Doubly fair priority-completion in assignment problems *

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Abstract

We consider the problem of assigning indivisible objects to the agents prioritized within their affiliated institutions. An example is the assignment of student exhange programs to students who are prioritized only in their own departments. Since students of different departments are incomparable, the problem is formalized as a priority-based indivisible goods allocation problem with incomplete priority. We show that each weak core allocation is attained by priority rule with a priority-completion, and vice versa. Moreover, we advocate a class of completions satisfying two fairness notions: interpersonal and interinstitutional fairness.

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1 Introduction

Suppose that the administration office of a university faces a problem of assigning indivisible objects, e.g., scholarship, student exchange programs, to the students who

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Econ.	Priority	Law	Priority
e_1	1	l_1	3
		l_2	4
e_2	2	l_3	5
		l_4	6

(i)

Econ.	Priority	Law	Priority
e_1	2	l_1	1
		l_2	3
e_2	5	l_3	4
		l_4	6

(ii)

Econ.	Priority	Law	Priority
e_1	5	l_1	1
		l_2	2
e_2	6	l_3	3
		l_4	4

(iii)

Table 1: Example of completions

are prioritized only in their own departments. For example, assume that students e_1, e_2 belonging to the department of economics and l_1, l_2, l_3, l_4 of department of law are searching for an opportunity to study abroad. Assume also that each department gives higher-priority for students with the smaller index, i.e., $e_1 \succ e_2$ and $l_1 \succ \ldots \succ l_4$. This situation looks similar to the so-called "school choice" problem, but exhibits the following two features:

Feature 1. Priority is **incomplete** because students of different departments are incomparable.

Feature 2. Priority is **common**, i.e., each exchange program gives the priority to students in the same manner. This is because the administration office refers to the priority order submitted by departments.

As long as the administration office is able to bear the cost to have an additional interview or examination, the incomparablility of students could be resolved by creating new data. However, the cost to extract such additional data could be enormous as the number of applicants becomes large. In this paper, we persue the market design without such cost. The second feature is typically observed in the problems that the administration office assigns indivisible objects to students, e.g., dorm room assignment (Abdulkadiroğlu and Sönmez, 1999).

We formalize the problem with above features as a priority-based indivisible goods allocation problem with incomplete priority (Balbuzanov and Kotowski, 2019). Showing the relationship among several core concepts (Theorem 1 and Proposition 1), we prove that the weak core is characterized by the range of priority rule, also known as serial-dictatorship rule, with completions of priority (Theorem 2), where a completion is a complete priority order consistent with the given incomplete priority. As is shown in Table 1, the class of completions contains wide range of possibility in terms of fairness. For example, the completions shown in (i) and (iii) are completely favorable to one of the departments, while the middle completion (ii) shows a balanced distribution of the opportunity. Among the various completions, we advocate a method to pick a plausible class based on two fairness notions: interpersonal and interinstitutional fairness (Theorem 3).

1.1 Related literature

In the one-to-one matching problems, a model with social endowments is called a house allocation problem (Hylland and Zeckhauser, 1979). The model with a private ownership is called a housing market (Shapley and Scarf, 1974), and that with mixed ownership is termed a house allocation problem with existing tenants (Abdulkadiroğlu and Sönmez, 1999). Balbuzanov and Kotowski (2019) generalize these models to an indivisible goods allocation problem with incomplete priority structure. To attain an efficient allocation, the Gale's top-trading cycles algorithm (Shapley and Scarf, 1974), or its variat, is applied to these problems. For a special class of these problems with homogeneous priority, priority rule is utilized to attain efficiency (Svensson, 1999).

