Rational Exuberance and Bubbles[*]

Yu Awaya[†] and Vijay Krishna[‡]

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Abstract

We study a model of dynamic adverse selection in which a large group of sellers sell an asset of uncertain quality to a larger group of buyers. The quality is known to the sellers but unknown to the buyers. There is, however, the possibility that if the asset is of low quality, this will be revealed via public news at a random time. We show that there is a unique equilibrium satisfying forward induction. In this equilibrium a bubble develops. Even a worthless asset is traded at rapidly increasing prices. This is because in the absence of bad news, buyers become more and more optimistic—they exhibit rational exuberance.

1 Introduction

In June 2020, the Nikola Motor Company—a start up aimed at manufacturing trucks powered by alternative fuels—went public. Interest in its forthcoming products was intense. By August, Nikola's share price had risen so steeply that its market value exceeded that of Ford Motors. But doubts about the company's products began to emerge soon thereafter. In September 2020, it was reported that a promotional video of a prototype Nikola hydrogen-electric truck driving down a highway was actually filmed while the truck rolled down an incline without any power! Within two days, the price of Nikola's stock fell 36%.[1]

A speculative bubble occurs when the price of an asset exceeds its fundamental value—the discounted sum of future dividends. Typically, this over-pricing—the gap between prices and fundamentals—grows over time as traders become more and more optimistic. Eventually, this *exuberance* must face reality and the bubble may then

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[†]Department of Economics, University of Rochester, E-mail: yuawaya@gmail.com.

[‡]Department of Economics, Penn State University, E-mail: vkrishna@psu.edu.

[1]The saga of Theranos, similar in many respects, is well-known and even the subject of a movie.

collapse. Often, as in the case of Nikola, the collapse is triggered by the arrival of public news that the asset is overvalued. But whether or not there is a bubble can only be assessed *ex post*. Much like the price of Nikola's stock, the stock price of another electric vehicle manufacturer, Tesla, also rose very quickly when it was fledgling manufacturer. Today, of course, Tesla is one of biggest car manufacturers in the US and looking back, one would be hard pressed to describe the high price of its stock in early days as a bubble. The fact that bubbles are difficult to recognize has troubled policymakers.[2]

The increasing optimism that drives bubbles is usually attributed to irrationality on the part of traders. Shiller (2015) coined the term "irrational exuberance" to describe this phenomenon. In this paper, we introduce a model of dynamic adverse selection in which exuberance is *rational*. An asset of uncertain quality is traded over time. If the asset is of low quality, there is the possibility that news, revealing this fact, will arrive. Such news comes stochastically and so in its absence, buyers become more and more optimistic. This rational exuberance leads to a speculative bubble that bursts only when, and if, bad news arrives. The rapid growth in prices is not due to any growth in fundamentals; rather it stems from exuberance of buyers.

In our model, a large group of sellers, each of whom holds one unit of an asset, faces a large group of short-lived buyers. The quality of the asset is uncertain—it may be valuable or worthless. The sellers themselves know the true, *common* quality but this is unknown to the buyers, Buyers value the high quality asset more than sellers do—there are gains from trade—while the low quality asset is worthless for all. Each seller decides when to sell over an infinite horizon but may be hit by a shock that compels her to liquidate her holdings immediately. At any instant, the market price of the asset is competitively determined. Past and current prices may be observed by buyers but quantities are not. Finally, as mentioned above, if the asset is worthless, there is the possibility that this will be revealed to all at some random time, thereby collapsing the bubble.[3]

Our main result is that for all parameter values[4]:

Theorem *There is a unique equilibrium satisfying forward induction.*[5] *A bubble develops—even a worthless asset trades at exponentially increasing prices until bad news arrives. The bubble has a maximum life span that is endogenous.*

In equilibrium, a worthless asset is gradually sold over a time interval $[0, T]$ where $T > 0$ is endogenously determined. These sales are comprised of both planned sales

[2] Bernanke and Gertler (2012) write: "Trying to stabilize asset prices *per se* is problematic for a variety of reasons, not the least of which is that it is nearly impossible to know for sure whether a given change in asset values results from fundamental factors, non-fundamental factors, or both."

[3] The bubble in our model is not "strong" in the sense defined by Allen, Morris and Postlewaite (1993). The fact that the asset is worthless is not mutually known amoug all traders.

[4] See Propostions 3.1 and 4.2 for precise statements.

[5] Specifically, we use the "never-a-weak best response" (NWBR) refinement of Kohlberg and Mertens (1986) or the D1 refinement of Cho and Kreps (1987). In our model, these are equivalent.

and forced sales due to liquidity shocks. A valuable asset is intentionally sold only at time T but again there are always some forced, liquidity driven, sales prior to this. The equilibrium price path is the same regardless of whether the asset is valuable or worthless. Thus, prices do not reveal the asset's quality. As time goes on and no bad news arrives, buyers become increasingly optimistic that the asset is, in fact, valuable. Thus, they are willing to pay more for the asset and this causes the market price to rise. Buyers' exuberance is completely rational.

The key to the equilibrium is that while all sellers discount future profits at a common rate, say r, when the asset is worthless sellers face the additional risk that bad news will arrive. If the Poisson arrival rate of such news is λ, then sellers effectively discount future profits at the risk-adjusted rate $\lambda + r$. Thus, sellers are less patient when the asset is worthless than when it is valuable. Put another way, when the asset is valuable, sellers are willing to wait longer to sell than when it is not.

In equilibrium, sellers "ride the bubble" unless forced to bail out because of liquidity needs. When the asset is valuable, all sellers wait until T as there is no risk that prices will collapse earlier because of bad news. When the asset is worthless, however, the incentive to wait is mitigated by the risk that the bubble will collapse because of bad news. In this case, sellers gradually bail out of the market before T.

Can the bubble last forever? In other words, can $T = \infty$? The answer is no. If sellers are willing to wait indefinitely when the asset is worthless, then by the argument above it must be that sellers are also willing to wait indefinitely when it is valuable. Since liquidity shocks ensure that there are positive sales at any instant, buyers must believe that there is at least some chance that the asset is valuable. Thus, prices must be also be positive at any instant. But this means that it is dominated for sellers to wait forever.

Conversely, why must there be a bubble at all? In other words, can $T = 0$? Indeed, there may be no-bubble equilibrium in which all sellers sell at time 0 regardless of the asset's quality. This is sustained, for example, by buyers' out-of-equilibrium beliefs that if any sales are observed after time 0, then the asset is deemed worthless. Such beliefs discourage sellers from waiting to sell. But such beliefs are implausible because as argued above, the incentive to delay sales is greater when the asset is valuable then when it is not. Forward induction, based on the idea that all deviations are purposeful rather than mistakes, then requires that on observing a delayed sale buyers assign high probability to events where such deviations are more profitable. Thus, on observing a delayed sale buyers should in fact believe that the asset is valuable, thereby disturbing the no-bubble equilibrium. As the result claims, in fact, forward induction rules out all but one equilibrium and in this equilibrium, $T > 0$.

Relative to the literature, our model of bubbles captures many appealing features simultaneously in a simple set up. The information structure is rather straightforward and captures the fact that many real-world bubble collapse because of public news. There is a *unique* (refined) equilibrium with the feature that a bubble emerges for all parameter values. Thus, the model is *robust.* Moreover, not only is a worthless asset over-priced but the *over-pricing grows* over time driven solely by rational exuberance. This last feature is unique to our model.

In our model, the bubble delays trade and so results in an *inefficiency*.[6] The fact that there is a unique (refined) equilibrium allows us to consider the implications of various "bubble-bursting" policies that have been proposed. Chief among these are interest rate policies as well as policies that result in more information about the true asset value.

We find that an increase in the interest rate decreases the maximum duration T of a bubble and so also the probability that it will burst before T. But an increase in the interest rate also decreases the present value of the gains from trade. In fact, because of this second, negative effect, an increase in the interest rate actually *decreases* welfare. Indeed, Bernanke and Gertler (2001) have argued that when considering interest rate policies to fight bubbles, one should take into account other economy-wide consequence of such policies. Our stark model is inadequate to address economy-wide consequences but nevertheless, in a limited sense, echoes this sentiment.

A related question is whether high interest rates mitigate asset price bubbles by deterring the *creation* of worthless assets. In Section 6, we extend our basic model to include a prior stage in which assets can be created by investors at some fixed cost. Again, it is assumed that investors know whether or not the asset is valuable prior to investing. If the asset is created, then it is traded in the same fashion as outlined above. We show that this two-stage model has a unique (refined) equilibrium as well. In this equilibrium, for intermediate levels of costs, there is positive investment regardless of quality but is lower if the asset is of low quality. Not surprisingly, increasing the interest rate decreases the level of investment in the low quality asset. This positive effect on welfare must now be weighed against the finding that in the second, trading stage, high interest rates decrease welfare because they reduce the present value of gains from trade. Examples show that the overall effect on welfare is now ambiguous.

Policies that increase transparency—that is, those that result in greater scrutiny of asset values—have unambiguous benefit. In our model, one may think of such policies as resulting in an increase in the arrival rate of bad news. We show that this increase is beneficial not only at the trading stage but also in the two-stage model with investment.

Other literature There is a vast literature on bubbles with different mechanisms surveyed ably by Brunnermeir (2009) and Barlevy (2018). Our paper falls into the category where the mechanism is asymmetric information.

Allen, Morris and Postlewaite (1993) develop a model of so-called "greater-fools" bubbles. Traders are asymmetrically informed with information partitions that become finer over time. In specific circumstances, prices exceed fundamentals. This is

[6] This is in contrast to many other models of bubbles where worthless assets, such as fiat money, improve efficiency by facilitating trade. See Section 5 for details.

because given his or her information, a trader is willing to buy the asset in the hope of selling to a greater fool later. There may be situations—called a strong bubble—in which it is mutual, but not common, knowledge that the asset is overpriced. In the work of Allen et al., the no-trade theorems of Milgrom and Stokey (1982) and Tirole (1982) are circumvented by introducing positive gains from trade, sometimes via insurance motives and sometimes via heterogeneous priors. Interestingly, Conlon (2004) shows that such bubbles can arise even when traders knowledge of over-pricing is of higher orders than being simply mutual. A different model of greater fools was introduced by Awaya, Iwasaki and Watanabe (2022) in which an asset is traded from a seller via a chain of middlemen to a final buyer. There is a common prior and positive gains from trade. The information of middlemen closer to the ultimate buyer is finer than that of those who are further away in the chain. For all parameter values, there is a unique equilibrium in which even a worthless asset is traded at positive prices. These increase over time as the asset is traded up the chain.

