Minimal Enforceability and Indirect Domination Relations in the Shapley-Scarf Economy*

Ryo Kawasaki†

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Abstract

We consider the von Neumann-Morgenstern (vNM) stable sets based on a farsighted version of the weak domination relation for the barter model with indivisible goods of Shapley and Scarf (1974). More recent research on farsighted stable sets has focused on coalitional sovereignty, a term used by Ray and Vohra (2015) regarding what coalitions can do to other coalitions, and they note the importance of this issue and how they may affect the results in characteristic function form games. However, defining a farsighted version of weak domination introduces an additional issue of what coalitions can or cannot do from the inside, an issue which needed not be analyzed for the stronger versions of indirect domination nor for myopic domination relations in the literature. We first impose a minimality condition on the coalitions that can deviate in just one step, as coalitions in this model represent trading cycles, where such a minimality condition is often imposed. Under the domain of preferences defined by Klaus, Klijn, and Walzl (2010), each Pareto efficient allocation in the core and its Pareto indifferent allocations constitute a vNM stable set defined by a farsighted weak domination that respects the minimal enforceability condition, and those are the only essentially singleton such vNM stable sets.

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†Department of Industrial Engineering and Economics, School of Engineering, Tokyo Institute of Technology, 2-12-1 O-okayama Meguro-ku, Tokyo 152-8552, Japan.

1 Introduction

In this paper, we consider the stable outcomes in the exchange economy with indivisible goods first defined in Shapley and Scarf (1974), when the agents are sufficiently farsighted. Specifically, we look at the von Neumann-Morgenstern (vNM) stable set, first introduced in von Neumann and Morgenstern (1953), that is defined by a domination relation which is the farsighted analogue of weak domination. We then look at properties of these sets and compare them to concepts in the literature.

Shapley and Scarf (1974) first consider the core, which is defined as the set of allocations that are not dominated, and they proceed to show that the core is always nonempty. Specifically, a core allocation is immune from a coalition redistributing their endowments that makes all members in the coalition better off. Moreover, the strict core, which is defined as the set of allocations that are not weakly dominated, can be empty. The main difference between weak and strong domination is that weak domination only requires one member of the coalition to be better off, while the others need not be worse off. Despite the possibility of the strict core being empty, Roth and Postlewaite (1977) argue that the core has its own shortcoming in that some core allocations may not persist since it is not immune to further redistributions after the allocation is decided upon.[1] The strict core, on the other hand, is immune to this type of redistribution, and this observation leads Roth and Postlewaite (1977) to advocate the strict core as a more suitable solution concept. Roth and Postlewaite (1977) nonetheless show that the strict core is nonempty when each agent's preferences are strict, that is, preferences do not admit any indifferences between two distinct objects. A necessary and sufficient condition for the nonemptiness of the strict core is given by Quint and Wako (2004).

Another important concept that has been extensively analyzed in this model is the competitive allocation. Its existence in every market is shown in Shapley and Scarf (1974) through the top trading cycles method attributed to David Gale, which in turn is used to establish the nonemptiness of the core. Roth and Postlewaite (1977) show that the strict core allocation coincides with the unique competitive allocation when agents' preferences are strict, and Wako (1984) show that in general, every strict core allocation is a competitive allocation, but this inclusion may be strict.

In addition to the core, there are some papers that have looked into the stable set of von Neumann and Morgenstern (1953) defined on the domination relations mentioned

[1] Roth and Postlewaite (1977) use the term "stable" to describe this property, but we avoid using this term here to distinguish this stability property from the one considered here. However, it should be noted that this stability concept is related to that of von Neumann and Morgenstern (1953). See Kawasaki (2015) for details.

above. The stable set defined by strong domination may not exist in general even when the number of agents is three as is documented in Kawasaki, Wako, and Muto (2020). The example is rather striking as there is an allocation that gives each agent his/her most favorite good, and each agent has strict preferences. On the other hand, the stable set defined by weak domination (wdom stable set) exists under the domain of strict preferences, as Roth and Postlewaite (1977) show that the strict core is the unique wdom stable set. In general, Wako (1991) shows that as long as the strict core is nonempty, it is the unique wdom stable set. Furthermore, Wako, Matsumoto, and Irisawa (2007) show that in general, the existence of wdom stable sets can be established when the number of agents is less than or equal to four. They also provide an example with five agents where only agent exhibits indifferent between two goods in the preference list, and as a result there is no wdom stable set. Moreover, Wako et al. (2007) also provide an example with a unique wdom stable set that consists of allocations violating individual rationality, that is, some agent is assigned a good that he/she likes less than the initial endowment.

The aforementioned results in the literature use domination relations that are inherently myopic; each player only takes into account the outcome (in this case, allocation) that is immediately realized as a result of that player's actions. In this paper, we focus on the farsighted analogue of the domination relations above to analyze the situation in which players are farsighted that they can take into account the possibility of other players forming coalitions to redistribute their endowments. Harsanyi (1974) and then later by Chwe (1994) for a more general model, define the concept of indirect domination to incorporate farsightedness of the agents into the model. Informally, an allocation indirectly dominates another if the former can be reached from the latter through possibly multiple trades among agents, and agents engaged in these trades are better off in doing so at the final outcome. Klaus et al. (2010) show that the set of competitive allocations also coincides to the farsighted version of the strong domination under a very weak assumption on preferences. Moreover, this concept has also been applied to similar models such as Klaus, Klijn, and Walzl (2011) for roommate problems and Mauleon, Vannetelbosch, and Vergote (2011) for two-sided matching problems. In those two papers, the results connect the singleton vNM farsightedly stable sets to core matchings.

This paper attempts to complete the missing parts of the analysis involving farsighted behavior in order to provide a farsighted analogue of the analysis in Greenberg (1992), which analyze myopic domination relations and how they affect the vNM stable sets in an abstract system, and in Roth and Postlewaite (1977), which analyze the different domination relations in Shapley and Scarf (1974). We first note that the farsighted analogue of weak domination allows some undesirable sequences of deviations. Specifically, since weak domination only requires one agent to be better off, we can have a coalition consisting of multiple trading cycles deviating, almost all of whom may be indifferent between the final allocation and the current allocation but may be required to deviate according to the definition of the farsighted domination relation based on weak domination. Therefore, we impose a minimality condition of coalitions that can deviate much like there is one for the definition of trading cycles and implicitly one for myopic domination relations, which have in mind deviations by a coalition consisting of one trading cycle. We find that this minimality condition does affect the underlying indirect domination relation based on weak domination but not on farsighted analogues of other domination relations considered in Kawasaki (2010) and Klaus et al. (2010).

After imposing this minimality condition, we analyze the vNM stable sets defined by the farsighted analogue of weak domination that satisfies the minimal enforceability condition. We show that under a weak condition on preferences used in Klaus et al. (2010), any generically singleton set is a vNM stable set of this domination relation if and only if it is a Pareto efficient allocation in the core defined by strong domination. This result establishes the existence of vNM stable sets defined on this indirect domination. Also, it gives a game-theoretic interpretation of the set of allocations that are in the core and Pareto efficient – some of which have been obtained through the modifications of the top trading cycle as given by Alcalde-Unzu and Molis (2011), Jaramillo and Manjunath (2012), and Aziz and De Keijzer (2012) and some through the nonconstructive method of Sotomayor (2005).

