Principal-agent problems with hidden savings in continuous time

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Abstract

In this paper, we consider a continuous-time principal-agent problem with hidden savings. The agent's problem, which is non-Markovian, is formulated using the stochastic HJB equation. Without loss of generality, attention is restricted to those contracts for which the agent optimally chooses zero savings. Then, the principal's problem can be expressed as maximizing her expected profit subject to two SDEs: one equation describing the agent's continuation utility process, and the other being the Euler equation concerning the agent's marginal utility process. It coincides with the formulation obtained under the first-order approach.

Keywords: moral hazard; hidden savings; continuous time; weak formulation; first-order approach; stochastic HJB equations.
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1 Introduction

In this paper, we consider a dynamic moral hazard problem in continuous time, in which the agent can save / borrow without being observed by the principal. We show that the principal's problem can be formulated as maximizing her expected profit subject to two stochastic differential equations (SDE's)–one equation describing the evolution of the agent's continuation utility, and the other corresponding to the agent's Euler equation for his utility maximization problem. Since this has a Markovian form, it can be solved using a standard method.

Our contribution is to obtain such a formulation without appealing to the "first-order approach," which replaces the agent's incentive compatibility constraint with his first-order conditions. Indeed, the formulation of the principal's problem that we obtain is identical with what one would obtain by using the first-order approach. We attain it by characterizing the agent's problem by the stochastic Hamilton-Jacobi-Bellman (HJB) equation, and by focusing on those contracts which induce the agent to choose zero savings.

The validity of the first-order approach has been considered to be crucial for tractability in many principal-agent problems. However, as shown by Rogerson (1985) and Jewitt (1988), even in the static moral hazard problem, the conditions required for the validity of the first-order approach are stringent. If we wish to consider dynamic problems, the required conditions would become even harder to satisfy.

It is not necessarily the case if the model is formulated in continuous time. A continuous-time moral hazard problem is first studied by Holmstrom and Milgrom (1987), where the agent has exponential utility and is subject to Brownian shocks. The validity of the first-order approach in the Holmstrom-Milgrom model is established later by Schättler and Sung (1993). Further extension is conducted by Sannikov (2008), who considers a general utility function for the agent, and establishes the validity of the first-order approach in his model. A textbook treatment of moral-hazard problems in continuous time is given by Cvitanić and Zhang (2013).

However, all these results are obtained under the assumption that the principal can observe the agent's consumption (equivalently, his savings). The observability of the agent's consumption (savings) implies that the principal can effectively control the agent's consumption process. Such an assumption may not be realistic in many applications.

An important question is thus whether or not we can extend the validity of the first-order approach (in continuous time) to the case where the principal does not observe the agent's savings. Williams (2008) is a seminal paper that addresses this question (see also Williams (2015)). His finding is unfortunately negative. The moral hazard model with hidden savings violates the assumptions for the sufficiency theorem of the stochastic maximum principle, and thus the validity of the first-order approach is not warranted.

In this paper, we do not intend to establish the sufficiency of the first-order conditions for the agent's problem. Instead, we express the agent's utility maximization problem as a dynamic programming problem. We should emphasize that the agent's problem is not Markovian, because payments from the principal are allowed to depend on the whole history of outcomes. Hence, the optimality condition obtained from the agent's problem is not a standard HJB equation, but a stochastic HJB equation. We use the original result by Peng (1992), as well as some from Øksendal and Sulem (2019).

To illustrate the robustness of our approach, we consider two models of dynamic moral hazard. In the first one, the agent supplies unobserved effort to produce output. As discussed above, Holmstrom and Milgrom (1987) is the seminal work, which is generalized by Sannikov (2008). In the second model, the agent does not provide effort. Instead, he manages an asset for the principal, whose return is only observed by the agent. This problem is considered in the continuous-time framework by DeMarzo and Sannikov (2006), and Biais et al. (2007).

For both of these models, we demonstrate how to apply stochastic-HJB-equation approach, and show that the principal's problem can be formulated using the two SDEs, one for the agent's utility process and the other for his marginal utility process, as long as attention is restricted (without loss of generality) to those contracts for which the agent optimally chooses to save nothing.

In the existing literature, two papers have addressed a closely related question. First, Williams (2015) considers a hidden effort model as Sannikov (2008) but allows for hidden savings. As in Holmstrom and Milgrom (1987), Williams (2015) assumes the exponential utility function for the agent. In that framework, he considers a relaxed problem in which the principal maximizes her expected profit subject to the first-order condition (Euler equation) of the agent, without establishing its sufficiency for the agent's problem. It is then verified that the solution to the relaxed problem is indeed incentive compatible for the agent. Such an "ex-post-verification approach" is often used in application (e.g., Farhi and Werning (2013) and Golosov et al. (2016)).

Second, Di Tella and Sannikov (2021) consider the asset management problem as in DeMarzo and Sannikov (2006), and Biais et al. (2007). They allow for hidden savings in the model where the agent's utility function is of the CRRA form. Then they establish that the first-order approach is valid under some additional condition for the contract offered by the principal, which is shown to be satisfied at optimum.

Our results are extensions of those obtained by Williams (2015) and Di Tella and Sannikov (2021), since we do not make parametric assumptions on the agent's utility function. In addition, our results clarify why the "ex-post-verification approach," such as the one used by Williams (2015), works. Even if the first-order approach is not valid in the true sense, the principal's problem can be written in the same way as the one that is implied by the first-order approach.

The rest of the paper is organized as follows. Section 2 considers the model with hidden effort. Section 3 studies the model with hidden returns. Section 4 concludes and discusses potential lines of future research.

2 Hidden effort

In this section, we consider a version of the dynamic principal-agent model of Sannikov (2008), modified so that the agent can save/borrow, without being observed by the principal.

2.1 The model

Time is continuous and indexed by $t \in [0, T]$, where $0 < T < \infty$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, on which a standard Brownian motion $B : [0, T] \times \Omega \to \mathbb{R}$ is defined. Let $\mathbb{F} = \{\mathscr{F}_t\}_{0 \le t \le T}$ be the augmented filtration generated by B. Without loss, we let $\mathscr{F} = \mathscr{F}_T$. All stochastic processes considered in this paper are assumed to be progressively measurable with respect to $\mathbb{F}$. Let $\mathbb{E}$ denote the expectation operator associated with $\mathbb{P}$.

Let $X : [0, T] \times \Omega \to \mathbb{R}$ be the cumulative output process, which is observable both by the principal and by the agent. We employ the *weak formulation*,[1] and assume that

$$X_t = \sigma B_t,$$

with $\sigma > 0$ is a constant. The process B is a standard Brownian motion under the benchmark probability measure $\mathbb{P}$, but not under different measures. As described below, the agent's

[1]For the weak formulation, see, for instance, Section 10.4 of Cvitanić and Zhang (2013) and Chapter 9 of Zhang (2017).

effort affects the probability measure, and thus the distribution of B (see, e.g., equation (4) below). Note also that the augmented filtration generated by X coincides with $\mathbb{F}$. It would be straightforward to consider a more general Itô process for X.

2.1.1 The agent

The agent provides effort, which affects the probability measure on $(\Omega, \mathscr{F})$. Let $N : [0,T] \times \Omega \to \mathcal{N}$ denote the process of the agent's effort, where $\mathcal{N} = [0, \overline{N}]$ with $\overline{N} > 0$. The effort process is the agent's private information, and is not observable by the principal.

The effort process N changes the probability measure on $(\Omega, \mathscr{F})$ from $\mathbb{P}$ to $\mathbb{P}^N$ in the following way. Given N, define the associated process M^N by

$$M_t^N := \exp\left(\int_0^t \frac{N_s}{\sigma} \, dB_s - \frac{1}{2} \int_0^t \frac{N_s^2}{\sigma^2} \, ds \right). \tag{1}$$

Since $\mathcal{N}$ is compact, M^N is a martingale (under $\mathbb{P}$), so that $\mathbb{E}[M_T^N] = 1$.[2] The probability measure induced by the effort process N, $\mathbb{P}^N$, is then defined as:

$$d\mathbb{P}^N := M_T^N \, d\mathbb{P}. \tag{2}$$

Let $\mathbb{E}^N$ be the expectation operator corresponding to $\mathbb{P}^N$.

By the Girsanov theorem,[3]

$$B_t^N := B_t - \int_0^t \frac{N_s}{\sigma} \, ds \tag{3}$$

is a standard Brownian motion under $\mathbb{P}^N$. It follows that the cumulative output process can be expressed as

$$dX_t = \sigma \, dB_t = N_t \, dt + \sigma \, dB_t^N. \tag{4}$$

The effort process N affects the probability measure $\mathbb{P}^N$ by affecting the drift of the cumulative output process X.

