the case of firm-specific skill.

In order to help understand how workers are assigned to firms under the assortative assignment constraint consistent with condition 4.1, <Figure 3> illustrates graphs of the inverse function of m_k. The horizontal axis represents skill levels of workers $m = 0, 1, \cdots$, while the vertical axis records type numbers of firms $k = 0, 1, \cdots$. The dotted line with its slope 45 degree starting from the origin corresponds to the case of firm-specific skill. The blue straight line displays the assortative assignment constraint that is implied by equation (4.2). In this figure, workers with n different skill levels are randomly assigned to each (positive) k type firms for $k = 1, \cdots$, while workers with $n - 1$ different skill levels are assigned to type 0 firms.

It is possible to obtain a closed-form representation of relative price distortion when two conditions 4.1.A and 4.1.B hold. An immediate consequence of this result is a simplified closed-form characterization of the social planner's problem as well as a closed-form solution. It will be then shown that the optimal gross inflation is proportional to the ratio of current relative price distortion to lagged relative price distortion, while its constant of proportionality depends on parameters of both skill and price distributions as summarized in the following proposition.

Proposition 4.1 Suppose that conditions 4.1.A and 4.1.B hold. The assortative assignment constraint then leads to a recursive representation of relative price distortion:

$$\Delta_t = \frac{\alpha}{\bar{\gamma}}\Pi_t^{\epsilon}\Delta_{t-1} + (1 - \frac{\alpha}{\bar{\gamma}})(\frac{1 - \alpha\Pi_t^{\epsilon-1}}{1 - \alpha})^{\frac{\epsilon}{\epsilon-1}} \tag{4.3}$$

where $\bar{\gamma} = \gamma^n$. The gross inflation that minimizes relative price distortion in period t can be written as follows.

$$\Pi_t = \bar{\gamma}\frac{\Delta_t}{\Delta_{t-1}} \tag{4.4}$$

In addition, current-period's relative price distortion is determined by the following equation

$$\Delta_t = \bar{\gamma}^{-1}(\alpha\Delta_{t-1}^{1-\epsilon} + (1 - \alpha)^{\epsilon}(\bar{\gamma} - \alpha)^{1-\epsilon})^{-\frac{1}{\epsilon-1}} \tag{4.5}$$

for a given lagged value of relative price distortion, Δ_{t-1}.

Proposition 4.1 summarizes important results for the optimal inflation rate that can be derived from the theoretic model of this section, while its proof can be found in appendix. It would be now worthwhile to summarize implications of proposition 4.1 for the steady-state optimal inflation rate. It follows from equation (4.4) that the steady-state optimal gross inflation is $\Pi = \bar{\gamma}$. To the extent which $\gamma > 1$ and $n > 0$, it means that the steady-state optimal inflation rate is positive. In addition, the magnitude of the steady-state optimal inflation rate depends on parameters of skill and price distributions.

For the purpose of comparison, the case of $\gamma = 1$ corresponds to the case of technologically homogenous firms and workers, which in turn leads to a zero steady-state optimal inflation rate as shown in Woodford (2003) and Yun (2005). As a result, one can confirm that the introduction of the assignment of heterogenous workers with different skill levels to firms into an otherwise standard New Keynesian model with technologically homogenous firms and workers leads to the optimality of a positive steady-state inflation rate.

4.2 The Optimal Steady-State Inflation Rate in the United States

Having presented a theoretic model of the optimal steady-state inflation rate, the next topic is to discuss how one can use it to compute the optimal steady-state inflation rate in the United States. In particular, such a computation relies on model calibration that requires numerical values for parameters of skill and price distributions, which determines the magnitude of the steady-state optimal inflation rate.

Let us begin with the discussion of how to assign numerical values to parameters of skill distribution such as γ and ω. In doing so, one can exploit the feature of theoretical models that productivity levels of individual firms are solely determined by skill levels of their workers. But this feature does not necessarily require the equality between productivity and skill distributions, while productivity distribution can have useful information about skill distribution. In order to clarify conditions under which productivity distribution can have useful information about skill distribution, the following set of conditions is added to the model.

Condition 4.2 A: Individual firms survive with a constant probability of θ in each period.[4]
B: Conditional on survival, firms whose minimum skill level of workers at period t is $Z_{m,t}$ are subject to a minimum skill level of workers at period $t+1$ of $Z_{m+1,t+1}$.
C: Let $N(x)$ denote the measure of firms whose minimum skill level is x. $N(x)$ is a log-linear function of x as can be seen below:

$$\log N(x) = \kappa - \phi \log x \tag{4.6}$$

where κ and ϕ are positive constants.

Condition 4.2.A implies that the average life-span of individual firms is finite, while θ can be interpreted as a transition probability from current period's survival state to next period's survival

[4]In order to be make firm's optimization consistent with this condition, two mutually independent exogenous signals are supposed to affect the transition of firm's state between two consecutive time periods. Each period, an exit signal with a probability of $(1-\theta)$ and then an adjustment signal conditional upon survivals with a probability of $(1-\tilde{\alpha})$ are sent to individual firms. Hence the probability of each firm's fixing price is $\alpha = \theta\tilde{\alpha}$ as will be seen in figure.

state. The empirical rationale behind this condition is 'Establishment Age and Survival BED (Business Employment Dynamics) Data' published at the home page of Bureau of Labor Statistics. Specifically, the sample average (for the period of 2011-2022) of 'survival rates of previous year's survivors' for firms whose opening years belong to the period of 1994-2003 is 0.963. The average life-span of individual firms is about 27 years when $\theta = 0.963$.

Condition 4.2.B reflects the assumption that individual firms are subject to minimum skill levels for workers in order to attain efficient operations of production facilities, while these minimum skill levels increase over time conditional upon their survivals. Condition 4.2.C is associated with the recent literature on international trade such as Melitz and Redding (2014 and 2015). In these works, productivity levels of individual firms are determined by a Pareto distribution whose cumulative distribution function $F(x)$ with a positive shape parameter ϕ is

$$F(x) = 1 - (\frac{\underline{x}}{x})^{\phi} \tag{4.7}$$

for a positive value $x \in [\underline{x}, \infty]$ and $\underline{x} > 0$.

Condition 4.2.C can be interpreted as a sufficient condition to guarantee that the productivity distribution of individual firms is a Pareto distribution specified in equation (4.7). In order to show how this result works, let $S(x)$ $(= 1 - F(x))$ denote on the survival function of productivity distribution $F(x)$. The survival function can be interpreted as the fraction of firms whose productivity is greater than or equal to x. Looking at definitions of two functions $N(x)$ and $S(x)$, one can find that there is a relation between these two functions such as $S(x) = N(x)/N(\underline{x})$. If this relation holds, then the log-linearity of $N(x)$ as specified in equation (4.6) leads to equation (4.7) because of $S(x) = (\underline{x}/x)^{\phi}$.

Lemma 4.1 Suppose that conditions 4.2.A, 4.2.B, and 4.2.C hold. In addition, suppose that skill levels of workers at the steady state are determined by a constant log-difference model such as $Z_{m+1} = \gamma Z_m$ for $m = 0, \cdots$. If all of these conditions are satisfied, then survival functions of productivity distribution (4.7) imply that the following relation for parameters holds:

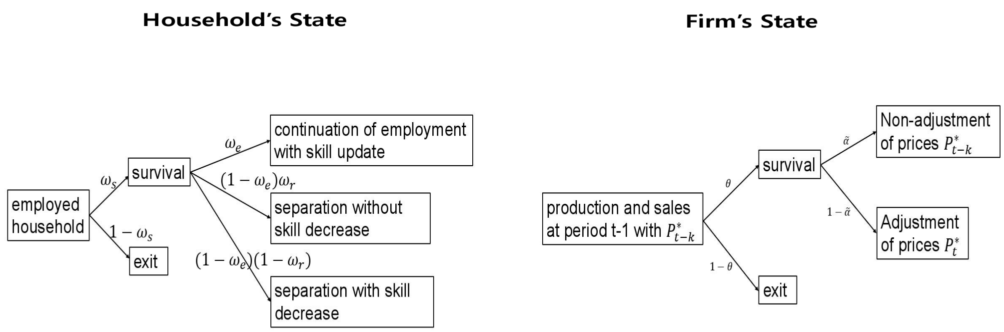
$$\theta = \gamma^{-\phi} \tag{4.8}$$

Proof: Both of condition 4.2.A and condition 4.2.B imply that $S(Z_{m+1,t+1}) = \theta S(Z_{m,t})$ for $m = 0, \cdots$. Condition 4.2.C also leads to $S(Z_{m+1,t+1}) = (Z_{m+1,t+1}/Z_{m,t})^{-\phi} S(Z_{m,t})$ for $m = 0, \cdots$. Hence steady-state versions of these two equations can be summarized as follows.

$$\begin{array}{rcl} S(Z_{m+1}) & = & \theta S(Z_m) \\ S(Z_{m+1}) & = & \gamma^{-\phi} S(Z_m) \end{array} \tag{4.9}$$

Dividing both sides of the first-line equation by those of the second-line equation leads to $\theta = \gamma^{-\phi}$, which is identical to equation (4.8). **QED**

Figure 4: Evolution of Household and Firms's States



Lemma 4.1 implies that it is possible to obtain a numerical value of γ once values of θ and ϕ are known. The value of ϕ is set to be $\phi = 4.25$ in Melitz and Redding (2014 and 2015). Recall that $\theta = 0.963$ as discussed above. Hence the value of γ is set to be $\gamma = 1.009$.

Let us turn to the discussion of how to choose a value of n. Recall that $n = \log\alpha/\log\omega$. Hence setting a value of n amounts to setting values of two parameters such as α and ω. For this reason, let us begin with the value of α that determines the degree of nominal price rigidity. The value of α is set to be $\alpha = (0.593)^4$, which means that firms tend to fix prices on average for 7.4 months. This value is consistent with estimation results of Ascari and Sbordone (2014) but slightly higher than the value used in Coibion, Gorodinichenko, and Wieland (2012) and Adam and Weber (2019).

The determination of ω depends on the nature of worker's skill, either firm-specific or general or both. When skill is equally useful to all firms, a worker's skill level does not need to depend on where he or she works. In this case, an experienced worker remains at the same skill level no matter whether he and she moves into a new workplace. But workers have both firm-specific and general skills on average in actual real world. So one can argue that a realistic skill distribution should reflect this feature. The left panel of <Figure 4> shows how a household's individual state evolves over time. In this figure, three independent exogenous random events are supposed to affect the transition of a household's state between time periods t and $t+1$. The first exogenous random event determines survivals of individual households by sending a death signal to each household with a constant (exogenously determined) probability of $(1-\omega_s)$. The second one determines a set of workers who separate from current workplaces by sending a separation signal to each worker with a constant (exogenously determined) probability of $(1-\omega_e)$. The third exogenous random event determines skill losses among separated workers by sending a skill-loss signal to each separated worker with a constant (exogenously determined) probability of $(1 - \omega_r)$. Specifically, a fraction of workers ω_r switch jobs whose production technologies are not fundamentally different without having any change in skill level, whereas the other fraction of workers $(1 - \omega_r)$ relocate to jobs at which previous job experiences are no longer useful. In the latter, worker's skill level returns to that of the youngest workers in the labor force. The measure of type m workers is then determined as a solution of a linear difference equation as can be seen below.

$$\Gamma_m = \omega_s \omega_e \Gamma_{m-1} + \omega_s (1 - \omega_e) \omega_n \Gamma_m \tag{4.10}$$

In this case, equation (4.10) implies that $\omega = \omega_s \omega_e / (1 - \omega_s (1 - \omega_e) \omega_r)$ and $\Gamma_0 = (1 - \omega_s (1 - (1 - \omega_e)(1 - \omega_r))) / (1 - \omega_s (1 - \omega_e) \omega_r)$.[5]

Let us move onto the discussion of how to choose parameter values. It should be noted that a typical worker is supposed to spend $1/(1 - \omega_s)$ years in the labor force with $1/(1 - \omega_e)$ years of average employee tenure at a workplace. Hence, if a typical US worker tends to spend 45 years in the labor market starting from 20 years old through 65 years old, then it means that $\omega_s = 0.978$, which is consistent with the value used in Gertler (1987). In addition, if a typical US worker's employee tenure is 4 years, then one can set $\omega_e = 0.75$, which is consistent with observed values of employee tenure published by the Bureau of Labor Statistics.[6] Hence values of ω_s and ω_e are set to be $\omega_s = 0.978$ and $\omega_e = 0.75$ respectively.

In order to choose a numerical value of ω_r, it might be possible to exploit the idea that a longer-spell of non-employment experience can reduce a worker's production efficiency, leading to a larger-decrease of wage. On the basis of this idea, ω_r can be interpreted as the fraction of separated workers with both zero and a short duration of non-employment. In this context, one can rely on recent empirical works on job-to-job flows including Hahn, Hyatt, Janicki, and Tibbets (2017), Hyatt and McEntarfer (2012), and Fallick, Haltiwanger and McEntarfer (2012) that emphasize the importance of non-employment durations among job changers for their earnings outcomes.

In theses studies, job-to-job separations are defined as main job separations where workers find new jobs within the same quarter and in the following quarter after their separations take place. In particular, Fallick, Haltiwanger and McEntarfer (2012) report that the fraction of job-to-job separations among all job separations is slightly higher than 60% for the years of 1995, 1999, and 2001. Reflecting empirical results discussed above, the value of ω_r is set to be $\omega_r = 0.6$ in this paper. The numerical values of parameters discussed above are summarized in <Table 4.1>.

[5] The case of purely firm-specific skill leads to a simpler expression for the measure of type m workers: $\Gamma_m = \omega_s \omega_e \Gamma_{m-1}$. In this case, $\Gamma_0 = 1 - \omega_s \omega_e$ so that $\omega = \omega_s \omega_e$.

[6] Specifically, employee tenure is defined as the median number of years that wage and salary workers had been with their current employer. Refer to 'Employee Tenure in 2020, New Release, Bureau of Labor Statistics' for the detailed discussion of how to estimate employee tenure.

Table 4.1: Benchmark Parameter Values

Parameters	Values	Description and Definition
ϕ	4.250	Shape parameter of productivity distribution
θ	0.963	Survival probability of firms
α	$(0.593)^4$	Fraction of non price-adjusting firms
$(1-\omega_s)^{-1}$	45	Average years of being in the labor force
$(1-\omega_e)^{-1}$	4	Average years of employee tenure
ω_r	0.6	Fraction of job-to-job separations
ϵ	7	Price elasticity of demand
b	1	Coefficient of leisure utility function

The next topic is how to compute values of parameters associated with skill distribution and assignment of workers to firms such as ω and n. First, the value of ω is set by substituting numerical values of ω_s, ω_e, and ω_r into equation (4.10), which in turn leads to $\omega = 0.314$. Second, the value of n is set on the basis of equation (4.2) i.e. $n = \log\alpha/\log\omega$, which also leads to $n = 2.54$.