The rest of the paper is organized as follows. In Section 2, we define the model and the domination relations used in the analysis. In Section 3, we present the main results of the paper. In Section 4, we show that the minimality condition only plays a key role with weak domination, as the results are unchanged for the other domination relations in the literature. Section 5 provides concluding remarks.

2 The Model and Definitions

Let $N = \{1, 2, \cdots, n\}$ be the set of agents, each endowed with an indivisible good. Because the endowment is fixed throughout the paper, with a slight abuse of notation, let N also denotes the set of indivisible goods in the market where, for example, good 1 is interpreted as the good initially endowed to agent 1.

An *allocation* x is then defined to be a bijection defined on the set N mapped to itself where $x(i) = j$ denotes that agent i is assigned the good initially owned by agent j. Let A be the set of allocations. Each allocation x partitions the set N into trading cycles, where a trading cycle is denoted by the set $K = \{i_0, i_1, \cdots, i_{k-1}, i_k = i_0\}$ where each agent in the set K is distinct (except i_k) and $x(i_j) = i_{j+1}$ for each $j = 0, \cdots, k-1$ modulo k. Each agent i is then included in exactly one trading cycle. Denote this trading cycle, defined by allocation x, by TC_i^x. Denote by e, the allocation that assigns to each agent his/her initial endowment. That is, $e(i) = i$ for all i. An allocation x is said to be individually rational if $x(i) \succsim_i i$ for all $i \in N$.

For a coalition $S \subseteq N$, define $x(S) = \{x(i) : i \in S\}$ the set of goods allocated to S in x. An allocation x is *dominated* by another allocation y in the classical sense if $y(S) = S$ and $y(i) \succ_i x(i)$ for all $i \in S$.[2] An allocation x is *weakly dominated* by allocation y in the classical sense if $y(S) = S$ and $y(i) \succsim_i x(i)$ for all $i \in S$ and $y(i) \succ_i x(i)$ for some $i \in S$. Furthermore, since S typically denotes agents in one trading cycle, it is often assumed, sometimes implicitly, that S is a minimal coalition satisfying $y(S) = S$ – that is, there is no proper subset T of S satisfying $y(T) = T$. If no confusion occurs, we drop the phrase "in the classical sense" throughout the paper, but it should be noted that these myopic domination relations are not equivalent to one-step deviations of the indirect domination relations that will be defined later.

The *core* is defined as the subset C of allocations such that no allocation in C is dominated. The *strict core* is defined as the subset SC of allocations such that no allocation in SC can be weakly dominated. The nonemptiness of the core holds in all markets and can be proved by the top trading cycles method, which is attributed to Gale and is explained in Shapley and Scarf (1974). In fact, the method is known to generate the set of competitive allocations, which is a subset of the core and contains the strict core.

Further definitions regarding allocations are also used in this paper. An allocation x is *Pareto efficient* if there is no allocation y such that $x(i) \precsim_i y(i)$ for all $i \in N$ with the relation holding strictly for some $i \in N$. Equivalently, x is Pareto efficient if it is not weakly dominated by another allocation y via N. Two allocations x and y are said to be *Pareto indifferent* if $x(i) \sim_i y(i)$ for all $i \in N$. If x and y are not Pareto indifferent, then they are said to be *Pareto distinct*. For each allocation x, denote by $I(x)$ the set of allocations y such that x and y are Pareto indifferent. That is, $I(x) = \{y \in A : x(i) \sim_i y(i),\ \forall i \in N\}$.

To define the farsighted counterparts of the domination relations, we need the concept of enforceability, which dictates what allocations each coalition can induce by themselves. The following definition is from Kawasaki (2010) and also used in Klaus et al. (2010).

[2]This terminology is from Klaus et al. (2010) in referring to the domination relation used in the literature, such as in Shapley and Scarf (1974).

For each $S \subseteq N$, the enforceability relation $\to_S$ is a binary relation defined on the set of allocations such that $x \to_S y$ if the following three conditions hold:

(E1) $y(S) = S$

(E2) $y(i) = i$ if $i \in N \setminus S$ and $TC_i^x \cap S \neq \emptyset$

(E3) $y(i) = x(i)$ if $i \in N \setminus S$ and $TC_i^x \cap S = \emptyset$

The first condition states that allocation y must be such that coalition S forms a trading cycle under the new allocation y. The second and third conditions lay down rules for the allocations of those outside S. Those affected by the move by S, that is, those players who were in the same trading cycle as someone from S in allocation x receives his/her own endowment. Those not affected by S, which are players who were not in the same trading cycle as someone in S, receive the same good as before in x. The third condition reflects the concept of coalitional sovereignty coined by Ray and Vohra (2015).

Using this binary relation, we can define the concept of indirect domination using the enforceability relation $\to$ that satisfies (E1)-(E3) along the same lines as Chwe (1994). Denote the resulting indirect domination relation by $\triangleright$, where $y \triangleright x$ denotes that y indirectly dominates x. The relation $y \triangleright x$ holds if there exist a sequence of coalitions $T^0, T^1, \cdots, T^{K-1}$ and allocations $x^0, x^1, \cdots, x^K$ such that $x^0 = x$, $x^K = y$, and for each $k = 0, 1, \cdots, K-1$ the following two conditions hold: (a) $x^k \to_{T^k} x^{k+1}$, and (b) $x^k(i) \precsim_i y(i)$ for all $i \in T^k$ and $x^k(j) \prec_j y(j)$ for some $i \in T^k$.[3]

Before moving on, we note that weak domination in the classical sense is not equivalent to the relation $\triangleright$ holding with a one-step sequence (that is, $K = 1$). The difference is due to condition (E3), which reflects coalitional sovereignty. See Kawasaki (2010) for the motivating example for that paper under a different domination relation, which will be discussed in Section 4. A similar point is also made for two-sided matching problems in Herings, Mauleon, and Vannetelbosch (2017).

As was mentioned in the introduction and the previous paragraphs in this section, while this relation does capture the coalitional sovereignty regarding the allocations to the agents outside the deviating coalition, the above enforceability relation does not impose many conditions to the deviating coalition itself. In the context of the problem considered here, it does not distinguish between S being just one trading cycle or a union of separate trading cycles. As opposed to the classical domination relation which usually require the

[3] Implicitly, in this sequence, the endowment of the market is fixed throughout. Instead, one can consider a sequence where the endowment changes in each step according to the trades that are carried out. A version of such domination relation restricted to bilateral trades is considered in Kamijo and Kawasaki (2010).

deviating coalition to be a trading cycle, it is not made clear whether the coalition of agents satisfying (E1) is a minimal one or not.

To make this distinction much clearer, we consider an alternative enforceability relation, denoted by $\to'$ which in addition to conditions (E1)-(E3) also satisfies a minimality condition. Formally, $x \to'_S y$ holds if $x \to_S y$ and the following minimality condition (E4) holds.

(E4) There does not exist $T \subsetneq S$ such that $y(T) = T$.

Therefore, if $x \to'_S y$ were to hold, there cannot exist a sequence such as $x \to'_{S_1} \cdots \to'_{S_k} y$ where $\{S_1, S_2, \cdots, S_k\}$ is a partition of S, since otherwise, this would contradict the minimality condition (E4) with respect to S.