At time 0, the principal offers the agent a contract (Y, S_T) that specifies payments to the agent. The first item of the contract, $Y : [0,T] \times \Omega \to \mathbb{R}_+$, specifies continuous payments at each point in time, and the second one, $S_T : \Omega \to \mathbb{R}_+$, corresponds to the lump-sum payment at the last date T. The continuous payment Y is $\mathbb{F}$-progressively measurable and the terminal payment S_T is $\mathscr{F}_T$-measurable. In words, the payment at any time t, Y_t, is based (only) on

[2] See, for instance, Lemma 2.6.1 of Zhang (2017).

[3] For instance, Theorem 2.6.4 of Zhang (2017).

the principal's observation of output until time t, $(X_s; 0 \le s \le t)$, and the terminal payment S_T is based on the whole history of output, $(X_s; 0 \le s \le T)$. The principal can commit to the contract that she offers.

The agent can save/borrow at the (constant) risk-free rate $r > 0$. In addition to the level of effort, the amounts of savings and consumption are the agent's private information. Let $C : [0, T] \times \Omega \to \mathbb{R}_+$ denote the consumption process.

Let $\tilde{A} : [0, T] \times \Omega \to \mathbb{R}$ be the process of the holdings of the risk free asset, which is allowed to be negative. The initial asset of the agent is zero: $\tilde{A}_0 = 0$. The flow budget constraint for the agent is given by

$$d\tilde{A}_t = (r\tilde{A}_t + Y_t - C_t)\, dt, \qquad \text{with } \tilde{A}_0 = 0.$$

The agent derives utility from the consumption process C, the effort process N, and his net terminal wealth $\tilde{A}_T + S_T$. His expected utility is given by

$$\mathbb{E}^N \left[\int_0^T e^{-\rho t} u(C_t, N_t)\, dt + e^{-\rho T} U(\tilde{A}_T + S_T) \right]$$

where ρ is the agent's subjective time discount rate.

For the agent's wealth, we find it convenient to work with its discounted value, A_t:

$$A_t := e^{-rt}\tilde{A}_t,$$

Then, the flow budget constraint is rewritten as:

$$dA_t = e^{-rt}(Y_t - C_t)\, dt, \qquad \text{with } A_0 = 0, \tag{5}$$

and the agent's expected utility becomes:

$$\mathbb{E}^N \left[\int_0^T e^{-\rho t} u(C_t, N_t)\, dt + e^{-\rho T} U(e^{rT} A_T + S_T) \right] \tag{6}$$

In what follows, we express the agent's problem in terms of the discounted value of his savings, A_t, rather than $\tilde{A}_t$.

2.1.2 Assumptions

For a technical reason, we assume that both consumption and effort processes are bounded. Let $\boldsymbol{N}$ be the set of $\mathbb{F}$-progressively measurable processes $N : [0, T] \times \Omega \to \mathcal{N}$; and $\boldsymbol{C}$ be the set of $\mathbb{F}$-progressively measurable processes $C : [0, T] \times \Omega \to \mathcal{C}$, where $\mathcal{C} = [0, \overline{C}]$ with $\overline{C} > 0$. We say (C, N) is *feasible* if $(C, N) \in \boldsymbol{C} \times \boldsymbol{N}$.

We assume that the flow utility function $u : \mathbb{R}_+^2 \to \mathbb{R}$ is continuous, twice continuously differentiable, monotonic with $u_c > 0$ and $u_n < 0$, and strictly concave; and that the terminal utility function $U : \mathbb{R} \to \mathbb{R}$ is increasing, continuous, and twice continuously differentiable with bounded derivatives. Note that u and its derivatives are all bounded on its effective domain $\mathcal{C} \times \mathcal{N}$.

Let $\boldsymbol{Y}$ be the set of $\mathbb{F}$-progressively measurable processes $Y : [0,T] \times \Omega \to \mathcal{Y}$, where $\mathcal{Y} = [0, \overline{Y}]$ with $\overline{Y} \geq \overline{C}$; and $\boldsymbol{S} = \mathbb{L}^2(\mathscr{F}_T, \mathbb{P})$, i.e., the set of all $\mathscr{F}_T$ measurable, square-integrable random variables. A contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$ is called *feasible*.

Under our assumptions, $\boldsymbol{C}, \boldsymbol{N}, \boldsymbol{Y}$ are all subsets of $\mathbb{L}^2(\mathbb{F}, \mathbb{P})$, i.e., the set of all $\mathbb{F}$-progressively measurable processes φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \, dt\right] < \infty$.

Given a feasible contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$, the agent's problem is to maximize his expected utility (6) subject to the budget constraint (5). For simplicity, we assume the existence of a solution to the agent's problem. The sup operator in what follows can thus be interpreted as the max operator. By this assumption, we can avoid the technical complication related to the measurability of the value function.

2.1.3 The principal

Given a contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$, a pair of consumption and effort processes $(C, N) \in \boldsymbol{C} \times \boldsymbol{N}$ is said to be *incentive compatible* if it maximizes the agent's expected utility (6) subject to the budget constraint (5). It is said to satisfy the participation constraint if the agent's expected utility (6) is greater than or equal to a given level of reservation utility.

The principal's objective is to choose a feasible contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$ so as to maximize her expected profit:

$$\mathbb{E}^N\left[\int_0^T e^{-rt}\,(dX_t - Y_t\,dt) - e^{-rT} S_T\right] = \mathbb{E}^N\left[\int_0^T e^{-rt}(N_t - Y_t)\,dt - e^{-rT} S_T\right]. \tag{7}$$

subject to the incentive compatibility and participation constraints for the agent.

2.2 First order necessary conditions

In this subsection, we derive the first-order necessary conditions of the agent's utility maximization problem for a given contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$.

Let $(C, N) \in \boldsymbol{C} \times \boldsymbol{N}$ be any feasible processes of consumption and effort. The associated utility process of the agent, W_t^A, is given by

$$W_t^A := \mathbb{E}_t^N \left[\int_t^T e^{-\rho s} u(C_s, N_s)\, ds + e^{-\rho T} U(e^{rT} A_T + S_T) \right],$$

By the extended martingale representation theorem (e.g., Lemma 10.4.6 in Cvitanić and Zhang (2013)), there exists $Z^A \in \mathbb{L}^2(\mathbb{F}, \mathbb{P}^N)$ such that[4]

$$W_t^A = e^{-\rho T} U(e^{rT} A_T + S_T) + \int_t^T e^{-\rho s} u(C_s, N_s)\, ds - \int_t^T e^{-\rho s} \sigma Z_s^A\, dB_s^N.$$

Since $dB_t^N = dB_t - N_t/\sigma\, dt$, one can view (W^A, Z^A) as the solution to the following backward stochastic differential equation (BSDE):

$$W_t^A = e^{-\rho T} U(e^{rT} A_T + S_T) + \int_t^T e^{-\rho s} \left[u(C_s, N_s) + Z_s^A N_s \right] ds - \int_t^T e^{-\rho s} \sigma Z_s^A\, dB_s. \tag{8}$$

In fact, under our assumptions, this BSDE has a unique solution $(W^A, Z^A) \in \mathbb{L}^2(\mathbb{F}, \mathbb{P}) \times \mathbb{L}^2(\mathbb{F}, \mathbb{P})$ (e.g., Theorem 4.3.1 of Zhang (2017)).

The agent's problem is to choose the consumption and effort processes $(C, N) \in \boldsymbol{C} \times \boldsymbol{N}$ so as to maximize W_0^A subject to (8) and (5). A standard technique can be applied to derive the first-order necessary conditions for the agent's problem (e.g., Section 10.2 of Cvitanić and Zhang (2013)).

Proposition 1. *Let a contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$ be given. Consider a pair of consumption and effort processes $(\hat{C}, \hat{N}) \in \boldsymbol{C} \times \boldsymbol{N}$. Let $\hat{A}$ be the associated wealth process:*

$$d\hat{A}_t = e^{-rt}(Y_t - \hat{C}_t)\, dt, \qquad \hat{A}_0 = 0.$$

Let $(\hat{W}^A, \hat{Z}^A)$ be the solution to BSDE (8):

$$d\hat{W}_t^A = -e^{-\rho t} \left[u(\hat{C}_t, \hat{N}_t) + \hat{Z}_t^A \hat{N}_t \right] dt + e^{-\rho t} \sigma \hat{Z}_t^A\, dB_t, \qquad \hat{W}_T^A = e^{-\rho T} U(e^{rT} A_T + S_T),$$

so that, $\hat{W}^A$ is the agent's utility process associated with $(\hat{C}, \hat{N})$. Let $\hat{M}^N$ and $(\hat{\Gamma}, \hat{Z}^\Gamma)$ be the solutions to the SDE and BSDE given, respectively, by:

$$d\hat{M}_t^N = \hat{M}_t^N \frac{\hat{N}_t}{\sigma}\, dB_t, \qquad \hat{M}_0^N = 1,$$
$$d\hat{\Gamma}_t = e^{-\rho t} \sigma \hat{Z}_t^\Gamma\, dB_t, \qquad \hat{\Gamma}_T = \hat{M}_T^N e^{(r-\rho)T} U'(e^{rT} \hat{A}_T + S_T).$$

[4]Note that $Z^A \in \mathbb{L}^2(\mathbb{F}, \mathbb{P}^N)$ if and only if $(e^{-\rho t} \sigma Z_t^A; 0 \le t \le T) \in \mathbb{L}^2(\mathbb{F}, \mathbb{P}^N)$.

