Choice against Phantoms

Kazuhiro Hara*† Marcel Menezes de Andrade Prado‡

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Abstract

We study the extent to which individual choice behaviors can be explained as equilibrium strategies in games. We view choice problems as a set of strategies given to the decision-maker, and the chosen alternatives are considered equilibrium strategies in a game. We assume that players other than the decision-maker (phantoms) and their set of strategies are unobservable. We will refer to this framework as "choice against phantoms." We demonstrate that in a general environment, without any assumptions about the phantoms, all choice behaviors can be explained as equilibrium strategies. Under the assumption that the set of phantom's strategies is fixed, this model is characterized by Sen's alpha axiom. We also show that a version that uses the strict Nash equilibrium as the equilibrium notion is characterized by WWARNI. As an application, we provide an alternative representation for choice functions satisfying Weak WARP. We also provide an alternative representation for Rational Shortlist methods.

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1 Introduction

We propose and study models of choice that interpret individual choice behaviors as equilibrium strategies in a game. We characterize choice behaviors that can be explained under this model. The decision-maker under consideration is a player in a strategic game. As an outside observer, we observe the set of strategies available to him and the choices he made. However, we assume that the other players in the game cannot be observed. We refer to these unobserved players as "phantoms." In essence, the decision-maker is in a strategic situation against phantoms where Nash equilibria are played, but we only observe the projection of the game and equilibria onto the decision-maker. We call this framework "choice

*Waseda University, School of Commerce, Nishi-Waseda, Shinjuku, Tokyo, 169-8050.
†Email: kazuhiro.hara@waseda.jp
‡EPGE Escola Brasileira de Economia e Finanças, Praia de Botafogo 190, Botafogo, Rio de Janeiro, 22250-900.

against phantoms." Choice against phantoms can arise in different contexts. For example, an individual in a strategic situation may agree to disclose their choices while other individuals involved refuse to do so for privacy concerns. Alternatively, a delusional individual may put him or herself in a hypothetical strategic situation and play a game against phantoms.

We characterize the choice behaviors that can be explained as "choice against phantoms." As this is a fairly general framework, various versions can be considered depending on the nature of the phantoms (number of phantoms, set of strategies, and payoff functions). Firstly, we consider the situation where multiple phantoms exist and the set of strategies of the phantoms is dependent on the choice problem:

- There is a set N of phantoms. Each phantom has a correspondence Γ_i that determines his strategy sets, and a payoff function u_i. The decision-maker has a payoff function u_d. Given a choice problem S, the choice is determined by

  $$c(S) = \{x \in S : (x, x_N) \in NE(S, (\Gamma_i(S))_{i \in N}, u_d, (u_i)_{i \in N}) \text{ for some } x_N \in \prod_{i \in N} \Gamma_i(S)\}$$

  where NE stands for the set of Nash equilibrium.

In Proposition 1, we demonstrate that when making almost no assumptions about the phantoms, anything goes. That is, any pattern of choice can be represented as above.

Seeking positive results, we consider the version where the phantom's set of strategies is fixed:

- There is one phantom with a fixed set of strategies Y. Each player i has a payoff function u_i. Given a choice problem S, the choice is determined by

  $$c(S) = \{x \in S : (x, y) \in NE(S, Y, u_1, u_2) \text{ for some } y \in Y\}.$$

In Theorem 3, we show that this model is characterized by Sen's Alpha Axiom. We show in Theorem 4 that, under suitable conditions, the payoff functions in this model can be continuous.

We then explore a version using strict Nash equilibrium as the equilibrium concept:

- There is one phantom with a fixed set of strategies Y. Each player i has a payoff function u_i. Given a choice problem S, the choice is determined by

  $$c(S) = \{x \in S : (x, y) \in SNE(S, Y, u_1, u_2) \text{ for some } y \in Y\},$$

  where SNE stands for the set of strict Nash equilibrium.

In Theorem 6, we show that this model is characterized by WWARNI, a condition introduced by Ribeiro and Riella (2017). This condition is equivalent to

α-axiom and expansion if there are only finitely many alternatives and the choice from every choice set is observed.

The decision-makers described in these models are highly rational in the sense that they choose equilibrium strategies. In this regard, Theorem 3 suggests that α-axiom may be an indication of high rationality. On the other hand, however, these models may violate the Weak Axiom of Revealed Preference (WARP) and, therefore, may not be written as a preference maximization. In this sense, they can be categorized as models of boundedly rational choice. Indeed, some of the existing models of boundedly rational choice can be expressed as games.

As an application of the main results, we provide alternative representations for choice functions that satisfy Weak WARP. We show that Rationalization by Cherepanov, Feddersen, and Sandroni (2013) and Categorization and choice by Manzini and Mariotti (2012) can be viewed as a selection from equilibrium strategies of a strategic game. We also show that the Rational Shortlist Method by Manzini and Mariotti (2007) can be expressed as a selection from strict equilibrium strategies.