Having completed the discussion of how to fix parameter values, let us move onto numerical predictions of the theoretic model of this section for the U.S. optimal inflation rate. The key point in this light is that the solution to the social planner's optimization problem can be summarized as a simple one-line equation: ($\Pi = \bar{\gamma}$ with $\bar{\gamma} = \gamma^n$). By substituting two parameter values of γ and n into this equation, one can find that the U.S. steady-state optimal inflation rate is 2.29%, which is slightly higher than the Federal Reserve's 2% inflation target. A caveat of this numerical result is to ignore an integer constraint for values of n associated with equation (4.2). In order to respect such an integer constraint, one can choose a natural number closest to $n = \log\alpha/\log\omega$. In this case, the U.S. steady-state optimal inflation rate turns out to be 2.70%.

4.3 Robustness of Numerical Result

Having computed the U.S optimal steady-state inflation rate under the baseline set of parameter values, the next topic is to check the robustness of the baseline numerical result shown above by deviating from baseline parameter values summarized in <Table 4.1>. The first case is to keep track of how the optimal inflation rate responds to a change in the specification of productivity distribution. In this case, productivity distribution is assumed to be a geometric distribution. Lemma 4.1 is no longer valid when productivity distribution is a geometric distribution. Hence one needs an alternative calibration strategy to fix a value of γ. For this reason, the following lemma replaces lemma 4.1.

Table 4.2: U.S. Optimal Inflation Rate

Specification of Productivity Distribution \ Degree of Nominal Price Rigidity	$\alpha = (0.593)^4$	$\alpha = (0.7292)^4$
Benchmark Specification: Pareto Distribution	2.26%	1.36%
Alternative Specification: Geometric Distribution		
Lower Bound: LIQR_{TFP}	4.06%	2.75%
Upper Bound: LIQR_{APL}	8.08%	4.80%

Note: LIQR_{TFP} means the log interquartile range of total factor productivity and LIQR_{APL} means the log interquartile range of average product of labor.

Lemma 4.2 Suppose that conditions 4.2.A and 4.2.B hold. In addition, the measure of firms whose minimum skill of workers is Z_m is $(1-\theta)\theta^m$ with $Z_{m+1} = \gamma Z_m$ for $m = 0, \cdots$. If all of these conditions are satisfied, then the following relation holds:

$$\gamma = \theta^{-LIQR/\log 3} \tag{4.11}$$

where $LIQR$ denotes the log interquartile range of productivity distribution:

$$LIQR = \log Z_{m_{0.75}} - \log Z_{m_{0.25}}. \tag{4.12}$$

Proof: The measure of firms whose minimum skill level at period t of workers is higher than $Z_{m,t}$ is θ^{m+1}, while the measure of firms whose minimum skill level at period t of workers is lower than or equal to $Z_{m,t}$ is $1-\theta^{m+1}$. Hence the value of m that corresponds the third quartile of firms ($= m_{0.75}$) is $m_{0.75} = -2\log 2/\log\theta - 1$. The value of m that corresponds to the first quartile of firms ($= m_{0.25}$) is $m_{0.25} = m_{0.75} + \log 3/\log\theta$. The substitution of these two equations into the definition of log interquartile range specified in equation (4.12) then leads to equation (4.11). **QED.**

Lemma 4.2 implies that estimates of $LIQR$ can be used to obtain a numerical value of γ given a value of θ. In order to measure firm-level productivity dispersion within and across industries, estimates of $LIQR$ are included in Dispersion Statistics on Productivity (DiSP) jointly published by U.S. Bureau of Labor Statistics and U.S. Census Bureau. For example, Cunningham et al. (2021) report (<Table 3> in their discussion paper) that estimates of $LIQR$ within industries are 0.520 for total factor productivity (hereafter $LIQR_{TFP}$) and 0.828 for average product of labor (hereafter $LIQR_{APL}$) respectively during the sample period of 1997-2016. The substitution of these estimates into equation (4.11) leads to $\gamma = 1.018$ in the former case and $\gamma = 1.031$ in the latter case respectively. Hence the optimal steady-state inflation rate in the United States is 4.60%

in the case of $LIQR_{TFP}$ and 8.08% in the case of $LIQR_{APL}$ respectively.[7]

The second case is to see how the degree of nominal price rigidity affects the optimal inflation rate. The optimal steady-state inflation rate lowers as the degree of nominal price rigidity increases. It is because the value of n decreases as the degree of nominal price rigidity increases. In addition, it is possible to find a wide range of estimates for the degree of nominal price rigidity reflecting the fact that there are a lot of microeconomic empirical studies on frequencies of price changes since seminal works of Bils and Klenow (2004) and Nakamura and Steinsson (2008).

In this literature, estimates of price duration are affected by empirical definitions of price changes as well as price data sets. For example, it is reported in Eichenbaum, Jaimovich, and Rebelo (2011) that reference prices, most quoted prices within a given time period, have an average duration of 3.70 quarters, which in turn leads to $\alpha = 0.7297$. Moreover, if the value of α is set to be $\alpha = 0.7297$, then the U.S. optimal steady-state inflation rate is 1.36% in the case of the Pareto distribution specified in equation (4.7), 2.75% in the case of $LIQR_{TFP}$, and 4.80% in the case of $LIQR_{APL}$ respectively as can be seen in <Table 4.2>.

The third case is to allow for the possibility that the probability of price adjustment for individual firms fluctuates over time. The motivation behind this one is that a model with time-varying probabilities of price adjustment is consistent with empirical studies on price-adjustment frequencies of individual firms including Nakamura, Steisson, Sun, and Villar (2018). In this case, let α_t denote the probability of price adjustment in period t, while the social planner is supposed to take it as given over time.

The result of such a modification of the benchmark model is reflected in the law of motion for relative price distortion as can be seen below.

$$\Delta_t = \frac{\alpha_t}{\bar{\gamma}_t}\Pi_t^{\epsilon}\Delta_{t-1} + (1 - \frac{\alpha_t}{\bar{\gamma}_t})(\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{\epsilon}{\epsilon-1}} \tag{4.13}$$

In this equation, $\bar{\gamma}_t$ is defined as follows.

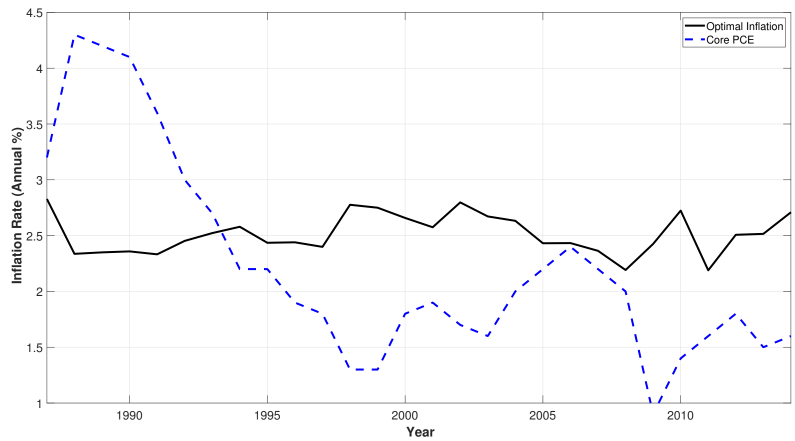
$$\bar{\gamma}_t = \gamma^{n_t}, \quad n_t = \frac{\log \alpha_t}{\log \omega} \tag{4.14}$$

where one can find a detailed discussion of how to obtain equations (4.13) and (4.14) in appendix.

The social planner is now supposed to find the aggregate inflation rate by minimizing relative price distortion in each period $t = 0, \cdots$. The result of the social planner's minimization problem is summarized in the following proposition whose proof is included in appendix.

[7] It follows from lemma 4.1 and lemma 4.2 that values of γ are determined by productivity dispersion and survival probability of individual firms. As a result, determinants of the optimal U.S. inflation in the present analysis differ from those in Adam and Weber (2019) where the optimal steady-state inflation equals the ratio of the average productivity growth of old incumbent firms to that of new entrant firms.

Figure 5: Optimal Inflation Dynamics: Counterfactual Simulation



Proposition 4.2 Given that the social planner takes as given the probability of price adjustment ($= \alpha_t$), the social planner's minimization at period t of relative price distortion specified in equation (4.13) leads to the following determination of inflation and relative price distortion.

$$\Pi_t = \bar{\gamma}_t \frac{\Delta_t}{\Delta_{t-1}} \tag{4.15}$$

$$\Delta_t = \bar{\gamma}_t^{-1}(\alpha_t \Delta_{t-1}^{1-\epsilon} + (1-\alpha_t)^{\epsilon}(\bar{\gamma}_t - \alpha_t)^{1-\epsilon})^{\frac{-1}{\epsilon-1}} \tag{4.16}$$

given a series of $\{\alpha_t\}_{t=0}^{\infty}$ with initial values of Δ_{-1} and s_{-1}.

A system of two equations (4.15) and (4.16) (presented in proposition 4.2) can be interpreted as a model of the optimal inflation rate that allows for the possibility that the probability of price adjustment varies over time. In order to simulate such a model, a time-series of α_t is taken from the empirical work of Nakamura, Steisson, Sun, and Villar (2018) on price-adjustment frequencies of firms for the period of 1987 - 2014, while the other parameter values are the same as those in <Table 4.1>. But it should be admitted that the simulation used in the section does not require numerical values of n_t to be positive integers. For this reason, its numerical values are approximate values obtained on the basis of a log-linear interpolation between two positive integer points.

<Figure 5> summarizes the optimal inflation path generated by simulating the model of two equations presented in proposition 4.2. The black solid and dotted lines correspond to the optimal inflation rate and the blue dotted line is the U.S. core PCE inflation rate. The black solid line fluctuates around 2 percent, while the black dotted line shows a persistent decline over the sample period from slightly above 3 percent to around 2 percent. The core PCE inflation rate also displays a persistent decline over the sample period starting from significantly higher than 3 percent and

then falling down to around 1.5 percent in early 2010s. It should be noted that the key difference between the two black lines depends on the existence of a trend in the skill level of workers who enter the labor market for the first time. The black solid line corresponds to the absence of such a trend, whereas the black dotted line allows for the impact of such a trend on the optimal inflation path.

5 Implementation of the Optimal Inflation Rate

Reflecting that theoretical results of the previous section are obtained without having a full description of how private agents make their decisions, this section presents a small-scale dynamic stochastic general equilibrium model that leads to the same optimal inflation rate presented in the previous section.

5.1 Economic Environment

The model of this section builds on the following three distinct features. The first one is that all workers have finite lives in the similar way as is done in Blanchard (1985) and Yarri (1965). The second one is that the whole set of workers is partitioned into an infinite number of distinct skill cohorts whose marginal products of labor are not the same. Hence there are an infinite number of age cohorts indexed by $a \in \{0, 1, 2, \cdots, \infty\}$ and an infinite number of skill cohorts indexed by $m \in \{0, 1, 2, \cdots, \bar{m}\}$, where the value of $\bar{m}$ is either finite or infinite. The third feature is that the whole set of monopolistically competitive firms can be partitioned into an infinite number of different groups indexed by $k \in \{0, 1, 2, \cdots, \infty\}$ because firms set nominal prices at different times.

5.2 Equilibrium Conditions

In this model, all individual households are identified with their names under the restriction that the same name should not be given to different persons. Specifically all names of those who are alive at period t is stored in $\mathcal{H}_t$ whose elements are not identical over time with its measure fixed to be one.[8] The instantaneous utility function $u(C_{h,t}, H_{h,t})$ of a household $h \in \mathcal{H}_t$ at period t is then assumed to be a twice continuously differentiable function of its arguments where $C_{h,t}$ and $H_{h,t}$ are its consumption and labor supply at period t respectively. The first-order partial derivative of the first variable is positive, and that of the second is negative.

[8] In this model, individual households are not identical even within the same age cohort because they can have different skill level, Hence it is not possible to use a cohort-by-cohort identification of households widely used in the literature such as Nistico (2012) and Del Negro, Giannoi, Patterson (2023). In addition, Gertler (1999) allows for the possibility that an active worker becomes a retiree with a constant probability in each period. In this case, a cohort-by-cohort identification can be used for workers because workers within the same age cohorts have the marginal product of labor as can be seen in Fujiwara and Yuki (2008), Carvalho, Ferrero and Nechio (2016), Fujiwara, Hori, and Wari (2019), and Gali (2021).

The optimization problem at period t of a household h can be written as follows.

$$\max_{C_{h,t},H_{h,t},A_{h,t}} \{u(C_{h,t}, H_{h,t}) + \beta\omega_s E_t[v(A_{h,t})]\} \tag{5.1}$$

subject to the following budget constraint and where $v(A_{h,t})$ is the value function evaluated at the beginning of period $t+1$:

$$C_{h,t} + (1-\theta_A)\mathcal{Q}_{M,t}A_{h,t} = (1+\theta_H)w_{m_{h,t},t}\ H_{h,t} + \omega_s^{-1}(\mathcal{Q}_{M,t} + \mathcal{D}_{M,t})A_{h,t-1} - T_t \tag{5.2}$$

where $A_{h,t}$ is a worker h's mutual fund investment at period t, $\mathcal{Q}_{M,t}$ is the real price at period t of the mutual fund'share, $\mathcal{D}_{M,t}$ is the real dividend at period t of the mutual fund'share, $w_{m_{h,t},t}$ is the real wage at period t for worker h who belongs to a $m_{h,t}$ skill cohort. In this representation of household's budget constraint, there are investment tax credit (proportional to investment cost), employment subsidy (proportional to labor income), and lump-sum tax, where θ_A is the ratio of investment tax credit to household's investment cost, θ_H is the ratio of employment subsidy to labor income, and T_t is the lump-sum tax at period t.[9] In this budget constraint, assets of households are managed by a mutual fund under annuity contracts which are supposed to distribute the total wealth of non-survivals equally to survivors in each period, following Blanchard (1985),

In order to provide a concrete example of household's optimization problem, household h's instantaneous utility function is assumed to be additively separable between consumption and leisure and linear in leisure:

$$u(C_{h,t}, H_{h,t}) = \log C_{h,t} + b(\bar{H} - H_{h,t}) \tag{5.3}$$

where $\bar{H}$ denotes a limit on the amount of hours worked by individual households and b is a positive constant. In this case, the corresponding utility-maximization condition for labor supply can be rewritten as follows.

$$bC_{m,t} = (1+\theta_H)w_{m,t} \tag{5.4}$$

The optimization condition for mutual-fund investment is

$$1 = E_t[\frac{\Lambda_{h,t\to t+1}R_{M,t+1}}{1-\theta_A}] \tag{5.5}$$

where $\Lambda_{h,t\to t+1}$ is the household h's inter-temporal marginal rate of substitution between periods t and $t+1$ and $R_{M,t}$ ($= (\mathcal{Q}_{M,t} + \mathcal{D}_{M,t})/\mathcal{Q}_{M,t-1}$) represent the realized gross rate of return at period t on household's investment on mutual funds

[9]It is shown in Sattinger (1975) that relative abilities of two workers must be equal to their relative wages in the absence of comparative advantage. In line with this result, the ratio of wages paid to workers with different skill levels equals the ratio of their skill levels as can be seen from budget constraint (5.2). But it does not mean that labor incomes of individual workers depends solely on their skill levels as will be seen later.