Abreu and Brunnermeir (2003) initiated research on "riding-the-bubble" models. Their seminal paper studies a situation in which holders of an asset decide when to optimally bail out of the market in the face of exogenous, rising prices. If sufficiently many traders bail out, however, the market crashes. Ideally, one would like to bail out just before a crash but there is uncertainty about exactly when this will happen. Traders receive *private* signals that conclusively reveal that the asset is overpriced but these signals are dispersed over time and so the over-pricing is not common knowledge. In the unique trading equilibrium, rational traders hold the asset even when they know it is over-priced. An obvious critique of the Abreu-Brunnermeir model is that the increase in asset prices is due to the presence of behavioral traders and so is exogenous.

In important work, Doblas-Madrid (2012, 2016) addresses the exogenous-price critique of the Abreu-Brunnermeir model by adding an explicit trading mechanism which determines prices. There is a second source of uncertainty—the possibility that an asset holder will be forced to sell because of liquidity needs. Doblas-Madrid delineates circumstances in which a bubble grows—prices rise—until, at a random time, it collapses. Rather than exuberance, in his model asset prices increase because of the exogenous growth in buyers' wealth. Typically, there are multiple equilibria—in some cases, an equilibrium with a bubble can coexist with one without.

In our model asset prices are endogenously determined. But, in our work, the rise in prices is not due to any change in the fundamentals—they rise solely because of increased optimism on the part of buyers—what we call exuberance. A second feature of our model is that in the unique (refined) equilibrium, a bubble must necessarily arise and this conclusion holds for *all* parameter values.[7]

While our work does not fall neatly into either of the two categories of models—

[7]Matsushima (2013) studies a variant of the Abreu-Brunnermeir model in which traders do not have any private information but are uncertain whether other traders are rational or behavioral. He identifies conditions under which there is a unique *Nash* equilibrium.

"greater fools" or "riding bubbles"—it is closer to the latter. As in those models, the information traders get consists of conclusive news that the asset is overpriced. An important difference is that in our model the news is *public* rather than private.

The bubble in our model is not strong in the sense that during the bubble, buyers do not know that the asset is overpriced. Thus over-pricing is not mutual knowledge among all traders. But whether or not a bubble is strong cannot be ascertained by an outside observer and even more important, by a policymaker.

The basic structure of our model resembles that of models of dynamic adverse selection as in the work of Hörner and Vieille (2009), Hwang (2018) and Kaya and Kim (2018) but in these papers, new information is either private or absent. The paper of Daley and Green (2012) is particularly close, as in their work adverse selection is also mitigated over time by *public* signals that arrive during the course of trading. The nature of the news is quite different, however. It is modelled as a Brownian process and so is not conclusive. As a result, prices go up and down and there are even episodes of no trade. In our work, prices rise monotonically and trade takes place continuously. Daley and Green look at stationary equilibria and show that more informative news processes may in fact decrease welfare.[8]

The remainder of this paper is organized as follows. The next section formally introduces the basic model. Section 3 exhibits an equilibrium in which there is a bubble of endogenous duration. Section 4 shows that all equilibria must have the same basic structure. Prices rise at an exponential rate. When worthless, the asset is sold gradually over a finite interval of time. We then show that only one equilibrium—the one constructed in Section 3—survives forward induction restrictions on out-of-equilibrium beliefs. Section 5 then examines the effects of "bubble-bursting" policies, notably interest rate policies and those that promote greater transparency. In Section 6 we extend the basic model in order to consider the incentives to create assets, worthless or not. This allows us to consider the effects of policies on these incentives. In Section 7 we show that the analysis is robust to there being positive gains from trade when the asset is of low quality as well as to the possibility of *good* news as well as bad news. Two appendices contain details of the uniqueness argument of Section 4.

2 Model

There is a continuum of long-lived sellers of mass one, each of whom holds one unit of an asset. Each seller decides on the time $t \in [0, \infty)$ when she wants to sell the asset. Time runs continuously and there is an infinite horizon. At any instant t, there is a

[8] In Section 6 of their paper, Daley and Green (2012) briefly consider a model like ours with Poisson news. But there are no liquidity shocks and the nature of equilibrium in the two models is quite different.

continuum of short-lived buyers with mass greater than one.[9] Thus, at any instant the mass of buyers is greater than the mass of sellers.

The asset is of uncertain quality. Specifically, there are two states of nature ω, either H or L, and the prior probability of H is $\rho \in (0, 1)$. In state H, the asset is of high quality and pays a flow dividend of w_H to each seller and a flow dividend of v_H to any buyer that purchases it. It is the case that $v_H > w_H \geq 0$ and so that in state H, the asset has greater value to buyers than to sellers—there are gains from trade. In state L, the asset pays no dividend and so is worthless for all.[10] Note that in each state, the asset has a *common value* to all sellers while they hold it and a, perhaps different, common value to all buyers who buy it.

All agents discount future payoffs at the rate $r > 0$.

Liquidity shocks Each seller may suffer from a "liquidity shock" that causes her to sell immediately at the current market price (as in Diamond and Dybvig, 1983 or Doblas-Madrid, 2012). Liquidity shocks are *independent* across sellers and arrive according to a Poisson process with parameter $\theta > 0$. Thus, the probability that a seller will be forced to liquidate her asset holdings before time t is $1 - e^{-\theta t}$.

Information All sellers know the true state of nature, H or L, while buyers are initially uninformed about the state. In state L, however, a *public* signal is generated, according to a Poisson process with parameter λ, so that the probability that the public signal will arrive before time t is $1 - e^{-\lambda t}$. In state H, no public signal is generated.[11] Thus, the signal carries conclusive "bad news" about the state. If bad news about the state arrives at time t, the state L becomes commonly known to all traders, buyers and sellers, at t and at all subsequent times.

Buyers at t can observe past prices but not the volumes of trade in the past.

Prices At any time t, the price of the asset is competitively determined.

Short sales Sellers cannot engage in short sales.

Behavior Rather than describing the strategies of the sellers explicitly, we will capture the sellers' decisions via a distribution function $F_\omega(\cdot) : \mathbb{R}_+ \to [0, 1]$ that, in state ω, determines the fraction $F_\omega(t)$ of sellers who plan to sell before time t in the absence of liquidity shocks.[12]

[9] By "short-lived" we mean that each buyer interacts with the market only at a single instant.

[10] In Section 7, we allow for the possibility that there are positive gains from trade in state L as well.

[11] In Section 7, we allow for the possibility of a public signal in state H as well.

[12] The same distribution function F_ω could result from different behaviors on the part of the sellers. For instance, it could be that each seller follows the same mixed strategy F_ω or it coud be that each seller with label $F_\omega(t)$ sells with probability one at t.

From this we can determine the fraction of sellers who sell before time t as

$$S_\omega(t) = 1 - e^{-\theta t}(1 - F_\omega(t)) \quad (1)$$

To see this, note that the fraction of sellers who sell after t is $1 - S_\omega(t)$ and this equals the fraction $e^{-\theta t}$ of sellers who have not been hit by a liquidity shock as yet times the fraction $1 - F_\omega(t)$ who plan to sell after t. Thus, S_ω is a *stopping-time distribution* in state ω.

Note that while $F_\omega(\cdot)$ is the distribution of *planned* selling times, $S_\omega(\cdot)$ is the distribution of *actual* selling times. S_ω takes into account the possibility that liquidity shocks may force a seller to sell before her planned selling time. Of course, F_ω stochastically dominates S_ω. Since there is a one-to-one relationship between S_ω and F_ω, there is no loss in supposing that the sellers behave according to S_ω which includes both planned and forced sales.

Similarly, there is no need to explicitly specify the strategies of the buyers either. All that is important is the resulting market price of the asset at any instant. Since the mass of buyers exceeds the mass of sellers at any time t, the market price $p(t)$ at t must equal the *expected* present discounted value of the asset to the buyers. In other words,

$$p(t) = \beta(t) \times \frac{v_H}{r} \quad (2)$$

where $\beta(t)$ is the posterior belief of the buyers at time t that the state is H, given their information at t. In other words, prices are determined *solely* by the beliefs of buyers.

Beliefs The buyers' posterior beliefs $\beta(t)$ are determined by the sellers' behavior, S_L and S_H. When forming these beliefs, the buyers know (i) whether or not bad news has arrived; (ii) past prices (but not trading volume). In addition, a buyers condition on the fact that he was able to obtain the asset. This last feature is to avoid the equivalent of the "winners' curse."

In the absence of news, the belief must be the *same* in either state H or L. The reason is that past prices are themselves determined by the past beliefs of buyers via (2) and so cannot carry any information not available to buyers. Thus, in the absence of news, beliefs cannot depend on the state.

Precisely, if no news has arrived by t, then the beliefs are formed as follows. If both $S_L(\cdot)$ and $S_H(\cdot)$ have densities $s_L(t) > 0$ and $s_H(t) > 0$ at t,

$$\beta(t) = \frac{\rho s_H(t)}{\rho s_H(t) + (1-\rho) e^{-\lambda t} s_L(t)} \quad (3)$$

Note that this is based on the fact that since the supply at time t in state ω is just $s_\omega(t)\,dt$ and in state H there is no possibility of bad news,.the relative likelihood of being able to obtain the asset is at time t is $s_H(t) / e^{-\lambda t} s_L(t)$.