2 Preliminaries

Let X be a nonempty set and $\mathcal{A}$ be a nonempty collection of nonempty subsets of X. A choice correspondence is a map $c : \mathcal{A} \to 2^X \setminus \{\emptyset\}$ with $c(S) \subset S$ for all S in $\mathcal{A}$. It is a choice function if $c(S)$ is a singleton for every S in $\mathcal{A}$.

A game $(S_i, u_i)_{i \in N}$ is a collection of pairs of a nonempty set of strategies S_i and a payoff function u_i for each $i \in N$, where N is a nonempty set of players. A Nash equilibrium of the game is a strategy profile s such that for each $i \in N$, $u_i(s) \geq u_i(s'_i, s_{-i})$ for every $s'_i \in S_i$. We denote the set of Nash equilibria of the game by $NE((S_i, u_i)_{i \in N})$. A strategy profile s is a strict Nash equilibrium if for each $i \in N$, $u_i(s) > u_i(s'_i, s_{-i})$ for every $s'_i \in S_i \setminus \{s_i\}$. We denote the set of strict Nash equilibria of the game by $SNE((S_i, u_i)_{i \in N})$. A two-player game (S_1, S_2, u_1, u_2) is symmetric if $S_1 = S_2$ and $u_1(x, y) = u_2(y, x)$ for every (x, y) in $S_1 \times S_2$.

3 Models of choice against phantoms

3.1 Anything goes...

A model of choice against phantoms is determined by specifying which game each decision problem corresponds to. We consider a model that should be as general as possible as a starting point. We allow for any number of phantoms, and the set of strategies for each phantom can change depending on the choice problem. To this end, let N be the set of phantoms, and suppose the set of strategies for each phantom i is determined by a correspondence Γ_i. The correspondence Γ_i matches each choice problem with the set of strategies available to the phantom. That is, when the choice problem is S, $\Gamma_i(S)$ becomes the set of strategies available to the phantom.

Given the generality of this framework, it is not surprising that every choice behavior can be explained as equilibrium strategies. In fact, as can be seen from the following example, every choice correspondence can be explained using just one phantom.

Example 1. Let c be an arbitrary choice correspondence. For each choice problem S, let $\Gamma(S) = \{S\}$. Let u_2 be an arbitrary constant payoff function for the phantom. Let u_1 be the decision maker's payoff function given by $u_1(x, S) = \mathbf{1}_{c(S)}(x)$. Clearly, $c(S) = \arg\max\{u_1(x, S) : x \in S\}$ and hence $x \in c(S)$ iff (x, S) is a Nash equilibrium of $(S, \{S\}, u_1, u_2)$.

From this example, it is clear that the framework we are considering is too general. In particular, the problem is that the set of phantom's strategies can change without any constraints depending on the choice problem. Therefore, it is conceivable to impose some constraints on Γ_i. Here, we consider a monotonicity condition, where the set of phantoms' strategy profiles change monotonically depending on the choice problem. Formally, we say a collection $(\Gamma_i)_{i\in N}$ of correspondences is monotone if $S \subsetneq T$ implies $\prod_{i\in N} \Gamma_i(S) \subsetneq \prod_{i\in N} \Gamma_i(T)$. This condition means that if the decision-maker's budget set decreases, the phantoms' budget sets also decreases. Specifically, when there is only one phantom, if the decision-maker's choice set becomes smaller, the phantom's choice set will also become smaller. As the following proposition shows, even in the presence of monotonicity, every choice behavior can be explained as equilibrium choices.

Proposition 1. There is a set N and a collection of correspondences $(\Gamma_i)_{i\in N}$ that is monotone, and payoff functions $(u_i)_{i\in N}$ such that for every choice correspondence c on $2^X \setminus \{\emptyset\}$, there is a payoff function u_d with which

$$c(S) = \{x \in S : (x, x_N) \in NE(S, (\Gamma_i(S))_{i\in N}, u_d, (u_i)_{i\in N}) \text{ for some } x_N \in \prod_{i\in N} \Gamma_i(S)\}$$

for each $S \in 2^X \setminus \{\emptyset\}$.

