Comonotonic Allocation with Heterogeneous Agents and Non-Linear-Discounting Utility

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Abstract

In this study, we employ non-linear discounting (*NLD*) utility in a multi-consumer model and demonstrate that saving allocation in the stationary equilibrium is comonotonic to consumers ' attitudes toward future consumption. Time-additively separable (*TAS*) utility with a fixed discount factor is widely employed in dynamic macroeconomics because of its tractability. However, this yields unrealistic results in a multi-consumer model. The number of savers in the economy is only one, while the others live on a day-to-day basis. On the contrary, NLD utility allows consumers to vary their discounting of future utility depending on their consumption plan. Due to this property, a comonotonic saving allocation can occur.

Keywords: non-linear discounting utility; Comonotonic allocation

1 Introduction

Many macroeconomic models assume a representative consumer. The sum of consumption is seen as the result of the "representative agent." Some critics argue that the behavior of the representative consumer cannot be constructed from the results of each consumer's utility maximization problem. Even if the aggregability problem is solved, the multi-consumer model is significant in discussing the equity of allocation or the effect of redistributive policy.

Many scholars have discussed wealth allocation. Ramsey (1928) conjected that society will be divided into two classes, impatient laborers and patient capitalists. Becker (1980) and Mitra and Sorger (2013) formally proved this. They employed time-additively separable (*TAS*) utility $U(c_0, c_1 \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t)$ where $c_0, c_1 \ldots$ is a consumption stream, and the constant β is

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a discount factor. The discount factor of TAS utility remains constant and independent of the consumption plan. A consumer with a higher discount factor evaluates the utility from future consumption more than other consumers do. This means that they are more patient. Moreover, they maintain a high tendency to save even if their capital accumulation deepens. Hence, they can offer a lower interest rate, while the others are excluded from the capital market.

non-linear discounting (*NLD*) utility is described with $W(c, u) = v(c) + \delta(u)$ where c is immediate utility, and u is future utility. This is a subclass of recursive utility suggested by Koopmans (1960). Letting $\delta(u) = \beta u$, TAS utility is included in the class of NLD utility, although we limit "NLD" utility to that of non-TAS to avoid confusion of diction. A consumer's attitude toward the future is the response of lifetime utility to the variance of future utility. We define the *time perspective* as

$$\delta'(u) = \lim_{u' \to u} \frac{W(c, u') - W(c, u)}{u' - u}.$$

This discounts future utility as well as the discount factor of the TAS utility. A consumer who is promised future welfare spares a larger portion of the additional income to immediate consumption. Time perspective is typically considered to decrease in future utility u. This property enables each consumer to adjust their consumption plan to satisfy the necessary condition of utility maximization.

Some studies employing non-TAS utility, such as Lucas and Stokey (1984) and Epstein and Hynes (1983), demonstrate that saving is owned by multiple agents. The former submits minimal knowledge on who saves and consumes more money or goods. The latter specifies the utility function to the Epstein-Hynes type, which describes the time perspective as dependent on past consumption. In this study, we show that comonotonic saving allocation can be viewed as the result of a consumer's forward-looking behavior.

The rest of this paper is organized as follows. Section 2 introduces the model and several assumptions. Section 3 examines the qualitative properties of stationary competitive equilibrium. A unique stationary competitive equilibrium exists. The comonotonic relationship between saving and time preferences always holds in the stationary competitive equilibrium. In Section 4, we show an example of stationary allocation by specifying the utility function to the Koopmans-Diamond-Williamson (KDW) utility, which is one of the NLD utilities. The stationary consumption level of each consumer is determined by solving the Euler equation.

2 Model and Assumptions

We analyze a one-sector model and assume that time is discrete with index $t = 0, 1, 2 \ldots.$ There are H consumers with indexes $h = 1, 2 \ldots H$ and only one producer. This economy has capital and products markets, in which both of are assumed to be competitive.