If $S_L(\cdot)$ has a mass point at t, that is $\lim_{\tau\uparrow t} S_L(\tau) < S_L(t)$, while $S_H(\cdot)$ does not, then $\beta(t) = 0$. Similarly, if only $S_H(\cdot)$ has a mass point at t, then $\beta(t) = 1$. Finally if both have mass points m_L and m_H at t, then

$$\beta(t) = \frac{\rho m_H}{\rho m_H + (1-\rho) e^{-\lambda t} m_L}$$

Note that these definitions are valid only if t is in the union of the supports of S_L and S_H. Outside the union of the two supports, beliefs are arbitrary. In later sections, forward induction will place restrictions on beliefs.

If bad news has arrived by t, and so it is commonly known that the state is L, then $\beta(t) = 0$.

Equilibrium A (Nash) *equilibrium* is a price path $p : \mathbb{R}_+ \to \mathbb{R}_+$ and distributions $S_H(\cdot)$ and $S_L(\cdot)$ such that

1. Given p, S_ω maximizes the expected payoff of the sellers in state ω.
2. Given $S_H(\cdot)$ and $S_L(\cdot)$, the buyers' beliefs $\beta(t)$ are formed as above and prices are
$$p(t) = \beta(t) \times \frac{v_H}{r}$$

Two features of the definition should be noted. First, it specifies only the path of prices *in the absence of news.* If conclusive news arrives before t, then the price, of course, falls to zero after that. Thus the path of actual prices is random. Sellers maximize payoffs taking the price path p as well as the possibility of news into account.

Second, the definition takes into account the fact that in the absence of news, the prices in the two states must be the *same.* This in turn follows the fact that beliefs $\beta(t)$ must be the same in these circumstances. Thus, buyers as well as outside observers cannot infer the state from the prices.

3 Bubble Equilibrium

In this section, we exhibit an equilibrium with a bubble in which prices rise exponentially because of rational exuberance. There exists a time horizon T such that all sales are completed by then. In state L—when the asset is worthless—sellers slowly sell between 0 and T, some early, some late. Precisely, a positive density plan to sell at any $t \in [0, T]$. In state H—when the asset is valuable—all sellers plan to wait until T before selling and sell before then only if compelled by liquidity needs. As time goes by and no news arrives, buyers become increasingly optimistic that the asset is valuable. This rational exuberance results in prices rising at a pace that exceeds the interest rate. The price path is the same whether or not the asset is actually valuable.

Proposition 3.1 *There exists a unique $T > 0$ such that the following constitute an equilibrium:*
(i) in state H, sellers sell according to the distribution

$$S_H(t) = \begin{cases} 1 - e^{-\theta t} & t < T \\ 1 & t \geq T \end{cases}$$

(ii) in state L, sellers sell according to S_L with support $[0, T]$ and density

$$s_L(t) = \frac{\rho}{1-\rho}\theta\left(e^{(\lambda+r)T}e^{-(r+\theta)t} - e^{(\lambda-\theta)t}\right) \tag{4}$$

(iii) in the absence of news, the price path is

$$p(t) = \begin{cases} e^{(\lambda+r)(t-T)} \times \frac{v_H}{r} & t < T \\ \frac{v_H}{r} & t \geq T \end{cases}$$

Proof. Case 1: $\lambda \neq \theta$.

First, note that if s_L has the form given in (4), then

$$\begin{aligned} S_L(t) &= \int_0^t s_L(\tau)\,d\tau \\ &= \frac{\rho}{1-\rho}\theta\left(e^{(\lambda+r)T}\left(\frac{1-e^{-(r+\theta)t}}{r+\theta}\right) + \frac{1-e^{(\lambda-\theta)t}}{\lambda-\theta}\right) \end{aligned} \tag{5}$$

where T is treated as an unknown. There is a unique T such that $S_L(T) = 1$, or equivalently

$$\frac{\rho}{1-\rho}\theta\left(e^{(\lambda+r)T}\left(\frac{1-e^{-(r+\theta)T}}{r+\theta}\right) + \frac{1-e^{(\lambda-\theta)T}}{\lambda-\theta}\right) = 1 \tag{6}$$

This is because the left-hand side of the equation above is 0 when $T = 0$ and unbounded as $T \to \infty$. Moreover, it is increasing in T. Thus there is a unique T such that $S_L(T) = 1$.

Second, we show that given (iii), it is a best response for the sellers in state H to play according to S_H. The expected discounted payoff of a seller in state H who plans to sell at $t < T$ is

$$\begin{aligned} \pi_H(t) &= \int_0^t \theta e^{-\theta\tau}\left(\left(1-e^{-r\tau}\right)\frac{w_H}{r} + e^{-r\tau}p(\tau)\right)d\tau \\ &\quad + e^{-\theta t}\left(\left(1-e^{-rt}\right)\frac{w_H}{r} + e^{-rt}p(t)\right) \end{aligned}$$

The term with the integral is the payoff to a seller from a forced sale, due to a liquidity shock, before t. This includes the stream of dividends prior to the sale. The second term is the payoff from selling at t.

Routine calculations show that $\pi'_H(t) > 0$ and so for sellers in state H, it is better to postpone sales until T. Moreover, waiting after T is suboptimal because the prices after T are the same as those at T.

Third, we show that given (iii), selling at any $t < T$ is optimal for sellers in state L. The payoff of a seller in state L who plans to sell at $t < T$ is

$$\pi_L(t) = \int_0^t \theta e^{-\theta\tau} e^{-(\lambda+r)\tau} p(\tau)\, d\tau + e^{-\theta t} e^{-(\lambda+r)t} p(t)$$

This is similar to $\pi_H(t)$ except that in the low state there are no dividends and there is the possibility of bad news arriving before t. Since prices increase at the rate $\lambda + r$, $\pi_L(t)$ is a constant. Moreover, selling after T is suboptimal in state L again because prices after T cannot be higher than those at T.

Finally, given (i) and (ii), the beliefs of the buyers at any $t < T$

$$\beta(t) = \frac{\rho\theta e^{-\theta t}}{\rho\theta e^{-\theta t} + (1-\rho)\, e^{-\lambda t} s_L(t)}$$

and substituting for $s_L(t)$ from (4) results in

$$\beta(t) = e^{(\lambda+r)(t-T)}$$

and so the price path must be

$$p(t) = e^{(\lambda+r)(t-T)} \times \frac{v_H}{r}$$

At T, there are no sellers in state L while there is a positive mass of sellers in state H and so

$$\beta(T) = 1$$

so the price at T should be

$$p(T) = \frac{v_H}{r}$$

Case 2: $\lambda = \theta$

The analysis for this case is parallel where from (4), we have

$$s_L(t) = \frac{\rho}{1-\rho}\theta \left(e^{(\lambda+r)T} e^{-(r+\theta)t} - 1\right)$$

and so

$$S_L(t) = \frac{\rho}{1-\rho}\theta \left(e^{(\lambda+r)T}\left(\frac{1-e^{-(r+\theta)t}}{r+\theta}\right) - T\right)$$

■

It is useful to see the workings of Proposition 3.1 in an example.

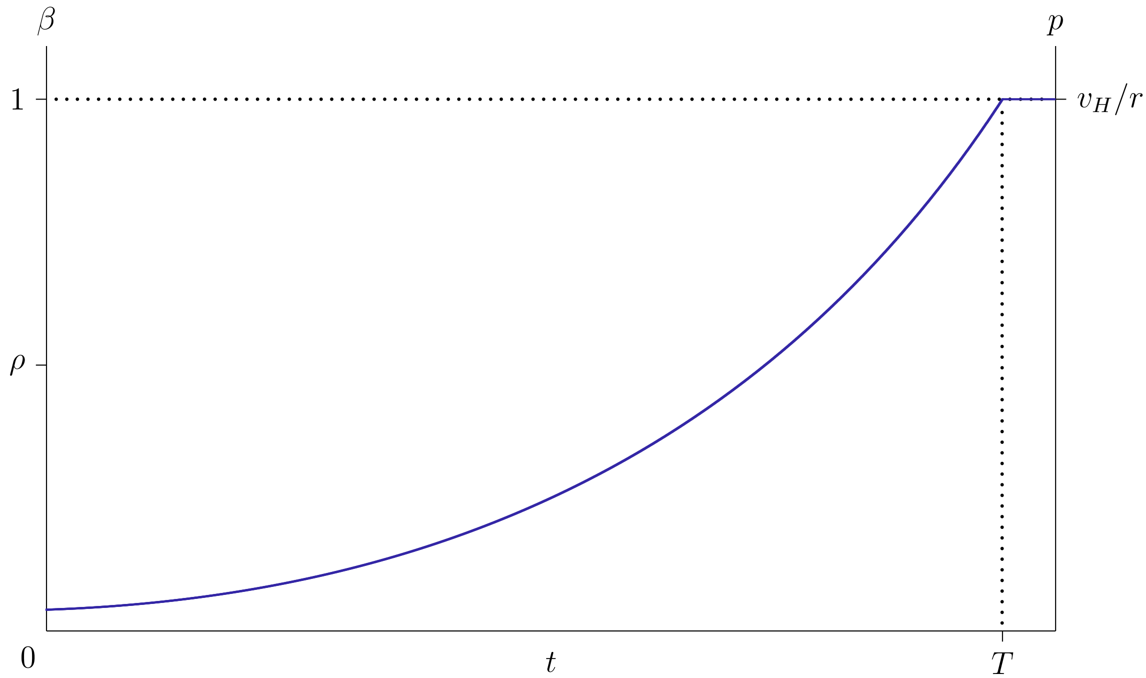

Figure 1: Path of Beliefs and Prices in Example 1

Example 1 *Suppose that the prior probability* $\rho = \frac{1}{2}$, *the interest rate* $r = 0.05$, *the arrival rate of bad news* $\lambda = 0.5$ *and the arrival rate of liquidity shocks* $\theta = 0.01$. *If the unit of time is one year, then these parameters correspond to a situation in which the bad news arrives every* 2 *years on average and liquidity shocks arrive every* 10 *years on average.*

For these parameters, the maximum duration of the bubble $T \simeq 6$ years. The probability that the bubble bursts before T is $1 - e^{-\lambda T} \simeq 0.95$ and the expected duration of the bubble is approximately 1.6 years.