Proof. We take X as the set of phantoms. For each $i \in X$, let $\Gamma_i(S) = \{0, 1\}$ if $i \in S$ and $\Gamma_i(S) = \{0\}$ otherwise. Note $(\Gamma_i)_{i\in N}$ is monotone. For each $x \in X$ and $z \in \{0, 1\}^X$, let $u_i(x, z) = z_i$. Let c be an arbitrary choice correspondence. Define u_d by $u_d(x, 0) = 0$ and $u_d(x, z) = \mathbf{1}_{c(\{i:z_i=1\})}(x)$ when $z \neq 0$. Let S be an arbitrary choice problem. If $x \in c(S)$ then (x, z^S), where $z_i^S = 1$ if $i \in S$ and $z_i^S = 0$ otherwise, is a Nash equilibrium of $(S, (\Gamma_i(S))_{i\in X}, u_,(u_i)_{i\in X})$. Conversely, if (x, z) is an equilibrium of $(S, (\Gamma_i(S))_{i\in X}, u_,(u_i)_{i\in X})$ then it must be the case that $z = z^S$. Therefore, $u_d(x, z) = \mathbf{1}_{c(S)}(x) = 1$ must hold for (x, z) to be an equilibrium. This implies $x \in c(S)$. □

When the number of alternatives is finite, in Proposition 1, the number of phantoms can be reduced to one.

Corollary 2. There is a monotone correspondence Γ, a phantom's payoff functions u_2 such that for every choice correspondence c on $2^X\setminus\{\emptyset\}$, there is a payoff function u_1 with which

$$c(S) = \{x \in S : (x, y) \in NE(S, \Gamma(S), u_1, u_2) \text{ for some } y \in \Gamma(S)\}$$

for each $S \in 2^X \setminus \{\emptyset\}$.

Proof. Construct Γ_i and u_i as in the proof of Proposition 1. Let $u_2 = \sum_{i \in X} u_i$ and $\Gamma = \prod_{i \in X} \Gamma_i$. Note Γ is monotone. For any choice correspondence c, define u_d as in the proof of Proposition 1 and let $u_1 = u_d$. Observe that (x, z) is an equilibrium of $(S, \Gamma(S), u_1, u_2)$ iff (x, z) is an equilibrium of $(S, (\Gamma_i(S))_{i \in X}, u_, (u_i)_{i \in X})$. Thus, $x \in c(S)$ iff (x, z^S) is an equilibrium of $(S, \Gamma(S), u_1, u_2)$. □

From these "anything goes" type of results, it becomes clear that stronger assumptions are needed to obtain meaningful models of "choice against phantoms". In the following sections, we will examine some special cases that have testable implications.

3.2 Nash Representation

In this model, we consider the decision maker who makes a choice against a player who has a fixed set of strategies. Specifically, we consider the following representation of a choice correspondence c on an arbitrary collection $\mathcal{A}$ of choice problems. There is a nonempty set Y and two real valued functions u_1 and u_2 defined on $X \times Y$ such that for each choice problem S in $\mathcal{A}$,

$$c(S) = \{x \in S : (x, y) \in NE(S, Y, u_1, u_2) \text{ for some } y \in Y\}. \tag{1}$$

It turns out that this model is characterized by Sen's α-axiom.

α-axiom. If $S \subset T$ then $c(T) \cap S \subset c(S)$.

This condition states that if an alternative chosen in a larger set is available in a smaller set, it must be chosen.

Theorem 3. A choice correspondence c on $\mathcal{A}$ satisfies α-axiom if and only if it admits a representation in (1).

Proof. It is easy to see that the representation in (1) implies α-axiom. We focus on the sufficiency part of the theorem. Let c be a choice correspondence that satisfies α-axiom. Let $Y := \mathcal{A} \cup \{\emptyset\}$ and $C := \bigcup_{S \in \mathcal{A}} c(S)$. Define $u_1 : X \times Y \to \mathbb{R}$ by

$$u_1(x, S) = \begin{cases} \mathbf{1}_{X \setminus S}(x) & S \neq \emptyset \\ \mathbf{1}_C(x) & S = \emptyset \end{cases}$$

Note $\mathbf{1}_T$ stands for the indicator function of the set T. Define $u_2 : X \times Y \to \mathbb{R}$ by

$$u_2(x, S) = \begin{cases} \mathbf{1}_{c(S)}(x) & S \neq \emptyset \\ \mathbf{1}_{X \setminus C}(x) & S = \emptyset \end{cases}$$

Now, let $x \in c(S)$. Then, $u_1(x, S) = 0 = u_1(z, S)$ for all $z \in S$. On the other hand, $u_2(x, S) = 1 \geq u_2(x, T)$ for all $T \in Y$. Thus, (x, S) is a Nash equilibrium of the game (S, Y, u_1, u_2).