The condition (E4) is closely related to the noncooperative version of the indirect domination for strategic games as considered in Nakanishi (2009) for the prisoner's dilemma game or considering only merging or splitting of two coalitions in coalition formation games as in Funaki and Yamato (2014). It also is the closest analogue to enforceability in the matching problems, roommate problems, and network formation games where only singletons and pairs are considered. Because there is no general consensus as to what the minimal set of players should be for the Shapley-Scarf model, we instead impose a minimality condition as a close substitute. We can then define an indirect domination relation using this modified enforceability relation $\to'$, denoted by $\blacktriangleright$. The relation $y \blacktriangleright x$ holds if the conditions for $y \rhd x$ hold using the relation $\to'$ instead of $\to$. In what follows, we focus on the relation $\blacktriangleright$ rather than $\rhd$. The reason for this will be made clear in the next section.

Given a domination relation, we can define the von Neumann-Morgenstern (vNM) stable set. A subset V of allocations is a *vNM stable set* with respect to the domination relation $\blacktriangleright$ or simply a *vNM* $\blacktriangleright$ *stable set* if it satisfies the following two conditions:

- For every $x, y \in V$, $x \blacktriangleright y$ does not hold.
- For every $x \notin V$, there exists $y \in V$ such that $y \blacktriangleright x$.

The first condition is typically called internal stability, while the second condition is typically called external stability. vNM stable sets defined by general domination relations may not exist in general. On the other hand, we show that certain types of vNM stable sets defined by the domination relation $\blacktriangleright$ do exist under a weak condition on preferences.

3 Results

In this section, we first note the difference between the two domination relations $\triangleright$ and $\blacktriangleright$ defined in the previous section. We first present a result that gives a necessary and sufficient condition for the relation $x \triangleright y$ to hold. Condition (a) below prevents the possibility of y to dominate x via some coalition S, while condition (b) prevents of y to weakly dominate x via N. This result resembles closely to analogous results shown in Lemma 1 of Mauleon et al. (2011) and Proposition 1 of Klaus et al. (2011) in characterizing the indirect domination relation for matching and roommate problems using a myopic domination relation.

Proposition 1. *Let x and y be individually rational allocations. Then, $x \triangleright y$ if and only if both of the following two conditions hold: (a) for every trading cycle TC_k^y of y, there exists some $i \in TC_k^y$ such that $x(i) \succsim_i y(i)$ and (b) for some $i \in N$, $x(i) \succ_i y(i)$.*

Proof. First, we show the "if" part. For each trading cycle TC_k^y, take each member i_k such that $x(i_k) \succsim_{i_k} y(i_k)$ with strict inequality for some i_k. Label the collection of these players as $S = \{i_1, i_2, \cdots, i_K\}$. This set of agents is nonempty by the conditions (a) and (b) above. Then, $y \to_S e$, where $e = (1, 2, \cdots, n)$ represents the initial endowment. This move is carried out by each player $i \in S$ choosing his/her own endowment. Now, since x is individually rational, $x(i) \succsim_i i$ for all $i \in N$. Moreover, since y is also individually rational and by (b), $x \neq e$ and that for some $i \in S$, $x(i) \succ_i y(i) \succsim_i i = e(i)$. Then, by adjoining $e \to_N x$ to the previous move establishes the sequence $y \to_S e \to_N x$ that satisfies the conditions for the relation $x \triangleright y$ to hold.

Conversely, suppose that $x \triangleright y$ holds. When condition (b) is not satisfied, then there cannot exist $i \in N$ with $x(i) \prec_i y(i)$, so that the conditions in the $\triangleright$ relation cannot be satisfied for the first coalition in the sequence. Thus, condition (b) must be satisfied. Now, suppose that condition (b) is satisfied, but condition (a) is not satisfied. That is, there exists some trading cycle T of y such that $x(i) \prec_i y(i)$. In order to induce an allocation z with $z(i) \neq y(i)$ for some $i \in T$, i must be included in some coalition in the sequence $T^0, \cdots, T^{K-1}$ with $z^k \to_{T^k} z^{k+1}$ for $k = 0, \cdots, K-1$. Let k^* be the first coalition that includes any member of T. Then, $z^{k^*}(i) = y(i)$ for all $i \in T$, according to the rules of $\to$. By definition of the $\triangleright$ relation, $z^{k^*}(i) = y(i) \precsim_i x(i)$ for all $i \in T^{k^*}$. By assumption, $y(i) \succ_i x(i)$, which is a contradiction. Thus, condition (a) must also be satisfied. □

Because $x \blacktriangleright y$ implies $x \triangleright y$, the "only if" part of Proposition 1 holds for $x \blacktriangleright y$ as well. We state the fact, without proof because of its straightforwardness, in the following corollary.

Corollary 1. *Let x, y be two individually rational allocations such that $x \blacktriangleright y$ holds. Then, conditions (a) and (b) of Proposition 1 hold.*

Now, let us examine the sequence used in the proof of the "if" part of Proposition 1. First, each member of S leaves the trading cycle that he/she is part of, by switching to his/her endowment, which by doing so members of S are weakly better off in the end by (a) with one agent better off by (b). Then, all the players form the trading cycles in x, with one player being strictly better off by condition (b). However, in the first step, it may be that the agent $i \in S$ with $x(i) \succ_i y(i)$ has $y(i) = i = e(i)$, so that in the first step, this agent i really does not "move," as he/she receives the same good before and after and is not involved in the move at all. However, this agent may be vital in fulfilling the condition, $x(i) \succ_i y(i)$, if i is the only such agent. This fact can be illustrated in the following example. It also highlights why the "if" part of Proposition 1 may fail for the relation $\blacktriangleright$.

Example 1. Consider the following market with $N = \{1, 2, 3\}$.

$$3 \succ_1 2 \succ_1 1$$
$$1 \succ_2 3 \succ_2 2$$
$$2 \succ_3 1 \succ_3 3$$

Consider the allocations $x = (1, 3, 2)$ and $y = (2, 3, 1)$. By Proposition 1, $y \rhd x$ holds. This fact can be checked explicitly by constructing the sequence

$$x \rightarrow_{\{1,2\}} e \rightarrow_N y.$$

This same sequence cannot be constructed using $\rightarrow'$ since in the first move, there exists a proper subset T of S (either $\{1\}$ or $\{2\}$) with $e(T) = T$.