If $v_H = 1$, then the resulting path of beliefs $\beta(t)$, in the absence of news, is depicted in Figure 1. Since prices are proportional to beliefs, the same figure also represents the path of prices $p(t)$. The two selling-time distributions S_L and S_H are depicted in Figure 2.

4 Uniqueness

In this section, we show that the bubble equilibrium derived in Proposition 3.1 is the only "plausible" equilibrium. Precisely, while there other Nash equilibria, these can only be supported by "implausible" out-of-equilibrium beliefs. A particularly simple Nash equilibrium (already discussed in the introduction) is a "no-bubble equilibrium" where in both states L and H, all sellers sell at time 0 at a price $p(0) = \rho \times v_H/r$. As long as $w_H < \rho v_H$, the equilibrium can be supported by buyers' out-of-equilibrium beliefs that if there are any sales after time 0, then the asset is worthless and so the

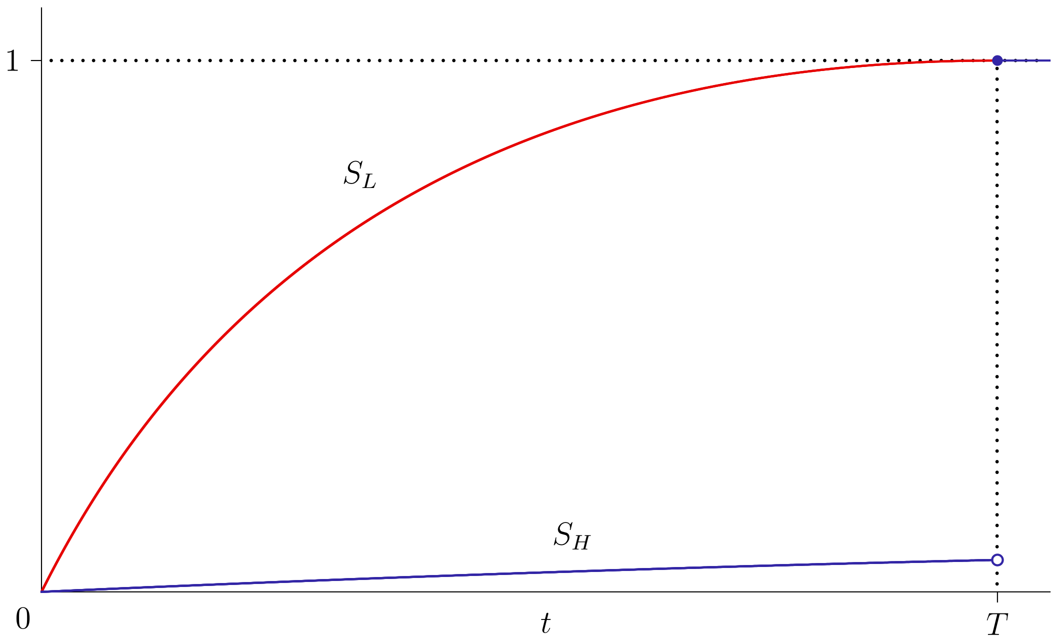

Figure 2: Selling-time Distributions in Example 1

price at any $t > 0$ will be 0. In what follows, we will use a specific forward induction notion of what plausible beliefs are. We begin with a characterization of all equilibria.

4.1 Structure of equilibria

Our first result, Proposition 4.1, shows that all equilibria have essentially the same structure as the one in Proposition 3.1: the price must grow at the rate $\lambda + r$ and sales in state H take place (stochastically) later than in state L. The only difference is that while the equilibrium derived in Proposition 3.1 is such that $S_L(\cdot)$ does not have a mass point at T, there may be other equilibria for which $S_L(\cdot)$ does. Note that different equilibria may have different time horizons—T may be different across equilibria.

Proposition 4.1 *If (p, S_L, S_H) constitute an equilibrium, then there exists a $T < \infty$ such that*
(i)

$$S_H(t) = \begin{cases} 1 - e^{-\theta t} & t < T \\ 1 & t \geq T \end{cases}$$

(ii) $S_L(\cdot)$ has a positive density on $[0, T)$ and (possibly) a mass point at T;
(iii) $p(t)$ grows at the rate $\lambda + r$ until T and $p(T) < v_H/r$ if $S_L(\cdot)$ has a mass point at T.

In Appendix B, we show that there are other equilibria besides the one in Proposition 3.1 or the "no-bubble" equilibrium in which $T = 0$.

Any equilibrium must have a structure similar to that of the one in Proposition 3.1 because given any price path, the incentive to wait is greater in state H than in state L. There are two reasons. First, in state H, sellers who wait receive a non-negative dividend of w_H while waiting. Second, and more important, in state L there is the possibility of bad news arriving and this adds to the cost of waiting. Precisely,

Lemma 4.1 *Given any price path p, if the payoff of sellers in state L from planning to sell at $t+\Delta$ when evaluated at t is positive, then the corresponding payoff of sellers in state H is strictly greater.*

Proof. The payoff of sellers in state L from selling at $t+\Delta$ evaluated at t is

$$\pi_L(t+\Delta, t) = \int_0^{\Delta} \theta e^{-\theta\tau} e^{-(\lambda+r)\tau} p(t+\tau)\, d\tau + e^{-\theta\Delta} e^{-(\lambda+r)\Delta} p(t+\Delta) \tag{7}$$

Since $\pi_L(t+\Delta, t) > 0$, it must be that either there is an open interval in $[t, t+\Delta]$ with positive prices or $p(t+\Delta)$ is positive.

The corresponding payoff in state H is

$$\begin{aligned} \pi_H(t+\Delta, t) &= \int_0^{\Delta} \theta e^{-\theta\tau} \left(\left(1 - e^{-r\tau}\right) \frac{w_H}{r} + e^{-r\tau} p(t+\tau) \right) d\tau \\ &\quad + e^{-\theta\Delta} \left(\left(1 - e^{-r\Delta}\right) \frac{w_H}{r} + e^{-r\Delta} p(t+\Delta) \right) \end{aligned}$$

and because $w_H \geq 0$,

$$\pi_H(t+\Delta, t) \geq \int_0^{\Delta} \theta e^{-\theta\tau} e^{-r\tau} p(t+\tau)\, d\tau + e^{-\theta\Delta} e^{-r\Delta} p(t+\Delta)$$

Finally, since $\lambda > 0$,

$$\pi_H(t+\Delta, t) > \pi_L(t+\Delta, t)$$

■

While the formal proof of Proposition 4.1 is in Appendix A, the basic argument is rather simple and runs as follows.

Suppose that the price path p together with $S_L(\cdot)$ and $S_H(\cdot)$ constitute an equilibrium where, as above, $S_\omega(t)$ denotes the fraction of sellers who sell before time t in state ω. As before let $F_\omega(t)$ denote the fraction of sellers who plan to sell before t. Recall that the distribution of planned sale times and actual sale times is related via (1).

For $\omega = L, H$, let $\operatorname{supp} F_\omega$ denote the support of the distribution F_ω, let $\underline{T}_\omega = \min \operatorname{supp} F_\omega$ and $\overline{T}_\omega = \sup \operatorname{supp} F_\omega$. Note that $\overline{T}_\omega = \infty$ is possible. Further, given an equilibrium (p, S_L, S_H), let $\beta(t)$ denote the belief of the buyers at time t derived from S_L and S_H.

First, in any equilibrium, $\overline{T}_L \leq \underline{T}_H$. Since sellers in H have a greater incentive to wait than those in L, they plan to sell at later times than sellers in L. Next, $\overline{T}_L = \overline{T}_H$ because after all the sellers in L have sold, there is no point for those in state H to wait any longer. Thus, since $\underline{T}_H = \overline{T}_H$ it is the case that in state H all sellers plan to sell at the same instant.

The second step is to show that in any equilibrium, the support of F_L is $\left[0, \overline{T}_L\right]$. This is done in two steps. If $\underline{T}_L > 0$, then in the interval $[0, \underline{T}_L]$ all sales come solely from those who have experienced a liquidity shock and it can be argued that the resulting beliefs and prices are such that it is better to sell at time 0. The second step is to show that there are no gaps in the support. The reason is similar because again, in the gap, sales come solely from those with liquidity needs and it is better to sell before the gap.

Third, F_L, and hence S_L, cannot have a mass point at any $t < \overline{T}_L$. The reason is that any such mass point would reveal that the state is L and then the belief at that point would fall to zero.

Finally, prices must rise at the rate $\lambda + r$ to keep sellers in L indifferent on $\left[0, \overline{T}_L\right]$. If S_L has a mass point at $\overline{T}_L$, then the beliefs of the buyers $\beta\left(\overline{T}_L\right) < 1$ and so the price $p\left(\overline{T}_L\right) < v_H/r$.

4.2 Forward induction

The game studied here has features in common with a dynamic signalling game as in Noldeke and van Damme (1990). As in their paper, *all* but one of the equilibria are vulnerable to *forward induction* arguments. Since all equilibria terminate at some time T and there is a positive density of sales at any time prior to and including T, the only out-of-equilibrium events are "delayed" sales that take place after T. Thus, for instance, equilibria could be sustained by out-of-equilibrium beliefs such that delayed sales result in low prices. In other words, buyers "threaten" sellers via beliefs that attribute any delayed sales to the state being L. A priori such out-of-equilibrium beliefs seem implausible because the incentive to delay sales is stronger in state H than in state L. In fact, it seems plausible that delayed sales should cause buyers to believe instead that the state is H.

To formalize "plausible" beliefs in our setting we use a property of stable sets of equilibria (Kohlberg and Mertens, 1986)—invariance to elimination of strategies that are never-a-weak best response to any equilibrium in the stable set (NWBR). In what follows, we use the definition of NWBR as formulated by Cho and Kreps (1987) for signalling games. This requires out-of-equilibrium beliefs to be concentrated on those types that have the strongest incentive to deviate. To apply this in our context, suppose that T is the terminal time in an equilibrium and let $\widetilde{p} : (T, \infty) \to \mathbb{R}$ be an "out-of-equilibrium" price path—that is, $\widetilde{p}(t)$ is the price that would emerge if there were delayed sales at a time $t > T$. Define

$$D_L = \{\widetilde{p} : \widetilde{\pi}_L(t, T) > p(T)\}$$

as the set of out-of-equilibrium price paths that, in state L, yield a payoff $\widetilde{\pi}_L$ is strictly higher than the equilibrium payoff. Note that the payoff $\widetilde{\pi}_L$ depends on the particular price path $\widetilde{p}$ (see (7) for a definition of $\pi_L(t,T)$). D_H is defined similarly.