Conversely, let $S \in \mathcal{A}$ and (x, T) be a Nash equilibrium of the game (S, Y, u_1, u_2). Suppose $T = \emptyset$. Then, $u_1(x, T) \geq u_1(z, T) = 1$ for any $z \in c(S)$ and hence $x \in C$. That is, $x \in c(A)$ for some $A \in \mathcal{A}$. But then, $u_2(x, A) > 0 = u_2(x, T)$. This is a contradiction. Thus, $T \neq \emptyset$. Suppose $x \notin C$. Then $u_2(x, \emptyset) = 1 > u_2(x, T)$, which is a contradiction. Thus, $x \in C$ and hence $u_2(x, T) \geq 1 = u_2(x, V)$ for some $V \in \mathcal{A}$. This implies $x \in c(T)$. Then, $u_1(x, T) = 0 \geq u_1(z, T)$ for all $z \in S$, which implies $S \subset T$. By α-axiom, $x \in c(S)$ as desired. □

An alternative representation for choice correspondences that satisfy α-axiom appears in Proposition 7 in Cherepanov, Feddersen, and Sandroni (2013). One may be able to construct preferences for each player in (1) by using their representation.

Under suitable conditions, one can take payoff functions in (1) to be continuous. To this end, let X be a nonempty compact metric space. Let $\mathbf{k}(X)$ denote the set of nonempty compact subsets of X endowed with the Hausdorff distance. The following condition is a standard upper hemicontinuity for choice correspondence.

Upper hemi-continuity. For every convergent sequences (x_n) in X and (S_n) in $\mathbf{k}(X)$, if $x_n \in c(S_n)$ for every n, then $\lim x_n \in c(\lim S_n)$.

By adding this condition, we obtain a representation with continuous payoff functions.

Theorem 4. Let X be a nonempty compact metric space. A choice correspondence c on $\mathbf{k}(X)$ satisfies α-axiom and Upper hemi-continuity if and only if there is a nonempty compact metric space Y and two continuous payoff functions u_1 and u_2 such that for each $S \in \mathbf{k}(X)$,

$$c(S) = \{x \in S : (x, y) \in NE(S, Y, u_1, u_2) \text{ for some } y \in Y\}.$$

Proof. It is easy to see that c satisfies α-axiom and upper hemicontinuity if it admits the representation. Conversely, let c be a choice correspondence that satisfies α-axiom and upper hemi-continuity. Let $Y := \mathbf{k}(X)$ and note that Y is a nonempty compact metric space. Define $u_1 : X \times Y \to \mathbb{R}$ by $u_1(x, T) = \min_{y \in T} d(x, y)$ where d is the metric on X. Observe that u_1 is a continuous function. Next, define the binary relation $\trianglerighteq_2$ on $X \times Y$ by $(x, S) \trianglerighteq_2 (y, T)$ if $(x, S) = (y, T)$, or $x = y$ and $x \in c(S)$. Observe that $\trianglerighteq_2$ is a preorder. Moreover, it is continuous by upper hemi-continuity of c. There is a continuous function u_2 on $X \times Y$ such that $(x, S) \trianglerighteq_2 (y, T)$ implies $u_2(x, S) \geq u_2(y, T)$ and $(x, S) \triangleright_2 (y, T)$ implies $u_2(x, S) > u_2(y, T)$.[1]

To complete the proof, let $x \in c(S)$. Then, $u_1(x, S) \geq u_1(z, S)$ for all $z \in S$ while $u_2(x, S) \geq u_2(x, T)$ for all $T \in Y$. That is, (x, S) is a Nash equilibrium of the game (S, Y, u_1, u_2). Conversely, let (x, T) be a Nash equilibrium of (S, Y, u_1, u_2). Then, $u_1(x, T) \geq u_1(z, T)$ for all $z \in S$ and $u_2(x, T) \geq u_2(x, A)$ for all $A \in Y$. In particular, $u_2(x, T) \geq u_2(x, \{x\})$ while $(x, \{x\}) \trianglerighteq_2 (x, T)$. This implies $(x, T) \trianglerighteq_2 (x, \{x\})$ and hence $x \in c(T)$. Then by definition of u_1, $u_1(x, T) = 0$ and

[1] See Levin (1983) and Corollary 1 in Evren and Ok (2011).

hence $u_1(z, T) = 0$ for all $z \in S$. Thus, S is a subset of T. By α-axiom, $x \in c(S)$ as desired. $\square$

Here are some examples of choice models that satisfy α-axiom and their representations of the form in (1).

Example 2. (Rational choice) Every choice correspondence that can be written as maximization of a preference relation satisfies α-axiom. Such choice correspondences can be written as in (1) by introducing a phantom with only one strategy.

Example 3. (Choice functions) A choice function c on $\mathcal{A}$ containing all sets with at most three elements satisfies α-axiom if and only if it is preference maximization. Hence, for such choice functions, the model in (1) coincides with preference maximization.