Let $\mathbf{c}^h = (c_0^h, c_1^h, \ldots) \in \mathbb{R}_+^\infty$ and $\mathbf{k}^h = (k_0^h, k_1^h \ldots) \in \mathbb{R}_+^\infty$ be the hth consumer's consumption and saving plans, respectively. We denote the consumption and saving paths from period t by ${}_t\mathbf{c} = (c_t, c_{t+1} \ldots)$ and ${}_t\mathbf{k} = (k_t, k_{t+1} \ldots)$, respectively. Let $\bar{\mathbf{c}} = (\bar{c}, \bar{c} \ldots)$ and $\bar{\mathbf{k}} = (\bar{k}, \bar{k} \ldots)$ be constant paths.

The producer lends capital $K_t \in \mathbb{R}_{++}$ to consumers with an interest rate $r_t \in \mathbb{R}_{++}$ at the beginning of period t. To analyze consumption and saving distributions, we exclude zero input and output. For simplicity, the producer is assumed to have linear production function $F(K) = AK$ $(A > 1)$. The interest rate is always $r = A - 1$; otherwise, the solution to the profit maximization problem does not exist. If the interest rate is $r = A - 1$, the producer's profit is always zero, and the capital is never accumulated in the economy. The producer's behavior is described by the solution to the static profit maximization problem (PMP)

$$\max_{K_t \in \mathbb{R}_{++}} AK_t - (1 + r_r)K_t$$
$$r_t \in \mathbb{R}_{++} : \text{given}$$

Consumers are given an initial endowment k_0^h in period 0. They lend saving k_t^h to the producer and obtain interest at rate r_t. We define set $X_{t=1}^\infty[0, A^t k_0]$ as

$$X_{t=1}^\infty[0, A^t k_0] = \{(c_0, c_1 \ldots) \in \mathbb{R}_+^\infty | (^\forall t = 1, 2 \ldots)\ 0 \leq c_t \leq A^t k_0\}$$

Any feasible consumption path starting from k_0^h is included in $X_{t=1}^\infty[0, A^t k_0^h]$. A lifetime utility function U^h defined over $\mathbb{R}_+^\infty$ is represented by

$$U^h(\mathbf{c}) = v^h(c_0) + \delta^h[U^h({}_1\mathbf{c})]$$

where $v^h; \mathbb{R}_+ \to \mathbb{R}$ and $\delta^h : X_{t=1}^{\infty}[0, A^t k_0^h] \to (\mathbb{R} \cup \{+\infty\})$.

Functions v^h, δ^h, and U^h are assumed to satisfy (U1)-(U7).

(U1) Strict Monotonicity

The functions v^h and δ^h are strictly increasing.

(U2) Existence and Negativeness of Second-Order Derivative

Functions v^h and δ^h are twice continuously differentiable over their domains. It holds that $(v^h)''(c) < 0$ and $(\delta^h)''(u) < 0$ for any $c \in \mathbb{R}_+$ and $u \in U^h\left[X_{t=1}^{\infty}[0, A^t k_0^h]\right]$.

Condition $(\delta^h)''(u) < 0$ implies that a consumer who is promised high future utility will increase immediate consumption rather than saving after obtaining additional one unit of income.

(U3) Real Valued

For any $u \in U^h\left[X_{t=1}^{\infty}[0, A^t k_0^h]\right]$, $\delta^h(u)$ takes a real value.

(U4) Postponing Damages Lifetime Utility

For any $u \in U^h\left[X_{t=1}^{\infty}[0, A^t k_0^h]\right]$ such that $u > U^h(\mathbf{0})$, it holds that $u > v^h(0) + \delta^h(u)$.

This implies that if a consumer postpones the beginning of their consumption plans and consumes nothing today, their lifetime utility decreases.

(U5) Compensation for Postponing

For any $u \in U^h\left[X_{t=1}^{\infty}[0, A^t k_0^h]\right]$, there exists $c' \in \mathbb{R}_+$ such that $u \leq v^h(c') + \delta^h(u)$.

The function v^h is strictly increasing; hence, such c' is unique. This implies that even if a consumer postpones the beginning of their consumption plan, sufficient consumption today can compensate for the decrease in lifetime utility caused by postponing.