If $(\hat{C}, \hat{N})$ is incentive compatible for the agent, then it satisfies the first-order conditions:

$$\hat{M}_t^N e^{-\rho t} \partial_c u(\hat{C}_t, \hat{N}_t) - \hat{\Gamma}_t e^{-rt} \begin{cases} \leq 0, & \text{if } \hat{C}_t = 0, \\ = 0, & \text{if } \hat{C}_t \in (0, \overline{C}), \\ \geq 0, & \text{if } \hat{C}_t = \overline{C}, \end{cases} \tag{9}$$

$$\partial_n u(\hat{C}_t, \hat{N}_t) + \hat{Z}_t^A \begin{cases} \leq 0, & \text{if } \hat{N}_t = 0, \\ = 0, & \text{if } \hat{N}_t \in (0, \overline{N}), \\ \geq 0, & \text{if } \hat{N}_t = \overline{N}. \end{cases} \tag{10}$$

Proof. The proof is a straightforward application of the argument in Cvitanić and Zhang (2013). Under our assumptions, Assumptions 10.2.1-10.2.2 of Cvitanić and Zhang (2013) are satisfied. Let $(\hat{C}, \hat{N})$ be a given consumption-effort process. We see that one of the adjoint processes given in equation (10.24) in Cvitanić and Zhang (2013) coincides $\hat{M}_t^N$:

$$\hat{M}_t^N = 1 + \int_0^t \hat{M}_s^N \frac{\hat{N}_s}{\sigma} \, dB_s$$

Let $\hat{\Gamma}$ denote the other adjoint process, which is given by

$$\hat{\Gamma}_t = \hat{M}_T^N e^{(r-\rho)T} U'(e^{rT} A_T + S_T) - \int_t^T e^{-\rho s} \sigma \hat{Z}_s^\Gamma \, dB_s.$$

Then the claim in Proposition 1 follows from Lemma 10.2.4 of Cvitanić and Zhang (2013), noting our constraint that $C_t \in [0, \overline{C}]$ and $N_t \in [0, \overline{N}]$. □

Since $M_t^N > 0$ for all t, we can eliminate it from the first-order conditions. Define

$$\Lambda_t := \frac{\Gamma_t}{M_t^N}.$$

Then, using the Itô formula, one can see that (Λ, Z^Λ) is the solution to the following BSDE:

$$d\Lambda_t = -N_t e^{-\rho t} Z_t^\Lambda \, dt + e^{-\rho t} \sigma Z_t^\Lambda \, dB_t; \qquad \Lambda_T = e^{(r-\rho)T} U'(e^{rT} A_T + S_T). \tag{11}$$

Then the first-order conditions in Proposition 1 are rewritten as:

$$e^{-\rho t} \partial_c u(\hat{C}_t, \hat{N}_t) - \hat{\Lambda}_t e^{-rt} \begin{cases} \leq 0, & \text{if } \hat{C}_t = 0, \\ = 0, & \text{if } \hat{C}_t \in (0, \overline{C}), \\ \geq 0, & \text{if } \hat{C} = \overline{C}, \end{cases} \tag{12}$$

$$\partial_n u(\hat{C}_t, \hat{N}_t) + \hat{Z}_t^A \begin{cases} \leq 0, & \text{if } \hat{N}_t = 0, \\ = 0, & \text{if } \hat{N}_t \in (0, \overline{N}), \\ \geq 0, & \text{if } \hat{N}_t = \overline{N}. \end{cases} \tag{13}$$

2.3 Difficulty to prove the sufficiency of the first-order conditions

The first-order approach replaces the incentive compatibility constraint of the agent by the first-order conditions (9)-(10) (or equivalently, by (12)-(13)). It is often indispensable for computational tractability of the principal's problem. However, it is known to be difficult to establish the sufficiency of those first-order conditions, on which the validity of the first-order approach relies.[5]

To see this point in our problem, consider the Hamiltonian associated with the agent's problem:

$$H(t, y, c, n, z^A, m^N, \gamma) := m^N e^{-\rho t}\big[u(c,n) + z^A n\big] + \gamma e^{-rt}(y - c). \tag{14}$$

The first-order conditions (9)-(10) can also be derived from maximizing the Hamiltonian for each t and $(Y_t, Z_t^A, M_t^N, \Gamma_t)$:

$$\max_{(c,n)\in\mathcal{C}\times\mathcal{N}} H(t, Y_t, c, n, Z_t^A, M_t^N, \Gamma_t)$$

Under our assumptions, the Hamiltonian H defined in (14) is strictly concave in (c, n). Then, it might appear to be straightforward to show the validity of the first-order approach using the stochastic maximum principle. Unfortunately, however, that is not true. It is because the sufficiency theorem of the stochastic maximum principle (e.g., Theorem 10.2.9 of Cvitanić and Zhang (2013)) requires the Hamiltonian H to be concave in (c, n, z^A), rather than in (c, n). This concavity property is not satisfied here, because of the multiplicative term nz^A in H.

The multiplicative term nz^A is due to the fact that the agent's choice of the effort process N affects the probability distribution of the cumulative output process X. Thus, this difficulty has the same root as the existing work on the static and discrete-time models encounters on the validity of the first-order approach. In what follows, we use the stochastic Hamilton-Jacobi-Bellman equation, rather than the stochastic maximum principle, to derive sufficient conditions for the agent's utility maximization problem. We then establish that, as long as we restrict our attention to those contracts that induce the agent zero savings, the principal's problem can be written *as if* the first-order approach were valid.

2.4 Stochastic HJB equation

To derive the sufficient conditions for the agent's utility maximization problem, we follow the approach of Peng (1992) and rely on the *stochastic Hamilton-Jacobi-Bellman equation*. It is

[5] For instance, Williams (2008), and Cvitanić and Zhang (2013).

also described in Section 5.4 of Øksendal and Sulem (2019), where Peng's (1992) approach is associated with the result on backward stochastic partial differential equations (BSPDE) by Ma et al. (2012).

Given a feasible contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$, let us define the value function $V^A : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$V^A(t,a,\omega) := \sup_{(C,N)\in\boldsymbol{C}\times\boldsymbol{N}} \mathbb{E}_t^N \left[\int_t^T e^{-\rho s} u(C_s, N_s)\, ds + e^{-\rho T} U(e^{rT} A_T + S_T) \right] \tag{15}$$

$$\text{s.t.} \quad A_T = a + \int_t^T e^{-rs}(Y_s - C_s)\, ds$$

Unlike in the Markovian case, the value function depends on ω, reflecting the non-Markovian nature of the problem. Relatedly, while in the Markovian case, the value function is characterized as a solution to the PDE, the HJB equation, here, it is characterized as a solution to the BSPDE, the stochastic HJB equation.

We begin with the property that the value function is Lipschitz continuous in (t,a).

Proposition 2. *For $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$, the value function $V^A(t,a,\omega)$ defined in (15) is Lipschitz continuous in (t,a) for all $\omega \in \Omega$.*