Example 4. (Choice against Nature) Let Ω be a state space, Y be a prize space, and $X := Y^\Omega$ be the set of acts. The decision maker has a utility function u and chooses an act to maximize expected utility $U(f, p) = \int_\Omega u(f)\, dp$ for a given prior p. On the other hand, the nature has a utility function v and chooses a prior on Ω from a set P of priors in order to maximize expected utility $V(f, p) = \int_\Omega v(f)\, dp$ for a given act f. The choice is determined by

$$c(S) = \{f \in S : (f, p) \in NE(S, P, U, V) \text{ for some } p \in P\}.$$

Note that the choice defined above is nonempty, for example, when Ω is finite, Y is a compact convex subset of $\mathbb{R}^n$, P is compact and convex, u and v are affine, and S is compact and convex. Moreover, if $v = -u$ and S are compact and convex, then the model coincides with the maxmin expected utility of Gilboa and Schmeidler (1989).

Example 5. (Path-Independent Choice) A choice correspondence c on $2^X \setminus \{\emptyset\}$ is path-independent if $c(S \cup T) = c(c(S) \cup c(T))$ for every S and T. It is know that path independence implies α-axiom. Models of choosing the two finalists studied by Eliaz, Richter, and Rubinstein (2011) are examples of path-independent choice.

An important special case of path-independent choice rule is q-acceptance rule. It is used in school choice to model a school's preference and is characterized by a linear order $\succeq$ over students and the number q of students the school can accept. If there are less than or equal to q students applying, the school accepts all of them. If there are more than q students applying, the school accepts the top q students among them according to $\succeq$.

Aizerman and Malishevski (1981) showed that, when X is finite, a choice correspondence c on $2^X \setminus \{\emptyset\}$ is path-independent if and only if there are linear orders $\succeq_i$ for $i = 1, \ldots, k$ for some k such that

$$c(S) = \{x \in S : x \succeq_i y \text{ for every } y \in S \text{ for some } i\}.$$

That is, choices from set S are maximizers of $\succeq_i$. Using this representation, one can easily construct a representation of the form (1). Indeed, let $Y := \{1, \ldots, k\}$ and u_2 be a constant function on $X \times Y$. For each $i = 1, \ldots, k$, let f_i be a utility function representing $\succeq_i$. Define u_1 on $X \times Y$ by $u_i(x, i) = f_i(x)$.

Example 6 (Threshold model)**.** A threshold model consists of a utility function u and a function $\delta : X \times \mathcal{A} \to \mathbb{R}_+$. The choice is determined by

$$c(S) = \{x \in S : \max u(S) \le u(x) + \delta(x, S)\}.$$

It is easy to see that c satisfies α-axiom if $S \subset T$ implies $\delta(x, T) \le \delta(x, S)$. Under this assumption, c can be written in the form (1) as follows. Let $Y := \mathcal{A}$, $u_2(x, S) = \mathbf{1}_{c(S)}(x)$, and

$$u_1(x, S) = \begin{cases} \max u(S) & x \in c(S) \\ u(x) & \text{otherwise} \end{cases}$$

3.2.1 Strictly Competitive Representation

There are no constraints imposed on games in Nash representation. One can consider various special cases of Nash representation corresponding to different classes of games. Here we deal with the case in which the payoffs of the decision maker and the phantom are in complete conflict in the following sense.

Definition. A game (X, Y, u_1, u_2) is strictly competitive if for each (x, y) and (x', y') in $X \times Y$, $u_1(x, y) \ge u_1(x', y')$ iff $u_2(x, y) \le u_2(x', y')$.

In short, a strictly competitive game is an ordinal zero-sum game. As the following theorem shows, every choice correspondence that can be expressed as Nash representation with strictly competitive game is rationalizable in the sense of preference maximization.

Theorem 5. If a choice correspondence c admits a representation in (1) where the corresponding game is strictly competitive, then it is rationalizable, i.e., there is a complete and transitive relation $\succsim$ such that $c(S) = \max(S, \succsim)$ for all $S \in \mathcal{A}$.

Proof. Let (X, Y, u_1, u_2) be a strictly competitive game and (1) holds for each $S \in \mathcal{A}$. We show that c satisfies the congruence axiom. Let $x_1 \in c(S_1)$, $x_2 \in S_1 \cap c(S_2)$, ..., $x_k \in S_{k-1} \cap c(S_k)$, $x_k \in c(T)$, and $x_1 \in T$. For each $i = 1, \dots, k$, let $y_i \in Y$ be such that $(x_i, y_i) \in NE(S_i, Y, u_1, u_2)$ and $y \in Y$ be such that $(x_k, y) \in NE(T, Y, u_1, u_2)$. It is enough to show that $(x_1, z) \in NE(T, Y, u_1, u_2)$. Then, $u_1(x_i, y_i) \ge u_1(x, y_i)$ and $u_2(x_i, y_i) \ge u_2(x_i, y)$ for all $(x, y) \in S_i \times Y$. By strict competitiveness, therefore, we have $u_1(x_i, y_i) \ge u_1(x_{i+1}, y_i) \ge u_1(x_{i+1}, y_{i+1})$ for $i = 1, \dots, k-1$. Thus, $u_1(x_1, y_1) \ge u_1(x_k, y_k)$. On the other hand, strict competitiveness implies $u_1(x_k, y_k) \ge u_1(x_k, z) \ge u_1(x_1, z) \ge u_1(x_1, y_1)$. Therefore, $u_1(x_1, z) = u_1(x_k, z) = u_1(x_1, y_1)$ and by strict competitiveness, $u_2(x_1, z) = u_2(x_k, z) = u_2(x_1, y_1)$. This means x_1 is a best response to z and z is a best response to x_1 in (T, Y, u_1, u_2). Hence, $(x_1, z) \in NE(T, Y, u_1, u_2)$ as desired. □