Reflecting that transitions of three individual states such as survival, employment, and skill affect the household's inter-temporal marginal rate of substitution, it is represented as a weighted average of three ratios of next-period's and current-period's marginal utilities as can be seen below.

$$\Lambda_{h,t\to t+1} = \omega_e \tilde{\Lambda}_{h,t\to t+1} + (1-\omega_e)\omega_r \hat{\Lambda}_{h,t\to t+1} + (1-\omega_e)(1-\omega_r)\bar{\Lambda}_{h,t\to t+1} \tag{5.6}$$

The first term in the right-hand side of this equation corresponds to the ratio of marginal utilities conditional on the continuation of employment between periods t and $t+1$ with an increase in the skill level:

$$\tilde{\Lambda}_{h,t\to t+1} = \omega_s \beta \frac{u_1(c(a_{h,t}+1, m_{h,t}+1, k_{h,t}+1, s_{a,t+1}), h(a_{h,t}+1, m_{h,t}+1, k_{h,t}+1, s_{a,t+1})}{u_1(c(a_{h,t}, m_{h,t}, k_{h,t}, s_{a,t}), c(a_{h,t}, m_{h,t}, k_{h,t}, s_{a,t}))}$$

The second one corresponds to the ratio of marginal utilities conditional on the absence of skill change between periods t and $t+1$:

$$\hat{\Lambda}_{h,t\to t+1} = \omega_s \beta \sum_{\hat{k}=0}^{\bar{k}} V_{t+1}(a_{h,t}+1, m_{h,t}, \hat{k}) \frac{u_1(c(a_{h,t}+1, m_{h,t}, \hat{k}, s_{a,t+1}), h(a_{h,t}+1, m_{h,t}, \hat{k}, s_{a,t+1}))}{u_1(c(a_{h,t}, m_{h,t}, k_{h,t}, s_{a,t}), h(a_{h,t}, m_{h,t}, k_{h,t}, s_{a,t}))}$$

where $V_{t+1}(m_{h,t}, \hat{k})$ is the probability at period $t+1$ that household h with skill level $m_{h,t}$ at period $t+1$ works at a $\hat{k}$ type firm at period $t+1$. The third one corresponds to the ratio of marginal utilities conditional on skill loss between periods t and $t+1$:

$$\bar{\Lambda}_{h,t\to t+1} = \omega_s \beta \sum_{\hat{k}=0}^{\bar{k}} V_{t+1}(0, \hat{k}) \frac{u_1(c(a_{h,t}+1, 0, \hat{k}, s_{a,t+1}), h(a_{h,t}+1, 0, \hat{k}, s_{a,t+1}))}{u_1(c(a_{h,t}, m_{h,t}, k_{h,t}, s_{a,t}), h(a_{h,t}, m_{h,t}, k_{h,t}, s_{a,t}))}$$

where $V_{t+1}(0, \hat{k})$ is the probability at period $t+1$ that household h with skill level 0 at period $t+1$ works at a $\hat{k}$ type firm at period $t+1$.

Let us define the aggregate inter-temporal marginal rate of substitution ($= \Lambda_{t,t+1}$) as a weighted average of individual households' intertemporal marginal rates of substitution divided by $(1-\theta_A)$ as follows:

$$\Lambda_{t,t+1} = (1-\theta_A)^{-1} \sum_{k=0}^{\bar{k}} \sum_{m=0}^{\bar{m}} V_t(m,k)\Lambda_{t,t+1}(m,k) \tag{5.7}$$

In particular, $\Lambda_{t,t+1}$ acts as the stochastic discount factor to measure the value at period t of one unit of consumption good at period $t+1$, which is proportional to the price at period t of a contingent claim to give one unit of consumption good conditional on the occurrence of an aggregate state at period $t+1$. Hence the gross rate of return on household's investment on mutual funds should satisfy the following equation:

$$1 = E_t[\Lambda_{t,t+1} R_{M,t+1}] \tag{5.8}$$

Let us turn to the discussion of mutual funds. First, there are a lot of perfectly competitive mutual funds. Second, the absence of arbitrage profits is also used to determine the market value of mutual funds in the centralized asset market with a complete set of (one-period ahead) state-contingent claims conditional on the whole set of aggregate states in each period $t = 0, \cdots$. The zero-profit condition at period t for perfectly competitive mutual funds requires the equality between cash inflows and outflows:

$$\mathcal{Q}_{M,t}A_t + \mathcal{D}_{F,t} = (\mathcal{Q}_{M,t} + \mathcal{D}_{M,t})A_{t-1} \tag{5.9}$$

where A_t is the total number of shares held at period t by households, and $\mathcal{D}_{F,t}$ is the aggregate dividend income from firms.

Before going further, it would be worthwhile to address the issue of how to determine the magnitude of sales subsidies used for the elimination of steady-state economic distortion caused by monopolistic competition in goods markets following Woodford (2003). For this reason, one should discuss whether the optimality of a positive inflation rate has any impact on the specification of sales subsidies to fix the economic distortion associated with monopolistic competition in goods markets. Specifically, subsidies at period t of individual firms whose prices are set at period $t-k$ are determined as follows.

$$S_{t-k,t} = ((1+\theta_F)\Pi^k - 1)D_{t-k,t} \tag{5.10}$$

where θ_F is a positive constant, Π is steady-state gross inflation, and $S_{t-k,t}$ represents sales subsidies at period t of firms whose prices are set at period $t-k$. The decentralization of the optimal allocation requires (steady-state) inflation-indexed sales subsidies in the case of a positive optimal steady-state inflation rate, whereas it disappears in the case of a zero optimal steady-state inflation rate.

Taking into account the impact of sales subsidies, one can present the following profit maximization problem at period t of firms that rest prices at period t.

$$\max_{P_t^*}\sum_{k=0}^{\infty}\alpha^k E_t[\Lambda_{t,t+k}\{(1+\theta_F)\Pi^k(\frac{P_t^*}{P_{t+k}})^{1-\epsilon} - mc_{t+k}(\frac{P_t^*}{P_{t+k}})^{-\epsilon}\}Y_{t+k}] \tag{5.11}$$

The first-order condition of the profit maximization problem (5.11) with respect to the nominal reset price can be written as follows.

$$\frac{P_t^*}{P_t} = \frac{S_t}{M_t} \tag{5.12}$$

where M_t and S_t are respectively defined as

$$\begin{aligned} M_t &= (1+\theta_F)Y_t + \alpha E_t[\Lambda_{t,t+1}(\Pi_{t+1}^{\epsilon-1}\Pi)M_{t+1}] \\ S_t &= \tfrac{\epsilon}{\epsilon-1}mc_tY_t + \alpha E_t[\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon}S_{t+1}] \end{aligned} \tag{5.13}$$

Having described optimizing behaviors of firms and households, let us turn to fiscal and monetary policies. In this model, the role of fiscal policy is to help implement the optimal allocation in a decentralized equilibrium by using employment subsidies, sales subsidies, and investment credits, while lump sum taxes are imposed on households. In addition, government's budget is supposed to be balanced in each period. Hence the government's budget constraint at period t can be written as

$$T_t = T_{F,t} + T_{H,t} + T_{A,t} \tag{5.14}$$

where aggregate sums of sales subsidies $T_{F,t}$, employment subsidies $T_{H,t}$ and mutual-fund investment credits $T_{A,t}$ are respectively defined as follows.

$$\begin{array}{rcl} T_{F,t} & = & (1-\alpha)\sum_{k=0}^{\infty}\alpha^k S_{t-k,t} \\ T_{H,t} & = & \theta_H w_t H_t \\ T_{A,t} & = & \theta_A Q_{M,t} A_t \end{array} \tag{5.15}$$

It would be worthwhile to emphasize two different roles of fiscal policy measures $\{T_{A,t},\, T_{H,t},\, T_{F,t}\}_{t=0}^{\infty}$ in the implementation process of the optimal allocation. The first one is to deal with the steady-state distortion associated with monopolistic competition at goods markets. The second one is to deal with the impact of heterogenous workers with different skill levels on the aggregate inter-temporal marginal rate of substitution and marginal cost of production. The motivation behind this one is that aggregation bias arises in the decentralization of the optimal allocation because of different time preferences and constraints facing individual households and the social planner. It also should be noted that the first role is well-known in the literature on the optimal monetary policy, whereas the second one is relatively new to the literature. The central bank's role is to achieve a sequence of gross aggregate inflations $\{\Pi_t\}_{t=0}^{\infty}$, while households and firms take as inflation targets.

Having described how optimizing behaviors of firms and households are affected by aggregate variables in previous subsections, the focus is now given on the characterization of aggregate equilibrium conditions. <Table 5.1> contains 8 aggregate equilibrium conditions for 8 aggregate endogenous variables such as $\{M_t,\, S_t,\, H_t,\, Y_t,\, p_t^*,\, mc_t,\, \Delta_t,\, \Lambda_{t,t+1}\}$, while the derivation of these equilibrium conditions is explained in appendix. The number of endogenous variables thus matches the number of equilibrium conditions given a series of exogenous aggregate productivity levels $\{Z_t\}_{t=0}^{\infty}$, and a set of fiscal and monetary policies such as $(\theta_H,\, \theta_F,\, \theta_A)$, and Π, $\{\Pi_t\}_{t=0}^{\infty}$. It also should be noted that this table includes a minimal self-sufficient set of equilibrium conditions for aggregate quantities and prices, rather than including all equilibrium conditions discussed in previous subsections.

5.3 Optimal Inflation Target

The analysis of this section is focused on closed-form solutions to a benevolent government's optimization problem without having any approximation, while the existence of closed-form solutions requires a set of simplifying assumptions as can be seen below. The first assumption is that an infinitely-lived benevolent government exists. The second assumption is that government's instantaneous utility function is defined as an equally weighted sum of instantaneous utilities of households. The government's life-time utility function at period 0 is then defined as the expected discounted

Table 5.1: Collection of Equilibrium Conditions

Description	Equation
Production Function	$Y_t = \frac{Z_t H_t}{\Delta_t}$
Relative Price Distortion	$\Delta_t = \frac{\alpha}{\bar{\gamma}}\Pi_t^{\epsilon}\Delta_{t-1} + (1-\frac{\alpha}{\bar{\gamma}})(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}}$
Price Level	$p_t^* = (\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{1}{1-\epsilon}}$
Real Reset Price	$p_t^* = \frac{S_t}{M_t}$
Profit Maximization A	$M_t = (1+\theta_F)Y_t + \alpha E_t[\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon-1}\Pi M_{t+1}]$
Profit Maximization B	$S_t = \frac{\epsilon}{\epsilon-1} mc_t Y_t + \alpha E_t[\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon}S_{t+1}]$
Labor Market	$Y_t = a_c mc_t Z_t$
IMRS	$\Lambda_{t,t+1} = \frac{a_\lambda \beta}{1-\theta_A}\frac{Y_t}{Y_{t+1}}$

Note: IMRS means the aggregate inter-temporal marginal rate of substitution. $a_c = \frac{(1+\theta_H)(1-\omega)^2\gamma}{b(\gamma-\omega)(1-\omega\gamma)}$ and $a_\lambda = \frac{(1-\omega)^2\omega_s((1-\omega\gamma)(\omega_e+\gamma(1-\omega_e)\omega_r))+\gamma(1-\omega_e)(1-\omega_r))}{\gamma(1-\omega\gamma)}$.

sum of its instantaneous utility functions with its time discount factor β. The third assumption is that an equally weighted average of instantaneous utility functions of all workers (who are alive at period t) is required to satisfy the following relation:

$$\int_{\mathcal{H}_t} u(C_t(h), H_t(h))dh = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} V_t(m,k)u(C_t(m,k), H_t(m,k)) \tag{5.16}$$

The third assumption helps obtain the existence of a closed-form solution to the government's optimization problem because it facilitates to aggregate preferences of individual households without taking into account survivals, skill levels and workplaces of individual households in future periods as will be seen below.

Given the restriction specified in equation (5.16), the infinitely-lived benevolent government's objective function at period 0 is defined as follows.

$$V_0 = \sum_{t=0}^{\infty}\beta^t E_0[\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} V_t(m,k)u(C_t(m,k), H_t(m,k))] \tag{5.17}$$

Here the government's objective function specified in equation (5.17) does not include the survival probability of an individual household as a discount factor. A concrete example of equation (5.17) is that household' instantaneous utility function (5.3) leads to the following representation of the social planner's objective function at period 0.

$$V_0 = \sum_{t=0}^{\infty}\beta^t E_0[\log C_t + b(\bar{H} - H_t)] + \hat{V} \tag{5.18}$$

where a constant term $\hat{V}$ is defined as follows.

$$\hat{V} = (1-\beta)^{-1}(\frac{\omega\log\gamma}{1-\omega} + \log(\frac{1-\omega\gamma}{1-\omega}))$$

In this representation, government's instantaneous utility function is defined as an equally weighted average of instantaneous utility functions of individual households, while its derivation is included in appendix.

Having discussed how to define government's objective function, let us move onto the characterization of constraints. The set of equilibrium conditions included in <Table 5.1> can be regarded as the set of constraints facing the benevolent government. But it should be noted that only a subset of these conditions are directly included in the government's optimization problem. The other set of equilibrium conditions should be used when the government should find a set of equilibrium prices to implement its solution in a decentralized equilibrium. In sum, the benevolent government's optimization problem can be written as follows.

$$\begin{array}{rcl} V(s_{a,t}) & = & \max_{\{C_t,H_t,\Pi_t,\Delta_t\}}\{\log C_t + b(\bar{H}-H_t) + \beta E_t[V(s_{a,t+1})]\} \\ & & \text{subject to} \\ \Delta_t & = & \frac{\alpha}{\bar{\gamma}}\Pi_t^{\epsilon}\Delta_{t-1} + (1-\frac{\alpha}{\bar{\gamma}})(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}} \\ C_t & = & \Delta_t^{-1}Z_tH_t \end{array} \tag{5.19}$$

given a value of the aggregate state vector $s_{a,t} = [\Delta_{t-1}\ Z_t]'$ and a conditional probability density function of the aggregate productivity level $g(Z_{t+1}|Z_t)$.

The following proposition summarizes the solution of the optimization problem specified in equation (5.19) while its proof is provided in appendix. The optimal inflation that is implied by the solution of the optimization problem specified in equation (5.19) is the same as that of proposition 4.1. On the basis of this result, the model of this section can be regarded as a dynamic general equilibrium model that produces the same optimal inflation rate as obtained in the previous section. In addition, a key point of this proposition is that the solution to the optimization problem specified in equation (5.19) can be attained in a decentralized equilibrium that reflects optimizing behaviors of households and firms under optimal fiscal and monetary policies.