Now we claim that $y \blacktriangleright x$ cannot hold. The first move cannot come from $\{1\}$ as that move cannot change the allocation x. With agent 3 not willing to be involved in the first move, the only candidates are $\{2\}$ and $\{1, 2\}$. The coalition $\{2\}$ does not satisfy the second condition in $\blacktriangleright$, which leaves $\{1, 2\}$ as the only candidate. Moreover, the only allocation that this coalition can induce from x is $x^1 = (2, 1, 3)$. Now, from x^1, following the same logic as above, the only coalition that can move according to the definition of $\blacktriangleright$ is $\{1, 3\}$, which leads to the allocation $x^2 = (3, 2, 1)$. From x^2, using the same logic, we can show that the only coalition that move this time is $\{2, 3\}$. However, the allocation that results then is x, where we first started. Thus, we cannot construct a sequence where $y \blacktriangleright x$ holds. Therefore, the "if" part of Proposition 1 does not hold for the relation $\blacktriangleright$. Note that allocations y and allocation x are individually rational but not in the core. However, whether the associated allocations are in the core or not is inconsequential as the same conclusions can be drawn if we replace agent 2's preferences with the following:

$$1 \sim_2 3 \succ_2 2.$$

In this case, $x = (1, 3, 2)$ and $y = (2, 3, 1)$ are both in the core with the relation $y \triangleright x$ still holding, while by the same logic as in the previous case, $y \blacktriangleright x$ still does not hold. Nonetheless, the first example is important in that a very strong assumption in strict preferences still cannot ensure the equivalence of the two relations $\triangleright$ and $\blacktriangleright$. □

Proposition 1 provides a nice connection between a farsighted dominance relation in $\triangleright$ and a myopic domination relation, and similar results in Mauleon et al. (2011) and Klaus et al. (2011) exploit this relationship to derive their main results. However, in the Shapley-Scarf model, this connection is grounded on an enforceability relation that may not be conceptually satisfying, the consequences of which we have seen in Example 1. Therefore, despite the technical difficulties that stem from Proposition 1 no longer holding, we now focus on the domination relation $\blacktriangleright$, which satisfies the minimality condition (E4). We show in the following that we can still obtain some nice results involving vNM stable sets with respect to $\blacktriangleright$.

The next statement shows that allocations in vNM $\blacktriangleright$ stable sets must be individually rational. This result is in sharp contrast to what is shown in Wako et al. (2007), where there is a unique vNM stable set with respect to weak domination where all of its allocations are not individually rational.

Proposition 2. *Every allocation in a vNM $\blacktriangleright$ stable set is individually rational.*

Proof. We provide a proof for $\blacktriangleright$, but the same proof can be used for the $\triangleright$ relation as well. Let V be a vNM $\blacktriangleright$ stable set that includes an allocation x that is not individually rational. Then, for some $i \in N$, we have $x(i) \prec_i i$. Consider the move by i described by the relation $x \rightarrow_i x^1$, where by (E1) of the enforceability condition, $x^1(i) = i$. Now, x^1 indirectly dominates x by this one-step sequence, which implies by internal stability of V, $x^1 \notin V$. By external stability, there exists $y \in V$ such that $y \blacktriangleright x^1$. Because $x^1(i) = i$, by condition (E3), if i is not included in the sequence that establishes the domination relation between y and x^1, then $y(i) = i$. By the same logic used in Chwe (1994), $y \blacktriangleright x$ must hold, which contradicts the internal stability of V.[4] Otherwise, if i is included in

[4]The formal statement of the fact that is used here is the following: "If allocations a and b with

the sequence, let k be the first integer such that $i \in T^k$ so that $x^k(i) = i$. By definition of $\blacktriangleright$, we must have $i = x^k(i) \precsim_i y(i)$, so that relating y and x, we have $x(i) \prec_i y(i)$. Once again, we have that $y \blacktriangleright x$, which is a contradiction. Thus, for every $x \in V$, we must have $x(i) \succsim_i i$. $\square$

Before moving on, we note a very simple fact, that will be used quite often in the following. We mention this result more as reference that is needed to keep the proof of the main result succinct.

Lemma 1. *Let x and y be individually rational allocations such that x is Pareto dominated by y. Then, there exists $z \in I(y)$ such that $z \blacktriangleright x$.*

Proof. Consider first the case in which for every trading cycle of y, there exists some i such that $y(i) \succ_i x(i)$. Labeling these trading cycles as $T_1, T_2, \cdots, T_k$, $y \blacktriangleright x$ by forming the trading cycles one by one. Note that since y Pareto dominates x, $y(i) \succsim_i x(i)$ for all $i \in T_1$ with strict preference holding for some i by assumption. The argument for the other trading cycles follows in the same way. Thus, $y \blacktriangleright x$ holds via the sequence $x \rightarrow'_{T_1} \cdots \rightarrow'_{T_k} y$.

For all other cases, let $\hat{S} \equiv \{i \in N : y(i) \succ_i x(i)\}$ and consider the set $TC^y(\hat{S}) \equiv \bigcup_{i \in \hat{S}} TC^y(i)$. In this case, $N \setminus TC^y(\hat{S}) \neq \emptyset$ holds so that for all $j \in N \setminus TC^y(\hat{S})$, $y(j) \sim_j x(j)$. Labeling the trading cycles $\{TC^y(i)\}_{i \in \hat{S}}$ as $T_1, T_2, \ldots, T_k$, form these trading cycles one by one from x just as in the previous case. Denote z as the resulting allocation. Then, $z(j) = y(j)$ for all $j \in TC^y(\hat{S})$, and for each $i \in T_k$, $z(i) = y(i) \succ_i x(i)$. For all $j \in N \setminus TC^y(\hat{S})$, we have $z(j) = x(j) \sim_j y(j)$. Thus, $z \in I(y)$ and $z \blacktriangleright x$ hold, proving the lemma. $\square$

An immediate consequence is the following.

Corollary 2. *Let x be an individually rational allocation such that $x \notin I(e)$. Then, there exists $y \in I(x)$ such that $y \blacktriangleright e$.*

In the next proposition, we show that the strict core, if nonempty, is a vNM $\blacktriangleright$ stable set. If it is empty, it cannot satisfy the external stability condition of a vNM stable set, and hence cannot be a vNM stable set with respect to any domination relation. This result is a direct parallel to what is shown in Wako (1991), where the strict core, if nonempty, is the unique vNM stable set with respect to weak domination. Note that we

coalition $S \subseteq N$ satisfy $a \rightarrow'_S b$ and if allocation c is such that $c \blacktriangleright b$ holds along with $a(i) \precsim_i c(i)$ for all $i \in S$ and $a(j) \prec_j c(j)$ for some $j \in S$, then $c \blacktriangleright a$ must also hold." This fact is an analogue of a property of a farsighted domination relation that is also noted in p. 303 of Chwe (1994) and can be proved similarly by using the definition of $\blacktriangleright$.

cannot use the external stability property of the result there as weak domination in the classical sense does not necessarily imply domination signified by $\blacktriangleright$. See the motivating example in Kawasaki (2010), which can be adapted for the weak domination as well.

Proposition 3. *The strict core, if nonempty, is a vNM* $\blacktriangleright$ *stable set.*

Proof. Suppose that the strict core SC is nonempty. First, note that for any $x \in SC$, $SC = I(x)$. The inclusion $SC \supseteq I(x)$ follows from the definition of the strict core, and $SC \subseteq I(x)$ follows from Wako (1991). Thus, internal stability follows since neither $x \blacktriangleright y$ nor $y \blacktriangleright x$ can hold.