3.3 Stric Nash Representation

Next, we consider a version of Nash representation in which strict Nash equilibrium is used rather than Nash equilibrium. In this model, the decision maker, taking the choice set as the set of strategies, plays a game against a phantom as in Nash representation. His or her choices are strict Nash equilibria of the game. Specifically, we study the following representation for choice correspondence c. There is a nonempty set Y and two payoff functions u_1 and u_2 defined on $X \times Y$ such that for each choice problem S in $\mathcal{A}$,

$$c(S) = \{x \in S : (x, y) \in SNE(S, Y, u_1, u_2) \text{ for some } y \in Y\}. \tag{2}$$

It is easy to see that this representation implies α-axiom. In fact, it satisfies the following slightly stronger condition .

WWARNI: If $x \in S$ and for every $z \in S$ there is T such that $\{x, z\} \subset T$ and $x \in c(T)$ then $x \in c(S)$.

This condition is employed by Ribeiro and Riella (2017) as part of conditions that characterize the maximalization of a preorder. It is a weaker version of the Weak Axiom of Revealed Inferiority (See Eliaz and Ok (2006)) and implies α-axiom. The following theorem shows that the representation in (2) is characterized by this condition.

Theorem 6. A choice correspondence c on $\mathcal{A}$ satisfies WWARNI if and only if it admits a representation in (2).

Proof. Suppose $c(S) = \{x \in S : (x, y) \in SNE(S, Y, u_1, u_2)\}$ for all $S \in \mathcal{A}$. Fix $S \in \mathcal{A}$ and $x \in S$ such that for each $z \in S$, there is T_z with $\{x, z\} \subset T_z$ and $x \in c(T_z)$. For each $z \in S \setminus \{x\}$, there is $y_z \in Y$ such that $(x, y_z) \in SNE(T_z, Y, u_1, u_2)$. As (x, y_z) is a strict equilibrium, $y_z = y_{z'}$ for each $z, z' \in S \setminus \{x\}$. Put $y := y_z$ for any $z \in S \setminus \{x\}$. Then, for each $z \in S \setminus \{x\}$, $u_1(x, y) > u_1(z, y)$ as $(x, y) \in SNE(T_z, Y, u_1, u_2)$. Hence, $(x, y) \in SNE(S, Y, u_1, u_2)$ and thus $x \in c(S)$.

To see the other direction, let c be a choice correspondence that satisfies WWRNI. For each $x \in X$, define S_x as follows: $S_x = \{x\}$ if there is no $T \in \mathcal{A}$ with $x \in c(T)$, otherwise

$$S_x = \bigcup \{T \in \mathcal{A} : x \in c(T)\}.$$

Note, WWARNI implies $x \in c(S)$ iff $x \in S \subset S_x$. We use X as the phantom's set of strategies and define $u_1 : X \times X \to \mathbb{R}$ and $u_2 : X \times X \to \mathbb{R}$ as follows:

$$u_1(x, y) = \begin{cases} 2 & x \notin S_y \\ 1 & x = y \\ 0 & x \in S_y \text{ and } x \neq y \end{cases} \qquad u_2(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

We show that $c(S) = \{x \in S : (x, y) \in SNE(S, Y, u_1, u_2)\}$ for all $S \in \mathcal{A}$. Let $x \in c(S)$. Then, $S \subset S_x$ and hence $u_1(x, x) = 1 > 0 = u_1(z, x)$ for all $z \in S \setminus \{x\}$. Similarly, $u_2(x, x) = 1 > 0 = u_2(x, y)$ for all $y \in X \setminus \{x\}$. Hence, $(x, x) \in SNE(S, Y, u_1, u_2)$.