Proposition 5.1 Inflation, real output, and relative price distortion are determined by the following dynamic system of three equations under the solution to the government's optimization problem specified in equation (5.19).

$$\begin{array}{rcl} \Pi_t & = & \bar{\gamma}\frac{\Delta_t}{\Delta_{t-1}} \\ Y_t & = & a_c\frac{Z_t}{\Delta_t} \\ \Delta_t & = & \bar{\gamma}^{-1}(\alpha\Delta_{t-1}^{1-\epsilon} + (1-\alpha)^{\epsilon}(\bar{\gamma}-\alpha)^{1-\epsilon})^{\frac{-1}{\epsilon-1}} \end{array} \tag{5.20}$$

given a series of $\{Z_t\}_{t=0}^{\infty}$ and an initial value Δ_{-1}. In addition, there are series of $\{T_{A,t},\ T_{H,t},\ T_{F,t}\}_{t=0}^{\infty}$ and values of $(\theta_H,\ \theta_F,\ \theta_A)$ that implement inflation and real output specified in equation (5.20) at a decentralized equilibrium.

It should be noted that the determination of inflation and relative price distortion specified in equation (5.20) is identical to that of two equations (4.4) and (4.5), which in turn means the same result for the optimal inflation rate in proposition 4.1 and proposition 5.1. It also should be noted that proposition 5.1 provides a policy prescription with the central bank in terms of its target variables such as inflation and output. Given that the optimal output in the case of full-price flexibility is defined to be $Y_t^* = a_c Z_t$ and the output gap is defined to be $x_t = \log Y_t - \log Y_t^*$, the relation between inflation and output that is implied by equation (5.20) can be written as

$$x_t = x_{t-1} - (\pi_t - \pi) \tag{5.21}$$

where π_t $(= \log \Pi_t)$ is the aggregate inflation rate and π $(= \log \Pi)$ is the steady-state inflation rate. If the central bank's long-term target is set equal to the steady-state optimal inflation, equation (5.21) implies that the central bank should contract current-period's aggregate demand relative to previous period's aggregate demand when inflation rate is higher than its long-term target and expand when inflation rate is lower than its long-term target. In this sense, the policy prescription of proposition 5.1 supports an inflation targeting regime.

Finally, turning to the model with time-varying probabilities of price adjustment, one might ask if results of lemma 4.2 can be attained in a decentralized equilibrium in the similar way as is shown in the model with a constant probability of price adjustment. The following position shows that the corresponding government's optimization problem leads to the same result for the optimal inflation, while its proof is included in appendix.

Proposition 5.2 Inflation, real output, and relative price distortion are determined by the following dynamic system of three equations under the solution to a benevolent government's optimization problem.

$$\begin{aligned} \Pi_t &= \bar{\gamma}_t \frac{\Delta_t}{\Delta_{t-1}} \\ Y_t &= a_{c,t} \frac{Z_t}{\Delta_t} \\ \Delta_t &= \bar{\gamma}_t^{-1} (\alpha_t \Delta_{t-1}^{1-\epsilon} + (1-\alpha_t)^{\epsilon} (\bar{\gamma}_t - \alpha_t)^{1-\epsilon})^{\frac{-1}{\epsilon-1}} \end{aligned} \tag{5.22}$$

given a series of $\{Z_t, \alpha_t\}_{t=0}^{\infty}$ and an initial value Δ_{-1}. In addition, there are series of $\{T_{A,t}$, $T_{H,t}$, $\theta_{H,t}$, $T_{F,t}\}_{t=0}^{\infty}$ and values of $(\theta_F,\ \theta_A)$ that implement inflation and real output specified in equation (5.22) at a decentralized equilibrium.

6 Discussion of Related Literature

The first topic of this section is a review of recent studies on the optimality of a positive inflation rate in recent DSGE models. It should be admitted that a lot of important works exist in the literature on the optimal inflation rate as discussed in Schmitt-Grohe and Uribe (2010) and Guido and Sbordone (2014).[10] But the focus is given on a set of recent works whose numerical results for

[10]The optimality of a positive inflation rate also can be obtained in models with nominal-wage downward rigidities such as Kim and Ruge-Murcia (2009), Benigno and Ricci (2011), Carlsson and Westmark (2016), Schmitt-Grohe and

the optimal inflation rate are relatively close to actual inflation targets of central banks in advanced countries.

In this direction, two sets of recent works emerge as follows. The first set shares in common the emphasis on the importance of an effective lower bound on the short-term nominal interest rate to generate the optimality of a positive inflation rate. The main mechanism behind this result is that a positive inflation target helps reduce the macroeconomic risk of being stuck at such an effective lower bound in the presence of significant declines in the natural real interest rate. In this case, a negative relationship between the natural real interest rate and the optimal inflation target arises because the welfare cost of the effective lower bound creates the welfare cost of a low inflation rate, as can be seen in Coibion, Gorodnichenko, and Wieland (2012) and Andrade, Gali, Le Bihan, and Matheron (2019).

The second set shares in common the emphasis on the importance of firm heterogeneity to generate the optimality of a positive inflation rate such as different productivity growth patterns between price-adjusting and non-adjusting firms in Adam and Weber (2018) and quality bias in a nonlinear aggregation of individual prices in Schmitt-Grohe and Uribe (2012). The set-up of the present paper is closer to that of the second set than the first one in the sense that it permits different labor productivity levels between price-adjusting and non-adjusting firms. But a novel feature of the present paper is that the optimality of a positive inflation rate is associated with interactions between firm heterogeneity and household heterogeneity through assignment of workers with different skill levels to firms with different output levels.

It also should be noted that numerical predictions on the optimal inflation rate remain within a range between 2 percent and 3 percent in the work of Andrade, Gali, Le Bihan, and Matheron and between 1 percent and 3 percent in the work of Adam and Weber. The present paper's numerical predictions on the optimal inflation rate also fluctuate within a range between 2 percent and 3 percent under the benchmark calibration as can be seen in Figure 5. It means that actual inflation targets of central banks in advanced countries are embedded in the optimal range of inflation target prescribed by the theoretical model of the present paper.

It would be worthwhile to mention that the productivity dispersion of individual firms has been widely incorporated into recent DSGE models for various reasons. The related literature in this direction can be classified into three distinct sets. The first one is firm-specific production input factors to improve the empirical performance of theoretical models such as Woodford (2005), de Walque, Smets, and Wouters (2006), Matheron (2006), and Altig, Christiano, Eichenbaum, and

Uribe (2013), models with endogenous entries of firms such as Bilbiie, Fujwari, and Ghironi (2014) and in models with financial frictions such as Brunnermeier and Sannikov (2016). But the present paper is focused on nominal price rigidity and exogenous entries and exits of firms.

Linde (2011). The second one is idiosyncratic productivity shocks in both state-dependent pricing models and time-dependent pricing models to help match observed frequencies and sizes of price changes in microeconomic data such as Dotsey, King, Wolman (1999), Golosov and Lucas (2007), Gertler and Leahy (2008), and Nakamura and Steisson (2008). The third one is to study impacts of asymmetries in sectoral disturbances and firm-level productivity trends (between new and old firms) on optimal policy prescriptions such as Aoki (2001), Wolman (2011), and Adam and Weber (2019).

The main difference between the present paper and these works mentioned above lies in the mechanism to create the productivity dispersion of firms. The reason behind this argument is that the skill dispersion of workers acts as the main source of the productivity dispersion of firms in the model of this paper. In particular, this mechanism works through the assignment of workers with different skill levels to firms with different output levels when production skills of workers are not firm-specific. As a result, technologically ex-ante homogeneous firms become heterogenous firms with different productivity levels, which in turn leads to the optimality of a positive inflation rate.

The theoretical prescription obtained from the minimization of relative price distortion is that workers with higher skill levels should work at firms with higher output levels. It means that worker's skill level should increase with firm size measured in the unit of value-added if government wants to attain an efficient allocation. Given this theoretic result, one might wonder how this normative prescription is associated with results of microeconomic empirical studies. The reason for this one reflects a concern that the theoretical result of this paper must be hardly implementable in actual economies if negative correlations between firm's size and worker's skill are widely observed in microeconomic data. But it should be pointed out that a set of empirical works report positive correlations between firm's size and firm's productivity and between firm's size and worker's wage such as Idson and Oi (1999) and Berlingieri, Calligaris, and Criscuolo (2018), Bloom, Guvenen, Smith, Song, and Wachter (2018).

7 Conclusion

It has been shown in this paper that the optimality of a positive inflation rate emerges in models with nominal price rigidity when workers with higher skill levels are assigned to firms larger output levels. The theoretic contribution of this paper is two-fold. The first one is to show that the central bank's inflation policy can play an active role in the determination of resource allocation through its impact on the assignment of workers to firms. In particular, it has been shown in this paper that inflation can be used as a government's tool to generate the issue of how to assign workers to firms even when ex-ante technologically homogenous firms set prices in a non-synchronized fashion.

The second one is closely associated with the first one in the sense that the optimal steady-state inflation rate is a function of real economic variables including productivity and skill distributions of firms and workers as well as frequencies of price changes. In this sense, the model of this paper can be regarded as a real economic model of the optimal long-term inflation rate that emphasizes the importance of productivity dispersion or skill dispersion, average life-span of firms, and average employment tenure as well as average price duration.

As a practical policy prescription of this theoretical result, it also has been shown that the long-term U.S. optimal inflation rate is slightly above 2 percent under the benchmark model calibration. In this sense, the present paper would present a theoretical framework to help rationalize a 2 percent inflation target of actual central banks on the basis of a standard staggered price-setting model in which the theoretical rationalization of a substantially positive optimal inflation rate has been deemed to be a challenging task. But a caveat is that the magnitude of the optimal inflation rate varies within a relatively wide range starting from less than 2 percent through about 8 percent depending on distributional assumptions of skill and price distributions.

It should be admitted that inefficient matches between workers and firms can weaken the social benefit of inflation discussed above. The reason for this argument is that more firms with large output levels tend to hire more workers with low skill levels as inefficient matches between workers and firms increase, while the model of this paper abstracts from such mismatches between workers and firms as well as unemployed workers and vacant jobs. The introduction of such frictions into the present theoretical framework thus can be a realistic extension to explore. Given the work of Shimer (2005) where an assortative matching between workers and firms arises as the result of the social planner's problem who is constrained by anonymity restrictions, one might want to conjecture that mismatch between workers and firms will not eliminates all the social benefit of inflation discussed above. However, the size of reduction in the benefit of inflation is still an open question. As a result, the extension of the present analysis in this direction can be regarded as an interesting future research topic normatively and positively as well.

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Appendix

A Proof of Lemma 3.2 in Chapter 3

A.1 Optimal Steady-State Inflation in a Symmetric Two-Period Taylor Model

It has been shown in the text that there are two different representations of relative price distortion depending on how workers are assigned to firms in a symmetric two-period Taylor model. Let us start with the first representation that corresponds to the case where skilled workers are assigned to firms with P^*_{t-1} and unskilled workers are assigned to firms with P^*_t. In this case, relative price distortion together with the determination of the aggregate price level can be summarized as follows.

$$\begin{aligned} \Delta_t &= \tfrac{1}{2}\tfrac{Z_t}{Z_{0,t}}(\tfrac{P^*_t}{P_t})^{-\epsilon} + \tfrac{1}{2}\tfrac{Z_t}{Z_{1,t}}(\tfrac{P^*_{t-1}}{P_t})^{-\epsilon} \\ 1 &= \tfrac{1}{2}(\tfrac{P^*_t}{P_t})^{1-\epsilon} + \tfrac{1}{2}(\tfrac{P^*_{t-1}}{P_t})^{1-\epsilon} \end{aligned} \tag{A.1}$$

The substitution of the second-line equation into the first-line equation and then the elimination of (P^*_{t-1}/P_t) in the resulting equation leads to the following representation of relative price distortion.

$$\Delta_t = \frac{1}{2}\frac{Z_t}{Z_{0,t}}(p^*_t)^{-\epsilon} + \frac{1}{2}\frac{Z_t}{Z_{1,t}}(2-(p^*_t)^{1-\epsilon})^{\frac{\epsilon}{\epsilon-1}} \tag{A.2}$$

where $p^*_t = P^*_t/P_t$. The partial differentiation of relative price distortion with respect to real reset price then leads to the following optimization condition

$$p^*_t = (\frac{2}{1+\frac{Z_{1,t}}{Z_{0,t}}})^{\frac{1}{1-\epsilon}} \tag{A.3}$$

By substituting equation (A.3) into equation (A.2), the resulting optimized value of relative price distortion in the first case is

$$\begin{aligned} \Delta_t &= \tfrac{1}{2}\tfrac{Z_t}{Z_{0,t}}(\tfrac{2}{1+\frac{Z_{1,t}}{Z_{0,t}}})^{\frac{\epsilon}{\epsilon-1}} + \tfrac{1}{2}\tfrac{Z_t}{Z_{1,t}}(2-\tfrac{2}{1+\frac{Z_{1,t}}{Z_{0,t}}})^{\frac{\epsilon}{\epsilon-1}} \\ &= \tfrac{1}{2}\tfrac{Z_t}{Z_{0,t}}(\tfrac{2Z_{0,t}}{Z_{0,t}+Z_{1,t}})^{\frac{\epsilon}{\epsilon-1}} + \tfrac{1}{2}\tfrac{Z_t}{Z_{1,t}}(\tfrac{2Z_{1,t}}{Z_{0,t}+Z_{1,t}})^{\frac{\epsilon}{\epsilon-1}} \end{aligned} \tag{A.4}$$

Turning to the second case, the second representation corresponds to the case where unskilled workers are assigned to firms with P^*_{t-1} and skilled workers are assigned to firms with P^*_t. In this case, relative price distortion together with the determination of the aggregate price level can be summarized as follows.

$$\begin{aligned} \Delta_t &= \tfrac{1}{2}\tfrac{Z_t}{Z_{1,t}}(\tfrac{P^*_t}{P_t})^{-\epsilon} + \tfrac{1}{2}\tfrac{Z_t}{Z_{0,t}}(\tfrac{P^*_{t-1}}{P_t})^{-\epsilon} \\ 1 &= \tfrac{1}{2}(\tfrac{P^*_t}{P_t})^{1-\epsilon} + \tfrac{1}{2}(\tfrac{P^*_{t-1}}{P_t})^{1-\epsilon} \end{aligned} \tag{A.5}$$

The substitution of the second-line equation into the first-line equation and then the elimination of (P^*_{t-1}/P_t) in the resulting equation leads to the following representation of relative price distortion.

$$\Delta_t = \frac{1}{2}\frac{Z_t}{Z_{1,t}}(p^*_t)^{-\epsilon} + \frac{1}{2}\frac{Z_t}{Z_{0,t}}(2-(p^*_t)^{1-\epsilon})^{\frac{\epsilon}{\epsilon-1}} \tag{A.6}$$

The partial differentiation of relative price distortion with respect to real reset price then leads to the following optimization condition

$$p_t^* = (\frac{2}{1+\frac{Z_{0,t}}{Z_{1,t}}})^{\frac{1}{1-\epsilon}} \tag{A.7}$$

By substituting equation (A.7) into equation (A.6), the resulting optimized value of relative price distortion in the second case is

$$\begin{aligned} \Delta_t &= \tfrac{1}{2}\tfrac{Z_t}{Z_{1,t}}(\tfrac{2}{1+\frac{Z_{0,t}}{Z_{1,t}}})^{\frac{\epsilon}{\epsilon-1}} + \tfrac{1}{2}\tfrac{Z_t}{Z_{0,t}}(2-\tfrac{2}{1+\frac{Z_{0,t}}{Z_{1,t}}})^{\frac{\epsilon}{\epsilon-1}} \\ &= \tfrac{1}{2}\tfrac{Z_t}{Z_{1,t}}(\tfrac{2Z_{1,t}}{Z_{0,t}+Z_{1,t}})^{\frac{\epsilon}{\epsilon-1}} + \tfrac{1}{2}\tfrac{Z_t}{Z_{0,t}}(\tfrac{2Z_{0,t}}{Z_{0,t}+Z_{1,t}})^{\frac{\epsilon}{\epsilon-1}} \end{aligned} \tag{A.8}$$

Comparing the second-line of equation (A.4) with that of equation (A.8), one can confirm the same optimized value of relative price distortion under the two cases, which is also identical to equation (3.5). This result therefore proves the first part of Lemma 3.2.