Proof. For any $(t,a) \in [0,T] \times \mathbb{R}$ and $(C,N) \in \boldsymbol{C} \times \boldsymbol{N}$, denote that

$$A_T^{(C,N),(t,a)} = a + \int_t^T e^{-rs}(Y_s - C_s)\, ds$$

For any $(t, a), (t', a') \in [0, T] \times \mathbb{R}$,

$$
\begin{aligned}
&\left|V^A(t,a,\omega) - V^A(t',a',\omega)\right| \\
&= \Bigg| \sup_{(C,N)\in \boldsymbol{C}\times\boldsymbol{N}} \mathbb{E}^N \left[\int_t^T e^{-\rho s} u(C_s, N_s)\, ds + e^{-\rho T} U(e^{rT} A_T^{(C,N),(t,a)} + S_T)\right] \\
&\qquad - \sup_{(C,N)\in \boldsymbol{C}\times\boldsymbol{N}} \mathbb{E}^N \left[\int_{t'}^T e^{-\rho s} u(C_s, N_s)\, ds + e^{-\rho T} U(e^{rT} A_T^{(C,N),(t',a')} + S_T)\right]\Bigg| \\
&\le \sup_{(C,N)\in \boldsymbol{C}\times\boldsymbol{N}} \Bigg| \mathbb{E}^N \left[\int_t^T e^{-\rho s} u(C_s, N_s)\, ds + e^{-\rho T} U(e^{rT} A_T^{(C,N),(t,a)} + S_T)\right] \\
&\qquad - \mathbb{E}^N \left[\int_{t'}^T e^{-\rho s} u(C_s, N_s)\, ds + e^{-\rho T} U(e^{rT} A_T^{(C,N),(t',a')} + S_T)\right]\Bigg| \\
&= \sup_{(C,N)\in \boldsymbol{C}\times\boldsymbol{N}} \Bigg| \mathbb{E}^N \Bigg[\int_t^{t'} e^{-\rho s} u(C_s, N_s)\, ds \\
&\qquad + e^{-\rho T} U(e^{rT} A_T^{(C,N),(t,a)} + S_T) - e^{-\rho T} U(e^{rT} A_T^{(C,N),(t',a')} + S_T)\Bigg]\Bigg| \\
&\le \sup_{(C,N)\in \boldsymbol{C}\times\boldsymbol{N}} \mathbb{E}^N \Bigg[\left|\int_t^{t'} e^{-\rho s} u(C_s, N_s)\, ds\right| \\
&\qquad + \left|e^{-\rho T} U(e^{rT} A_T^{(C,N),(t,a)} + S_T) - e^{-\rho T} U(e^{rT} A_T^{(C,N),(t',a')} + S_T)\right|\Bigg]
\end{aligned}
$$

Since $(C_s, N_s) \in \mathcal{C} \times \mathcal{N}$ for all s and u is continuous, it follows that there exists a constant L_1 such that

$$
\left|\int_t^{t'} e^{-\rho s} u(C_s, N_s)\, ds\right| < L_1 \left|t' - t\right|, \quad \forall \omega \in \Omega.
$$

Also, for all $\omega \in \Omega$,

$$
\begin{aligned}
\left|A_T^{(C,N),(t,a)} - A_T^{(C,N),(t',a')}\right| &= \left|a + \int_t^T e^{-rs}(Y_s - C_s)\, ds - a' - \int_{t'}^T e^{-rs}(Y_s - C_s)\, ds\right| \\
&\le \left|a' - a\right| + \left|\int_t^{t'} (Y_s - C_s)\, ds\right| \\
&\le \left|a' - a\right| + L_2 \left|t' - t\right|,
\end{aligned}
$$

where L_2 is a constant depending on $\overline{Y}$ and $\overline{C}$. Then, since U is continuously differentiable with bounded derivatives, there exists constants L_3 and L_4 such that

$$
\left|e^{-\rho T} U(e^{rT} A_T^{(C,N),(t,a)} + S_T) - e^{-\rho T} U(e^{rT} A_T^{(C,N),(t',a')} + S_T)\right| \le L_3 \left|a' - a\right| + L_4 \left|t' - t\right|
$$

Combining these inequalities, we conclude that there exists constants L_a and L_t such that for all $\omega \in \Omega$

$$
\left|V^A(t,a,\omega) - V^A(t',a',\omega)\right| \le L_a \left|a' - a\right| + L_t \left|t' - t\right|
$$

that is, V^A is Lipschitz continuous in (t, a). □

Hence, for any $(C, N) \in \boldsymbol{C} \times \boldsymbol{N}$, the process $V^A(t, A_t, \omega)$ is $\mathbb{F}$-progressively measurable, where the wealth process A is given by (5). In addition, it follows from Rademacher's theorem that $V^A(t, a, \omega)$ is differentiable in a almost everywhere.

The following proposition establishes that V^A satisfies the stochastic HJB equation:

$$\begin{aligned} dv(t,a,\omega) = &- \sup_{(c,n)\in\mathcal{C}\times\mathcal{N}} \Big\{ e^{-\rho t}\big[u(c,n) + z(t,a,\omega)n\big] + \partial_a v(t,a,\omega) e^{-rt}\big[Y(t,\omega) - c\big] \Big\}\, dt \\ &+ e^{-\rho t}\sigma z(t,a,\omega)\, dB_t, \qquad (16)\\ v(T,a,\omega) = &\; e^{-\rho T} U(e^{rT}a + S_T(\omega)). \end{aligned}$$

Thanks to the Lipschitz continuity of the value function, we can interpret that the above equation is satisfied in the weak sense.

Proposition 3. *Given $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$, the value function V^A satisfies the stochastic HJB equation* (16)*.*

Proof. As in Peng (1992), for any fixed $a \in \mathbb{R}$, express $V_t^A(a) = V^A(t, a, \omega)$ as a semimartingale

$$V_t^A(a) = e^{-\rho T} U(e^{rT}a + S_T) + \int_t^T g_s(a)\, ds - \int_t^T e^{-\rho t}\sigma z_s^A(a)\, dB_s$$

for some g and z^A. From the dynamic programming principle, we obtain

$$V_t^A(a) = \sup_{(C,N)\in\boldsymbol{C}\times\boldsymbol{N}} \mathbb{E}_t^N \left[\int_t^{t+\tau} e^{-\rho s} u(C_s, N_s)\, ds + V_{t+\tau}^A\left(A_{t+\tau}^{(C,N),(t,a)}\right)\right] \qquad (17)$$

Applying the Itô-Wentzell formula[6] to $V_s^A\left(A_s^{(C,N),(t,a)}\right)$, $s \in [t, t+\tau]$, we obtain

$$\begin{aligned} V_{t+\tau}^A\left(A_{t+\tau}^{(C,N),(t,a)}\right) &= V_t^A(a) + \int_t^{t+\tau} \Big\{ -g_s\left(A_s^{(C,N),(t,a)}\right) + \partial_a V_s^A\left(A_s^{(C,N),(t,a)}\right) e^{-rs}(Y_s - C_s)\Big\}\, ds \\ &\quad + \int_t^{t+\tau} e^{-\rho s}\sigma z_s^A\left(A_s^{(C,N),(t,a)}\right) dB_s \\ &= V_t^A(a) + \int_t^{t+\tau} \Big\{ -g_s\left(A_s^{(C,N),(t,a)}\right) \\ &\quad + \partial_a V_s^A\left(A_s^{(C,N),(t,a)}\right) e^{-rs}(Y_s - C_s) + e^{-\rho s} z_s^A\left(A_s^{(C,N),(t,a)}\right) N_s \Big\}\, ds \\ &\quad + \int_t^{t+\tau} e^{-\rho s}\sigma z_s^A\left(A_s^{(C,N),(t,a)}\right) dB_s^N \qquad (18) \end{aligned}$$

[6]See, for instance, Theorem 2.3.1 in Kunita (2019).

where we have used the relationship $dB_t = \frac{N_t}{\sigma}\,dt + dB_t^N$. Substituting (18) into (17), we obtain

$$\sup_{(C,N)\in \boldsymbol{C}\times\boldsymbol{N}} \mathbb{E}_t^N\Bigg[\int_t^{t+\tau}\Big\{e^{-\rho s}\big[u(C_s,N_s) + z_s^A\left(A_s^{(C,N),(t,a)}\right)N_s\big] + \partial_a V_s^A\left(A_s^{(C,N),(t,a)}\right)e^{-rs}(Y_s - C_s) - g_s\left(A_s^{(C,N),(t,a)}\right)\Big\}\,ds\Bigg] = 0$$

Then, by using the mean-value theorem and dominated convergence theorem, we get

$$g_t(a) = \sup_{(c,n)\in\mathcal{C}\times\mathcal{N}}\Big\{e^{-\rho t}\big[u(c,n) + z_t^A(a)n\big] + \partial_a V_t^A(a)e^{-rt}(Y_t - c)\Big\}$$

This proves that (V^A, z^A) solves the stochastic HJB equation (16). □

The next result is the verification theorem, establishing the sufficiency of the stochastic HJB equation for optimality. To state the result, let us consider feedback control processes $c(t,a,\omega)$ and $n(t,a,\omega)$, where $c : [0,T]\times\mathbb{R}\times\Omega \to \mathcal{C}$ and $n : [0,T]\times\mathbb{R}\times\Omega\to\mathcal{N}$. We assume that c and n are $\mathbb{F}$-progressively measurable for any fixed a. Notice that the agent's strategy is *not* required to be Markovian because c and n are allowed to depend on $\omega\in\Omega$.