Conversely, let $(x, y) \in SNE(S, Y, u_1, u_2)$. It is clear from the definition of u_2 that $x = y$. Suppose there is no $T \in \mathcal{A}$ with $x \in c(T)$. Then, there is $z \in S \setminus \{x\}$

and $u_1(z,x) = 2 > 1 = u_1(x,x)$. This is a contradiction. Hence, $x \in c(T)$ for some $T \in \mathcal{A}$. If there is $z \in S \setminus S_x$, then $u_1(z,x) = 2 > 1 = u_1(x,x)$. This is a contradiction. Hence, $S \subset S_x$ and thus $x \in c(S)$ by WWARNI. □

When X is finite, and $\mathcal{A}$ contains all choice problems, WWARNI can be decomposed into α-axiom and the following condition.

Expansion: If $x \in c(S)$ and $x \in c(T)$ then $x \in c(S \cup T)$.

Corollary 7. Let X be a nonempty finite set. A choice correspondence on $2^X \setminus \{\emptyset\}$ satisfies α-axiom and Expansion if and only if it admits a representation in (2).

Proof. By the previous Theorem, it is enough to show that WWARNI is equivalent to α-axiom and Expansion. Clearly, WWARNI implies α-axiom and Expansion. To see the converse, suppose c satisfies α-axiom and Expansion. Let S be a choice problem such that $x \in S$, and for every $y \in S$, there is T_y with $\{x,y\} \subset T_y$ and $x \in c(T_y)$. Let $T := \bigcup_{y \in S} T_y$ and note that $S \subset T$. Since X is finite, $x \in c(T)$ by Expansion. By α-axiom, $x \in c(S)$. □

Here are some examples of choice models that satisfy WWARNI.

Example 7 (Maximization of an acyclic relation)**.** Sen (1971) showed that, when X is finite, a choice correspondence c on $2^X \setminus \{\emptyset\}$ satisfies α-axiom and Expansion if and only if there is a complete and acyclic relation $\succsim$ on X such that $c(S) = \max(S, \succsim)$ for each S. The above corollary states that such a choice correspondence can be expressed as a game in the sense of (2).

Example 8 (Maximalization of an preorder)**.** Let $\succsim$ be a preorder on a finite set X and c be a choice correspondence defined by $c(S) = \mathrm{MAX}(S, \succsim)$. It is easy to show that c satisfies WWARNI.

Example 9 (Maximization of an interval order)**.** A special case of threshold models (See Example 6) in which the threshold function δ depends only on x corresponds to the maximization of an interval order. The choice is determined by

$$c(S) = \{x \in S : \max u(S) \leq u(x) + \delta(x)\}.$$

This model satisfies WWARNI.

4 Application

4.1 Weak WARP

The following weaker version of the weak axiom of revealed preference for choice function has been studied recently by Manzini and Mariotti (2007), Manzini and Mariotti (2012), and Cherepanov, Feddersen, and Sandroni (2013).

Weak WARP: If $x \neq y$, $\{x, y\} \subset S \subset T$ and $c(\{x, y\}) = c(T) = x$ then $c(S) \neq y$.

We provide an alternative representation for models consistent with Weak WARP. The following theorem shows that all choice functions that satisfy Weak WARP can be expressed as a selection from equilibrium strategies.

Theorem 8. Let X be a finite set and c a choice function on $2^X \setminus \{\emptyset\}$. Then, c satisfies Weak WARP if and only if there is a nonempty set Y and functions $u_1 : X \times Y \to \mathbb{R}$, $u_2 : X \times Y \to \mathbb{R}$, and an irreflexive binary relation $\succ$ on X such that

$$c(S) = \max\left(\{x \in S : (x, z) \in NE\left(S, Y, u_1, u_2\right) \text{ for some } z \in Y\}, \succ\right)$$

for each $S \in 2^X \setminus \{\emptyset\}$.

Proof. First, assume a choice function c admits a representation in the statement. Let $x \neq y$, $\{x, y\} \subset S \subset T$, and $c(\{x, y\}) = c(T) = x$. If $(y, z) \notin NE\left(S, Y, u_1, u_2\right)$ for all $z \in Y$ then, $c(S) \neq y$. If $(y, z) \in NE\left(S, Y, u_1, u_2\right)$ for some $z \in Y$, then $x \succ y$ as $c(\{x, y\}) = x$. Since $c(T) = x$, $(x, z') \in NE\left(T, Y, u_1, u_2\right)$ and hence $(x, z') \in NE\left(S, Y, u_1, u_2\right)$ for some $z' \in Y$. Therefore, $c(S) \neq y$.