School choice is a house allocation problem with multiple copies of objects (Abdulkadiroğlu and Sönmez, 2003). The vast literature on school choice contains a research under priority with ties (Erdil and Ergin, 2008; Ehlers and Erdil, 2010), which looks similar, but different to the incomplete priority case. A difference is the following: ties are exogenously given in the former case, while be not in the latter case. This makes the set of completions far richer than that of tie-breakers. For example, in Table 1, since e_2 is not comparable with students of law department, e_2 could be prioritized to l_1 in some completion (Table 1(i)). However, in another completion (Table 1(iii)), l_1 could be prioritized to e_1, who is explicitly prioritized to e_2 in the given priority order. This never happens under a complete priority with ties. Thus, an appropriate research direction to find a plausible completion may start with how to form ties under the given incomplete priority (Hatakeyama and Kurino, 2022). Apportionment is a related topic in this line (Balinski and Young, 2001). As is discussed in detail in Section 4, our approach takes another direction. We transform the problem of finding a completion into the allocation problem of quantified opportunity (point) to establish a system under which agents with higher points get higher priority. The transformation enables us to define two fairness concepts, interpersonal and interinstitutional fairness, which in turn are helpful to establish an intuitive system to allocate objects.

The contributions of the current paper could be summarized as follows: Under the assumption of common priority (Feature 2), we show the following.

- Extending the Balbuzanov and Kotowski's priority-based formalization of property right to the many-to-one setting, we show the relationship among several core concepts in terms of set-theoretic inclusion (Theorem 1 and Proposition 1).
- The weak core is characterized by the range of priority rule with priority-completions (Theorem 2).
- We formalize the concept of interpersonal and interinstitutional fair point allocations to pick a class of conpletions. Moreover, we advocate a concrete method, the (generalized) midpoint rule, to implement it. We provide a characterization of the generalized midpoint rule based mainly on the two fairness notions (Theorem 3).

The rest of the paper organized as follows. Section 2 formalizes the priority-based indivisible goods allocation problem with an incomplete priority structure. In section 3, generalizing the property right formalized in Balbuzanov and Kotowski (2019), we define the several core concepts. Then, we provide two characterization results of weak core (Theorem 1 and 2). In section 4, we first consider the refinement of weak core in terms of fairness notions in the ex-post sense. Then, we turn to the design of completion selection rule through point allocations. We propose a rule called the midpoint rule, and provide a characterization of a class of rules including it. Proofs are relegated to Appendix B.

2 Model

We introduce an indivisible goods allocation problem with incomplete priority. It is the one in Balbuzanov and Kotowski (2019) with multiple copies of objects.[1] Let

[1] Balbuzanov and Kotowski (2019) deals with two matching problems: a simple economy with the initial endowment structure, and a relational economy with the priority structure. The model we borrow is the latter.

$N = \{1, 2, \ldots, n\}$ be the set of agents. Let $\mathcal{O} = \{o_1, o_2, \ldots, o_m\}$ be the set of real objects. Each real object $o \in \mathcal{O}$ has $q_o \in \mathbb{N}$ copies. Let $q := (q_o)_{o \in \mathcal{O}} \in \mathbb{N}^m$ be the quota vector. We assume that the existence of null object $o_0 \notin \mathcal{O}$, which represents no consumption. For each $o \in \mathcal{O}$, a binary relation $\succeq_o$ on N is given, which represents the priority to consume o. We assume that $\succeq_o$ is reflexive, transitive and anti-symmetric, but not necessarily complete.[2] Letting $\succeq := (\succeq_o)_{o \in \mathcal{O}}$, we call $\succeq$ the priority structure. Each agent $i \in N$ has a preference represented by a complete, transitive and anti-symmetric binary relation R_i on $\mathcal{O} \cup \{o_0\}$. Let $\mathcal{R}$ be the set of preference relations. Let $\mathcal{R}^N$ be the set of preference profiles.

A **problem** is a 5-tuple $(N, \mathcal{O} \cup \{o_0\}, \succeq, q, R)$, where $R \in \mathcal{R}^N$. Throughout this paper, we fix $N, \mathcal{O} \cup \{o_0\}, \succeq$ and q. Thus, each problem is simply denoted by a preference profile R. An allocation is a function from $a : N \to \mathcal{O} \cup \{o_0\}$ such that for each $o \in \mathcal{O}$, $|a^{-1}(o)| \leq q_o$. Let $\mathcal{A}$ be the set of allocations.

In this paper, we concentrate on the class of problems satisfying the following condition.