Associated with (c,n), (A, W^A, Z^A) is given by the solution to the forward backward stochastic differential equations (FBSDE):

$$dA_t = e^{-rt}\Big\{Y(t,\omega) - c(t,A_t,\omega)\Big\}\,dt, \qquad A_0 = 0; \tag{19}$$

$$dW_t^A = -e^{-\rho t}\Big\{u\big(c(t,A_t,\omega), n(t,A_t,\omega)\big) + Z_t^A n(t,A_t,\omega)\Big\}\,dt + e^{-\rho t}\sigma Z_t^A\,dB_t,$$

$$W_T^A = e^{-\rho T}U(e^{rT}A_T + S_T(\omega)). \tag{20}$$

Let us assume that the above FBSDE is well-posed, i.e., it has a unique solution (A, W^A, Z^A). Since it is decoupled (i.e., the forward equation (19) does not depend on (W^A, Z^A)), a sufficient condition for its well-posedness is that c and n are uniformly Lipschitz continuous in a.[7]

Next, again given (c,n), consider the following FBSDE on $[t,T]$:

$$dA_s^{t,a} = e^{-rs}\Big\{Y(s,\omega) - c(s,A_s,\omega)\Big\}\,ds, \qquad A_t = a; \tag{21}$$

$$dW_s^{A,t,a} = -e^{-\rho s}\Big\{u\big(c(s,A_s^{t,a},\omega), n(s,A_s^{t,a},\omega)\big) + Z_s^{A,t,a}n(s,A_s,\omega)\Big\}\,ds + e^{-\rho s}\sigma Z_s^{A,t,a}\,dB_s,$$

$$W_T^{A,t,a} = e^{-\rho T}U(e^{rT}A_T^{t,a} + S_T(\omega)). \tag{22}$$

Then one can define $w^{c,n} : [0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$ by

$$w^{c,n}(t,a,\omega) := W_t^{A,t,a}(\omega)$$

[7] It follows, for instance, from Theorem 8.3.4 of Zhang (2017).

That is, $w^{c,n}(t, A_t, \omega)$ is the expected (remaining) utility of the agent at (t, ω) and with wealth A_t, when he uses a feedback control (c, n).

As in Ma et al. (2012), $w^{c,n}$ can be characterized as (a part of) the solution to a BSPDE as follows. Suppose that for each fixed $a \in \mathbb{R}$, $(w^{c,n}, z^{c,n})$ is the solution of the BSDE:

$$dw^{c,n}(t,a,\omega) = -G^{c,n}(t,a,\omega)\,dt + e^{-\rho t}\sigma z^{c,n}(t,a,\omega)\,dB_t, \tag{23}$$
$$w^{c,n}(T,a,\omega) = e^{-\rho T}U(e^{rT}a + S_T(\omega)), \tag{24}$$

where the function $G^{c,n} : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is determined in equation (27) below.

By applying the Itô-Wentzell formula to $w^{c,n}(t, A_t, \omega)$, we obtain

$$\begin{aligned} dw^{c,n}(t,A_t,\omega) = \Big\{ &-G^{c,n}(t,A_t,\omega) + \partial_a w^{c,n}(t,A_t,\omega)e^{-rt}\big[Y(t,\omega) - c(t,A_t,\omega)\big]\Big\}\,dt \\ &+ e^{-\rho t}\sigma z^{c,n}(t,A_t,\omega)\,dB_t, \end{aligned} \tag{25}$$

where $\partial_a w^{c,n}(t,a,\omega) := \frac{\partial w^{c,n}}{\partial a}(t,a,\omega)$.[8] Comparing (20) and (25), we obtain

$$Z_t^A = z^{c,n}(t, A_t, \omega) \tag{26}$$

and

$$\begin{aligned} G^{c,n}(t,a,\omega) = e^{-\rho t}\Big\{u\big[c(t,a,\omega), n(t,a,\omega)\big] + z^{c,n}(t,a,\omega)n(t,a,\omega)\Big\} \\ + \partial_a w^{c,n}(t,a,\omega)e^{-rt}\big[Y(t,\omega) - c(t,a,\omega)\big] \end{aligned} \tag{27}$$

With $G^{c,n}(t, a, \omega)$ given by (27), the system of equations given by (23)-(24) define a BSPDE. The unique existence of a (regular weak) solution $(w^{c,n}, z^{c,n})$ to this BSPDE is guaranteed by Theorem 6.1 of Ma et al. (2012).

Proposition 4. *Let a feasible contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$ be given. Consider a feedback control process (c^*, n^*). Let $(A^*, W^{A,*}, Z^{A,*})$ be the associated solution to FBSDE: (19)-(20), and define (C^*, N^*) by $C_t^* = c^*(t, A_t^*, \omega)$ and $N_t^* = n^*(t, A_t^*, \omega)$. Let $(w^{A,*}, z^{A,*})$ be the solution to the BSPDE (23)-(24). Assume that for each (t, a, ω), $(c^*(t, a, \omega), n^*(t, a, \omega))$ is a maximizer of $G^{c,n}(t, a, \omega)$:*

$$\begin{aligned} \big(c^*(t,a,\omega), n^*(t,a,\omega)\big) \in \arg\max_{(c,n)\in\mathcal{C}\times\mathcal{N}} \Big\{ &e^{-\rho t}\big[u(c,n) + z^{A,*}(t,a,\omega)n\big] \\ &+ \partial_a w^{A,*}(t,a,\omega)e^{-rt}\big[Y(t,\omega) - c\big]\Big\}, \end{aligned} \tag{28}$$

Then $w^{A,*} = V^A$ and (C^*, N^*) is an optimal control process.

[8] The derivatives can be interpreted in the weak sense. See, for instance, Remark 5.12 of Øksendal and Sulem (2019).

Proof. We follow the proof of the verification theorem in Section 3.2 of Peng (1992). Define $G^*(t,a,\omega)$ by

$$G^*(t,a,\omega) = e^{-\rho t}\Big\{u\big[c^*(t,a,\omega), n^*(t,a,\omega)\big] + z^{A,*}(t,a,\omega)n^*(t,a,\omega)\Big\} + \partial_a w^{A,*}(t,a,\omega)e^{-rt}\big[Y(t,\omega) - c^*(t,a,\omega)\big]$$

Let $(C,N) \in \boldsymbol{C} \times \boldsymbol{N}$ be any feasible control process. Let A be the associated wealth process:

$$A_t = \int_0^t e^{-rs}\Big\{Y(s,\omega) - C(s,\omega)\Big\}\,ds$$

Applying the Itô-Wentzell formula to $w^{A,*}(t,A_t,\omega)$, we obtain

$$dw^{A,*}(t,A_t,\omega) = \Big\{-G^*(t,A_t,\omega) + \partial_a w^{A,*}(t,A_t,\omega)e^{-rt}\big[Y_t - C_t\big]\Big\}\,dt + e^{-\rho t}\sigma z^{A,*}(t,A_t,\omega)\,dB_t,$$

that is,

$$w^{A,*}(0,0,\omega) = e^{-\rho T}U(e^{rT}A_T + S_T) + \int_0^T \Big\{G^*(t,A_t,\omega) - \partial_a w^{A,*}(t,A_t,\omega)e^{-rt}\big[Y_t - C_t\big]\Big\}\,dt - \int_0^T e^{-\rho t}\sigma z^{A,*}(t,A_t,\omega)\,dB_t$$

By the hypotheses of the proposition,

$$G^*(t,A_t,\omega) \geq e^{-\rho t}\big[u(C_t,N_t) + z^{A,*}(t,A_t,\omega)N_t\big] + \partial_a w^{A,*}(t,A_t,\omega)e^{-rt}\big[Y_t - C_t\big]$$

holds for all t a.s. Therefore,

$$\begin{aligned} w^{A,*}(0,0,\omega) &\geq e^{-\rho T}U(e^{rT}A_T + S_T) + \int_0^T e^{-\rho t}\big[u(C_t,N_t) + z^{A,*}(t,A_t,\omega)N_t\big]\,dt \\ &\quad - \int_0^T e^{-\rho t}\sigma z^{A,*}(t,A_t,\omega)\,dB_t \\ &= e^{-\rho T}U(e^{rT}A_T + S_T) + \int_0^T e^{-\rho t}u(C_t,N_t)\,dt - \int_0^T e^{-\rho t}\sigma z^{A,*}(t,A_t,\omega)\,dB_t^N \end{aligned}$$

where $B_t^N = B_t - \int_0^t N_s/\sigma\,ds$ is the Brownian motion associated with $\mathbb{P}^N$. Taking expectations $\mathbb{E}^N$ on both sides, we obtain

$$w^{A,*}(0,0,\omega) \geq \mathbb{E}^N\left[e^{-\rho T}U(e^{rT}A_T + S_T) + \int_0^T e^{-\rho t}u(C_t,N_t)\,dt\right]$$

On the other hand, under the hypotheses of the proposition, we have

$$w^{A,*}(0,0,\omega) = \mathbb{E}^{N^*}\left[e^{-\rho T}U(e^{rT}A_T^* + S_T) + \int_0^T e^{-\rho t}u(C_t^*,N_t^*)\,dt\right]$$

Since (C,N) is an arbitrary control process, this shows that the control process (C^*,N^*) is optimal and its value is given by $w^{A,*}$. □