Note again that the "if" part of Proposition 1 cannot be used, so we need to construct a different sequence to establish the relation. Let x be an allocation in SC and take any $y \notin SC$. Since, x is in the strict core, x is also in the core, and for every trading cycle of y, there exists j such that $y(j) \precsim_j x(j)$. Moreover, since x is also Pareto efficient by being in the strict core, for some $i \in N$, $y(i) \prec_i x(i)$. Let S be the coalition that takes all such agents, and form a trading cycle, within which there is no proper subset that can constitute a trading cycle. Call the resulting allocation y^1 so that we now have $y^1(S) = S$. If S is a trading cycle under x, then assign the goods under S and then construct the other trading cycles under x unless there is a trading cycle in which every agent is indifferent between the good assigned by x and the initial endowment.[5] Note that since x is in the strict core, we have either that for some $j \in S$, $y^1(j) \prec_j x(j)$ or for all $j \in S$, $y^1(j) \sim_j x(j)$. In the former case, the relation $x' \blacktriangleright y$ can then be established by the sequence for some $x' \in I(x) = SC$:

$$y \to_S y^1 \to_j e \to \cdots \to x'.$$

The rest of the above sequence involves constructing trading cycles according to the proof of Corollary 2.

Now, consider the latter case. Under the assumption that S is not a trading cycle or a union of trading cycles under x, there exists $j \in N \setminus S$ such that $x(j) \in S$. However, if for all $j \in N \setminus S$ with $x(j) \in S$ we have $x(j) \sim_j j$, then assign to all j the good that he/she initially owned and construct all other trading cycles according to x to obtain an allocation $x' \in I(x)$ with $x' \blacktriangleright y$. Otherwise, for some j, we have $x(j) \succ_j j$. Label this j as j_1 and $x(j)$ as j_0 and let $\{j_0, j_1\}$ induce an allocation y^2 from y^1 with $y^2(j_1) = j_0$ and $y^2(j_0) = j_1$

[5] This possibility of this exception arises every time we consider reconstructing the trading cycles under x, but since this yields another allocation $x' \in I(x)$, which is in the set we are analyzing, in the remainder of the paper, we omit this phrase regarding possible indifferences between the allocation and endowment whenever it is sufficiently clear.

and $y^2(k) = k$ for all other k. Note that since $y^2(j_1) = x(j_1)$ by construction, we must have $y^2(j_0) \precsim_{j_0} x(j_0)$. Otherwise, y^2 weakly dominates x via $\{j_0, j_1\}$ contradicting that $x \in SC$. In all subsequent parts of this proof, if (a) the constructed trading cycle is in x or if (b) some player (in the current case, j_0) strictly prefers what he/she obtains at x over the current one, then the proof is complete either by constructing all the other trading cycles in x in the former case as in Corollary 2, or by inducing the initial endowment and then forming all the trading cycles in x using Corollary 2 in the latter case. This leaves out the case in which for the trading cycle considered where $y^2(j_1) \sim_{j_1} x(j_1)$ and $y^2(j_0) \sim_{j_0} x(j_0)$ with $y^2(j_0) \neq x(j_0)$.

By the same logic as before, there must exist $j \in N \setminus \{j_0, j_1\}$ where $x(j) \in \{j_0, j_1\}$. Since $x(j_1) = j_0$, $x(j) = j_1$. Label this new j as j_2 and consider the allocation y^3 with $y^3(j_1) = y^2(j_1) = j_0$, $y^3(j_2) = j_1$, and $y^3(j_0) = j_2$. As before, we must have $y^3(j_0) \precsim_{j_0} x(j_0)$. Otherwise, y^3 weakly dominates x via $\{j_0, j_1, j_2\}$, contradicting that $x \in SC$. If either condition (a) or (b) in the previous paragraph is satisfied, then the proof is complete by the same logic. So, if neither is satisfied, we must have $y^3(j_0) \sim_{j_0} x(j_0)$ with $y^3(j_0) \neq x(j_0)$. Therefore, there must exist some $j \in N \setminus \{j_0, j_1, j_2\}$ with $x(j) = j_2$.

If this above process is repeated while neither condition (a) nor (b) is satisfied, the process must terminate when all players are involved in this process. However, that would mean that allocation x results, and the resulting sequence of allocations $y^1, y^2, \cdots$ are part of the sequence that establishes $x \blacktriangleright y$. Thus, external stability holds. □

While the result is not entirely surprising, the proof is very involved primarily due to the minimality condition (E4) imposed on the relation $\blacktriangleright$ and the coalitional sovereignty condition (E3). One caveat to this result is that it does not say that the strict core, when nonempty, is the unique vNM stable set with respect to $\blacktriangleright$. In fact, in Example 2 below, we show that there exists another vNM $\blacktriangleright$ stable set.

Example 2. Let $N = \{1, 2, 3\}$, and consider the following preferences of the agents.

$$3 \succ_1 2 \succ_1 1$$
$$3 \succ_2 1 \succ_2 2$$
$$2 \succ_3 1 \succ_3 3$$

Consider the allocations $x = (1, 3, 2)$ and $y = (2, 3, 1)$, where $x \in SC$ but $y \notin SC$. Nonetheless, $y \blacktriangleright x$ indeed holds. Using the same logic as in Example 1, the first move involves $x \rightarrow_{\{1,2\}} x^1 = (2, 1, 3)$. Note that $y(1) = x^1(1)$, while $y(i) \succ_i x^1(i)$ for $i = 2, 3$. Thus, the sequence $x \rightarrow_{\{1,2\}} x^1 \rightarrow_N y$ establishes $y \blacktriangleright x$. Likewise it can be shown that

$\{y\}$ is a vNM $\blacktriangleright$ stable set. $\square$

Now, under a weak assumption on preferences, we can show a stronger result, that an essentially singleton set consisting of a Pareto efficient core allocation is a vNM $\blacktriangleright$ stable set, and moreover, they are the only such essentially singleton vNM $\blacktriangleright$ stable sets.

We make the following assumption on preferences, also used in Klaus et al. (2010):

$$\text{for every } i, j \in N \ (i \neq j), \ j \succ_i i \text{ or } j \prec_i i. \tag{R}$$

This restriction on preferences is not particularly strong, as strict preferences – one in which each individually finds no two distinct goods to be indifferent – satisfies this restriction as well. Moreover, results for two-sided matching (Mauleon et al. (2011)) and roommate problems (Klaus et al. (2011)) are proved under the restriction that preferences are strict. In the literature related to the housing market, the 0-1 preferences defined in the pairwise kidney exchange model of Roth, Sönmez, and Ünver (2005) also satisfy this condition. The (-1)-0-1 preferences in Wako et al. (2007), which is an extended version of the 0-1 preferences including the possibility of a good being less preferred to the initial endowment, also satisfies this condition. Wako et al. (2007) show that if all individuals have (-1)-0-1 preferences, then the housing market always admits a wdom stable set.

The condition (R) alone is not sufficient to restore Proposition 1 for the relation $\blacktriangleright$, as is shown in Example 1. Nonetheless, Example 2 showed that for allocation$y$, $y \blacktriangleright x$ still holds. Notice that allocation y in Example 2 is a core allocation that is also Pareto efficient. We generalize this result under the preference domain (R) below. That is, we show in the following that if x is an allocation that is in the core and also Pareto efficient, then $I(x)$ is a vNM $\blacktriangleright$ stable set. Moreover, those are the only essentially singleton vNM $\blacktriangleright$ stable sets.[6] These results hold despite the fact that the "if" part of Proposition 1 still fails under the preference assumption in (R).