(U6) Asymptotic Boundedness

It holds that

$$e^* = \overline{\lim_{c \to +\infty}} \frac{\ln\{v^h(c) + \delta^h(0)\}}{\ln c} \text{ is finite}$$

$$\bar{d} = \bar{\sup_{(c,u)}} \sup_{u' \in (u,+\infty)} \frac{W(c,u') - W(c,u)}{u' - u} < 1$$

$$(d^*)A^{e^*} < 1$$

$$\text{where } d^* = \overline{\lim_{u \to +\infty}} \sup_{c} \sup_{u' \in (u,+\infty)} \frac{W(c,u') - W(c,u)}{u' - u}$$

This condition and linear production function compose a sufficient condition of *Lower Convergence* suggested by Streufert (1990). Lower Convergence can be described as follows:

For any $k_0 \in \mathbb{R}_+$ and $\mathbf{c} \in X_{t=0}^{\infty}[0, A^t k_0]$, it holds that

$$\lim_{t \to \infty} U^h(\mathbf{c}_t, \mathbf{0}) = U^h(\mathbf{c}).$$

This implies that an agent evaluates their distant future utilities much less.

(U7) Range of Time Perspective

There exists some $u' \in U^h \left[X_{t=1}^{\infty}[0, A^t k_0^h]\right]$ such that

$$(\delta^h)'(u') < \frac{1}{A}$$

It holds that

$$\frac{1}{A} < (\delta^h)'[U^h(\mathbf{0})] < 1$$

This implies that there exists an attainable future utility level u' that satisfies the Euler equation $(\delta^h)'(u')A = 1$.

Lemma 2.1. *Given that (U2) is satisfied, the utility function $U^h(\mathbf{c})$ is partially differentiable with respect to any c_t, and the partial derivative is*

$$D_{c_t}U(\mathbf{c}) = \prod_{s=0}^{t-1} (\delta'[U({}_{s+1}\mathbf{c})]) \cdot v'(c_t)$$

Proof)
This is derived by repeatedly applying the chain rule.

□

Consumers are assumed to perfectly foresee future interest rates. Thus, the behavior of each consumer is described by the solution to the dynamic utility maximization problem (UMP)

$$\begin{aligned}
\max_{\mathbf{c}^h,\mathbf{k}^h\in\mathbb{R}^\infty_+} \; & U^h(\mathbf{c}^h) \\
\text{sub.to } & ({}^\forall t = 0,1\ldots)\ c^h_t + k^h_{t+1} \leq (1+r_t)k^h_t \\
& ({}^\forall t = 0,1,\ldots)\ r_t \in \mathbb{R}_{++} : \text{ given} \\
& k^h_0 \in \mathbb{R}_+ : \text{ given}
\end{aligned}$$

The inner solution to the UMP is characterized by the Euler equation

$$\prod_{s=0}^{t} (\delta'[U({}_{s+1}\mathbf{c})]) \cdot v'(c_{t+1})(1+r_t) = \prod_{s=0}^{t-1} (\delta'[U({}_{s+1}\mathbf{c})]) \cdot v'(c_t)$$

In particular, in the stationary state, the equation is reduced to

$$\delta'[U(\bar{\mathbf{c}})] = \frac{1}{1+\bar{r}}.$$

3 Stationary Competitive Equilibrium

If the combination $< (\mathbf{c}^h)^H_{h=1}, (\mathbf{k}^h)^H_{h=1}, \mathbf{K}, \mathbf{r} >$ satisfies (1)–(5), then it is a *competitive equilibrium*.

(1) Non Negativeness of Variables
In all periods t and for all consumers h, the h^{th} consumer's consumption c^h_t and saving k^h_t are non-negative. The capital input K_t and interest rate r_t are positive for all $t = 0, 1, \ldots$.

(2) Consumers' Optimality
For all $h = 1, 2 \ldots H$, $(\mathbf{c}^h, \mathbf{k}^h)$ is the solution for the h-th consumer's *UMP*.

(3) Producer's Optimality

For all $t = 0, 1 \ldots$, K_t is the solution for the *PMP*.

(4) Balance of the Capital Market
For all $t = 0, 1 \ldots$, it holds that $\sum_{h=1}^{H} k_t^h = K_t$.

(5) Balance of Products Market
For all $t = 0, 1 \ldots$, it holds that $\sum_{h=1}^{H} c_t^h + \sum_{h=1}^{H} k_{t+1}^h = F(K_t)$.