A.2 Optimal Steady-State Inflation in a Calvo Model

Let us turn to the second equation of Lemma 3.2. Let us start with the first case where unskilled workers are assigned to price-adjusting firms and skilled workers are assigned to non-adjusting firms in the case of a positive inflation. In this case, current and lagged relative price distortions can be written as follows.

$$\begin{aligned} \Delta_t &= \tfrac{(1-\alpha)Z_t}{Z_{0,t}}((\tfrac{P_t^*}{P_t})^{-\epsilon} + \alpha\gamma^{-1}(\tfrac{P_{t-1}^*}{P_t})^{-\epsilon} + \gamma^{-1}\textstyle\sum_{k=2}^{\infty}\alpha^k(\tfrac{P_{t-k}^*}{P_t})^{-\epsilon}) \\ \alpha\Pi_t^{\epsilon}\Delta_{t-1} &= \tfrac{(1-\alpha)Z_{t-1}}{Z_{0,t-1}}(\alpha(\tfrac{P_{t-1}^*}{P_t})^{-\epsilon} + \gamma^{-1}\textstyle\sum_{k=2}^{\infty}\alpha^k(\tfrac{P_{t-k}^*}{P_t})^{-\epsilon}) \end{aligned} \tag{A.9}$$

In the presence of a stationary skill distribution, the definition of the aggregate labor productivity leads to the following relations.[11]

$$\begin{aligned} Z_t^{-1} &= Z_{0,t}^{-1}(1-\alpha+\gamma^{-1}\alpha) \\ Z_{t-1}^{-1} &= Z_{0,t-1}^{-1}(1-\alpha+\gamma^{-1}\alpha) \end{aligned} \tag{A.10}$$

By substituting equation (A.10) into equation (A.9) and then subtracting the second-line from the first-line in the resulting equation leads to the following representation of relative price distortion.

$$\Delta_t = \alpha\Pi_t^{\epsilon}\Delta_{t-1} + \frac{1-\alpha}{1-\alpha+\gamma^{-1}\alpha}((\frac{P_t^*}{P_t})^{-\epsilon} + \alpha(\gamma^{-1}-1)(\frac{P_{t-1}^*}{P_t})^{-\epsilon})$$

In this case, steady-state relative price distortion is

$$\Delta = \frac{(1-\alpha)(1+\alpha(\gamma^{-1}-1)\Pi^{\epsilon})}{(1-\alpha\Pi^{\epsilon})(\alpha\gamma^{-1}+1-\alpha)}(\frac{1-\alpha\Pi^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}} \tag{A.11}$$

[11]For the simplicity of the analysis, measures of firms and workers are assumed to be the same.

Let us turn to the second case where skilled workers are assigned to price-adjusting firms and unskilled workers are assigned to non-adjusting firms in the case of a negative inflation rate. In this case, current and lagged relative price distortions are given by

$$\begin{aligned} \Delta_t &= \tfrac{Z_t}{Z_{1,t}}((1-\alpha)(\tfrac{P_t^*}{P_t})^{-\epsilon} + \alpha(1-\alpha)\gamma(\tfrac{P_{t-1}^*}{P_t})^{-\epsilon} + (1-\alpha)\gamma\textstyle\sum_{k=2}^{\infty}\alpha^k(\tfrac{P_{t-k}^*}{P_t})^{-\epsilon}) \\ \alpha\Pi_t^{\epsilon}\Delta_{t-1} &= \tfrac{Z_{t-1}}{Z_{1,t-1}}(\alpha(1-\alpha)(\tfrac{P_{t-1}^*}{P_t})^{-\epsilon} + (1-\alpha)\gamma\textstyle\sum_{k=2}^{\infty}\alpha^k(\tfrac{P_{t-k}^*}{P_t})^{-\epsilon}) \end{aligned} \quad \text{(A.12)}$$

In the presence of a stationary skill distribution, the definition of the aggregate labor productivity leads to the following relations.

$$\begin{aligned} Z_t^{-1} &= Z_{1,t}^{-1}(1-\alpha+\gamma\alpha) \\ Z_{t-1}^{-1} &= Z_{1,t-1}^{-1}(1-\alpha+\gamma\alpha) \end{aligned} \quad \text{(A.13)}$$

By substituting equation (A.13) into equation (A.12) and then subtracting the second-line from the first-line in the resulting equation leads to the following representation of relative price distortion.

$$\Delta_t = \alpha\Pi_t^{\epsilon}\Delta_{t-1} + \frac{1-\alpha}{\alpha\gamma+1-\alpha}((\frac{P_t^*}{P_t})^{-\epsilon} + \alpha(\gamma-1)(\frac{P_{t-1}^*}{P_t})^{-\epsilon})$$

In this case, steady-state relative price distortion is

$$\Delta = \frac{(1-\alpha)(1+\alpha(\gamma-1)\Pi^{\epsilon})}{(1-\alpha\Pi^{\epsilon})(\alpha\gamma+1-\alpha)}(\frac{1-\alpha\Pi^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}} \quad \text{(A.14)}$$

It follows from equations (A.11) and (A.14) that one can obtain a unified representation of steady-state relative price distortion as follows

$$\Delta = \frac{(1-\alpha)(\phi+(1-\phi)\Pi^{\epsilon})}{1-\alpha\Pi^{\epsilon}}(\frac{1-\alpha\Pi^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}} \quad \text{(A.15)}$$

where ϕ is defined as

$$\begin{aligned} \phi &= 1-\alpha(\gamma^{-1}-1) \qquad \text{if} \quad \Pi > 1 \\ &= 1-\alpha(\gamma-1) \qquad \text{if} \quad \Pi < 1 \end{aligned} \quad \text{(A.16)}$$

The requirement of $(1 < \gamma < \alpha^{-1}+1)$ leads to $\phi > 1$ when $\Pi > 1$ and $1 > \phi > 0$ when $\Pi < 1$. In addition, the value of ϕ does not affect the value of relative price distortion when $\Pi = 1$. The partial differentiation of steady-state relative price distortion with respect to gross inflation leads to the following equation.

$$\frac{\partial\Delta}{\partial\Pi} = 0 \quad \rightarrow \quad (1-\phi(1-\alpha))(\Pi-1) + (1-\phi)(1-\alpha\Pi^{\epsilon})^2 = 0 \quad \text{(A.17)}$$

Comparing equation (A.17) with equation (3.6), one can confirm that the two equations are the same. Hence this result proves the second part of lemma 3.2.

B Proofs of Lemmas and Propositions in Chapter 4: The Optimal Inflation Rate in the United States

This appendix summarizes proofs of propositions and lemmas used to calibrate the model in chapter 4 where there are two propositions that summarize the law of motion for relative price distortion under a assortative assignment and the corresponding solution to the social planner's optimization problem in models with constant and time-varying probabilities of price adjustment. In addition, there are two lemmas to be used for the model calibration of the U.S. optimal steady-state inflation rate.

B.1 Proof of Proposition 4.1

The first part of this proof is to derive the law of motion for relative price distortion under assortative assignment specified in equation (4.3). In particular, condition 4.1.A. implies the following representation of relative price distortion.

$$\Delta_t = (1-\alpha)\sum_{k=0}^{\infty}\alpha^k \frac{Z_t}{\bar{Z}_{k,t}}(\frac{P^*_{t-k}}{P_t})^{-\epsilon} \tag{B.1}$$

where $\bar{Z}_{k,t}$ is defined as a harmonic mean of skill levels assigned to all type k firms:

$$\bar{Z}_{k,t} = (\sum_{m=m_{k-1}+1}^{m_k} \bar{\omega}^{(k)}_m Z^{-1}_{m,t})^{-1}, \quad \bar{\omega}^{(k)}_m = \frac{\Gamma_m}{\sum_{i=m_{k-1}+1}^{i=m_k}\Gamma_i} \tag{B.2}$$

where $\bar{\omega}^{(k)}_m$ is the weight given to the skill level of type m workers who work at type k firms.

Let us begin with the discussion of how $Z_t/\bar{Z}_{k,t}$ is determined. The substitution of equation (4.2) into the definition of $\bar{\omega}^{(k)}_m$ leads to the following equation.

$$\bar{\omega}^{(k)}_m = \frac{(1-\omega)\omega^m}{\omega^{nk}(1-\omega^n)} \tag{B.3}$$

The substitution of equation (B.3) into equation (B.2) implies that the average skill level of workers who work at type k firms is determined as follows.

$$\bar{Z}_{k,t} = \frac{(1-\omega^n)(1-\omega\gamma^{-1})}{(1-\omega)(1-\omega^n\bar{\gamma}^{-1})}\bar{\gamma}^k Z_{0,t} \tag{B.4}$$

where $\bar{\gamma} = \gamma^n$. In addition, the substitution of condition 4.1.B into the definition of the aggregate labor productivity leads to the following relation between the aggregate productivity and initial skill level:

$$Z_t = \frac{1-\omega\gamma^{-1}}{1-\omega}Z_{0,t} \tag{B.5}$$

The substitution of equations (B.4) and (B.5) into equation (B.1) leads to the following decomposition of relative price distortion:

$$\Delta_t = \frac{(1-\alpha)(1-\omega^n\bar{\gamma}^{-1})}{1-\omega^n}(\frac{P^*_t}{P_t})^{-\epsilon} + \sum_{k=1}^{\infty}\frac{(1-\alpha)(1-\omega^n\bar{\gamma}^{-1})}{1-\omega^n}(\frac{\alpha}{\bar{\gamma}})^k(\frac{P^*_{t-k}}{P_t})^{-\epsilon}.$$

The coefficient of the first-term in the right-hand side of this equation can be rewritten as

$$\omega^n = \alpha \quad \rightarrow \quad \frac{(1-\alpha)(1-\omega^n\bar{\gamma}^{-1})}{1-\omega^n} = 1 - \frac{\alpha}{\bar{\gamma}}.$$

The substitution of this equation into the decomposition of relative price distortion specified above leads to the following representation.

$$\Delta_t = (1-\frac{\alpha}{\bar{\gamma}})(\frac{P_t^*}{P_t})^{-\epsilon} + \sum_{k=1}^{\infty}\frac{(1-\alpha)(1-\omega^n\bar{\gamma}^{-1})}{1-\omega^n}(\frac{\alpha}{\bar{\gamma}})^k(\frac{P_{t-k}^*}{P_t})^{-\epsilon}. \quad \text{(B.6)}$$

The first-term in the right-hand side of equation (B.6) reflects the impact of current-period's real reset price on relative price distortion and the second-term summarizes the impact of lagged real reset prices on relative price distortion. Meanwhile it should be noted that the second-term in the right-hand side of equation (B.6) is simplified by the use of one-period lagged relative price distortion as can be seen below.

$$\frac{\alpha}{\bar{\gamma}}\Pi_t^{\epsilon}\Delta_{t-1} = \sum_{k=1}^{\infty}\frac{(1-\alpha)(1-\omega^n\bar{\gamma}^{-1})}{1-\omega^n}(\frac{\alpha}{\bar{\gamma}})^k(\frac{P_{t-k}^*}{P_t})^{-\epsilon}. \quad \text{(B.7)}$$

By subtracting equation (B.7) from equation (B.6) and then substituting the definition of the aggregate price level into the resulting equation, one can obtain a recursive representation of relative price distortion as follows.

$$\Delta_t = \frac{\alpha}{\bar{\gamma}}\Pi_t^{\epsilon}\Delta_{t-1} + (1-\frac{\alpha}{\bar{\gamma}})(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}} \quad \text{(B.8)}$$

Now one can confirm that equation (B.8) is equal to the first equation of proposition 4.1 specified in equation (4.3).

Having completed the discussion of how to specify the law of motion for relative price distortion, let us move onto the minimization of relative price distortion. This one corresponds to the second part of proposition 4.1. The first-order partial differentiation of relative price distortion with respect to the aggregate gross inflation is

$$\Pi_t\Delta_{t-1} = \frac{\bar{\gamma}-\alpha}{1-\alpha}(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{1}{\epsilon-1}} \quad \text{(B.9)}$$

In order to compute the impact of lagged relative price distortion on current-period's relative price distortion, let us substitute the first-order condition (B.9) into equation (B.8), which in turn leads to the following representation.

$$\Delta_t = \frac{\bar{\gamma}-\alpha}{\bar{\gamma}(1-\alpha)}(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{1}{\epsilon-1}} \quad \text{(B.10)}$$

The substitution of equation (B.10) into the first-order condition (B.9) leads to the following relation between the aggregate gross inflation and growth of relative price distortion

$$\Pi_t = \bar{\gamma}\frac{\Delta_t}{\Delta_{t-1}} \quad \text{(B.11)}$$

In particular, this equation is equal to the second equation of proposition 4.1 specified in equation (4.4).

Let us turn to the third equation of proposition 4.1. By substituting equation (B.11) into the law of motion for relative price distortion (B.8) (to eliminate gross inflation) and then rearranging the resulting equation, one can obtain the optimal law of motion for relative price distortion as follows.

$$\Delta_t^{1-\epsilon} = \bar{\gamma}^{\epsilon-1}(\alpha\Delta_{t-1}^{1-\epsilon} + (1-\alpha)^{\epsilon}(\bar{\gamma}-\alpha)^{1-\epsilon}) \tag{B.12}$$

Here this equation is equal to the third equation of proposition 4.1. This warps up the derivation of three equations of proposition 4.1. In addition, the substitution of equation (B.10) into the definition of the aggregate price index (2.3) together with $\Phi_k = (1-\alpha)\alpha^k$ also leads to the following representation of the optimal real reset price.

$$p_t^* = \frac{\bar{\gamma}-\alpha}{\bar{\gamma}(1-\alpha)}\Delta_t^{-1} \tag{B.13}$$

B.2 Proof of Proposition 4.2

The first part of this appendix is to show how to obtain the law of motion for relative price distortion in the model with time-varying probabilities of price adjustment. The second part is to present the proof of proposition 4.2 on the basis of the first part's result. Recall that a fraction of firms $(1-\alpha_t)$ reset prices and the other fraction of firms α_t do not in each period $t = 0, \cdots$. In this case, relative price distortion is defined as follows.

$$\Delta_t = \frac{(1-\alpha_t)Z_t}{\bar{Z}_{0,t}}(\frac{P_t^*}{P_t})^{-\epsilon} + \sum_{k=1}^{\infty}(\prod_{i=0}^{k-1}\alpha_{t-i})\frac{(1-\alpha_{t-k})Z_t}{\bar{Z}_{k,t}}(\frac{P_{t-k}^*}{P_t})^{-\epsilon} \tag{B.14}$$

In the same way as is done in the model with a constant probability of price adjustment, the average skill level at period t of workers who work at type k firms, $\bar{Z}_{k,t}$, is defined as a harmonic mean of their skill levels:

$$\bar{Z}_{k,t} = (\sum_{m=m_{k-1,t}+1}^{m_{k,t}}\bar{\omega}_{m,t}^{(k)}Z_{m,t}^{-1})^{-1}, \quad \bar{\omega}_{m,t}^{(k)} = \frac{\Gamma_m}{\sum_{i=m_{k-1,t}+1}^{i=m_{k,t}}\Gamma_i} \tag{B.15}$$

where $\bar{\omega}_{m,t}^{(k)}$ is the weight at period t given to the skill level of type m workers who work at type k firms and $m_{k,t}$ is the upper bound at period t of skill levels of workers who work at type k firms:

$$\omega^{m_{k,t}+1} = \prod_{i=0}^{k}\alpha_{t-i} \tag{B.16}$$

for $k = 0, \cdots$.