Also, define

$$D_L^0 = \{\widetilde{p} : \widetilde{\pi}_L(t,T) = p(T)\}$$

The NWBR criterion requires that if

$$D_L^0 \subseteq D_H$$

then buyers must believe that any sales after T occur only in state H.

Since $D_L^0 \subseteq D_H$ is guaranteed by Lemma 4.1, in any equilibrium satisfying NWBR, prices after T must equal v_H/r. But in any equilibrium where S_L has a positive mass at its terminal date, the highest price in equilibrium $p(T) < v_H/r$ and the discontinuity in prices at T means that in either state, sellers would, for a small Δ, prefer to delay selling to $T+\Delta$ rather than selling at T. Thus, we reach the conclusion that any equilibrium with a positive mass of sellers at T in state L fails NWBR.

The equilibrium of Proposition 3.1 is not vulnerable to NWBR because the terminal price $p(T) = v_H/r$ and there is no discontinuity. Sellers have no incentive to delay sales after T since prices are already as high as possible. We summarize this as:

Proposition 4.2 *The equilibrium of Proposition 3.1 is the only equilibrium satisfying NWBR.*

NWBR is a demanding requirement and it is reasonable to wonder if weaker refinements also rule out $m > 0$ equilibria. The Cho and Kreps (1987) D1 criterion, for instance, requires that if

$$D_L \cup D_L^0 \subseteq D_H$$

then buyers must believe that any sales after the equilibrium T occur only in state H. It is easy to verify that in the game studied here, D1 and NWBR are equivalent.

A still weaker criterion is *belief monotonicity.* This property underlies the "divinity" concept of Banks and Sobel (1987) and requires that since the incentive to delay is greater in state H than in state L, any delayed sales should cause buyers to believe that H is no less likely than before. This, of course, is weaker than NWBR and D1 which require that delayed sales cause buyers believe that the state is H for sure. It can be shown that the belief monotonicity requirement is unable to rule out equilibria where in state L, there is a small mass of sellers at the terminal date. Thus belief monotonicity does not select the equilibrium of Proposition 3.1 uniquely.[13]

[13] The "intuitive criterion" of Cho and Kreps (1987) is difficult to apply in our game because of its dynamic and stochastic nature.

5 Bubble policies

It is commonly argued that rapid and unwarranted increases in asset prices—bubbles—destabilize the economy and governments should try to prevent or shorten the duration of such bubbles (see for example, Barlevy, 2018). While our bare-bones model does not address the economy-wide effects of a bubble, it is nevertheless useful in that we can analyze whether and how policy can counter rapid asset price increases. Here we examine two channels. The first, commonly suggested, channel is for the central bank to raise interest rates. The second is greater transparency, that is, improving the information available to buyers. Here we are assuming, of course, that the policymaker does not know the state H or L either.

Before examining the role of policy, it is worth noting that in our model the bubbles delay trade and so are inefficient. This is counter to what happens in most OLG-based models where worthless assets with a positive price facilitate trade and so bubbles can be Pareto improving.[14] The same is true in some "greater-fools" models where overpriced assets are beneficial for risk-averse agents as they provide insurance. Barlevy (2018) has criticized models in which bubbles are beneficial since they suggest that policymakers should fuel bubbles rather than puncture them.

Interest rate policies Recall from (5) that the (maximum) duration T is the solution to

$$\frac{\rho}{1-\rho}\theta\left(\frac{e^{(\lambda+r)T}-e^{(\lambda-\theta)T}}{r+\theta}+\frac{1-e^{(\lambda-\theta)T}}{\lambda-\theta}\right)=1$$

Some routine calculations then show that[15]

$$\frac{dT}{dr}=-\frac{\left((r+\theta)\,T-\left(1-e^{-(r+\theta)T}\right)\right)}{(\lambda+r)\,(r+\theta)\,(1-e^{-(r+\theta)T})}<0 \tag{8}$$

Thus, T is decreasing in the interest rate r and $\lim_{r\to 0} T<\infty$. The probability that bad news arrives before T, thereby causing the bubble to burst, is just $1-e^{-\lambda T}$ and this probability also increases as r increases. In this narrow sense, an increase in the interest rate decreases both the size and duration of a bubble.

But how does r affect overall welfare? In our simple model, welfare equals the present value of the gains from trade. In state L, there are no gains from trade. In state H, there are gains from trade and a bubble causes a delay in when these gains are reaped. An increase in the interest rate has two effects. First, as above, raising r decreases the delay T and so is beneficial. On the other hand, raising r decreases the discounted present value of the gains from trade and so is detrimental. We now study the net effect of raising r.

[14] Notable exceptions are Grossman and Yanagawa (1993) and Guerron-Quintana et al. (2022). The source of inefficiency in these models is that asset bubbles crowd out investment. In Allen, Barlevy and Gale (2022) bubbles are inefficient because they cause costly defaults.

[15] Details of these calculations are available from the authors.

The net effect is easiest to see when θ is very small, that is, when liquidity shocks are very rare. In this case, the welfare in state H is

$$W_H \approx \left(1 - e^{-rT}\right) \frac{w_H}{r} + e^{-rT} \frac{v_H}{r} \tag{9}$$

since until T, the asset is held by sellers who get a dividend of w_H and after T, it is held by buyers who get a dividend of v_H. Then

$$\frac{dW_H}{dr} = \frac{\partial W_H}{\partial r} + \frac{\partial W_H}{\partial T} \frac{dT}{dr}$$

and using (8), it may be verified that

$$\begin{aligned} \frac{dW_H}{dr} &= -\frac{w_H}{r^2} - \left(\left(\frac{1}{r} + T \right) - \frac{rT - \left(1 - e^{-rT}\right)}{(\lambda + r)\left(1 - e^{-rT}\right)} \right) e^{-rT} \left(\frac{v_H - w_H}{r} \right) \\ &< -\frac{1}{r} \left(1 + rT - \frac{rT - \left(1 - e^{-rT}\right)}{1 - e^{-rT}} \right) e^{-rT} \left(\frac{v_H - w_H}{r} \right) \end{aligned}$$

where the inequality follows by neglecting the first term and setting $\lambda = 0$. But now it is easy to see that $dW_H/dr < 0$, that is, an increase in the interest rate decreases welfare. This suggests that while raising the interest rate decreases the delay in reaping the gains from trade, this positive effect is wiped out by the decrease in the present value of those gains.

In fact, it can be shown that $dW_H/dr < 0$ even if there are liquidity shocks are not rare.[16] Thus, in our model interest rate increases always *decrease* welfare.

Bernanke and Gertler (2001) have cautioned that raising interest rates has other detrimental effects in dampening economic activity and so perhaps central banks should not raise interest rates in a knee-jerk manner in response to bubbles. Our finding echoes, in a limited sense, this sentiment.

Information policies A start up that wishes to go public is subject to detailed scrutiny by the Securities and Exchange Commission (SEC). This process is cumbersome and time consuming—taking up to 18 months. In recent years, start ups have avoided this detailed scrutiny by being acquired by, or merging with, an existing public company whose only purpose is to acquire the start up! This vehicle, appropriately called a Special Purpose Acquisition Company (SPAC), thus offers a legal channel to avoid detailed scrutiny. For instance, instead of going public itself, in 2020 Nikola merged with the publicly traded SPAC called VectolQ Acquisition Company.

In the context of our model, the existence of vehicles like SPACs can be roughly translated into a decrease in the amount of information available to the market, that is, a decrease in λ.

[16] Again, details of these calculations are available from the authors.

The effect of a change in λ on the maximum duration T is rather obvious. Since bad news causes the bubble to collapse, if such news arrives earlier, this can only decrease the maximum duration of the bubble. Now from (9) it follows immediately that an increase in λ increases welfare as well.

Our bare-bones model thus suggests that instead of using interest rates to combat bubbles, policies that lead to greater transparency of asset quality may be better.

The effects of improved public information are also studied by Asako and Ueda (2014) in a "riding-the-bubble" context. They show that depending on its nature, sometimes public warnings of a bubble may fail to prevent one. Conlon (2015) and Holt (2019) study such policies in the "greater fools" context and shows that information policies designed to burst bubbles may be detrimental, sometimes because they prevent mutual insurance.

6 A model with asset creation

In a bubble an asset is traded at prices that exceed its fundamental value. This may result in another kind of inefficiency—over-investment in the creation of such assets such as a worthless start up. Policies that discourages investment in low-quality assets will be socially beneficial. The model studied in previous sections can be extended to address this second source of inefficiency.

Suppose that prior to any trading, there is a stage in which agents can create an asset by incurring some costs. The sequence of events is as follows. First, the state H or L is determined according to the prior probabilities ρ_0 and $1-\rho_0$, respectively. Second, knowing the state, sellers decide whether or not to start a new company at a cost $c > 0$.[17] After this, shares of the new company are traded over an infinite horizon exactly as in the model of Section 2. If the asset is traded, then buyers know, of course, that it was created but are unaware of the amount of investment.

As before, in state L the asset is worthless. But as we will see, in the unique equilibrium of this extended game, there will be investment in creating a worthless asset. One goal of a planner may be to prevent or reduce such inefficient investment.

6.1 Equilibrium with asset creation

Suppose that in state L the fraction of sellers who invest in creating the asset is α_L and in state H, this fraction is α_H. Knowing this, *prior to trading*, the buyers update their beliefs so that the probability that the state is H becomes

$$\rho(\alpha_L) \equiv \frac{\rho_0 \alpha_H}{\rho_0 \alpha_H + (1-\rho_0)\alpha_L} \tag{10}$$

We then have the following:

[17] For simplicity, we are assuming that the cost is independent of the state. This is easily amended so that the costs are c_H and c_L with $c_H > c_L > 0$.