The first-order conditions associated with the maximization problem in (28) are

$$e^{-\rho t}\partial_c u\big[c^*(t,a,\omega), n^*(t,a,\omega)\big] - \partial_a w^{A,*}(t,a,\omega)e^{-rt} \begin{cases} \leq 0, & \text{if } c^*(t,a,\omega)=0, \\ =0, & \text{if } c^*(t,a,\omega)\in(0,\overline{C}), \\ \geq 0, & \text{if } c^*(t,a,\omega)=\overline{C}, \end{cases} \tag{29}$$

$$\partial_n u\big[c^*(t,a,\omega), n^*(t,a,\omega)\big] + z^{A,*}(t,a,\omega) \begin{cases} \leq 0, & \text{if } n^*(t,a,\omega)=0, \\ =0, & \text{if } n^*(t,a,\omega)\in(0,\overline{N}), \\ \geq 0, & \text{if } n^*(t,a,\omega)=\overline{N}. \end{cases} \tag{30}$$

Since (C^*, N^*) is an optimal control, it satisfies the first-order conditions (12)-(13). In particular, there exists a solution $(A^*, W^{A,*}, Z^{A,*}, \Lambda^*, Z^{\Lambda,*})$ to the following FBSDE

$$A_t^* = \int_0^t e^{-rs}(Y_s - C_s^*)\, ds \tag{31}$$

$$W_t^{A,*} = e^{-\rho T}U(e^{rT}A_T^* + S_T) + \int_t^T e^{-\rho s}\big[u(C_s^*, N_s^*) + Z_s^{A,*}N_s^*\big]\, ds - \int_t^T e^{-\rho s}\sigma Z_s^{A,*}\, dB_s, \tag{32}$$

$$\Lambda_t^* = e^{(r-\rho)T}U'(e^{rT}A_T^* + S_T) + \int_t^T N_s^* e^{-\rho s} Z_s^{\Lambda,*}\, ds - \int_t^T e^{-\rho s}\sigma Z_s^{\Lambda,*}\, dB_s, \tag{33}$$

such that[9]

$$W_t^{A,*} = w^{A,*}(t, A_t^*, \omega), \quad Z_t^{A,*} = z^{A,*}(t, A_t^*, \omega), \quad \Lambda_t^* = \partial_a w^{A,*}(t, A_t^*, \omega) \tag{34}$$

2.5 Principal's problem

In the previous subsection, it is shown that incentive compatible strategies of the agent are characterized by the stochastic HJB equation (28). However, it can be computationally intractable, if one formulates the principal's problem using the agent's stochastic HJB equation as its constraint. In this subsection, we show that, by focusing on those contracts for which the agent chooses to hold no assets, the stochastic HJB equation is effectively reduced to a system of two BSDEs. It follows that the principal's problem can be formulated as maximizing her expected profit subject to those two BSDEs, which is solvable using a standard method. Indeed, this formulation of the principal's problem coincides with what is obtained using the first-order approach. It is *as if* the first-order approach were valid, although it may not be in the true sense.

As argued, for instance, in Williams (2015) and Di Tella and Sannikov (2021), without loss of generality, we may restrict attention to those contracts for which the agent chooses to neither save nor borrow:

$$C_t = Y_t, \quad \text{and} \quad A_t = 0, \qquad \forall t \in [0,T], \text{ a.s.}$$

[9] We can also show that $Z_t^{\Lambda,*} = \partial_a z^{A,*}(t, A_t^*, \omega)$.

This simplifies the principal's problem significantly, because we only have to compute $Y(\cdot,\cdot) = c(\cdot, 0, \cdot)$, $n(\cdot, 0, \cdot)$, $V^A(\cdot, 0, \cdot)$, and $\partial_a V^A(\cdot, 0, \cdot)$.

Now, abusing notation, we define $y(t, p, z)$ and $n(t, p, z)$ by

$$e^{-\rho t}\partial_c u\big[y(t,p,z), n(t,p,z)\big] - pe^{-rt} \begin{cases} \leq 0, & \text{if } y(t,p,z) = 0, \\ = 0, & \text{if } y(t,p,z) \in (0, \overline{C}), \\ \geq 0, & \text{if } y(t,p,z) = \overline{C}, \end{cases} \tag{35}$$

$$\partial_n u\big[y(t,p,z), n(t,p,z)\big] + z \begin{cases} \leq 0, & \text{if } n(t,p,z) = 0, \\ = 0, & \text{if } n(t,p,z) \in (0, \overline{N}), \\ \geq 0, & \text{if } n(t,p,z) = \overline{N}. \end{cases} \tag{36}$$

Under our assumptions, y and n are Lipschitz in (p, z).

Then, consider the following BSDE:

$$\begin{aligned} W_t^A &= e^{-\rho T}U(S_T) + \int_t^T e^{-\rho s}\Big\{u\big(y(s,\Lambda_s,Z_s^A), n(s,\Lambda_s,Z_s^A)\big) + Z_s^A n(s,\Lambda_s,Z_s^A)\Big\}\, ds \\ &\quad - \int_t^T e^{-\rho s}\sigma Z_s^A\, dB_s, \\ \Lambda_t &= e^{(r-\rho)T}U'(S_T) + \int_t^T n(s,\Lambda_s,Z_s^A)e^{-\rho s}Z_s^\Lambda\, dt - \int_t^T e^{-\rho s}\sigma Z_s^\Lambda\, dB_s. \end{aligned}$$

Using $dB_t^n = dB_t - n(t,\Lambda_t,Z_t^A)/\sigma\, dt$, they are rewritten as

$$W_t^A = e^{-\rho T}U(S_T) + \int_t^T e^{-\rho s}u\big(y(s,\Lambda_s,Z_s^A), n(s,\Lambda_s,Z_s^A)\big)\, ds - \int_t^T e^{-\rho s}\sigma Z_s^A\, dB_s^n, \tag{37}$$

$$\Lambda_t = e^{(r-\rho)T}U'(S_T) - \int_t^T e^{-\rho s}\sigma Z_s^\Lambda\, dB_s^n. \tag{38}$$

Under our assumptions, they have a unique solution $(W^A, \Lambda, Z^A, Z^\Lambda)$ (e.g., Theorem 4.3.1 of Zhang (2017)), and in addition,

$$V^A(t, 0, \omega) = W_t^A, \quad \text{and} \quad \partial_a V^A(t, 0, \omega) = \Lambda_t$$

Thus, we can obtain the value function at $A_t = 0$ by solving the BSDE. The result is summarized in the following proposition.

Proposition 5. *Given W_0^A, the principal's problem is to choose a contract $(Y, S_T) \in \boldsymbol{Y} \times \boldsymbol{S}$ so as to maximize her expected profit:*

$$\mathbb{E}^N\left[\int_0^T e^{-rt}(N_t - Y_t)\, dt - e^{-rT}S_T\right] \tag{39}$$

subject to (37)-(38).

One can solve the above problem in different ways. Here, we illustrate the approach based on dynamic programming. For this purpose, we need to rewrite the equation for W^A as a forward equation, rather than the backward equation as in (37). Suppose that $U^{-1}:\mathbb{R}\to\mathbb{R}$ is well-defined. Then (37)-(38) are rewritten as a Markovian FBSDE:

$$\begin{aligned}
W_t^A &= W_0^A - \int_0^t e^{-\rho s} u\big(y(s,\Lambda_s,Z_s^A), n(s,\Lambda_s,Z_s^A)\big)\,ds + \int_0^t e^{-\rho s}\sigma Z_s^A\,dB_s^n;\\
\Lambda_t &= e^{(r-\rho)T}U'(S_T(W_T^A)) - \int_t^T e^{-\rho s}\sigma Z_s^\Lambda\,dB_s^n,
\end{aligned}$$

where $S_T(W_T^A)$ is the transfer at the terminal date, determined by W_T^A:

$$S_T(W_T^A) := U^{-1}(e^{\rho T}W_T^A)$$

Under our assumption, the assumptions of Theorem 8.3.5 of Zhang (2017) are satisfied. Therefore, there exists a Lipschitz continuous function $\Lambda(t,w)$ such that $\Lambda_t = \Lambda(t, W_t^A)$. It follows that the forward equation for W^A is given by

$$W_t^A = W_0^A - \int_0^t e^{-\rho s} u\big(y(s,\Lambda(s,W_s^A),Z_s^A), n(s,\Lambda(s,W_s^A),Z_s^A)\big)\,ds + \int_0^t e^{-\rho s}\sigma Z_s^A\,dB_s^n$$

Then, the rest is standard, as described, for instance, in section 5.4.4 of Cvitanić and Zhang (2013). Let $V^P(t,w)$ denote the value function for the principal. Then the HJB equation for the principal's problem is

$$\begin{aligned}
\sup_z\Big\{ & e^{-rt}\big[n(t,\Lambda(t,w),z) - y(t,\Lambda(t,w),z)\big] + \partial_t V^P(t,w)\\
& - \partial_w V^P(t,w) e^{-\rho t} u\big(y(t,\Lambda(t,w),z), n(t,\Lambda(t,w),z)\big) + \frac{1}{2}\partial_{ww}V^P(t,w)(e^{-\rho t}\sigma z)^2\Big\} = 0
\end{aligned}$$

Remark. Proposition 5 might be extended to the case where shocks follow a Markov chain, rather than a Brownian motion. Then it would apply to the optimal unemployment insurance problem considered, for instance, by Hopenhayn and Nicolini (1997), Kocherlakota (2004), Mitchell and Zhang (2010).