Assumption 1 (Common priority)**.** *The priority structure $\succeq$ is **common** if*

$$\forall o, o' \in \mathcal{O}, \succeq_o = \succeq_{o'} .$$

Hereafter, if there is no confusion, the symbol $\succeq$ not only represents a profile of priority relations, but also the individual priority relation $\succeq_o$ all objects share.

Balbuzanov and Kotowski (2019) deals with another class of problems with acyclic priority structure. The priority structure $\succeq$ is **acyclic** *if* for each $o \in \mathcal{O}$, and each $\{i, j, h\} \subseteq N$, if $i \succ_o j$ and $i \not\succeq_o h$, then $h \succ_{o'} j$ for all $o' \in \mathcal{O} \setminus \{o\}$. As is shown in the following example, when $|N| \geq 3$ and $|\mathcal{O}| \geq 2$, acyclicity and Assumption 1 are independent, i.e., one of the two conditions does not imply the other condition. Thus, the results in Balbuzanov and Kotowski (2019) and that of ours are independent.

Example 1 (Common priority structure may not be acyclic.)**.** Suppose that $|N| \geq 3$ and $|\mathcal{O}| \geq 2$. Let $i, j, h \in N$. Suppose that for each $o \in \mathcal{O}$, $i \succ_o j$ and h is not

[2] A binary relation on X, denoted as $\geq$, is reflexive if for each $x \in X$, $x \geq x$. A binary relation $\geq$ on X is complete if for each $\{x, y\} \subseteq X$, $x \geq y$ or $y \geq x$. A binary relation $\geq$ on X is transitive if for each $\{x, y, z\} \subseteq X$, $x \geq y$ and $y \geq z$ imply $x \geq z$. A binary relation $\geq$ on X is anti-symmetric if for each $\{x, y\} \subseteq X$, $x \geq y$ and $y \geq x$ imply $x = y$.

comparable with i and j under $\succeq_o$. Then, for any $o' \in \mathcal{O}$, not $h \succ_{o'} j$. Thus, the common priority structure above is not acyclic. $\diamond$

3 Result I: Core under the common priority

In this section, we investigate three types of the core under the common priority structure. Roughly speaking, an allocation belongs to the core if it is impossible for any coalition $C \subseteq N$ to be better off by reallocating the objects "owned" by C. Thus, to define a concept of the core, we need to establish a notion of the ownership.

The concept of the property right we adopt in this paper is based on a primitive data of the priority structure. A fomulation is proposed in Balbuzanov and Kotowski (2019): given an allocation $a \in \mathcal{A}$, a coalition C owns (more precisely, has the conditional right to exclusively use) objects assigned to the agents who are subordinate to at least one member in C.[3] To establish this in the setup with multiple copies of real objects, we need to extend the concept of an object and an allocation. The set of **extended objects** is defined as $\bar{\mathcal{O}} := \{o_{k\ell} \mid 1 \leq k \leq m \text{ and } 1 \leq \ell \leq q_{o_k}\}$. The set of **extended allocations** is defined as $\bar{\mathcal{A}} := \{\bar{a} : N \to \bar{\mathcal{O}} \cup \{o_0\} \mid \forall o \in \bar{\mathcal{O}}, |\bar{a}^{-1}(o)| \leq 1\}$.

Given an allocation $a \in \mathcal{A}$, the corresponding extended allocation $\bar{a} \in \bar{\mathcal{A}}$ is defined as follows: i) if $a(i) \in \mathcal{O}$(say o_k), let $\bar{a}(i) = o_{k\ell}$, where $\ell = |\{j \in N \mid j < i \text{ and } a(j) = o_k\}| + 1$, and ii) if $a(i) = o_0$, let $\bar{a}(i) = o_0$. Conversely, given an extended allocation $\bar{a} \in \bar{\mathcal{A}}$, the corresponding allocation $a \in \mathcal{A}$ is defined as follows: i) if $\bar{a}(i) \in \bar{\mathcal{O}}$(say $o_{k\ell}$), let $a(i) = o_k$, and ii) if $\bar{a}(i) = o_0$, let $a(i) = o_0$.