Theorem 1. *Suppose that each agent's preferences satisfies (R). Then, the set of allocations $V = I(x)$ is a vNM $\blacktriangleright$ stable set if and only if x is Pareto efficient and lies in the core.*

Proof. Suppose that $V = I(x)$ is a vNM stable set with respect to $\blacktriangleright$. Fix any $y \notin I(x)$. By external stability, there exists $x' \in I(x)$ such that $x' \blacktriangleright y$. By Corollary 1, conditions (a) and (b) of Proposition 1 hold. By (a), x' cannot be dominated by any allocation. By (b), x' cannot be weakly dominated by an allocation via N. Since $x' \in I(x)$, both of

[6] The term essentially singleton is from Sönmez (1999) in describing such sets.

these statements also hold for x. Thus, x must be a core allocation that is also Pareto efficient.

Now, suppose that x is Pareto efficient and is in the core. We now show that $V = I(x)$ is a vNM stable set with respect to $\blacktriangleright$. The internal stability of $I(x)$ is clear, so it suffices to prove external stability, that is, for any $y \notin I(x)$, there exists $x' \in I(x)$ with $x' \blacktriangleright y$.

As noted in the parts preceding this theorem, the "if" part of Proposition 1 cannot be used, so we need to construct a different sequence to establish the relation. Since, x is in the core, for every trading cycle of y, there exists j such that $y(j) \precsim_j x(j)$. Moreover, since $y \notin I(x)$ and x is Pareto efficient, for some $i \in N$, $y(i) \prec_i x(i)$. Let S be the coalition that takes all such agents, and form a trading cycle, within which there is no proper subset that can constitute a trading cycle. Note that $S \neq N$ must hold since x is Pareto efficient. Call the resulting allocation y^1 so that we now have $y^1(S) = S$. If S is also a trading cycle in x, then choose such y^1 where $y^1(i) = x(i)$ for all $i \in S$. In that case, the proof is complete by using Corollary 2 and the individual rationality of the core allocation x to the housing market restricted to the set $N \setminus S$. The same logic can be used to complete the proof in the case when $y^1(i) \sim_i x(i)$ for all $i \in S$. It then remains to consider the case when S is not a trading cycle in x and for some $i \in S$, $y^1(i) \sim_i x(i)$ does not hold.

Suppose now that S itself is not a trading cycle in x, but it is a union of smaller trading cycles in x. In this case $x(N \setminus S) = N \setminus S$ holds. Also, for allocation y^1 defined above, we must have $y^1(j) = j$ for all $j \notin S$, by condition (E2) and the fact that S includes at least one member from each non-singleton trading cycle in y. Since x is Pareto efficient, there must exist some $i \in S$ such that $y^1(i) \prec_i x(i)$. Otherwise, if for all $i \in S$, we have $y^1(i) \succsim_i x(i)$ with strict preference holding for some i (which we have assumed from the previous paragraph), then by the concatenation of y^1 with x by assigning goods in y^1 for those in $i \in S$ and x for $i \in N \setminus S$ yields an allocation that Pareto dominates x, which is a contradiction. Now, for agent i with $y^1(i) \prec_i x(i)$, have i then choose to break off of this trading cycle by claiming his/her endowment, so that now, every agent in S has his/her endowment. Recalling that every agent in $N \setminus S$ has his/her endowment from the first move, we are at allocation e as a result. By Corollary 2, the proof for this case is complete.

Another simple case arises when y^1 is not individually rational so that for some $i \in S$, $y^1(i) \prec_i i$. From y^1, have i induce e and use Corollary 2 again. This logic holds whenever any of the allocations that we consider here is not individually rational. Thus, in the following, assume that all of the allocations that we consider in the sequence we are building are individually rational.

For all other cases, because x is a core allocation, we must have some $i \in S$ such that $y^1(i) \precsim_i x(i)$. Otherwise, we have $y^1(S) = S$ and $y^1(i) \succ_i x(i)$ for all $i \in S$, which is a restatement of y^1 dominating x. Note that this holds for any choice of y^1 that satisfies $y^1(S) = S$. Now, take such $i \in S$ and label as i_0.

One simple case is when $y^1(i_0) \prec_{i_0} x(i_0)$. In this case, the next move is for i_0 to break off the trading cycle S to form a singleton trading cycle, thereby inducing the allocation e. The proof is then complete by using Corollary 2. In the following, we assume that $y^1(i_0) \sim_{i_0} x(i_0)$ for i_0 and all similar cases to that of i_0.

If there is some $j \in N \setminus S$ such that $\{i_0, j\}$ is a trading cycle in x, then form that coalition, and the remainder of the sequence involves forming trading cycles in x since $y^1(j) = j \prec_j x(j)$, thereby establishing the desired result (see the proof of Corollary 2). At any point in the following argument, if we can find such a doubleton set, then the proof is complete by using this logic. Thus, in what follows, we assume that we cannot find such doubletons that are themselves trading cycles in x.

First, note that there is some $j \in N \setminus \{i_0\}$ such that $j \prec_j x(j)$. Otherwise, if for all $j \in N \setminus \{i_0\}$, $j \sim_j x(j)$ holds, then by the Pareto efficiency of x, either $y^1(i) \sim_i x(i)$ for all $i \in S$ or there exists $i \in S$ with $y^1(i) \prec_i x(i)$. The former case implies $y^1 \in I(x)$ completing the proof, while the latter case also yields our conclusion immediately as stated before.

Now, there are two cases to consider for agent j: $j \notin S$ and $j \in S$.

Case 1: Suppose that $j \notin S$. Let y^2 be such that $y^2(i_0) = j$, $y^2(j) = i_0$, and $y^2(h) = h$ for $h \notin \{i_0, j\}$. Then, the proof is complete if also $j \precsim_{i_0} x(i_0)$. To see this, since i_0 and j must ultimately be in different trading cycles, each of those trading cycles contains some agent receiving his/her endowment under y^2, which is strictly less preferred to the good allocated under x by the domain restriction (R). Apply Corollary 2 to form the trading cycles to obtain x.

Suppose that this is not the case, and let $j = i_1$. Then, we have the following

$$i_0 \precsim_{i_1} x(i_1),$$
$$i_1 \succ_{i_0} x(i_0).$$

Now, consider i_1. By the same logic as before with i_0, there exists $j \neq i_1$ such that $i_1 \precsim_j x(j)$. If we have $j \precsim_{i_1} x(i_1)$, then we are done. Thus, suppose that $j \succ_{i_1} x(i_1)$, relabel $j = i_2$, and consider the allocation y^2 with $y^2(i_1) = i_2$, $y^2(i_2) = i_1$, and $y^2(h) = h$ for all $h \notin \{i_1, i_2\}$. As before, for i_2, there exists $j \neq i_2$ with $i_2 \precsim_j x(j)$. If we also have $j \precsim_{i_2} x(i_2)$, then we are done. Otherwise, label $j = i_3$ and proceed on.