Having specified the definition of relative price distortion in the model with time-varying probabilities of price adjustment, let us move onto the discussion of how to obtain the law of motion for relative price distortion. In doing so, the first task is to obtain the law of motion for the average skill level of firms that do not reset prices, which amounts to finding an a relation between $\bar{Z}_{k,t}$ and $\bar{Z}_{k-1,t-1}$. It follows from equation (B.15) that weights of type m workers who work for type k firms at period t and type $k-1$ firms at period $t-1$ can be written as

$$\begin{aligned} \omega_{m,t}^{(k)} &= \frac{(1-\omega)\omega_m}{\omega^{m_{k-1,t}+1}(1-\omega^{m_{k,t}-m_{k-1,t}})} \\ \omega_{m,t-1}^{(k-1)} &= \frac{(1-\omega)\omega_m}{\omega^{m_{k-2,t-1}+1}(1-\omega^{m_{k-1,t-1}-m_{k-2,t-1}})} \end{aligned} \quad \text{(B.17)}$$

In the same way as is done in the model with a constant probability of price adjustment, the value of n_{t-i} ($= \log\alpha_{t-i}/\log\omega$) is a positive constant for $i = 0, \cdots$. In addition, equation (B.16) is used to show that the following relations hold.

$$\begin{aligned} m_{k-i,t} - m_{k-(i+1),t} &= n_{t-k} \\ m_{k-i,t} - m_{k-(i+1),t-1} &= n_t \end{aligned} \quad \text{(B.18)}$$

for $k = 2, \cdots$ and a non-negative integer i less than $k-1$. The substitution of equations (B.17) and (B.18) into equation (B.15) leads to the following representations of average skill levels at periods $t-1$ and t of workers who work at firms that reset prices at period $t-k$.

$$\begin{aligned} \bar{Z}_{k,t}^{-1} &= \frac{(1-\omega)(\sum_{m=0}^{n_{t-k}}(\frac{\omega}{\gamma})^m)}{1-\alpha_{t-k}} Z_{m_{k-1,t}+1}^{-1} \\ \bar{Z}_{k-1,t-1}^{-1} &= \frac{(1-\omega)(\sum_{m=0}^{n_{t-k}}(\frac{\omega}{\gamma})^m)}{1-\alpha_{t-k}} Z_{m_{k-2,t-1}+1}^{-1} \end{aligned} \quad \text{(B.19)}$$

Dividing the first-line equation of (B.17) by the second-line equation and then rearranging the resulting equation, one obtain a law of motion for the average skill of workers for firms whose prices are reset at period $t-k$.

$$\bar{Z}_{k,t} = v_t \bar{Z}_{k-1,t-1} \quad \text{(B.20)}$$

for $k = 1, \cdots$ and where v_t is defined as follows.

$$v_t = \gamma^{n_t}\frac{Z_{0,t}}{Z_{0,t-1}}. \quad \text{(B.21)}$$

Having described how the average skill of workers who work at each type firms evolves over time, let us turn to the discussion of how to obtain the recursive representation of relative price distortion. Equation (B.14) implies that current-period's relative price distortion is

$$\frac{\bar{Z}_{0,t}\Delta_t}{Z_t} = (1-\alpha_t)(\frac{P_t^*}{P_t})^{-\epsilon} + \frac{\alpha_t\bar{Z}_{0,t}}{\bar{Z}_{1,t}}((1-\alpha_{t-1})(\frac{P_{t-1}^*}{P_t})^{-\epsilon} + \frac{(1-\alpha_{t-2})\alpha_{t-1}\bar{Z}_{1,t}}{\bar{Z}_{2,t}}(\frac{P_{t-2}^*}{P_t})^{-\epsilon} + \cdots) \quad \text{(B.22)}$$

The substitution of equation (B.20) into equation (B.22) leads to the following equation.

$$\frac{\bar{Z}_{0,t}\Delta_t}{Z_t} = (1-\alpha_t)(\frac{P_t^*}{P_t})^{-\epsilon} + \frac{\alpha_t\bar{Z}_{0,t}}{\bar{Z}_{1,t}}((1-\alpha_{t-1})(\frac{P_{t-1}^*}{P_t})^{-\epsilon} + \frac{(1-\alpha_{t-2})\alpha_{t-1}\bar{Z}_{0,t-1}}{\bar{Z}_{1,t-1}}(\frac{P_{t-2}^*}{P_t})^{-\epsilon} + \cdots) \quad \text{(B.23)}$$

Equation (B.14) also implies that one-period lagged relative price distortion is

$$\frac{\Pi_t^{\epsilon}\bar{Z}_{0,t-1}\Delta_{t-1}}{Z_{t-1}} = (1-\alpha_{t-1})(\frac{P_{t-1}^*}{P_t})^{-\epsilon} + \frac{(1-\alpha_{t-2})\alpha_{t-1}\bar{Z}_{0,t-1}}{\bar{Z}_{1,t-1}}(\frac{P_{t-2}^*}{P_t})^{-\epsilon} + \cdots \quad \text{(B.24)}$$

The substitution of equation (B.24) into equation (B.23) leads to the following equation.

$$\frac{\bar{Z}_{0,t}\Delta_t}{Z_t} = (1-\alpha_t)(\frac{P_t^*}{P_t})^{-\epsilon} + \frac{\alpha_t \bar{Z}_{0,t}}{\bar{Z}_{1,t}} \frac{\Pi_t^\epsilon \bar{Z}_{0,t-1}\Delta_{t-1}}{Z_{t-1}} \tag{B.25}$$

Dividing both sides of equation (B.25) by $\bar{Z}_{0,t}/Z_t$ and then rearranging the resulting equation, one can obtain the following representation of relative price distortion.

$$\Delta_t = \frac{\alpha_t}{\bar{\gamma}_t}\Pi_t^\epsilon \Delta_{t-1} + \frac{1-\alpha_t}{s_t}(\frac{P_t^*}{P_t})^{-\epsilon} \tag{B.26}$$

where s_t and $\bar{\gamma}_t$ are defined as follows

$$\begin{aligned} s_t &= \tfrac{\bar{Z}_{0,t}}{Z_t} \\ \bar{\gamma}_t &= \tfrac{s_t \bar{Z}_{1,t}}{s_{t-1}\bar{Z}_{0,t}} \end{aligned} \tag{B.27}$$

In order to simplify the expression of relative price distortion in equation (B.26), one can use an equilibrium relation between s_t and γ_t as will shown below. The price distribution is used to show that the aggregate labor productivity can be written as follows.

$$\begin{aligned} \tfrac{1}{Z_t} &= (1-\alpha_t)\tfrac{1}{\bar{Z}_{0,t}} + \alpha_t(\tfrac{(1-\alpha_{t-1})}{\bar{Z}_{1,t}} + \tfrac{(1-\alpha_{t-2})\alpha_{t-1}}{\bar{Z}_{2,t}} + \cdots) \\ &= (1-\alpha_t)\tfrac{1}{\bar{Z}_{0,t}} + \tfrac{\alpha_t}{v_t}(\tfrac{(1-\alpha_{t-1})}{\bar{Z}_{0,t-1}} + \tfrac{(1-\alpha_{t-2})\alpha_{t-1}}{\bar{Z}_{1,t-1}} + \cdots) \\ &= (1-\alpha_t)\tfrac{1}{\bar{Z}_{0,t}} + \tfrac{\alpha_t}{v_t Z_{t-1}} \end{aligned} \tag{B.28}$$

By substituting equation (B.20) into the final line of equation (B.28) and then substituting the definitions of s_t into the resulting equation, one can obtain a law of motion for s_t as can be seen below.

$$\frac{\bar{Z}_{0,t}}{Z_t} = 1-\alpha_t + \alpha_t \frac{\bar{Z}_{0,t}}{\bar{Z}_{1,t}}\frac{\bar{Z}_{0,t-1}}{Z_{t-1}} \ \rightarrow \ s_t = 1-\alpha_t+\alpha_t\frac{\bar{Z}_{0,t}}{\bar{Z}_{1,t}}s_{t-1} \tag{B.29}$$

By dividing both sides of equation (B.28) by s_t and then substituting the definition of $\bar{\gamma}_t$ into the resulting equation, one can find that the coefficient of the second term in the right-hand side of equation (B.26) can be written as follows.

$$s_t = 1-\alpha_t+\alpha_t\frac{\bar{Z}_{0,t}}{\bar{Z}_{1,t}}s_{t-1} \quad \rightarrow \quad \frac{1-\alpha_t}{s_t} = 1 - \frac{\alpha_t}{\bar{\gamma}_t}$$

The substitution of this equation into the right-hand side of equation (B.26) leads to the following representation of relative price distortion.

$$\Delta_t = \frac{\alpha_t}{\bar{\gamma}_t}\Pi_t^\epsilon \Delta_{t-1} + (1-\frac{\alpha_t}{\bar{\gamma}_t})(\frac{P_t^*}{P_t})^{-\epsilon} \tag{B.30}$$

In order to replace real reset price by gross inflation in the right-hand side of equation (B.30), let us turn to the definition of the aggregate price index in the model with time-varying probabilities of price adjustment as follows.

$$\begin{aligned} P_t^{1-\epsilon} &= (1-\alpha_t)(P_t^*)^{1-\epsilon} + \alpha_t((1-\alpha_{t-1})(P_{t-1}^*)^{1-\epsilon} + (1-\alpha_{t-2})\alpha_{t-1}(P_{t-2}^*)^{1-\epsilon} + \cdots) \\ &= (1-\alpha_t)(P_t^*)^{1-\epsilon} + \alpha_t P_{t-1}^{1-\epsilon} \end{aligned}$$

Diving both sides of this equation by $P_t^{1-\epsilon}$ leads to the following equation.

$$\frac{P_t^*}{P_t} = (\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{1}{1-\epsilon}} \tag{B.31}$$

By substituting equation (B.31) into equation (B.30), one can obtain the following representation of relative price distortion.

$$\Delta_t = \frac{\alpha_t}{\bar{\gamma}_t}\Pi_t^{\epsilon}\Delta_{t-1} + (1-\frac{\alpha_t}{\bar{\gamma}_t})(\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{\epsilon}{\epsilon-1}} \tag{B.32}$$

given an initial value of Δ_{-1} and a series of $\{\alpha_t\}_{t=0}^{\infty}$. By substituting equations (B.20) and (B.21) into the second line of equation (B.27), it is also possible to obtain a simplified representation of $\bar{\gamma}_t$.

$$\bar{\gamma}_t = v_t\frac{Z_{t-1}}{Z_t} \quad \rightarrow \quad \bar{\gamma}_t = \gamma^{n_t}\frac{s_{0,t}}{s_{0,t-1}}$$

where $s_{0,t}$ $(= Z_{0,t}/Z_t)$ is the ratio of the lowest skill level to the aggregate average skill level at period t. The presence of a stationary skill distribution of workers implies that $s_{0,t} = s_{0,t-1}$ as can be seen from equation (B.4). In this case, $\bar{\gamma}_t$ is determined as follows.

$$\bar{\gamma}_t = \gamma^{n_t} \tag{B.33}$$

Let us move onto the minimization of relative price distortion. The minimization condition can be simplified as follows.

$$\Pi_t\Delta_{t-1} = \frac{\bar{\gamma}_t-\alpha_t}{1-\alpha_t}(\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{1}{\epsilon-1}} \tag{B.34}$$

The substitution of (B.34) into equation (B.32) leads to the following equation.

$$\Delta_t = \frac{\bar{\gamma}_t-\alpha_t}{\bar{\gamma}_t(1-\alpha_t)}(\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{1}{\epsilon-1}} \tag{B.35}$$

Dividing both sides of equation (B.34) by those of equation (B.35), one can find that the optimal gross inflation is proportional to the growth of relative price distortion as can be seen below.

$$\Pi_t = \bar{\gamma}_t\frac{\Delta_t}{\Delta_{t-1}} \tag{B.36}$$

In addition, substituting equation (B.36) into equation (B.32) in order to eliminate gross inflation and then rearranging the resulting equation, one can obtain the optimal law of motion for relative price distortion.

$$\Delta_t^{1-\epsilon} = \bar{\gamma}_t^{\epsilon-1}(\alpha_t\Delta_{t-1}^{1-\epsilon} + (1-\alpha_t)^{\epsilon}(\bar{\gamma}_t-\alpha_t)^{1-\epsilon}) \tag{B.37}$$

The substitution of equation (B.36) into the definition of the aggregate price index (B.31) leads to the following representation of the optimal real reset price.

$$p_t^* = \frac{\bar{\gamma}_t-\alpha_t}{\bar{\gamma}_t(1-\alpha_t)}\Delta_t^{-1} \tag{B.38}$$

C Aggregate Equilibrium Conditions and Implementation of Optimal Allocation

C.1 Aggregate Equilibrium Conditions: Constant Probability of Price Adjustment

Let us begin with the derivation of 8 equilibrium conditions included in <Table 5.1>. The first line corresponds to the aggregate production function (2.10). The second line is the law of motion for relative price distortion specified in equation (4.3). The third line comes from the determination of the aggregate price index specified in equation (2.3) under the assumption that $\Phi_k = (1-\alpha)\alpha^k$ for $k = 0, \cdots$. Since the discussion of how to obtain these three equations can be found in the text, this appendix begins with the fourth line of <Table 5.1>. The fourth line through sixth line of Table 5.1 are derived from the profit-maximization of firms that reset prices in period t. In this table, the profit-maximization condition consists of three equations with a recursive representation, while it is also possible to use the alternative one-line representation of the profit-maximization condition in terms of an expected discounted sum of an infinite number of future terms. It will be shown in the next subsection that firm's marginal production cost is independent of its output level.