3 Hidden returns

In this section, we consider a version of the principal-agent model studied by Di Tella and Sannikov (2021). Here, we do not assume CRRA preferences for the agent nor impose the restriction on the volatility of the compensation process, unlike Di Tella and Sannikov (2021). The proofs of the propositions in this section are similar to those given in the previous section, and hence are omitted.

3.1 The model

The agent manages risky capital delegated by the principal. The instantaneous return of capital reported to (observed by) the principal is

$$dR_t = (r + \alpha - N_t)\,dt + \sigma\,dB_t^N$$

where r is the risk-free rate; $\alpha > 0$ is the risk premium; $\sigma > 0$ is the volatility of the return; $N_t \geq 0$ is the hidden action that the agent takes to divert returns for his private benefits; and B^N is a standard Brownian motion defined by equation (42) below.

As in the previous section, we state the model in the framework of weak formulation. Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, on which a standard Brownian motion B is defined; $\mathbb{F} = \{\mathscr{F}_t\}_{0\leq t\leq T}$ is the augmented filtration generated by B; and $\mathbb{E}$ is the expectation operator associated with $\mathbb{P}$.

The cumulative return process is defined as the strong solution to

$$dR_t = (r + \alpha)\,dt + \sigma\,dB_t$$

The diversion action N affects the probability distribution of the return process. As in the previous model, let $\boldsymbol{N}$ be the set of $\mathbb{F}$-progressively measurable processes $N : [0, T] \times \Omega \to \mathcal{N}$, where $\mathcal{N} = [0, \overline{N}]$.

Then, for $N \in \boldsymbol{N}$, the process M^N defined by

$$M_t^N := \exp\left(-\int_0^t \frac{N_s}{\sigma}\,dB_s - \frac{1}{2}\int_0^t \frac{N_s^2}{\sigma^2}\,ds\right) \tag{40}$$

is a martingale; the probability measure $\mathbb{P}^N$ defined by

$$d\mathbb{P}^N := M_T^N\,d\mathbb{P} \tag{41}$$

is the measure induced by action N; and B^ℓ defined by

$$B_t^N := B_t + \int_0^t \frac{N_s}{\sigma}\,ds \tag{42}$$

is a standard Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P}^N)$. The return process R is then expressed as

$$dR_t = (r + \alpha)\,dt + \sigma\,dB_t = (r + \alpha - N_t)\,dt + \sigma\,dB_t^N \tag{43}$$

A contract offered by the principal is $(Y, S_T, K) \in \boldsymbol{Y} \times \boldsymbol{S} \times \boldsymbol{K}$, where Y_t is the payment to the agent at each point in time t; S_T is the terminal payment at date T; K_t is the amount of capital that the agent is delegated to manage at time t; and $\boldsymbol{K}$ is the set of progressively measurable processes $K : [0,T] \times \Omega \to \mathcal{K}$ with $\mathcal{K} = [0, \overline{K}]$. Diversion of $N_t \geq 0$ gives the agent a flow of funds $\phi N_t K_t$, where $\phi \in (0,1)$.

Without being observed by the principal, the agent can freely borrow and lend at the risk-free interest rate r. Let $\tilde{A}_t$ be the risk-free asset owned by the agent at time t, and let $A_t := e^{-rt}\tilde{A}_t$. Then, the budget constraint of the agent is given by

$$dA_t = e^{-rt}(Y_t + \phi K_t N_t - C_t)\,dt, \qquad A_0 = 0, \tag{44}$$

where $C \in \boldsymbol{C}$ is the consumption process of the agent.

Given a contract $(Y, S_T, K) \in \boldsymbol{Y} \times \boldsymbol{S} \times \boldsymbol{K}$, the agent chooses $(C, N) \in \boldsymbol{C} \times \boldsymbol{N}$ so as to maximize his expected utility

$$\mathbb{E}^N \left[\int_0^T e^{-\rho t} u(C_t)\,dt + e^{-\rho T} U(e^{rT} A_T + S_T) \right] \tag{45}$$

subject to the budget constraint (44). Here, the flow utility function u is a function of C_t only. We make similar assumptions on u and U as in the previous section. A pair (C, N) is said to be incentive compatible with respect to (Y, S_T, K) if it solves the agent's utility maximization problem. We continue to assume that the agent's problem has a solution for any feasible contracts.

3.2 First-order necessary conditions

Associated with a consumption process $C \in \boldsymbol{C}$, the agent's utility process W^A is defined as

$$W_t^A := \mathbb{E}_t^N \left[\int_t^T e^{-\rho s} u(C_s)\,ds + e^{-\rho T} U(e^{rT} A_T + S_T) \right]$$

As in the previous section, (W^A, Z^A) satisfies the BSDE given by

$$W_t^A = e^{-\rho T} U(e^{rT} A_T + S_T) + \int_t^T e^{-\rho s} \big[u(C_s) - Z_s^A N_s \big]\,ds - \int_t^T e^{-\rho s} \sigma Z_s^A\,dB_s. \tag{46}$$

The agent's problem is to choose $(C, N) \in \boldsymbol{C} \times \boldsymbol{N}$ to maximize W_0^A subject to (44) and (46).

Of particular interest is the condition under which "no diversion," $N \equiv 0$, is an incentive compatible choice of the agent. The next proposition describes a necessary condition for its optimality. Note that when $N = 0$, the adjoint process $M_t^N = 1$ for all t, and thus is dropped from the optimality conditions.

Proposition 6. *Let $(Y, S_T, K) \in \boldsymbol{Y} \times \boldsymbol{S} \times \boldsymbol{K}$ be a given contract. Consider a pair of consumption and no-diversion processes $(\hat{C}, 0) \in \boldsymbol{C} \times \boldsymbol{N}$ satisfying the budget constraint, so that the associated wealth process $\hat{A}$ satisfies (44):*

$$d\hat{A}_t = e^{-rt}(Y_t - \hat{C}_t)\, dt, \qquad \hat{A}_0 = 0.$$

The associated utility process $(\hat{W}^A, \hat{Z}^A)$ is the solution to the BSDE (46):

$$d\hat{W}_t^A = -e^{-\rho t} u(\hat{C}_s)\, ds + e^{-\rho t}\sigma Z_t^A\, dB_t, \qquad \hat{W}_T^A = e^{-\rho T} U(e^{rT}\hat{A}_T + S_T).$$

The adjoint process for $\hat{A}$ is given by the solution $(\hat{\Gamma}, \hat{Z}^\Gamma)$ to the BSDE:

$$d\hat{\Gamma}_t = e^{-\rho t}\sigma \hat{Z}_t^\Gamma\, dB_t, \qquad \hat{\Gamma}_T = e^{(r-\rho)T} U'(e^{rT}\hat{A}_T + S_T).$$

Then, necessary conditions for $(\hat{C}, 0)$ to be incentive compatible for the agent are given by the first-order conditions

$$e^{-\rho t} u_c(\hat{C}_t) - \hat{\Gamma}_t e^{-rt} \begin{cases} \leq 0, & \text{if } \hat{C}_t = 0, \\ = 0, & \text{if } \hat{C}_t \in (0, \overline{C}), \\ \geq 0, & \text{if } \hat{C}_t = \overline{C}, \end{cases}$$
$$-e^{-\rho t}\hat{Z}_t^A + \hat{\Gamma}_t e^{-rt}\phi K_t \leq 0.$$

3.3 Stochastic HJB equation

Just as in the hidden-effort model discussed in the previous section, the sufficiency theorem of the stochastic maximum principle (e.g. Theorem 10.2.9 of Cvitanić and Zhang (2013)) cannot be used to establish the sufficiency of the first-order conditions in Proposition 6. We, again, employ the dynamic programming approach.

Given a contract $(Y, S_T, K) \in \boldsymbol{Y} \times \boldsymbol{S} \times \boldsymbol{K}$, the value function $V^A : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is defined by

$$V^A(t, a, \omega) := \sup_{(C,N)\in \boldsymbol{C}\times\boldsymbol{N}} \mathbb{E}_t^N \left[\int_t^T e^{-\rho s} u(C_s)\, ds + e^{-\rho T} U(e^{rT} A_T + S_T) \right] \tag{47}$$
$$\text{s.t.} \quad A_T = a + \int_t^T e^{-rs}(Y_s + \phi K_s N_s - C_s)\, ds$$

As in the previous section, we can show that the value function is Lipschitz continuous and satisfies the stochastic HJB equation.