Proposition 6.1 *There is a unique equilibrium of the two-stage game that satisfies NWBR.*[18] *If $c < \frac{v_H}{r}$, then in state H, all sellers invest, that is, $\alpha_H = 1$ while in state L, a fraction $\alpha_L > 0$ invest.*

Proof. Let $\Gamma(\rho)$ denote the trading game of Section 2 and let π_L and π_H denote the resulting profits in state L and H, respectively.

First, there cannot be an equilibrium in which $\alpha_L = 0$, that is, there is no investment in state L. If $\alpha_H > 0$, then $\rho(\alpha_L) = 1$ and so $\pi_H = v_H/r > c$. Now in state L, if a seller invests it can also get a profit of v_H/r since buyers are sure that the state is H. So there cannot be an equilibrium in which there is positive investment only in state H. Now suppose $\alpha_H = 0$ as well, that is, there is no investment in either state. We In the game $\Gamma(\rho)$ the profit of a seller in state H exceeds the profit in state L. Thus, after observing that a positive measure of sellers are in the market, NWBR requires that buyers believe that the state is H. This will cause sellers in both states to invest. Thus, there cannot be an equilibrium with the NWBR property such that there is zero investment in both states.

Next, we argue that in any equilibrium, $\alpha_H = 1$. Again, the profit of a seller in state H exceeds the profit in state L and this is true no matter what ρ is. Since $\alpha_L > 0$ means that $\pi_L \geq c$, we have $\pi_H > c$ and so in state H, all sellers invest.

We now show that there is a unique $\alpha_L > 0$ such that α_L and $\alpha_H = 1$ are part of an equilibrium. Let T_0 be the maximum duration of the bubble in the game $\Gamma(\rho_0)$, that is, T_0 is the unique solution to

$$\frac{\rho_0}{1-\rho_0}\theta\left(\frac{e^{(\lambda+r)T}-e^{(\lambda-\theta)T}}{r+\theta}+\frac{1-e^{(\lambda-\theta)T}}{\lambda-\theta}\right)=1$$

Case 1: $c \leq e^{-(\lambda+r)T_0} \times (v_H/r)$

Note that the sellers' profit $\Gamma(\rho_0)$ when the state is L is $e^{-(\lambda+r)T_0} \times (v_H/r)$. Thus, if $c \leq e^{-(\lambda+r)T_0} \times (v_H/r)$ then the unique equilibrium is for all sellers in both states to invest.

Case 2: $e^{-(\lambda+r)T_0} \times (v_H/r) < c < (v_H/r)$

For any $\alpha \in (0, 1]$, let $T(\alpha)$ be the unique solution to

$$\frac{\rho(\alpha)}{1-\rho(\alpha)}\theta\left(\frac{e^{(\lambda+r)T}-e^{(\lambda-\theta)T}}{r+\theta}+\frac{1-e^{(\lambda-\theta)T}}{\lambda-\theta}\right)=\alpha$$

This condition is analogous to (6) and ensures that the total mass of sellers in state L is α. Using (10), this can be rewritten as

$$\frac{\rho_0}{1-\rho_0}\theta\left(\frac{e^{(\lambda+r)T}-e^{(\lambda-\theta)T}}{r+\theta}+\frac{1-e^{(\lambda-\theta)T}}{\lambda-\theta}\right)=\alpha^2$$

[18] Again, in this model NWBR is equivalent to the D1 criterion.

As before, the left-hand side is an increasing and continuous function of T that is zero when $T = 0$ and unbounded as T increases. Thus there is unique $T(\alpha)$. Notice also that the solution $T(\alpha)$ is a continuous and monotonic function of α.

We claim that there exists a unique α such that

$$e^{-(\lambda+r)T(\alpha)} \times \frac{v_H}{r} = c$$

This is because when $\alpha = 1$, $T(1) = T_0$ and we have assumed that

$$e^{-(\lambda+r)T_0} \times \frac{v_H}{r} < c$$

Moreover, as $\alpha \to 0$, $T(\alpha) \to 0$ and

$$\frac{v_H}{r} > c$$

Thus there exists an $\alpha_L \in (0,1)$ such that

$$e^{-(\lambda+r)T(\alpha_L)} \times \frac{v_H}{r} = c \tag{11}$$

■

6.2 Policies to discourage worthless assets

Using the equilibrium derived in Proposition 6.1, we can study how different policies affect (a) investment in low quality assets; and (b) the gains from trade.

Interest rate policies with investment It is easy to see that an increase in the interest rate discourages wasteful investment in state L. In the equilibrium derived above in Proposition 6.1, in state H these profits are strictly greater than the cost of investment and so a small change in r does not affect investment in state H. In state L, however, profits are exactly the same as costs and only a fraction $\alpha_L \in (0,1)$ invest (assuming costs are moderate). So in state L, a small increase in the interest rate r decreases profits and so decreases α_L as well. To see this, note that a decrease in α_L causes profits to rise because buyers are now more optimistic about the state and so profits are higher. Thus to maintain indifference between investing and not in state L, an increase in r is compensated by a decrease in α_L. Thus, an increase in r discourages wasteful investment in state L.

But interest rate increases have other effects as well. As in the previous section, an increase in r reduces the duration and size of the bubble. But as we saw, this benefit is not enough to compensate for the decrease in the present value of the gains from trade.

To summarize, in the game with investment, interest rate increases have the following effects:

1. they discourage wasteful investment in state L when the asset is worthless;
2. they reduce the duration and size of the bubble in the trading stage;
3. they reduce the gains from trade.

While the first two effect are positive, the third is negative.

Note that expected welfare

$$W^* = \rho_0 (W_H - c) - (1 - \rho_0) \alpha_L c$$

The first term is the net welfare from trading and this occurs only in state H. The second term represents wasteful investment and this occurs only in state L.

The overall effect on welfare is ambiguous as shown in the example below.

Example 2 *Suppose that the prior probability $\rho_0 = \frac{1}{3}$, the arrival rate of bad news $\lambda = 0.7$ and the arrival rate of liquidity shocks $\theta = 0.5$ Further, suppose that the flow dividend $v_H = 1$ and the cost of investment $c = 1$.*

It can be shown that if $r < 0.17$, then $\alpha_L = 1$, that is, all investors invest in both states. If $r > 1$, then no one invests in either state. If $0.17 < r < 1$ the unique equilibrium is such that $\alpha_L \in (0, 1)$. In other words, there is positive investment in state L but only a fraction α_L invest.

For small values of r in the relevant range, say $r = 0.2$, a small increase in r results in a *decrease* in welfare. The reduction in the present value of the gains from trade is large relative to the savings from reducing wasteful investment.

For very high values of r in the relevant range, say $r = 0.8$, a small increase in r results in an *increase* in welfare. When r is very high, the present value of the gains from trade is very small and in fact, welfare is negative. In other words, society would be better-off if there were no investment and hence no trade in this asset. An increase in r now increases welfare (which is negative) as the gains from decreasing wasteful investment are high.

Information policies with investment The effects of increased scrutiny—so that bad news arrives sooner—are unambiguous. We have already seen that in the trading game $\Gamma(\rho)$, an increase in λ decreases maximum duration T and improves welfare in state H. It can be shown that an increase in λ also decreases the profit of sellers in state L.[19] Thus, greater scrutiny naturally decreases the incentive to invest when the asset is of low quality. This has direct benefit as wasteful investment is lower. The decrease in wasteful investment also has an indirect effect—it decreases the probability that buyers will face a low quality seller. This increase in the posterior probability that the state is H is a further benefit as also decreases T.

Once again, increasing λ thus has an unambiguously positive effect on overall expected welfare.

[19] Again, details of these calculations are available from the authors.

7 Other extensions

Here we show that the model and results stay intact if we relax two of our simplifying assumptions: the fact that the asset if worthless to all in state L and the fact that the only news that arrives is bad news.

Gains from trade In the analysis so far we have assumed that $v_L = w_L = 0$, that is, there are no gains from trade when the asset is of low quality. This assumption can be easily relaxed with no qualitative changes to the analysis. So suppose that $v_L > w_L = 0$ so that buyers derive positive dividends even in state L.

With this change, the price at time t

$$p(t) = \beta(t)\frac{v_H}{r} + (1 - \beta(t))\frac{v_L}{r}$$

that is, the expected value of the asset to buyers with belief $\beta(t)$ that the state is H.

This results in only a minor change to Proposition 3.1 in that the expression for s_L, the density of sales in state L, becomes

$$s_L(t) = \frac{\rho}{1-\rho}\theta e^{(\lambda-\theta)t}\left(\frac{(v_H - v_L)}{e^{(\lambda+r)t}e^{-(\lambda+r)T}v_H - v_L} - 1\right)$$

Everything else remains unchanged. In particular, prices still grow at the rate $\lambda + r$.

It is routine to verify that an increase in v_L decreases the maximum duration of the bubble.

Good news and bad news We have assumed that the only news that comes is conclusive *bad* news that the asset is of low quality. Suppose that in addition of bad news, there is also the possibility of conclusive *good* news that the asset is of high quality. Specifically, suppose that as in previous sections, in state L a conclusive signal arrives at the Poisson rate $\lambda_L > 0$ but in addition, in state H there is a distinct signal that arrives at a Poisson rate $\lambda_H > 0$. Suppose also that $\lambda_L > \lambda_H$.

This means that in the absence of *any* news, good or bad, the beliefs are

$$\beta(t) = \frac{\rho\theta e^{-\theta t}}{\rho\theta e^{-\theta t} + (1-\rho)e^{-(\lambda_L - \lambda_H)t}s_L(t)}$$

Of course, if bad news arrives, then $\beta(t) = 0$ thereafter, and if good news arrives, then $\beta(t) = 1$ thereafter. Again Proposition 3.1 needs to be amended only slightly. The expression for s_L, the density of sales in state L, becomes (assuming again that $v_L = 0$)

$$s_L(t) = \frac{\rho}{1-\rho}\theta\left(e^{(\lambda_L+r)T}e^{-(\lambda_H+r+\theta)t} - e^{(\lambda_L-\lambda_H-\theta)t}\right)$$

Once again, everything else remains unchanged. Prices grow at the rate $\lambda_L + r$. The behavior of sellers in state H remains the same—the incentive to wait increases because of the possibility of good news.