C.2 Derivation of Marginal Production Cost

The key reason why marginal production costs of individual firms are independent of their output levels is that real wages of individual workers are proportional to their skill levels. In order to prove that this argument is correct, one can start with the definition of the aggregate labor income. The aggregate labor income is defined as the aggregate sum of labor incomes of individual workers in each period. It is possible to rely on either household (labor) income surveys or establishment payroll surveys in the aggregation of labor incomes of individual workers. For this reason, one can come up with the following relation for the definition of the aggregate real income.

$$\begin{aligned} w_t H_t &= \int_{\mathcal{H}_t} w_{h,t} H_{h,t} dh \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} V_t(m,k) w_t(m,k) H_t(m,k) \end{aligned} \tag{C.1}$$

It should be noted that equation (C.1) can be used to define the aggregate real wage given the definition of the aggregate hours worked by workers.

Now one can rely on a "guess-and-verification" method to show how real wages of individual workers are determined under the assumption that their real wages are proportional to skill levels as can be seen below.

$$w_t(m,k) = Z_{m,t} X_t \tag{C.2}$$

where X_t does not depend on m and k in each period $t = 0, \cdots$. The substitution of equation (C.2) into the second-line equation of equation (C.1) together with equations (2.4) and (2.10) leads to the following representation of X_t

$$X_t = \frac{w_t}{Z_t}\frac{\Delta_t}{\Delta_{p,t}} \tag{C.3}$$

By substituting equation (C.3) into equation (C.2), one can find that real wages of individual workers are affected by aggregate variables such as the aggregate real wage, aggregate labor productivity, and relative price distortion as well as their skill levels:

$$w_t(m,k) = \frac{Z_{m,t}}{Z_t}\frac{\Delta_t}{\Delta_{p,t}}w_t \quad \rightarrow \quad w_{m,t} = \frac{Z_{m,t}}{Z_t}\frac{\Delta_t}{\Delta_{p,t}}w_t \tag{C.4}$$

where $w_{m,t}$ is the real wage at period t of type m workers.

An important consequence of equation (C.4) is that marginal costs of production are independent of output levels of individual firms. Since labor is the only one production input in the model of this paper, the substitution of equation (C.4) into the total production cost for type k firms leads to the following representation.

$$w_{m,t}H_t(m,k) = \frac{w_t}{Z_t}\frac{\Delta_t}{\Delta_{p,t}}Y_{k,t} \tag{C.5}$$

It follows from equation (C.5) that real marginal costs ($= mc_t$) is independent of output levels of individual firms.

$$mc_t = \frac{w_t}{Z_t}\frac{\Delta_t}{\Delta_{p,t}} \tag{C.6}$$

C.3 Derivation of Labor-Market Equilibrium Condition

Let us move onto the equilibrium condition of the labor market. In general, the aggregate consumption is defined as follows.

$$C_t = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} V_t(m,k)C_t(m,k) \tag{C.7}$$

Before going further, recall that the definition of the aggregate labor productivity leads to the following equation:

$$Z_t = \frac{\gamma - \omega}{\gamma(1-\omega)}Z_{0,t} \tag{C.8}$$

Given that $Z_{m,t} = \gamma^m Z_{0,t}$, equation (C.8) implies that the skill level of type m workers can be written as follows.

$$Z_{m,t} = \frac{\gamma^{m+1}(1-\omega)}{\gamma - \omega}Z_t \tag{C.9}$$

Let us turn to the utility optimization condition of households. By substituting equation (C.4) into equation (5.4), one can show that a type m worker' consumption is affected by his or her skill level.

$$C_{m,t} = \frac{(1+\theta_H)Z_{m,t}\Delta_t}{bZ_t\Delta_{p,t}}w_t \tag{C.10}$$

The substitution of equation (C.9) into equation (C.10) also leads to the following equation:

$$C_{m,t} = \frac{\gamma^{m+1}(1+\theta_H)(1-\omega)\Delta_t}{b(\gamma-\omega)\Delta_{p,t}} w_t \tag{C.11}$$

By substituting equation (C.11) into equation (C.7), one can find that the aggregate consumption is linear to the aggregate real wage.

$$C_t = a_c \frac{\Delta_t}{\Delta_{p,t}} w_t \tag{C.12}$$

where coefficient a_c is defined as follows.

$$a_c = \frac{\gamma(1+\theta_H)(1-\omega)^2}{b(\gamma-\omega)(1-\omega\gamma)}. \tag{C.13}$$

In this representation of the aggregate consumption function, the positivity of the aggregate consumption is guaranteed by the condition of $0 < \gamma\omega < 1$ together with $\omega < 1 < \gamma$. The substitution of equation (C.6) into equation (C.12) leads to the seventh equation of <Table 5.1>.

$$C_t = a_c mc_t Z_t \tag{C.14}$$

C.4 Derivation of Aggregate Inter-temporal Marginal Rate of Substitution

Let us move onto the final equilibrium condition of <Table 5.1>, which corresponds to the characterization of the aggregate inter-temporal marginal rate of substitution in terms of the growth of the aggregate consumption. In order to do this one, equations (C.11) and (C.12) are used to show that a type m household's consumption level is proportional to the aggregate consumption level as can be seen below.

$$C_{m,t} = \frac{\gamma^m(1-\omega\gamma)}{1-\omega} C_t \tag{C.15}$$

Now let us pick a household $h \in \mathcal{H}_t$ whose skill level and workplace are $m_{h,t} = m$ and $k_{h,t} = k$. By substituting equation (C.15) into definitions of IMRS (inter-temporal marginal rate of substitution) included in the right-hand side of equation (5.6), one can obtain three different representations of household h's IMRS reflecting three different cases:

$$\begin{aligned} \tilde{\Lambda}_{h,t\to t+1} &= \tfrac{(1-\omega)\omega_s\beta C_t}{\gamma C_{t+1}} \\ \hat{\Lambda}_{h,t\to t+1} &= \tfrac{(1-\omega)\omega_s\beta C_t}{C_{t+1}} \\ \bar{\Lambda}_{h,t\to t+1} &= \tfrac{(1-\omega)\omega_s\beta\gamma^m C_t}{C_{t+1}} \end{aligned} \tag{C.16}$$

By substituting equation (C.16) into equation (5.6), a type m worker's IMRS can be written as follows.

$$\Lambda_{m,t\to t+1} = \beta(1-\omega)\omega_s(\gamma^{-1}\omega_e + (1-\omega_e)\omega_r + (1-\omega_e)(1-\omega_r)\gamma^m)\frac{C_t}{C_{t+1}} \tag{C.17}$$

The substitution of equation (C.17) into the definition of the aggregate inter-temporal marginal rate of substitution (5.7) leads to a closed-form representation of the aggregate intertemporal marginal rate of substitution as can be seen below.

$$\Lambda_{t,t+1} = \frac{a_\lambda \beta}{1-\theta_A} \frac{C_t}{C_{t+1}} \tag{C.18}$$

where a_λ is defined as

$$a_\lambda = \frac{(1-\omega)^2 \omega_s}{\gamma(1-\omega\gamma)}((1-\omega\gamma)(\omega_e + \gamma(1-\omega_e)\omega_r)) + \gamma(1-\omega_e)(1-\omega_r))$$

By comparing equation (C.18) with the eighth equation of <Table 5.1>, one can confirm that these two equations are the same.

C.5 Implementation of Government's Optimal Allocation: Constant Probability of Price Adjustment

In order to formulate a benevolent government's optimization problem, let us begin with the characterization of its objective function. The government's objective function is obtained by the aggregation of preferences of individual households. In the model of this paper, the aggregation of preferences can be facilitated by the result that consumption levels of individual households are determined solely by their skill levels. The substitution of equation (C.15) into the definition of the instantaneous social welfare function (5.16) implies that an equally weighted average of instantaneous utility functions of individual households can be written as follows.

$$\sum_{m=0}^{\infty} \Gamma_m(\log C_{m,t} + b(\bar{H} - H_{m,t})) = \log C_t + b(\bar{H} - H_t) + \frac{\omega \log \gamma}{1-\omega} + \log(\frac{1-\omega\gamma}{1-\omega}) \tag{C.19}$$

Let us turn to the optimality conditions obtained by using the social welfare function specified in equation (5.19). The optimization condition for the aggregate consumption is

$$bC_t = \frac{Z_t}{\Delta_t} \tag{C.20}$$

By substituting this condition into the aggregate production function, one can find that the optimal amount of the aggregate hours worked at period t is

$$H_t = b^{-1} \tag{C.21}$$

The optimization condition for gross inflation is

$$\Pi_t = \bar{\gamma} \frac{\Delta_t}{\Delta_{t-1}} \tag{C.22}$$

Table C.1: Optimal Fiscal Policy and Monetary Policies

Description	Equation
Sales Subsidies	$T_{F,t} = (\tau_{F,t} - 1)Y_t$
	$\tau_{F,t} = \alpha(\Pi_t^{\epsilon-1}\Pi)\tau_{F,t-1} + \theta_F(1-\alpha)(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}}$
Employment Subsidies	$T_{H,t} = \theta_H \Delta_{p,t} mc_t Y_t$
	$\Delta_{p,t} = \alpha\Pi_t^{\epsilon}\Delta_{p,t-1} + (1-\alpha)(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha})^{\frac{\epsilon}{\epsilon-1}}$
Investment Credits	$T_{A,t} = \theta_A \mathcal{W}_t$
	$\mathcal{W}_t = E_t[\frac{\mathcal{W}_{t+1}+D_{F,t+1}}{R_{M,t+1}}]$
	$\beta E_t[\frac{Y_t R_{M,t+1}}{Y_{t+1}}] = 1$
	$\mathcal{D}_{F,t} = Y_t(1 - \Delta_{p,t} mc_t) + T_{F,t}$
Government's Budget	$T_t = T_{A,t} + T_{F,t} + T_{H,t}$
Inflation	$\Pi_t = \bar{\gamma}\frac{\Delta_t}{\Delta_{t-1}}$

Note: In this table, initial values of $\tau_{F,-1}$, Δ_{-1} and $\Delta_{p,-1}$ are taken as given and the steady-state value of gross inflation is $\Pi = \bar{\gamma}$. The determination of Y_t and Δ_t is specified in equation (5.20). The real marginal cost is determined by $mc_t = a_c\Delta_t^{-1}$. $\mathcal{W}_t = Q_{M,t}A_t$ is the market value of mutual funds.

C.6 Optimal Prices Consistent with Government's Solution: Constant Probability of Price Adjustment

The substitution of equation (B.38) into profit-maximization condition (5.12) implies that the profit-maximization condition consistent with the optimal allocation can be written as

$$p_t^* = \frac{S_t}{M_t} \quad \rightarrow \quad \frac{S_t}{M_t} = \frac{\bar{\gamma} - \alpha}{\bar{\gamma}(1-\alpha)}\Delta_t^{-1} \tag{C.23}$$

By substituting equations (C.22) and (C.23) into a pair of forward-looking conditions specified in (5.13) and then subtracting both sides of resulting two equations, the real marginal cost consistent with the optimal allocation can be written as follows.

$$mc_t = (1+\theta_F)\frac{(\epsilon-1)(\bar{\gamma}-\alpha)}{\epsilon\bar{\gamma}(1-\alpha)}\Delta_t^{-1} \tag{C.24}$$

By substituting equation (C.24) into equation (C.6), the real wage consistent with the optimal allocation can be written as follows.

$$w_t = (1+\theta_F)\frac{(\epsilon-1)(\bar{\gamma}-\alpha)}{\epsilon\bar{\gamma}(1-\alpha)}\frac{\Delta_{p,t}}{\Delta_t}\frac{Z_t}{\Delta_t} \tag{C.25}$$

The substitution of equation (C.24) into equation (C.14) implies that the corresponding aggregate equilibrium consumption is

$$C_t = (1+\theta_F)\frac{a_c(\epsilon-1)(\bar{\gamma}-\alpha)}{\epsilon\bar{\gamma}(1-\alpha)}\frac{Z_t}{\Delta_t} \tag{C.26}$$

Comparing equation (C.20) with equation (C.26), one can find that the following condition is needed to attain the optimal allocation as a result of a decentralized equilibrium.

$$b^{-1} = (1+\theta_F)\frac{a_c(\epsilon-1)(\bar{\gamma}-\alpha)}{\epsilon\bar{\gamma}(1-\alpha)} \tag{C.27}$$

Let us move onto the discussion of how to determine θ_A, θ_F, θ_H. In the case of the intertemporal marginal rate of substitution (IMRS), the solution to government's optimization problem implies that the optimal IMRS is

$$\Lambda_{t,t+1} = \beta\frac{C_t}{C_{t+1}} \tag{C.28}$$

Comparing equation (C.18) with equation (C.28), one can find that the following condition is needed to ensure the equality between equilibrium and optimal IMRS:

$$\frac{a_\lambda}{1-\theta_A} = 1 \tag{C.29}$$

The optimal fiscal policies are defined as a set of taxes and subsidies (θ_A, θ_H, θ_F) to help implement the optimal allocation in a decentralized equilibrium. First, equation (C.29) can be used to choose a value of θ_A as can be seen below.

$$\theta_A = 1 - a_\lambda \tag{C.30}$$

Second, the substitution of equation (C.13) into (C.27) (for the elimination of a_c) leads to the following equation for pairs of (θ_H, θ_F):

$$\frac{\epsilon(1+\theta_F)}{\epsilon-1} = \frac{\bar{\gamma}-\alpha}{\bar{\gamma}(1-\alpha)}\frac{(1+\theta_H)(1-\omega)^2\gamma}{(\gamma-\omega)(1-\omega\gamma)} \tag{C.31}$$

Third, the role of sales subsidies is to fix the steady-state distortion associated with the existence of monopolistically competitive firms in goods markets in the same as is done in the literature such as Woodford (2003). In particular, these sales subsidies are used to restore the equality of 'price = marginal cost' in the model of full-price flexibility. Recall that $\alpha = 0$ corresponds to the case of full-price flexibility. In the case of $\alpha = 0$, the profit-maximization condition (5.12) is reduced to a simple representation: $P_t^*/MC_t = \epsilon/((\epsilon-1)(1+\theta_F))$. In this case, $P_t^* = MC_t$ is satisfied when the following condition holds.

$$\frac{\epsilon}{(1+\theta_F)(\epsilon-1)} = 1 \tag{C.32}$$

In sum, equations (C.30), (C.31), and (C.32) can be used to show how to set values of fiscal policy measures necessary to implement the optimal allocation in a decentralized equilibrium as can be seen below.

$$\begin{aligned} \theta_A &= 1 - a_\lambda \\ \theta_H &= \tfrac{(1-\alpha)(\gamma-\omega)(1-\omega\gamma)\bar{\gamma}}{\gamma(\bar{\gamma}-\alpha)(1-\omega)^2} - 1 \\ \theta_F &= \tfrac{1}{\epsilon-1} \end{aligned} \tag{C.33}$$

Let us move onto the discussion of how subsidies and taxes evolve over time in the model with a constant probability of price adjustment. Let us begin with sales subsidies of <Table C.1>. Recall that sales subsidies are defined as follows.

$$\begin{aligned} T_{F,t} &= Y_t((1+\theta_F)\sum_{k=0}^{\infty}(1-\alpha)(\alpha\Pi)^k(\frac{P_{t-k}^*}{P_t})^{1-\epsilon}-1) \\ \tau_{F,t} &= \frac{T_{F,t}}{Y_t}+1 \end{aligned}$$

Dividing both side of the first-line equation by Y_t and then substituting the second-line equation into the resulting equation leads to the following equation.

$$\begin{aligned} \tau_{F,t} &= (1+\theta_F)(1-\alpha)\sum_{k=0}^{\infty}(\alpha\Pi)^k(\frac{P_{t-k}^*}{P_t})^{1-\epsilon} \\ &= (1+\theta_F)(1-\alpha)((\frac{P_t^*}{P_t})^{1-\epsilon}+\sum_{k=1}^{\infty}(\alpha\Pi)^k(\frac{P_{t-k}^*}{P_t})^{1-\epsilon}) \end{aligned} \tag{C.34}$$

In addition, one-period lagged version of the first-line of equation (C.34) is

$$(\alpha\Pi)\Pi_t^{\epsilon-1}\tau_{F,t-1} = (1+\theta_F)(1-\alpha)\sum_{k=1}^{\infty}(\alpha\Pi)^k(\frac{P_{t-k}^*}{P_t})^{1-\epsilon} \tag{C.35}$$

Subtracting equation (C.35) from the second-line of equation (C.34) leads to the following representation.

$$\tau_{F,t} = (\alpha\Pi)\Pi_t^{\epsilon-1}\tau_{F,t-1} + (1-\alpha)(1+\theta_F)(\frac{P_t^*}{P_t})^{1-\epsilon}Y_t \tag{C.36}$$

The substitution of the definition of the aggregate price index into equation (C.36) the leads the same equation as the second equation of <Table C.1>.