Now, we extend a concept of ownership in Balbuzanov and Kotowski (2019) to the many-to-one setting. Given an allocation $a \in \mathcal{A}$, the **conditional endowment system** is the function $\omega_a : 2^N \to 2^{\bar{\mathcal{O}}}$ such that for each $C \in 2^N$, ω_a assigns the set of extended objects "owned" by C. Formally,

$$\omega_a(C) := \left\{o_{k\ell} \in \bar{\mathcal{O}} \mid \exists i \in C \text{ s.t. } i \succeq_{o_k} \bar{a}^{-1}(o_{k\ell})\right\} \cup \left(\bar{\mathcal{O}} \setminus \bar{a}(N)\right).$$[4]

In words, the first part indicates the set of extended objects assigned to agents subordinate to at least one member of C, and the second part indicates the set of extended objects disposed at a.

[3]Here, we use the word "conditional" because the ownership depends on the given allocation a.
[4]Balbuzanov and Kotowski (2019) call ω_a the weak conditional endowment system.

Given an allocation $a \in \mathcal{A}$ and a coalition $C \subseteq N$, we say that $a|_C$ is **achievable** on $\bar{X} \subseteq \bar{\mathcal{O}}$ if the number of agents in C who receive $o_k \in \mathcal{O}$ at a does not exceeds the number of o_k's copies in $\bar{X}$. Formally, for each $k \in \{1, 2, \dots, m\}$, $\big|\{i \in C \mid a(i) = o_k\}\big| \leq \big|\{x \in \bar{X} \mid \exists \ell \in \{1, 2, \dots, q_{o_k}\} \text{ s.t. } x = o_{k\ell}\}\big|$. To introduce three types of the core, we first define the concept of blocking in three ways.

Definition 1 (Weak block)**.** An allocation $a \in \mathcal{A}$ is weakly blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at $R \in \mathcal{R}^N$ *if*

(i) $\forall i \in C, b(i) \; R_i \; a(i)$,

(ii) $\exists i \in C$ s.t. $b(i) \; P_i \; a(i)$, and

(iii) $b|_C$ is achievable on $\omega_a(C)$.

Definition 2 (Strong block)**.** An allocation $a \in \mathcal{A}$ is strongly blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at $R \in \mathcal{R}^N$ *if*

(i) $\forall i \in C, b(i) \; P_i \; a(i)$, and

(ii) $b|_C$ is achievable on $\omega_a(C)$.

Definition 3 (Exclusion block)**.** An allocation $a \in \mathcal{A}$ is exclusion blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at $R \in \mathcal{R}^N$ *if*

(i) $\forall i \in C, b(i) \; P_i \; a(i)$, and

(ii) $\forall i \in N, [a(i) \; P_i \; b_i \Rightarrow \bar{a}(i) \in \omega_a(C)]$.

An allocation $a \in \mathcal{A}$ belongs to the **strong core** (Resp. **weak core**, **exclusion core**) at $R \in \mathcal{R}^N$ *if* no coalition $C \in 2^N \backslash \{\emptyset\}$ weakly (Resp. strongly, exclusion) blocks a through any allocation at R. Let $\mathcal{SC}(R)$, $\mathcal{WC}(R)$ and $\mathcal{EC}(R)$ be the strong core, weak core and exclusion core at R, respectively.[5] Our first result summarizes the relationship among three core concepts.

Theorem 1. *For each* $R \in \mathcal{R}^N, \mathcal{SC}(R) \subseteq \mathcal{EC}(R) = \mathcal{WC}(R)$.

[5]Balbuzanov and Kotowski (2019) call $\mathcal{EC}$ the direct exclusion core in the context of simple economy. In addition to $\mathcal{EC}$, they introduce another exclusion core concept based on an extension of the conditional endowment system. However, under Assumption 1, two exclusion core concepts coincide. See Appendix A for a detailed description.

Allocation	Blocking coalition	Allocation	Blocking coalition	Allocation	Blocking coalition
$o_1o_2o_3o_4$	$\{1,3\}$	$o_1o_3o_2o_4$	$\{1,4\}$	$o_1o_4o_2o_3$	$\{2,3\}$
$o_1o_2o_4o_3$	$\{3\}$	$o_1o_3o_4o_2$	$\{3\}$	$o_1o_4o_3o_2$	$\{3\}$
$o_2o_1o_3o_4$	$\{1\}$	$o_3o_1o_2o_4$	$\{1\}$	$o_4o_1o_2o_3$	$\{1\}$
$o_2o_1o_4o_3$	$\{1\}$	$o_3o_1o_4o_2$	$\{1\}$	$o_4o_1o_3o_2$	$\{1\}$

Table 2: Strong core may be empty.