Conversely, let c be a choice function that satisfies Weak WARP. Define a correspondence α by $\alpha(S) = \{y \in S : y = c(T) \text{ for some } T \supset S\}$ for each $S \in 2^X \setminus \{\emptyset\}$. Define $\succ$ by $x \succ y$ if $x \neq y$ and there are $S \subset T$ such that $\{x, y\} \subset S$, $x = c(S)$ and $y = c(T)$. Then, $\succ$ is asymmetric by Weak WARP and $c(S) = \max\left(\alpha(S), \succ\right)$. Notice that α satisfies α-axiom. By Theorem 3, there is a nonempty set Y and functions $u_1 : X \times Y \to \mathbb{R}$ and $u_2 : X \times Y \to \mathbb{R}$ such that

$$\alpha(S) = \{x \in S : (x, y) \in NE(S, Y, u_1, u_2) \text{ for some } y \in Y\}.$$

Thus, a desired representation is obtained. □

4.2 Rational Shortlist Method

Manzini and Mariotti (2007) proposed the Rational Shortlist Method, a model for choice function, and showed that it is characterized by Weak WARP and Expansion. Here, we propose an alternative model characterized by the same axioms. In this model, the decision maker makes a selection from equilibrium strategies in symmetric equilibria.

Theorem 9. Let X be a finite set and c a choice function on $2^X \setminus \{\emptyset\}$. Then, c satisfies Weak WARP and Expansion if and only if there is a nonempty set Y and functions $u_1 : X \times Y \to \mathbb{R}$, $u_2 : X \times Y \to \mathbb{R}$, and an irreflexive binary relation $\succ$ on X such that

$$c(S) = \max\left(\{x : (x, y) \in SNE\left(S, Y, u_1, u_2\right) \text{ for some } y \in Y\}, \succ\right)$$

for each $S \in 2^X \setminus \{\emptyset\}$.

Proof. First, assume c admits a representation in the statement. Define a correspondence α by $\alpha(S) = \{x : (x, y) \in SNE\,(S, Y, u_1, u_2)$ for some $y \in Y\}$. By Theorem 6, α satisfies α-axiom and Expansion. Hence, c satisfies Weak WARP. To see that c satisfies Expansion, let $x = c(S) = c(T)$. Then, as α satisfies Expansion, $x \in \alpha(S \cup T)$. On the other hand, for each $y \in \alpha(S \cup T)$, $y \in \alpha(S)$ or $y \in \alpha(T)$. Without loss of generality, let $y \in \alpha(S)$. Then, $x \succ y$ as $x = c(S)$. Hence, $x = c(S \cup T)$.

Conversely, let c be a choice correspondence that satisfies Weak WARP and Expansion. Define $\alpha(S) = \{y \in S : y = c(T)$ for some $T \supset S\}$ for each $S \in 2^X \setminus \{\emptyset\}$. Then, α satisfies α-axiom. Define $\succ$ by $x \succ y$ if $x \neq y$ and there are $S \subset T$ such that $\{x, y\} \subset S$, $x = c(S)$ and $y = c(T)$. Then, $\succ$ is asymmetric by Weak WARP and $c(S) = \max(\alpha(S), \succ)$. Notice that α satisfies Expansion. Indeed, if $x \in \alpha(S)$ and $x \in \alpha(T)$ then, there is $S' \supset S$ such that $x = c(S')$ and $T' \supset T$ such that $x = c(T')$. As c satisfies Expansion, $x = c(S' \cup T')$ and thus $x \in \alpha(S \cup T)$. Therefore, by Theorem 6, there is a nonempty set Y and functions $u_1 : X \times Y \to \mathbb{R}$, $u_2 : X \times Y \to \mathbb{R}$ such that $\alpha(S) = \{x : (x, y) \in SNE\,(S, Y, u_1, u_2)$ for some $y \in Y\}$. Hence, the desired representation is obtained. $\square$

5 Related Literature

This paper is related not only to the theory of boundedly rational choice but also to the emerging literature on the revealed preference approach to game theory. The primary focus of this literature is on the identification of testable restrictions of game-theoretic solution concepts. The literature can be classified into two parts depending on whether or not a game form is given.

Sprumont (2000) and Galambos (2004) studied the rationalization of choice correspondence in terms of Nash equilibria when there are given game forms and the observed choices are strategy profiles. In the same setup, Lee (2012) studied the rationalization by Nash equilibria of a zero-sum game. Ray and Zhou (2001) studied similar problems when extensive game forms with complete information are given. In contrast, we do not assume game forms as given. In addition, we assume that the choices of other players are unobservable.

The other approach takes choice correspondence as primitive and does not assume game forms to be observed. For example, Bossert and Sprumont (2013) showed that every choice function is rationalizable as an outcome of backward induction.[2] Rehbeck (2014) and Xiong (2014) extended the result to choice correspondences. Li and Tang (2017) showed a similar result for random choice rules. In these papers, each alternative corresponds to an outcome of a game, and the chosen alternatives are interpreted as equilibrium outcomes. In contrast, this paper considers the rationalization of choice correspondence in terms of strategic games. We interpret each alternative as a decision maker's strategy in an unobserved strategic environment, and chosen alternatives are understood as equilibrium strategies.

[2]See also Xu and Zhou (2007), and Horan (2012).

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