The possibility of good news ($\lambda_H > 0$) leads to an increase in the maximum duration of the bubble.

A Appendix: Proof of Proposition 4.1

Suppose that the price path p together with $S_L(\cdot)$ and $S_H(\cdot)$ constitute an equilibrium where, as above, $S_\omega(t)$ denotes the fraction of sellers who sell before time t in state ω. As before let $F_\omega(t)$ denote the fraction of sellers who plan to sell before t. Recall that the distribution of planned sale times and actual sale times is related by

$$S_\omega(t) = 1 - e^{-\theta t}(1 - F_\omega(t))$$

For $\omega = L, H$, let $\operatorname{supp} F_\omega$ denote the support of the distribution F_ω, let $\underline{T}_\omega = \min \operatorname{supp} F_\omega$ and $\overline{T}_\omega = \sup \operatorname{supp} F_\omega$. Note that $\overline{T}_\omega = \infty$ is possible. Further, given an equilibrium (p, S_L, S_H), let $\beta(t)$ denote the belief of the buyers at time t derived from S_L and S_H.

Lemma A.1 *In any equilibrium, the payoff in state L is positive.*

Proof. First, suppose $\min\left(\overline{T}_L, \overline{T}_H\right) > 0$.

At any time $t \leq \min\left(\overline{T}_L, \overline{T}_H\right)$, there is at least a positive density of sellers who are compelled to sell for liquidity reasons regardless of the price. Thus, at any such t, $\beta(t) > 0$ unless in state L there is a mass of sellers at t. But since in state L there can be at most a countable number of instances where there is a mass of sellers, there is an open set of times less than $\min\left(\overline{T}_L, \overline{T}_H\right)$ where $\beta(t) > 0$ and hence $p(t) > 0$ as well. By selling at any such time in state L, sellers can guarantee a positive payoff and so their equilibrium payoff must also be positive.

Next, suppose $\overline{T}_H = 0$. This means that in state H all sellers sell at time 0 and so the price $p(0) > 0$. Thus, by selling at 0 in state L sellers can guarantee a positive payoff. So again their equilibrium payoff must be positive as well.

Finally, suppose that $0 = \overline{T}_L < \overline{T}_H$. This means that if there are sales at some $t > 0$, then $\beta(t) = 1$ and again in state L, sellers can deviate and sell at $t > 0$ and so there cannot be such an equilibrium. ■

Lemma A.2 *In any equilibrium,*

$$\overline{T}_L \leq \underline{T}_H$$

Proof. Suppose, by way of contradiction, that $\underline{T}_H < \overline{T}_L$.

Lemma A.1 guarantees that $\pi_L\left(\overline{T}_L, \underline{T}_H\right) > 0$ and so from Lemma 4.1, the payoff in state H from selling at $\overline{T}_L$ evaluated at $\underline{T}_H$ is strictly greater than the corresponding payoff in state L, that is, $\pi_H\left(\overline{T}_L, \underline{T}_H\right) > \pi_L\left(\overline{T}_L, \underline{T}_H\right)$.

It is also the case that selling at $\underline{T}_H$ is a best-response in state H and so

$$p\left(\underline{T}_H\right) \geq \pi_H\left(\overline{T}_L, \underline{T}_H\right)$$

Finally, selling at $\overline{T}_L$ is a best response in state L and so

$$p\left(\underline{T}_H\right) \leq \pi_L\left(\overline{T}_L, \underline{T}_H\right)$$

which is impossible since $\pi_H\left(\overline{T}_L, \underline{T}_H\right) > \pi_L\left(\overline{T}_L, \underline{T}_H\right)$. Thus, $\overline{T}_L \leq \underline{T}_H$. ■

Lemma A.3 *In any equilibrium,*

$$\overline{T}_L = \overline{T}_H$$

that is, in state H all sellers sell at $\overline{T}_L$.

Proof. From Lemma A.2 above we know that $\overline{T}_L \leq \underline{T}_H$ and hence $\overline{T}_L \leq \overline{T}_H$ as well. Suppose, by way of contradiction, that $\overline{T}_L < \overline{T}_H$.

Let $t \in \left(\overline{T}_L, \overline{T}_H\right)$ and note that the belief at t, $\beta(t) = 1$ since in state H there is a positive fraction of seller who sell after t while in state L no sellers sell after t. But now, in state H, selling at t is better than selling at any $t' > t$ such that $t' \in \left(\overline{T}_L, \overline{T}_H\right)$. ■

Lemma A.4 *In any equilibrium,*

$$\operatorname{supp} F_L = [0, \overline{T}_L]$$

Proof. First, we will show that $\underline{T}_L = 0$. Suppose that $\underline{T}_L > 0$. Now there are two possibilities: either (i) $0 < \underline{T}_L = \overline{T}_L$; or (ii) $0 < \underline{T}_L < \overline{T}_L$.

Case 1: $0 < \underline{T}_L = \overline{T}_L$.

Now in both states, there are only involuntary sales at any $t < \underline{T}_L$ and $s_L(t) = s_H(t) = \theta e^{-\theta t}$. Thus, from (3) the buyers' belief at any $t < \underline{T}_L$ is

$$\beta(t) = \frac{\rho}{\rho + (1-\rho) e^{-\lambda t}}$$

Moreover, from Lemma A.3, in state H all sellers sell at $\overline{T}_L = \underline{T}_L$ and in state L all sellers sell at $\underline{T}_L$ as well. Thus, the belief at $\underline{T}_L$ is

$$\beta\left(\underline{T}_L\right) = \frac{\rho}{\rho + (1-\rho) e^{-\lambda \underline{T}_L}}$$

and a routine calculation shows that given the resulting price path $p(t) = \beta(t) \times v_H / r$, it is better to sell at 0 than at $\underline{T}_L$, that is, $\pi_L(0,0) > \pi_L\left(\underline{T}_L, 0\right)$.

Case 2: $0 < \underline{T}_L < \overline{T}_L$.

Now in both states, there are only involuntary sales at any $t < \overline{T}_L$ and $s_L(t) = s_H(t) = \theta e^{-\theta t}$. Thus, from (3) the buyers' belief at any $t < \overline{T}_L$ is

$$\beta(t) = \frac{\rho}{\rho + (1-\rho)e^{-\lambda t}}$$

If S_L has a mass point at $\underline{T}_L$, then $\beta(\underline{T}_L) = 0$ because S_H does not have a mass point there (Lemma A.3). So S_L has a positive density at $\underline{T}_L$, $s_L(\underline{T}_L) \geq \theta e^{-\theta \underline{T}_L}$ since the density of sellers at $\underline{T}_L$ (weakly) exceeds the density of involuntary sales. Thus

$$\begin{aligned} \beta(\underline{T}_L) &= \frac{\rho \theta e^{-\theta t}}{\rho \theta e^{-\theta t} + (1-\rho)e^{-\lambda t} s_L(\underline{T}_L)} \\ &\leq \frac{\rho}{\rho + (1-\rho)e^{-\lambda t}} \end{aligned}$$

and again a routine calculation shows that the payoff of sellers in state L is higher if they sold at $t = 0$ rather than at $\underline{T}_L$.

Thus we have shown that $\operatorname{supp} F_L \subseteq [0, \overline{T}_L]$.

Finally, we show that in fact $\operatorname{supp} F_L = [0, \overline{T}_L]$. If $\overline{T}_L = 0$, then the statement of the lemma is obvious. Otherwise, if $t < \overline{T}_L$ and $t \notin \operatorname{supp} F_L$ then there is a $t' > t$ such that $t' \in \operatorname{supp} F_L$. The belief at t' is such that

$$\beta(t') \leq \frac{\rho}{\rho + (1-\rho)e^{-\lambda t'}}$$

and again it is better to sell at time t. ■

So we have established that in any equilibrium, in state L the support of F_L is an interval $[0, T]$, while in state H, all sellers sell at T.

Lemma A.5 *In any equilibrium, S_L does not have a mass point at any $t < T$.*

Proof. Since in any equilibrium, S_H does not have a mass point at any $t < T$, if S_L, and hence F_L, were to have a mass point at t, then the resulting belief at time t, $\beta(t) = 0$ and hence the price $p(t) = 0$ as well. Thus it would not be a best-response to sell with positive probability at t. ■

Lemma A.6 *In any equilibrium, the price path is p is such that, for all $t < T$,*

$$p(t) = p(0)e^{(\lambda + r)t}$$

Proof. Note that the payoff of the seller in state L from planning to sell at time t is

$$\pi_L(t, 0) = \int_0^t \theta e^{-\theta \tau} e^{-(\lambda + r)\tau} p(\tau)\, d\tau + e^{-\theta t} e^{-(\lambda + r)t} p(t)$$

Since in any equilibrium S_L with support $[0, T]$, the seller must be indifferent everywhere in the support and so $\pi_L'(t, 0) = 0$ for all $t \in (0, T)$.

Thus,

$$\theta e^{-\theta t} e^{-(\lambda+r)t} p(t) - (\theta + \lambda + r) e^{-\theta t} e^{-(\lambda+r)t} p(t) + e^{-\theta t} e^{-(\lambda+r)t} p'(t) = 0$$

from which it follows that

$$-(\lambda + r) p(t) + p'(t) = 0$$

and so

$$\frac{d}{dt} \ln p(t) = \lambda + r$$

and thus the price path must exhibit exponential growth at the rate $\lambda + r$. ■

Lemma A.7 *In any equilibrium,* $p(T) = \lim_{t \uparrow T} p(t)$.

Proof. If $p(T) > \lim_{t \uparrow T} p(t)$, then for some small Δ, in state L the seller would be better off selling at T rather than at $T - \Delta$ contradicting the fact that selling at $T - \Delta$ is a best response in state L.

On the other hand, if $p(T) < \lim_{t \uparrow T} p(t)$, then for some small Δ, in state H the seller would be better off selling at $T - \Delta$ rather than at T contradicting the fact that selling at T is a best response in state H. ■

The lemmas above establish Proposition 4.1.