Note : The sequence $o_io_jo_ko_\ell$ indicates the allocation $a \in \mathcal{A}$ such that $a_1 = o_i, a_2 = o_j, a_3 = o_k$ and $a_4 = o_\ell$. We omit allocations in which some agents receive the null object because they are weakly blocked by a single agent. The table exhausts the allocations in which $\{1,2\}$ consumes $\{o_1, o_2\}$, $\{o_1, o_3\}$ or $\{o_1, o_4\}$. Since $\{1,2\}$ and $\{3,4\}$ are symmetric, the table exhausts all allocations.

Next, we consider the existence of the core. The following example shows that the existence of a strong core allocation is not guaranteed.

Example 2 (Strong core may be empty)**.** Suppose that $N = \{1,2,3,4\}$, $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$ and $q = (1,1,1,1)$. Suppose also that the priority structure is given as follows: $1 \succ 2$ and $3 \succ 4$. Let $R \in \mathcal{R}^N$ be such that for each $i \in N$, $o_1 \ R_i \ o_2 \ R_i \ o_3 \ R_i \ o_4 \ R_i \ o_0$. In Table 2, each allocation in the left column is weakly blocked by the corresponding coalition in the right column. Thus, $\mathcal{SC}(R) = \emptyset$. $\diamond$

In contrast to the strong core, the weak core (= exclusion core) is always non-empty. Moreover, they are reached by adjusting a quite simple algorithm: The priority rule. To define the priority rule, we first introduce a notation. A binary relation $\trianglerighteq$ on N is a **completion** of $\succeq$ *if* i) $\trianglerighteq$ is complete, transitive and anti-symmetric, and ii) for each $\{i,j\} \subseteq N$, if $i \succeq j$, then $i \trianglerighteq j$. Let $\Gamma(\succeq)$ be the set of completions of $\succeq$.

Given a completion $\trianglerighteq \in \Gamma(\succeq)$, the **priority rule with respect to** $\trianglerighteq$, $\varphi^{\trianglerighteq}$, is a function from $\mathcal{R}^N$ to $\mathcal{A}$ defined as follows: Let $i_1 \trianglerighteq i_2 \trianglerighteq \ldots \trianglerighteq i_n$. Let $R \in \mathcal{R}^N$. In step 1, agent i_1 picks the most favorite object from $\mathcal{O} \cup \{o_0\}$. If it is a real object, update the quota vector by subtracting 1 from the quota of the object. Letting $\varphi^{\trianglerighteq}_{i_1}(R)$ be the picked object, go to the next step. In step 2, agent i_2 picks the most favorite object from the remaining real objects under the updated quota and the null object. If it is a real object, update the quota vector by subtracting 1 from the latest quota of the object. Letting $\varphi^{\trianglerighteq}_{i_2}(R)$ be the picked object, go to the next step. Repeating this process n times, we get $\varphi^{\trianglerighteq}(R) := \left(\varphi^{\trianglerighteq}_{i_k}(R)\right)_{k=1}^{n} \in \mathcal{A}$.

The range of the priority rule with a completion is denoted as $\Phi(R)$, i.e., for each $R \in \mathcal{R}^N$, $\Phi(R) := \{\varphi^{\trianglerighteq}(R) | \trianglerighteq \in \Gamma(\succeq)\}$. Based on this, we give a complete characterization of the weak core as follows:

Theorem 2. *For each $R \in \mathcal{R}^N, \mathcal{WC}(R) = \Phi(R)$.*

4 Result II: Priority-completion on the basis of interpersonal and interinstitutional fairness

In this section, we work on the refinement of the weak core in terms of fairness. Fairness is the norm as important as efficiency to evaluate an allocation or a rule in wide range of allocation problems.[6] In the sequal, we go back to the original problem that the administration office of a university assigns indivisible objects to students prioritized only in their own department. To capture the problem, we adopt the following assumption in addition to Assumption 1. Note that the symbol $\succeq$ in the statement represents the priority relation all objects have in common.