Because the number of players is finite, this process yields a sequence $i_0, i_1, i_2, \cdots, i_L$ that has to cycle. Without loss of generality, suppose that $i_L = i_0$. This sequence satisfies the following properties:

$$\begin{aligned} i_l &\precsim_{i_{l+1}} x(i_{l+1}), \\ i_{l+1} &\succ_{i_l} x(i_l). \end{aligned}$$

Notice, however, that this implies that an allocation z defined by $z(i_l) = i_{l+1}$ modulo L can dominate x via the coalition $\{i_1, i_2, \cdots, i_L\}$, which contradicts that x is in the core. Thus, at some point in the sequence, say L, we have both i_{L-1} and i_L weakly preferring x over y^L and can induce others who may strictly prefer x to join them in the trading cycles that appear in x. Thus, $x \blacktriangleright y$ holds. Because the original allocation y was arbitrarily chosen outside of $I(x)$, this establishes the external stability of $I(x)$, and thus $I(x)$ is a vNM stable set with respect to $\blacktriangleright$.

Case 2: Now, suppose that $j \in S$. Because the choice of y^1 earlier was arbitrary as long as $y^1(S) = S$ was satisfied along with condition (E4), we can reassign the allocations for S as close to x as we can. In other words, choose as y^1 an allocation such that (E4) is satisfied and that contains a maximal chain of (distinct) agents $i_0, i_1, \cdots, i_K$ in S with $y^1(i_k) = x(i_k) = i_{k+1}$ for all $k = 0, 1, \cdots, K-1$ with maximum length. This chain cannot cycle since by assumption, S is not a trading cycle under x nor is it a union of trading cycles under x. Therefore, there exists $j \in N \setminus S$ with $x(j) = i_0$, or otherwise we can construct a longer chain by appending j to it. Now, $j \neq i_0$, so by (R), $x(j) \succ_j j$ has to hold. The rest of the proof then follows from Case 1 by using j and i_0. □

At this time, we have no clear results regarding the structure of vNM $\blacktriangleright$ stable sets other than those described in Proposition 3 and Theorem 1, other than they must contain multiple allocations, possibly those outside the core. Instead, we focus on necessary conditions for certain types of vNM $\blacktriangleright$ stable sets, if they were to exist.

By Corollary 1, for any pair of Pareto distinct allocations x and y in a set V, we know that if x and y dominate each other in the classical sense, then neither $x \blacktriangleright y$ nor $y \blacktriangleright x$ holds, and internal stability of V holds. If there is a set of agents not involved in any domination relation between the allocations in this set, then each allocation restricted to this set must be in the core of the housing market restricted to this set.

Proposition 4. *Let V be a vNM $\blacktriangleright$ stable set, consisting of $K+1$ Pareto distinct allocations, the representative of those given by $x, x^1, x^2, \cdots, x^K$. Suppose that x dom_{S_k} x^k in the classical sense where S_k are coalitions. Then, the restriction of x to the set*

$N \setminus \left(\bigcup_{k=1}^{K} S_k\right)$ *(if nonempty) must lie in the core of the market restricted to the set* $N \setminus \left(\bigcup_{k=1}^{K} S_k\right)$.

Proof. Let V be a vNM $\blacktriangleright$ stable set, and suppose by way of contradiction that $x \in V$ is an allocation such that there exists an allocation y in the restricted market induced by the set of players $N \setminus \left(\bigcup_{k=1}^{K} S_k\right)$ where y dominates x restricted to $N \setminus \left(\bigcup_{k=1}^{K} S_k\right)$ via some coalition T. Denote the restriction of x to $N \setminus \left(\bigcup_{k=1}^{K} S_k\right)$ by $x|_{N \setminus (\bigcup_{k=1}^{K} S_k)}$. Among the pairs y and T, choose one where T is minimal with respect to set inclusion and where $x|_{N \setminus (\bigcup_{k=1}^{K} S_k)} \to'_T y$ holds.[7] Now, construct z such that

$$z(i) = \begin{cases} x(i) & \text{if } i \in \bigcup_{k=1}^{K} S_k, \\ y(i) & \text{if } i \notin \bigcup_{k=1}^{K} S_k. \end{cases} \tag{1}$$

By the properties of x and t, z is an allocation. By Corollary 1 of Proposition 1, for $k = 1, 2, \cdots, K$, $x^k \blacktriangleright z$ cannot hold, since z dominates x^k for each k through S_k. Moreover, $z \neq x$ holds since z and x assign different goods outside of $\bigcup_{k=1}^{K} S_k$. Since z dominates x via coalition T, $x \blacktriangleright z$ cannot hold. Now notice that $x \to'_T z$ holds and with this sequence establishes $z \blacktriangleright x$. By internal stability, $z \notin V$ must hold. However, we have a contradiction involving the external stability of V, as $V = \{x, x^1, x^2, \ldots, x^K\}$ but neither $x \blacktriangleright z$ nor $x^k \blacktriangleright z$ for $k = 1, 2, \ldots, K$ holds. □

The previous result resembles closely to the structure of the absorbing sets of roommate problems in Inarra, Larrea, and Molis (2013). The set of agents involved in the domination within the set parallel the set of "dissatisfied agents" for the roommate problems, while those outside are "satisfied."

Also, worth noting is that the conditions in Proposition 4 are based on condition (a) in Proposition 1. Recall that Proposition 1 does not give a complete characterization of $\blacktriangleright$, but when it does, then the conditions in Proposition 4 correspond to the negation of condition (a) of Proposition 4, so the condition would be redundant.

4 Minimality Condition for Other Domination Relations

While we have seen that the minimality condition (E4) plays a significant role both in the results and in the conceptualization of the domination relation, we argue in this section that the minimality condition can be applied to results of Kawasaki (2010) and Klaus

[7]This is always possible since the domination relation is strict.

et al. (2010) without any significant changes. Thus, minimality only plays a crucial role when considering a farsighted version of weak domination.

It should also be noted that this issue does not arise in the literature that uses classical domination concepts. For these domination relations, it only mattered whether an allocation was dominated or not. Specifically, the only important information in the classical domination relation is the existence of a coalition under which an allocation is dominated so that restricting the constituents of that deviating coalition plays no significant role. Therefore, imposing the minimality condition has no effect on the results in the existing literature that uses the classical concepts.

The results of Kawasaki (2010) and Klaus et al. (2010) center around the set of competitive allocations, or equivalently, allocations that arise from the top trading cycles procedure of Gale, which is informally described in the following way. First, each agent $i \in N$ "points" to his/her most preferred house. By the finiteness of the number of agents involved, there must be a cycle, called a top trading cycle, where if we start from one agent and follow the finger-pointing, we arrive back at the original agent. For each agent in that cycle, assign the good owned by the agent that he/she is pointing to. Remove that cycle and repeat the process starting with agents now pointing to the most preferred among the remaining goods. This procedure is repeated until every agent has been assigned a good by being part of a top trading cycle. By the finiteness of the set of the players and the fact that in each step the number of players decrease, the top trading cycles procedure must terminate.

With indifferences in preferences, there may be multiple allocations that result from the top trading cycles method, but as Shapley and Scarf (1974) show, these allocations are in the core and are also what they call competitive allocations. Kawasaki (2010) and Klaus et al. (2010) show that the set of competitive allocations also constitutes a vNM stable set with respect to a domination relation that reflects farsightedness of the agents and is based on some myopic domination relation introduced in the literature.