B Appendix: Other equilibria

While Proposition 4.1 establishes the necessary structure that all equilibria must have, the reader may rightly wonder if there are equilibria other than the one from Proposition 3.1. Here we show that indeed there are other equilibria and typically, a continuum of other equilibria. This is most easily seen in the case where $w_H = 0$, that is, even in state H sellers do not receive any dividends. Without any belief restrictions, we have the following result:

Proposition B.1 *Suppose* $w_H = 0$. *For any* $m \in [0, 1]$, *there is a unique* $T_m > 0$ *such that the following constitute an equilibrium:*
(i) in state H, *sellers sell according to the distribution*

$$S_H(t) = \begin{cases} 1 - e^{-\theta t} & t < T_m \\ 1 & t \geq T_m \end{cases}$$

(ii) in state L, *sellers sell according to* S_L *with support* $[0, T_m]$ *and density*

$$s_L(t) = \frac{\rho}{1-\rho} \theta \left(e^{(\lambda+r)T_m} \frac{\rho + (1-\rho) e^{-\lambda T_m} m}{\rho} e^{-(r+\theta)t} - e^{(\lambda-\theta)t} \right) \tag{12}$$

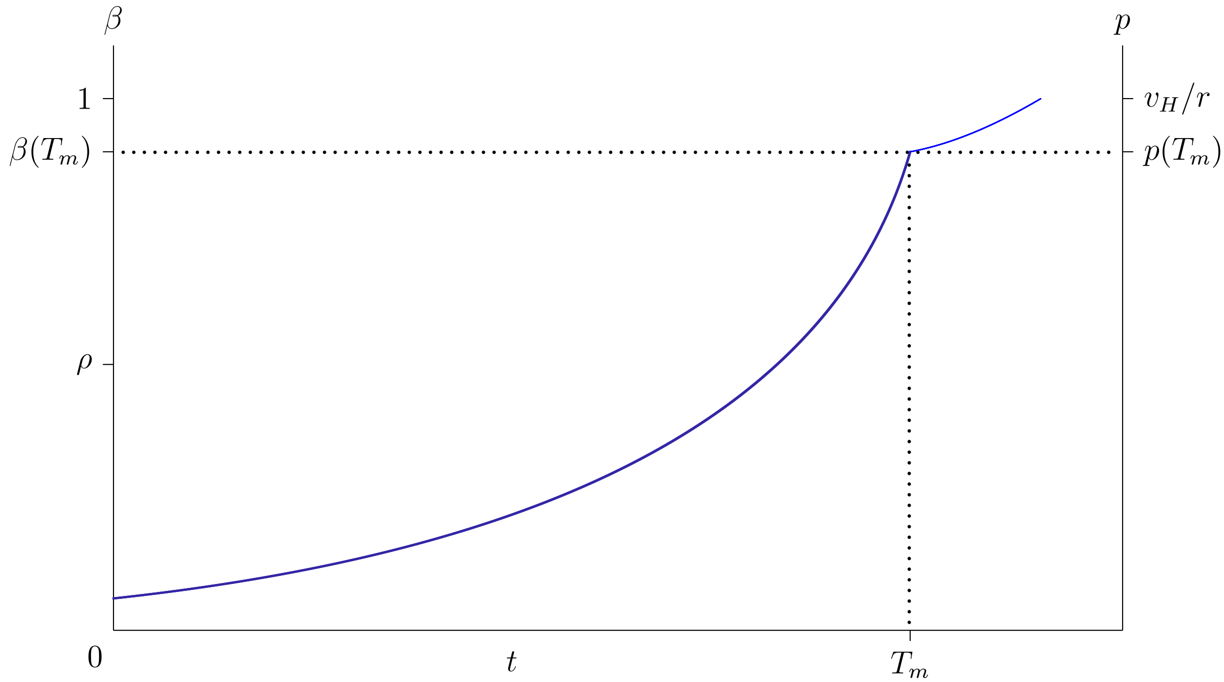

Figure 3: Path of Beliefs and Prices in Example 3

(iii) the price path is

$$\begin{array}{ll} p(t) = e^{(\lambda+r)(t-T_m)} \times \frac{\rho}{\rho+(1-\rho)e^{-\lambda T_m} m} \times \frac{v_H}{r} & t \leq T_m \\ p(t) = 0 & t > T_m \end{array}$$

Remark B.1 *Note that the "out-of-equilibrium" price path* $p(t)$*, for* $t > T_m$*, is the worst possible. There are many others that will support the equilibrium as well. But nevertheless, out-of-equilibrium prices cannot be too high. If for some* $t > T_m$*,* $\pi_H(t, T_m) > p(T_m)$ *then in state* H*, sellers would prefer to sell at* t *rather than at* T_m*.*

Remark B.2 *If* $m > 0$*, then* $p(T_m) < \frac{v_H}{r}$*.*

When combined with Proposition 4.1, the result above characterizes all equilibria.

Proof of Proposition B.1 As in the proof of Proposition 3.1, suppose that $\lambda \neq \theta$. If s_L has the form given in (12), then

$$\begin{aligned} S_L(t) &= \int_0^t s_L(\tau)\, d\tau \\ &= \frac{\rho\theta}{(1-\rho)} \left(e^{(\lambda+r)T_m} \frac{\rho + (1-\rho)\, e^{-\lambda T_m} m}{\rho} \frac{1 - e^{-(\theta+r)t}}{r+\theta} + \frac{1 - e^{(\lambda-\theta)t}}{\lambda - \theta} \right) \end{aligned}$$

where T_m is treated as an unknown. It is enough to show that there is a unique T_m such that $\lim_{t \uparrow T_m} S_L(t) = 1 - m$, or equivalently

$$\frac{\rho}{1-\rho}\theta \left(e^{(\lambda+r)T_m} \frac{\rho + (1-\rho)\, e^{-\lambda T_m} m}{\rho} \frac{1 - e^{-(\theta+r)T_m}}{r+\theta} + \frac{1 - e^{(\lambda-\theta)T_m}}{\lambda - \theta} \right) = 1 - m$$

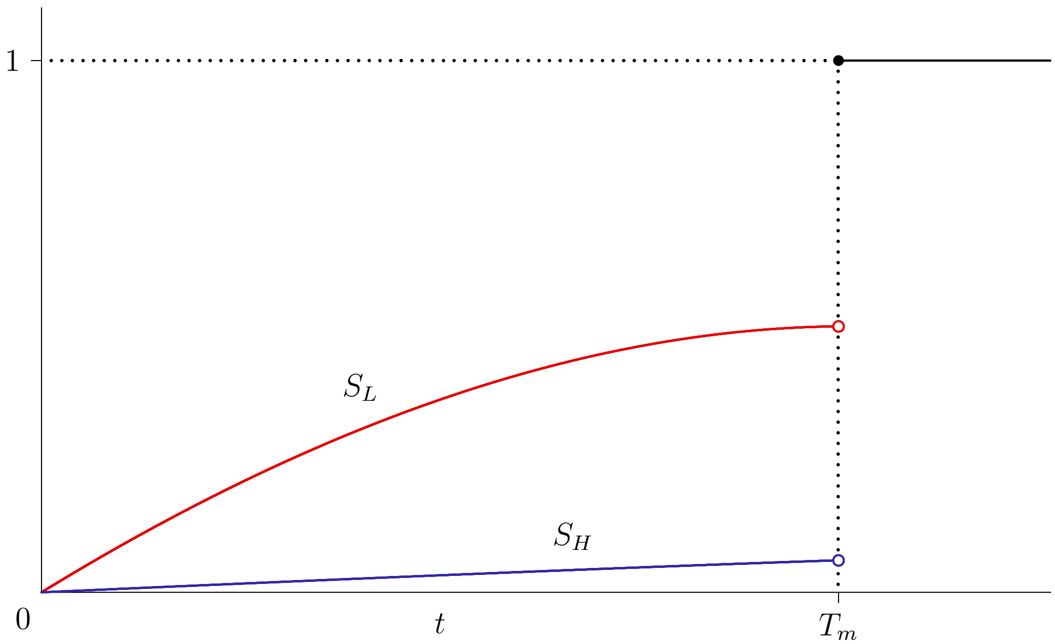

Figure 4: Selling-time Distributions in Example 3

The left-hand side of the equation above is 0 when $T_m = 0$ and unbounded as $T_m \to \infty$. Moreover, it is increasing and continuous in T_m. Thus there is a unique T_m such that $\lim_{t \uparrow T_m} S_L(t) = 1 - m$.

The remaining steps of the proof are identical to those in the proof of Proposition 3.1 and are omitted. ■

Remark B.3 *When $w_H > 0$, the equilibrium profit of the sellers in state H has to be high enough so that sellers are not better off holding on to the asset forever. If m is too high, then the price at the terminal date T_m is too low to guarantee this. But in any case, for any w_H, there is an upper bound $\overline{m} > 0$ such that there is an equilibrium for every $m < \overline{m}$. Thus even when $w_H > 0$, there is a continuum of equilibria. Of course, no equilibrium with $m > 0$ passes the NWBR requirement.*

Example 3 *The parameters are the same as in Example 1, that is, $\rho = \frac{1}{2}$, the interest rate $r = 0.05$, the arrival rate of bad news $\lambda = 0.5$ and the arrival rate of liquidity shocks $\theta = 0.01$.*

Figure 3 depicts the path of beliefs (and prices), in the absence of news, in an equilibrium in which $m = \frac{1}{2}$, that is, in state L, half the sellers wait until T_m to sell. The thick line indicates equilibrium beliefs $\beta(t)$ and observe that $\beta(T_m) < 1$. The thin line represents an upper bound to out-of-equilibrium beliefs $\beta(t)$, for $t > T_m$. If out-of-equilibrium beliefs were higher than this upper bound, then sellers in state H would wait beyond T_m. Thus, to sustain an equilibrium with $m = \frac{1}{2}$, out-of-equilibrium beliefs are bounded away from 1. Figure 4 depicts the resulting selling-time distributions.

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