Assumption 2. *Let $\mathcal{I}$ be a partition of N such that for each $I \in \mathcal{I}$,*

- *$\forall I \in \mathcal{I}$, the priority $\succeq$ is complete on I, and*
- *$\forall I, I' \in \mathcal{I}$ with $I \neq I'$, $\forall i \in I, \forall i' \in I'$, $i \not\succeq i'$.*

We call each $I \in \mathcal{I}$ an institution. To avoid the trivial case, we assume that $|\mathcal{I}| \geq 2$.

Establishing two fairness notions of interpersonal and interinstitutional fairness, we work on the refinement of the weak core.

4.1 Refinement of the core on the basis of ex-post fairness notions

In this subsection, we define interpersonal and interinstitutional fairness notion in the ex-post sense. The following is a standard notion of fairness in the matching literature. An allocation $a \in \mathcal{A}$ is interpersonally-fair at $R \in \mathcal{R}^N$ *if* no agent has justified envy, i.e., there exists no pair of agents $(i, j) \in N \times N$ such that i) $a_j \ P_i \ a_i$, and ii) $i \succ j$. Note that this property is achieved by the priority rule under

[6] As is shown in Lemma 3 in Appendix B, each weak core allocation is Pareto efficient.

any completion. In other words, every weak core allocation is interpersonally-fair (Theorem 2). Thus, this fairness notion alone is not helpful to refine the weak core.

To define an interinstitutional fairness notion, we need a tool to evaluate an allocation from the institution perspective. The following notation $\rho_k^I(a,R)$ denotes the ratio of agents in I who receive the k-th best object at a. Formally, the rank distribution of $a \in \mathcal{A}$ for $I \in \mathcal{I}$ at $R \in \mathcal{R}^N$ is $\rho_k^I(a,R) \in \mathbb{R}_+^{m+1}$ defined as follows: for each $k \in \{1,2,\ldots,m{+}1\}$, $\rho_k^I(a,R) := \frac{|\{i \in I | a_i \text{ is the } k\text{-th favorite object at } R_i.\}|}{|I|}$. An allocation $a \in \mathcal{A}$ is interinstitutionally-fair in the ex-post sense at $R \in \mathcal{R}^N$ *if* there exists no pair of institutions $(I,I') \in \mathcal{I} \times \mathcal{I}$ such that $\rho^I(a,R)$ stochastically dominates $\rho^{I'}(a,R)$, i.e., for all $k \in \{1,2,\ldots,m+1\}$, $\sum_{\ell=1}^k \rho_k^I(a,R) \geq \sum_{\ell=1}^k \rho_k^{I'}(a,R)$, and for some $k \in \{1,2,\ldots,m+1\}$, the inequality is strict.

Although the interinstitutional fairness notion above seems reasonable, it is too strong to practice as shown in the following example.

Example 3 (A problem in which no weak core allocation is interinstitutionally-fair in the ex-post sense)**.** Suppose that $|N| \geq 2$, $|\mathcal{O}| \geq 2$ and $q = (1,1,\ldots,1)$. Assume, without loss of generality, that for $I, I' \in \mathcal{I}$ with $I \neq I'$, $1 = \max_{\succeq} I$ and $2 = \max_{\succeq} I'$.

Let $R \in \mathcal{R}^N$ be such that i) $o_1 \; R_i \; o_2 \; R_i \; o_0 \; R_i \ldots$ for $i \in \{1,2\}$, and ii) $o_0 \; R_i \ldots$ for $i \geq 3$. Then, for any completion $\trianglerighteq \in \Gamma(\succeq)$, the corresponding priority rule allocation $\varphi^{\trianglerighteq}(R)$ assigns the best object for all members of one of I and I'. However, the other institution contains one member who receives the second-best object. Thus, $\varphi^{\trianglerighteq}(R)$ is not interinstitutionally-fair in the ex-post sense at R. Since $\trianglerighteq$ is arbitrary, all allocations in $\Phi(R) \, (= \mathcal{WC}(R))$ are not interinstitutionally-fair in the ex-post sense at R. ◇

In this paper, we do not persue the line of selecting fair allocations, i.e., fairness in the ex-post sense. Instead, we turn to the problem of selecting fair priority-completions to establish an allocation system that attains interpersonal and interinstitutional fairness in the ex-ante sense.