Proposition 7. *For $(Y, S_T, K) \in \boldsymbol{Y} \times \boldsymbol{S} \times \boldsymbol{K}$, the value function $V^A(t, a, \omega)$ defined in (47) is bounded, and Lipschitz continuous in (t, a) for all $\omega \in \Omega$.*

Proposition 8. *Given $(Y, S_T, K) \in \boldsymbol{Y} \times \boldsymbol{S} \times \boldsymbol{K}$, the value function V^A satisfies the stochastic HJB equation:*

$$\begin{aligned}
dv(t,a,\omega) &= -\sup_{(c,n)\in\mathcal{C}\times\mathcal{N}} \Big\{ e^{-\rho t}\big[u(c) + z^A(t,a,\omega)n\big] + \partial_a v(t,a,\omega)e^{-rt}\big[Y(t,\omega) + \phi K(t,\omega)n - c\big]\Big\}\, dt \\
&\qquad + e^{-\rho t}\sigma z^A(t,a,\omega)\, dB_t, \qquad (48)\\
v(T,a,\omega) &= e^{-\rho T}U(e^{rT}a + S_T(\omega)).
\end{aligned}$$

Thus, if no diversion, $(\hat{C}, 0)$, is optimal, the first-order conditions in Proposition 6 are expressed as

$$\begin{aligned}
& e^{-\rho t}u_c(\hat{C}_t) - \partial_a V^A(t, \hat{A}_t, \omega)e^{-rt} \begin{cases} \le 0, & \text{if } \hat{C}_t = 0, \\ = 0, & \text{if } \hat{C}_t \in (0, \overline{C}), \\ \ge 0, & \text{if } \hat{C}_t = \overline{C}, \end{cases} \\
& -e^{-\rho t}z^A(t, \hat{A}_t, \omega) + \partial_a V^A(t, \hat{A}_t, \omega)e^{-rt}\phi K_t \le 0.
\end{aligned}$$

Next, let us see the sufficiency for optimality of no diversion based on the stochastic HJB equation. Given $(Y, S_T, K) \in \boldsymbol{Y} \times \boldsymbol{S} \times \boldsymbol{K}$, consider a feedback control $(c(t,a,\omega), n(t,a,\omega))$, where $c : [0,T]\times\mathbb{R}\times\Omega \to \mathcal{C}$ and $n : [0,T]\times\mathbb{R}\times\Omega \to \mathcal{N}$, such that c and n are $\mathbb{F}$-progressively measurable for any fixed a, and the associated FBSDE:

$$dA_t = e^{-rt}\Big\{Y(t,\omega) + \phi K(t,\omega)n(t,A_t,\omega) - c(t,A_t,\omega)\Big\}\, dt, \qquad A_0 = 0, \tag{49}$$

$$dW_t^A = -e^{-\rho t}\Big\{u(c(t,A_t,\omega)) - Z_t^A n(t,A_t,\omega)\Big\}\, dt + e^{-\rho t}\sigma Z_t^A\, dB_t, \qquad W_T^A = e^{-\rho T}U(e^{rT}A_T + S_T). \tag{50}$$

is well-posed. Let $w^A(t,a,\omega)$ denote the decoupling field of this system. Then the BSPDE for w^A is derived as

$$dw^A(t,a,\omega) = -G^{c,n}(t,a,\omega)\, dt + e^{-\rho t}\sigma z^A(t,a,\omega)\, dB_t \tag{51}$$

$$w^A(T,a,\omega) = e^{-\rho T}U(e^{rT}a + S_T), \tag{52}$$

where

$$\begin{aligned}
G^{c,n}(t,a,\omega) := {}& e^{-\rho t}\Big\{u\big[c(t,a,\omega)\big] - z^A(t,a,\omega)n(t,a,\omega)\Big\} \\
& + \partial_a w^A(t,a,\omega)e^{-rt}\Big\{Y(t,\omega) + \phi K(t,\omega)n(t,a,\omega) - c(t,a,\omega)\Big\}
\end{aligned} \tag{53}$$

The next proposition describes the verification theorem for this model.

Proposition 9. *Let a contract $(Y, S_T, K) \in \boldsymbol{Y}\times\boldsymbol{S}\times\boldsymbol{K}$ be given. Consider a feedback control process with no diversion: $(c^*, 0)$. Let $(A^*, W^{A,*}, Z^{A,*})$ be the associated solution to FBSDE (49)-(50), and* *define* $C_t^* = c^*(t, A_t^*, \omega)$*. Let* $(w^{A,*}, z^{A,*})$ *be the solution to the BSPDE* (51)-(52)*. Assume that for each* (t, a, ω)*,* $(c^*(t, a, \omega), 0)$ *is a maximizer of* $G^{c,n}(t, a, \omega)$*:*

$$\big(c^*(t,a,\omega),0\big) \in \arg\max_{(c,n)\in\mathcal{C}\times\mathcal{N}} \Big\{e^{-\rho t}\big[u(c) - z^{A,*}(t,a,\omega)n\big] + \partial_a w^{A,*}(t,a,\omega)e^{-rt}\big[Y(t,\omega) + \phi K(t,\omega)n - c\big]\Big\} \tag{54}$$

Then $w^{A,*} = V^A$*, and* $(c^*, 0)$ *is an optimal control process.*

3.4 Principal's problem

Given a promised level of initial utility of the agent, W_0^A, the principal's objective is to minimize the expected cost of delivering W_0^A to the agent:

$$\mathbb{E}^N\left[\int_0^T e^{-rt}(Y_t - \alpha K_t)\,dt + e^{-rT}S_T\right]$$

where α is the risk premium in (43), and the principal chooses a feasible contract that induces $N \equiv 0$.

As in the previous model with hidden effort, we restrict attention to contracts such that the zero saving is optimal for the agent:

$$C_t = Y_t, \quad \text{and} \quad A_t = 0, \qquad \forall t \in [0,T], \text{ a.s.},$$

The following proposition establishes that, again, the principal's problem can be formulated as the minimization of her expected cost subject to two BSDEs, as far as attention is restricted to those contracts that induce the agent to choose zero savings. Thus, it is as if the first-order approach is valid.

Proposition 10. *Given* W_0^A*, the principal's problem is to choose a contract* $(Y, S_T, K) \in \boldsymbol{Y}\times\boldsymbol{S}\times\boldsymbol{K}$ *so as to minimize the expected cost:*

$$\mathbb{E}\left[\int_0^T e^{-rt}(Y_t - \alpha K_t)\,dt + e^{-rT}S_T\right] \tag{55}$$

subject to

$$W_t^A = e^{-\rho T}U(S_T) + \int_t^T e^{-\rho s}u(Y_s)\,ds - \int_t^T e^{-\rho s}\sigma Z_s^A\,dB_s, \tag{56}$$

$$\Gamma_t = e^{(r-\rho)T}U'(S_T) - \int_t^T e^{-\rho s}\sigma Z_s^\Gamma\,dB_s, \tag{57}$$

and

$$e^{-\rho t}\partial_c u(Y_t) - \Gamma_t e^{-rt} \begin{cases} \leq 0, & \text{if } Y_t = 0, \\ = 0, & \text{if } Y_t \in (0, \overline{C}), \\ \geq 0, & \text{if } Y_t = \overline{C}, \end{cases} \tag{58}$$

$$-e^{-\rho t} Z_t^A + \Gamma_t e^{-rt} \phi K_t \leq 0. \tag{59}$$

4 Conclusion

In this paper, we consider two models of moral hazard with hidden savings in continuous time. Instead of using the first-order approach, we use the stochastic HJB equation to characterize the optimality condition for the agent. Without loss of generality, we focus on those contracts for which the agent chooses zero savings. Given this, we show that the principal's optimization problem can be expressed as maximizing her expected profit subject to two SDEs: one equation describing the agent's continuation utility process, and the other being the Euler equation concerning the agent's marginal utility process. Such a formulation of the principal's problem coincides with the one obtained by assuming the validity of the first-order approach. Our result is an extension of those obtained by Williams (2015) and Di Tella and Sannikov (2021).

Our approach can be extended in a number of directions. For instance, it is of both theoretical and practical interest to extend it to an infinite-horizon setting. Our approach can also be applied to consider the problem of optimal unemployment insurance, where the shock follows a Poisson process. Other potentially interesting areas of application include, among others, dynamic mechanism design problems studied, e.g., in Pavan et al. (2014); the optimal taxation problem with private insurance such as Golosov and Tsyvinski (2007); the optimal taxation problem with human capital accumulation, e.g., Stantcheva (2017), Kapička and Neira (2019).

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