If $< (\mathbf{c}^h)_{h=1}^{H}, (\mathbf{k}^h)_{h=1}^{H}, \mathbf{K}, \mathbf{r} >$ satisfies (6), in addition to (1)–(5), this is called a *stationary competitive equilibrium.*

(6) Constant Level of Consumption, Capital, and Input
The paths $\mathbf{r}$, $(\mathbf{c}^h, \mathbf{k}^h)_{h=1}^{H}$, and $\mathbf{K}$ satisfy constancy.

"Stationarity" requires that all variables maintain the same level over time.

Due to the concavity of the production and utility functions, the inner stationary competitive equilibrium is likely characterized by the following conditions:

(I) Interest Rate: $\bar{r} = A - 1$

(II) Euler Equation: $({}^{\forall}h = 1, 2, \ldots H)$, $(\delta^h)'[U^h(\bar{\mathbf{c}})] = \dfrac{1}{A}$

(III) Budget Constraints

For all $h = 1, 2, \ldots H$, it holds that

$$\bar{k}^h = k_0^h$$
$$\bar{c}^h + \bar{k}^h = (1 + \bar{r})\bar{k}^h$$

(IV) Balance of the Capital Market: $\sum_{h=1}^{H} \bar{k}^h = \bar{K}$

Note that the balance of the products market is automatically achieved due to Walras ' Law.

Lemma 3.1. *Assume that the production function is $F(K) = AK$, and each consumer has a utility that satisfies $(U1)-(U7)$. Then, $< H, \mathbf{r}, (\bar{\mathbf{c}}^h, \bar{\mathbf{k}}^h)_{h=1}^{H}, \bar{\mathbf{K}} >$ satisfying (I)–(IV) is the unique inner stationary competitive equilibrium.*

This theorem asserts the uniqueness of "inner" and "stationary" competitive equilibrium. Hence, corner or nonstationary equilibria may also exist.

Proof)

First, we demonstrate the existence of the inner stationary competitive equilibrium. Choose $h \in \{1, 2, \ldots H\}$ arbitrarily. We demonstrate that there exists a unique $\bar{c}^h \in \mathbb{R}_{++}$ satisfying (II). (U2) ensures the continuity of the derivative δ'. According to (U7) and the intermediate value theorem, there exists $u^{h'} \in \mathbb{R}$ such that $(\delta^h)'(u^{h'}) = 1/A \ldots (a)$. From (U4), we have $v^h(0) < u^{h'} - \delta^h(u^{h'})$. From (U5), there exists $c' \in \mathbb{R}_+$ such that $u^{h'} - \delta^h(u^{h'}) \leq v^h(c')$. Thus, $v^h(0) < u^{h'} - \delta^h(u^{h'}) \leq v^h(c')$. According to the intermediate value theorem, there must be $\bar{c}^h \in \mathbb{R}_{++}$ such that $v^h(\bar{c}^h) = u^{h'} - \delta^h(u^{h'}) > v^h(0)$. Owing to the concavity of δ^h, we obtain

$$
\begin{aligned}
u^{h'} - U^h(\bar{\mathbf{c}}^h) &= \{v^h(\bar{c}^h) + \delta^h(u^{h'})\} - \{v^h(\bar{c}^h) + \delta^h[U^h(\bar{\mathbf{c}}^h)]\} \\
&\leq (\delta^h)'[U^h(\bar{\mathbf{c}}^h)]\{u^{h'} - U^h(\bar{\mathbf{c}}^h)\}
\end{aligned}
$$

By repeating this inequality, we obtain

$$
\begin{aligned}
u^{h'} - U^h(\bar{\mathbf{c}}^h) \leq & (\delta^h)'[U^h(\bar{\mathbf{c}}^h)]\{u^{h'} - U^h(\bar{\mathbf{c}}^h)\} \\
\leq & \{(\delta^h)'[U^h(\bar{\mathbf{c}}^h)]\}^2\{u^{h'} - U^h(\bar{\mathbf{c}}^h)\} \\
& \vdots \\
\leq & \{(\delta^h)'[U^h(\bar{\mathbf{c}}^h)]\}^t\{u^{h'} - U^h(\bar{\mathbf{c}}^h)\}
\end{aligned}
$$

for any $t = 3, 4 \ldots,$. Taking the limits of both sides with respect to t, then it holds that

$$
u^{h'} - U^h(\bar{\mathbf{c}}^h) \leq \lim_{t\to\infty} \{(\delta^h)'[U^h(\bar{\mathbf{c}}^h)]\}^t\{u^{h'} - U^h(\bar{\mathbf{c}}^h)\} = 0
$$