Turning to employment subsidies of <Table C.1>, recall that $T_{H,t} = \theta_H w_t H_t$ as specified in equation (5.15). Given this equation, equation (C.6) implies the third equation of <Table C.1> as can be seen below.

$$T_{H,t} = \theta_H mc_t \frac{Z_t H_t}{\Delta_t}\Delta_{p,t} \quad \rightarrow \quad T_{H,t} = \theta_H \Delta_{p,t} mc_t Y_t \tag{C.37}$$

Finally, the determination of mutual-fund investment credits consists of two forward-looking difference equations. One is the law of motion for the market value of mutual fund and the other is the asset pricing equation that holds for mutual fund's market price in the absence of arbitrage profits. In addition, by using equation (C.6) together with the definition of the aggregate production function, the aggregate dividend can be written as follows.

$$\mathcal{D}_{F,t} = Y_t - w_t H_t + T_{F,t} \quad \rightarrow \quad \mathcal{D}_{F,t} = Y_t(1-\Delta_{p,t} mc_t) + T_{F,t} \tag{C.38}$$

C.7 Aggregate Equilibrium Conditions: Time-Varying Probabilities of Price Adjustment

Let us move onto the discussion of how to obtain aggregate equilibrium conditions in the model with time-varying probabilities of price adjustment, which are comparable to those of <Table 5.1> in the model with a constant probability of price adjustment. The introduction of time-varying probabilities of price adjustment into the benchmark model alters specifications of subsidies (given to firms and households) that are needed to implement the optimal allocation obtained from the benevolent government's optimization. For example, sale subsidies given to firms are determined as follows.

$$S_{t-k,t} = ((1+\theta_F)(\prod_{i=0}^{k-1}\Psi_{t-k+i}) - 1)(\frac{P^*_{t-k}}{P_t})D_{t-k,t} \tag{C.39}$$

for $k = 1, \cdots$ and $S_{t,t} = \theta_F P^*_t D_{t-k,t}/P_t$. The aggregate sum at period t of sale subsidies $T_{F,t}$ is also defined as follows.

$$T_{F,t} = (1-\alpha_t)S_{t,t} + \sum_{k=1}^{\infty}(1-\alpha_{t-k})(\prod_{i=1}^{k}\alpha_{t-k+i})S_{t-k,t} \tag{C.40}$$

In addition, employment subsidies for households vary over time as well. For this reason, the ratio of the aggregate sum of employment subsidies to the aggregate labor income is specified as $\theta_{H,t} = T_{H,t}/(w_t H_t)$ in each period $t = 0, \cdots$.

Given these changes of subsidies discussed above, it is also necessary to modify optimization conditions of individual households. Despite of this result, one can check a set of equations (C.2), (C.3), (C.4), (C.5), and (C.6) to confirm that the labor market condition leads to the same relation between aggregate consumption and marginal cost of production except for the time-varying coefficient of marginal cost. Specifically, the substitution of $C_t = Y_t$ into the labor-market equilibrium condition leads to the following condition in the model with time-varying probabilities of price adjustment.

$$Y_t = a_{c,t} mc_t Z_t \tag{C.41}$$

where $a_{c,t}$ is defined as follows.

$$a_{c,t} = \frac{(1+\theta_{H,t})(1-\omega)^2\gamma}{b(\gamma-\omega)(1-\omega\gamma)}$$

Moreover, the functional form of the aggregate production function is the same as that of the model with a constant probability of price adjustment:

$$Y_t = \frac{Z_t H_t}{\Delta_t} \tag{C.42}$$

Recall that the determination of relative price distortion Δ_t becomes more complicated in the model with time-varying probabilities of price adjustment as shown in equation (B.31).

Let us turn to the profit-maximization problem of firms in the model with time-varying probabilities of price adjustment. Given sale subsidies for firms, the profit-maximization problem at period t of firms can be written as follows.

$$\max_{P^*_t}\{\Phi^*_t(\frac{P^*_t}{P_{t+k}}) + \sum_{k=1}^{\infty}E_t[\Lambda_{t,t+k}(\prod_{i=1}^{k}\alpha_{t+i})\Phi^*_{t+k}(\frac{P^*_t}{P_{t+k}})]\} \tag{C.43}$$

Table C.2: Collection of Equilibrium Conditions:
Time-varying Probabilities of Price Adjustment

Description	Equation
Production Function	$Y_t = \frac{Z_t H_t}{\Delta_t}$
Relative Price Distortion	$\Delta_t = \frac{\alpha_t}{\bar{\gamma}_t}\Pi_t^{\epsilon}\Delta_{t-1} + (1 - \frac{\alpha_t}{\bar{\gamma}_t})(\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{\epsilon}{\epsilon-1}}$
Price Level	$p_t^* = (\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{1}{1-\epsilon}}$
Real Reset Price	$p_t^* = \frac{S_t}{M_t}$
Profit Maximization A	$M_t = (1+\theta_F)Y_t + E_t[\alpha_{t+1}\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon-1}\Psi_{t+1}M_{t+1}]$
Profit Maximization B	$S_t = \frac{\epsilon}{\epsilon-1}mc_t Y_t + E_t[\alpha_{t+1}\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon}S_{t+1}]$
Labor Market	$Y_t = a_{c,t} mc_t Z_t$
IMRS	$\Lambda_{t,t+1} = \frac{a_\lambda \beta}{1-\theta_A}\frac{Y_t}{Y_{t+1}}$

Note: IMRS means the aggregate inter-temporal marginal rate of substitution. $a_{c,t} = \frac{(1+\theta_{H,t})(1-\omega)^2\gamma}{b(\gamma-\omega)(1-\omega\gamma)}$ and $a_\lambda = \frac{(1-\omega)^2\omega_s((1-\omega\gamma)(\omega_e+\gamma(1-\omega_e)\omega_r))+\gamma(1-\omega_e)(1-\omega_r))}{\gamma(1-\omega\gamma)}$.

Here the instantaneous profit flows at period $t+k$, represented by $\Phi_{t+k}^*(\frac{P_t^*}{P_{t+k}})$, is defined as follows.

$$\Phi_{t+k}^*(\frac{P_t^*}{P_{t+k}}) = ((1+\theta_F)(\prod_{i=1}^{k}\Psi_{t+i})(\frac{P_t^*}{P_{t+k}})^{1-\epsilon} - mc_{t+k}(\frac{P_t^*}{P_{t+k}})^{-\epsilon})Y_{t+k} \tag{C.44}$$

for $k = 1, \cdots$, and $\Phi_t^*(P_t^*/P_t) = ((1+\theta_F)(P_t^*/P_t)^{1-\epsilon} - mc_t(P_t^*/P_t)^{-\epsilon})Y_t$.

The first-order condition of the profit maximization problem (C.43) can be summarized as follows.

$$\frac{P_t^*}{P_t} = \frac{S_t}{M_t} \tag{C.45}$$

where M_t and S_t are defined as

$$\begin{aligned} M_t &= (1+\theta_F)Y_t + E_t[\alpha_{t+1}\Lambda_{t,t+1}(\Pi_{t+1}^{\epsilon-1}\Psi_{t+1})M_{t+1}] \\ S_t &= \tfrac{\epsilon}{\epsilon-1}mc_t Y_t + E_t[\alpha_{t+1}\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon}S_{t+1}] \end{aligned} \tag{C.46}$$

It also should be noted that an equilibrium relation between profit-maximizing price and aggregate inflation can be obtained by using the definition of the aggregate price index as can be seen in equation (B.30).

<Table C.2> contains 8 aggregate equilibrium conditions in the model with time-varying probabilities of price adjustment that are comparable to those of <Table 5.1> for 8 aggregate endogenous variables such as $\{M_t,\ S_t,\ H_t,\ Y_t,\ p_t^*,\ mc_t,\ \Delta_t,\ \Lambda_{t,t+1}\}$. In this table, the number of endogenous variables matches the number of equilibrium conditions given series of $\{Z_t,\ \alpha_t,\ \theta_{H,t},\ \Pi_t\}_{t=0}^{\infty}$ with $(\theta_F,\ \theta_A)$. The benevolent government's optimization problem is

$$\begin{aligned} V(s_{a,t}) &= \max_{\{C_t,H_t,\Pi_t,\Delta_t\}}\{\log C_t + b(\bar{H} - H_t) + \beta E_t[V(s_{a,t+1})]\} \\ &\quad \text{subject to} \\ \Delta_t &= \tfrac{\alpha_t}{\bar{\gamma}_t}\Pi_t^{\epsilon}\Delta_{t-1} + (1 - \tfrac{\alpha_t}{\bar{\gamma}_t})(\tfrac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{\epsilon}{\epsilon-1}} \\ C_t &= \Delta_t^{-1}Z_t H_t \end{aligned} \tag{C.47}$$

Table C.3: Optimal Fiscal Policy and Monetary Policies:
Time-varying Probabilities of Price Adjustment

Description	Equation
Sales Subsidies	$T_{F,t} = \tau_{F,t} Y_t$
	$\tau_{F,t} = \alpha_t \Psi_t \Pi_t^{\epsilon-1} \tau_{F,t-1} + (1+\theta_F)(1-\alpha_t)(\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{\epsilon}{\epsilon-1}}$
Employment Subsidies	$T_{H,t} = \theta_{H,t}\Delta_{p,t} mc_t Y_t$
	$\Delta_{p,t} = \alpha_t \Pi_t^{\epsilon}\Delta_{p,t-1} + (1-\alpha_t)(\frac{1-\alpha_t\Pi_t^{\epsilon-1}}{1-\alpha_t})^{\frac{\epsilon}{\epsilon-1}}$
Investment Credits	$T_{A,t} = \theta_A \mathcal{W}_t$
	$\mathcal{W}_t = E_t[\frac{\mathcal{W}_{t+1}+D_{F,t+1}}{R_{M,t+1}}]$
	$\beta E_t[\frac{Y_t R_{M,t+1}}{Y_{t+1}}] = 1$
	$\mathcal{D}_{F,t} = Y_t(1-\Delta_{p,t} mc_t) + T_{F,t}$
Government's Budget	$T_t = T_{A,t} + T_{F,t} + T_{H,t}$

given the aggregate state vector $s_{a,t} = [\Delta_{t-1}\ Z_t\ \alpha_t]'$ and the conditional joint probability density function of aggregate productivity level and price-adjustment probability $g(Z_{t+1}, \alpha_{t+1}|Z_t, \alpha_t)$. The partial differentiation of the social welfare function included in the social planner's optimization problem (C.47) with respect to consumption gives the following optimization condition:

$$bC_t = \frac{Z_t}{\Delta_t} \quad \rightarrow \quad Y_t = \frac{Z_t}{b\Delta_t} \tag{C.48}$$

The substitution of equation (C.48) into the aggregate production function specified in the first line of <Table C.2> implies that the optimal amount of the aggregate hours worked at period t is

$$H_t = b^{-1} \tag{C.49}$$

Recall that two optimality conditions for the aggregate gross inflation and relative price distortion are included in equations (B.36) and (B.37) respectively, while the optimality condition for the real reset price at period t is also included in equation (B.38).

Let us turn to the decentralization of the optimal allocation in the model with time-varying probabilities of price adjustment. The substitution of equation (B.38) into the profit maximization condition specified in the fourth line of <Table C.2> leads to the following equation.

$$p_t^* = \frac{\bar{\gamma}_t - \alpha_t}{\bar{\gamma}_t(1-\alpha_t)}\Delta_t^{-1} \quad \rightarrow \quad \frac{S_t}{M_t} = \frac{\bar{\gamma}_t - \alpha_t}{\bar{\gamma}_t(1-\alpha_t)}\Delta_t^{-1} \tag{C.50}$$

The substitution of equation (C.50) into the sixth line of <Table C.2> and then subtracting the resulting equation from the fifth line leads to the following equation.

$$\begin{aligned} mc_t &= \frac{(1+\theta_F)(\epsilon-1)(\bar{\gamma}_t-\alpha_t)}{\epsilon\bar{\gamma}_t(1-\alpha_t)}\Delta_t^{-1} \\ \Psi_{t+1} &= \bar{\gamma}_t^{-1}\frac{(1-\alpha_t)(\bar{\gamma}_{t+1}-\alpha_{t+1})}{(1-\alpha_{t+1})(\bar{\gamma}_t-\alpha_t)} \end{aligned} \tag{C.51}$$

By substituting the first line of equation (C.50) into the seventh line of <Table C.2> and the comparing the resulting equation with the aggregate production function with $H_t = b^{-1}$ leads to the following equation.

$$b^{-1} = a_{c,t} \frac{(1+\theta_F)(\epsilon-1)(\bar{\gamma}_t - \alpha_t)}{\epsilon\bar{\gamma}_t(1-\alpha_t)} \tag{C.52}$$

In sum, equations (C.30), (C.32), and (C.52) can be used to show how to set values of fiscal policy measures necessary to implement the optimal allocation in a decentralized equilibrium as can be seen below.

$$\begin{array}{lcl} \theta_A & = & 1 - a_\lambda \\ \theta_{H,t} & = & \frac{(1-\alpha_t)\bar{\gamma}_t}{\bar{\gamma}_t - \alpha_t} \frac{(\gamma-\omega)(1-\omega\gamma)}{(1-\omega)^2\gamma} - 1 \\ \theta_F & = & \frac{1}{\epsilon-1} \end{array} \tag{C.53}$$