Kawasaki (2010) considers a farsighted version of the antisymmetric weak domination relation defined by Wako (1999). An allocation y indirectly antisymmetrically weakly dominates x (denoted by $y \rhd_a x$) if there exists a sequence of coalitions $T^0, T^1, \cdots, T^{K-1}$ and allocations $x^0, x^1, \cdots, x^K$ such that $x^0 = x$, $x^K = y$, and for each $k = 0, 1, \cdots, K-1$ the following two conditions hold: (a) $x^k \to_{T^k} x^{k+1}$, (b) $x^k(i) \precsim_i y(i)$ for all $i \in T^k$ and $x^k(j) \prec_j y(j)$ for some $i \in T^k$, and (c) if $x^k(i) \sim_i y(i)$ for some $i \in T^k$, then $x^k(i) = x^{k+1}(i)$. Now, a domination relation respecting condition (E4) can also be defined in this case, and when that is the case, denote the relation by $y \blacktriangleright_a x$.

Kawasaki (2010) proves that the set of competitive allocations, which can be obtained from the top trading cycles method of Gale (see Shapley and Scarf (1974)), is the unique vNM $\triangleright_a$ stable set for all housing markets. Because internal stability holds for the $\triangleright_a$ relation, it must also hold for $\blacktriangleright_a$. External stability is proved by successively constructing the trading cycles in the competitive allocation. Since trading cycles, by definition, satisfy the minimality condition (E4), the same proof can be used to show that the set of competitive allocations is also the unique vNM $\blacktriangleright_a$ stable set.

On the other hand, Klaus et al. (2010) considers the farsighted version of the (stronger) domination relation, which is more conventional. An allocation y indirectly dominates x (denoted by $y \triangleright_s x$) if there exists a sequence of coalitions $T^0, T^1, \cdots, T^{K-1}$ and allocations $x^0, x^1, \cdots, x^K$ such that $x^0 = x$, $x^K = y$, and for each $k = 0, 1, \cdots, K-1$ the following two conditions hold: (a) $x^k \to_{T^k} x^{k+1}$ and (b) $x^k(i) \prec_i y(i)$ for all $i \in T^k$. Denote the relation by $y \blacktriangleright_s x$ when the above holds using the relation $\to'$ that also satisfies (E4). Although the proof of Klaus et al. (2010) does not strictly adhere to the minimality of trading cycles that we impose in this section, we can show that by an inspection of their proof, only a slight modification in the proof is needed to establish their result.

We give a brief description of their procedure. Let y be any allocation that is not a competitive allocation. Since y cannot be obtained via the top trading cycles method, at some step, there exists a trading cycle of y that is not a top trading cycle. Let x be a competitive allocation that maximizes the number of consecutive appearances from the beginning of common top trading cycles in the top trading cycles procedure between x and y. The "Top" procedure in the proof in Klaus et al. (2010) essentially spots these top trading cycles and removes them from further consideration in the proof. Because y is not competitive, there must exist a trading cycle of y that is not a top trading cycle at any step. While Klaus et al. (2010) use the notation S^* to denote the set of trading cycles that are top trading cycles in the remaining set of agents that do not appear in y. There must exist an agent $i \in S^*$ with $y(i) \prec_i x(i)$ as the trading cycles in y are not top trading cycles. Let S^{**} be the set of such agents. The sequence in Klaus et al. (2010) calls for a direct move from $y \to_{S^{**}} x^1$ with $x^1(i) = i$ for all $i \in S^*$, induced by all agents in S^{**} choosing his/her endowment and by doing so, all agents in S^* now are assigned their endowment in the new allocation x^1. Then, from x^1, x^2 can be induced by S^* where x^2 is an allocation which ultimately satisfies $x^2(i) = x(i)$ for all $i \in S^*$. This last step uses the domain restriction (R). These are the moves described in the "Trade" procedure. This process is repeated starting with the "Top" procedure for allocation x^2, until every trading cycle is deemed a top trading cycle by the "Top" procedure, which implies that the resulting allocation is a top trading cycle allocation or competitive allocation. Note that S^{**} and S^* are in general not minimal coalitions.

However, this sequence can be easily modified even if we impose our minimality condition (E4) simply by having each agent that prefers the good in x over y, shift to his/her endowment in succession. The resulting allocation should still be x^1. From x^1 to x^2, have each trading cycle in S^* induce this move one-by-one. This is possible without any complications since in x^1, each $i \in S^*$ is receiving his/her endowment. The trading cycles are minimal by definition, and thus this part of the sequence also can be decomposed into successive enforceability relations satisfying (E4). Repeating this process along the lines of the procedure in Klaus et al. (2010), in the end, the allocation x results, and we have a sequence where $x \blacktriangleright_s y$ holds.

To summarize, the previous literature on farsighted stability of allocations in the model of Shapley and Scarf (1974) did not depend on whether condition (E4) was satisfied or not. Therefore, the issue of minimality of the coalitions in the enforceability condition only affect the farsighted analogue of weak domination. The issue that arises from not considering the minimality condition then is not an issue in other farsighted domination relations previously considered.

We end this section by giving an example where the vNM $\blacktriangleright$ stable sets and the vNM $\blacktriangleright_s$ stable sets are disjoint. The following example from Sotomayor (2005), which is attributed to Jun Wako, highlights the difference between allocations supported by a vNM $\blacktriangleright$ stable set and competitive allocations, which are supported by vNM $\blacktriangleright_s$ and $\blacktriangleright_a$ stable sets.

Example 4. (Example 1 of Sotomayor (2005))

$$\begin{array}{ccccc} 2 & \succ_1 & 3 & \succ_1 & 1 \\ 1 & \sim_2 & 3 & \succ_2 & 2 \\ 2 & \succ_3 & 1 & \succ_3 & 3 \end{array}$$

The core consists of the following four allocations: $x^1 = (2, 1, 3)$, $x^2 = (1, 3, 2)$, $x^3 = (2, 3, 1)$, and $x^4 = (3, 1, 2)$. The set of competitive allocations is $\{x^1, x^2\}$, which is the unique vNM $\blacktriangleright_s$ stable set and also the unique vNM $\blacktriangleright_a$ stable set. However, neither competitive allocation is Pareto efficient as x^3 wdom$_N$ x^1 and x^4 wdom$_N$ x^2. By Theorem 1, $\{x^3\}$ and $\{x^4\}$ are vNM $\blacktriangleright$ stable sets, and moreover, it can be checked that they are the only ones.

5 Concluding Remarks

We have shown in this paper that using a farsighted analogue of weak domination introduces additional issues aside from those on coalitional sovereignty that have already been discussed in the literature. Specifically, for the house barter model of Shapley and Scarf (1974), whether coalitions in the form of multiple trading cycles can simultaneously deviate or not is a crucial issue, especially when considering that weak domination requires only one agent to be strictly better off. We have analyzed the effect of limiting such simultaneous deviations and imposed a minimality condition which effectively allows only one trading cycle to deviate in a particular step. Even with this restriction, we show that a core allocation that is Pareto efficient is a essentially singleton vNM stable set defined by the farsighted weak domination relation. This result is often found in the literature on matching problems that support the core outcome as a singleton vNM stable set defined by a farsighted domination relation. While the result may not be surprising, the technique used in the literature could not be directly applied. We also have mentioned that the minimality restriction imposed in this paper does not have consequences in the literature using a stronger domination relation. Through this research we would have hoped to shed light on possible difficulties that may arise when using a farsighted version of a weak domination relation that may not arise in other domination relations.

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