Due to (a), $(\delta^h)'(u^{h'})$ is less than 1. We obtain

$$
U^h(\bar{\mathbf{c}}^h) - u^{h'} \leq \lim_{t\to\infty} \{(\delta^h)'(u^{h'})\}^t\{U^h(\bar{\mathbf{c}}^h) - u^{h'}\} = 0
$$

by reversing $u^{h'}$ and $U^h(\bar{\mathbf{c}}^h)$. Thus, we obtain $u^{h'} = U^h(\bar{\mathbf{c}}^h)$ and (II)

$$
(\delta^h)'[U^h(\bar{\mathbf{c}})] = \frac{1}{A}
$$

Set the interest rate to satisfy the equation (I). The producer's optimality automatically holds under condition (I). Therefore we can take each consumer's savings to satisfy the equation (III). Capital input can also be set to satisfy the balance of the capital market in the equation (IV).

The uniqueness of the stationary competitive equilibrium is derived directly from the condition $(\delta^h)'' < 0$.

□

Theorem 3.1. *Assume that there uniquely exists an inner stationary competitive equilibrium* $< (\bar{\mathbf{c}}^h)_{h=1}^H, (\bar{\mathbf{k}}^h)_{h=1}^H, \bar{\mathbf{K}}, \bar{\mathbf{r}} >$. *If it holds that*

$$(\delta^1)'[U^1(\tilde{\mathbf{c}})] < (\delta^2)'[U^2(\tilde{\mathbf{c}})] < \ldots < (\delta^H)'[U^H(\tilde{\mathbf{c}})] \ldots (1)$$

for any $\tilde{\mathbf{c}} = (\tilde{c}, \tilde{c}, \ldots) \in \mathbb{R}_{++}^{\infty}$ *satisfying* $({}^{\forall}h = 1, 2, \ldots H)$ $U^h(\tilde{\mathbf{c}}) < +\infty$, *then the allocation is comonotonic to the order in (1), that is,*

$$0 < \bar{c}^1 < \bar{c}^2 < \ldots < \bar{c}^H$$
$$0 < \bar{k}^1 < \bar{k}^2 < \ldots < \bar{k}^H.$$

Inequality (1) implies that if each consumer takes the same consumption plan $\tilde{\mathbf{c}}$, a consumer with a larger index exhibits a larger time perspective.

Proof)

Choose two consumers with indexes $i, j (i < j)$ arbitrarily. In Lemma 3.1, we show that $\bar{\mathbf{c}}^i$ and $\bar{\mathbf{c}}^j$ must satisfy

$$(\delta^i)'[U^i(\bar{\mathbf{c}}^i)] = (\delta^j)'[U^j(\bar{\mathbf{c}}^j)] = \frac{1}{A} \ldots (a)$$

If they take the same consumption path $\mathbf{c}^i$, (1) implies that

$$(\delta^i)'[U^i(\bar{\mathbf{c}}^i)] < (\delta^j)'[U^j(\bar{\mathbf{c}}^i)] \ldots (b)$$

(a) and (b) yield

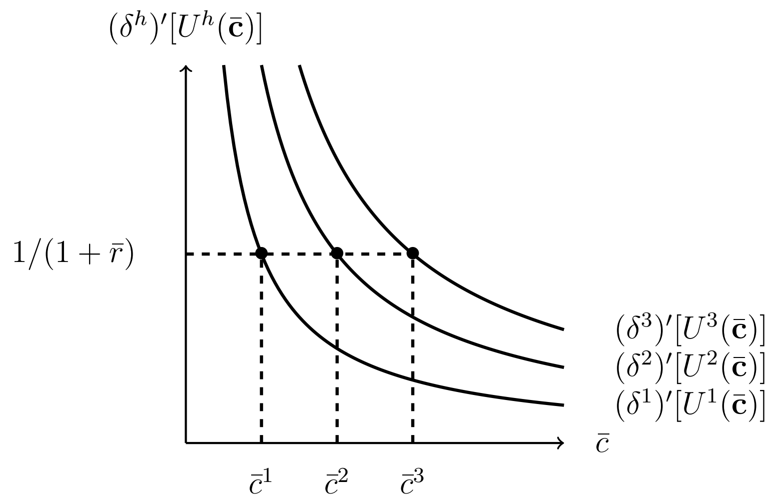
$$(\delta^j)'[U^j(\bar{\mathbf{c}}^j)] < (\delta^j)'[U^j(\bar{\mathbf{c}}^i)]$$

From $\delta'' < 0$, it holds that $U^j(\bar{\mathbf{c}}^i) < U^j(\bar{\mathbf{c}}^j)$. From (U1), $\bar{c}^i < \bar{c}^j$. The equation (c) in Lemma 3.1 implies that it also holds that $\bar{k}^i < \bar{k}^j$.

□

The assertion of this theorem in a three-consumer economy is summarized in Figure 1. The downward slope of each δ^h curve originates from $(\delta^h)'' < 0$. Moreover, δ^3 is the highest level owing to the order in (1).

Figure 1: Consumer's time perspective $D_u W(c,u)$ in a three-consumers economy



4 An Example of Stationary Competitive Equilibrium

We assume that each consumer has KDW utility

$$U^h(\mathbf{c}^h) = (c_0^h)^a + \lim_{t\to\infty} d_h \log[1 + (c_1^h)^a + d_h \log\{1 + (c_2^h)^a + \ldots + d_h \log(1 + (c_t^h)^a)\}]$$
$$\text{where } a \in (0,1), d_h \in (1/(A - \log A), 1).$$

Parameter d_h represents the intensity of a consumer's time preference. A consumer with a larger d_h evaluates future utility more. A range of d_h is required to guarantee an inner solution. The second term of the right-hand side is monotonically increasing in t; therefore, a limit exists if ∞ is permitted for the limit value. The KDW utility has an aggregator function $W^h(c,u) = c^a + d_h \log(1+u)$ and a time perspective $D_u W^h(c,u) = d_h/(1+u)$. Thus, it satisfies (U1)-(U7).

Assuming that $d_1 < d_2 < \ldots < d_H \ldots$(b), The equation (1) in Theorem 3.1 holds.

The Euler equation at the stationary competitive equilibrium is

$$\frac{d_h}{1+U^h(\bar{\mathbf{c}}^h)}(1+\bar{r}) = 1$$

The interest rate is $r = A - 1$; hence, we have $U^h(\bar{\mathbf{c}}^h) + 1 = d_h A$. We substitute it into $U^h(\bar{\mathbf{c}}^h) = (\bar{c}^h)^a + d_h \log\{U^h(\bar{\mathbf{c}}^h) + 1\}$, and then, we obtain

$$\begin{aligned} d_h A - 1 &= (\bar{c}^h)^a + d_h \log(d_h A) \\ \Rightarrow \bar{c}^h &= \{d_h A - 1 - d_h \log(d_h A)\}^{1/a} \ldots (c) \end{aligned}$$

$d_h \in (0,1)$ implies that $d_h \log(d_h A) < \log A$, (d) yields

$$d_h A - 1 - d_h \log(d_h A) > d_h A - 1 - \log A \geq 0$$

Hence, each consumer ' s stationary consumption level is strictly positive.

Next, we check for comonotonicity. By differentiating both sides of (c) with respect to d_h, it follows that

$$D_{d_h}\bar{c}^h = \frac{1}{a}\{d_h A - 1 - d_h \log(d_h A)\}^{1/a-1}\{A - 1 - \log(d_h A)\}$$

(d) and $d_h \in (0,1)$ result in $A-1-\log(d_h A) > 0$. Therefore, the derivative $D_{d_h}\bar{c}^h$ is positive. This implies that consumers with higher d_h enjoy a larger consumption *that is*

$$0 < \bar{c}^1 < \bar{c}^2 \ldots < \bar{c}^H$$

From (III), we obtain $\bar{k}^h = \bar{c}^h/\bar{r}$. Thus, the order of saving follows that of consumption, *that is,*

$$0 < \bar{k}^1 < \bar{k}^2 < \ldots < \bar{k}^H$$

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