4.2 Point allocation approach to the priority-completion

To clarify the difficulty in the selection of a fair completion, let us consider the following example. Suppose that there are two institutions $I = \{i_1, i_2\}$ and $I' = \{i'_1, \ldots, i'_4\}$ such that $i_1 \succ i_2$ and $i'_1 \succ i'_2 \succ i'_3 \succ i'_4$. Table 3 shows three examples

I	Priority	I'	Priority
i_1	1	i'_1	2
		i'_2	3
i_2	6	i'_3	4
		i'_4	5

$\trianglerighteq_1$

I	Priority	I'	Priority
i_1	2	i'_1	1
		i'_2	3
i_2	5	i'_3	4
		i'_4	6

$\trianglerighteq_2$

I	Priority	I'	Priority
i_1	3	i'_1	1
		i'_2	2
i_2	4	i'_3	5
		i'_4	6

$\trianglerighteq_3$

Table 3: Example of completions

of completion that may be regarded as "interinstitutionally-fair" in the following two sense: (1) the balance of the size of tiers, and (2) the distribution of favorable treatment between tiers.

The completions in Table 3 have the following features: The completion $\trianglerighteq_1$ is favorable to I in the upper tier, while it is to I' in the lower tier. In the completion $\trianglerighteq_3$, the favorable treatment in the upper and lower tiers is reversed compared with $\trianglerighteq_1$. The completion $\trianglerighteq_2$ is favorable to neither I or I' in both tiers. Thus, using one of these completion, we may attain an interinstitutionally fair state in terms of (1) and (2). The difficulty arises when we generalize the criteria (1) and (2) to the cases with many institutions pertaining various numbers of members (Imagine $\mathcal{I} = \{I_3, I_5, I_7, I_{11}, I_{13}, I_{17}, I_{19}\}$ with $|I_k| = k$). Let us call the senario of choosing a completion by generalizing (1) and (2) the direct selection (DS).

In this study, we persue another senario, which we call the indirect selection (IS), to choose a completion. We generate a completion through an alloocation of quantified opprotunity: a point allocation. A **point allocation** is a function $\alpha : N \to [0, 1]$ such that

$$\forall i, j \in N, i \succ j \Rightarrow \alpha(i) > \alpha(j).$$

Let $\mathcal{A}^P$ be the set of point allocations. The IS is a procedure to pick a point allocation $\alpha \in \mathcal{A}^P$ to choose a completion $\trianglerighteq \in \Gamma(\succeq)$ such that

$$\forall i, j \in N, \alpha(i) > \alpha(j) \Rightarrow i \vartriangleright j.$$ [7]

Then, by using the selected completion $\trianglerighteq$, we utilize the priority rule $\varphi^{\trianglerighteq}$ to determine

[7] Ties are broken arbitrarily.

an object allocation.[8] We consider that the IS has the following two advantages compared with the DS.

- First, the IS is simpler. While the DS accompanies with the difficulties pertaining to discrete resource allocation problems, e.g., the generalization of (1) and (2), it is easier to define fairness notions under the IS.

- Second, the IS is more flexible. As is shown in the discussion below, the system equipped with the quantified opportunity makes it easier for institutions to express their preferences on completions. For example, if institution I' in Table 3 would like to be more supportive to the students in the uppaer tier, it may prefers $\trianglerighteq_3$ to the others. This policy could be captured by a point allocation with extra points for i'_1 and i'_2.

Based on the idea of the point allocation, it is possible to formalize an interinstitutional fairness notion in a natural way. The following axiom requires that the assignment of the quantified opprtunity to institutions should be proportional to the number of members each institution has.

Definition 4. A point allocation $\alpha \in \mathcal{A}^P$ is **interinstitutionally-fair (IIF)** *if*

$$\forall I, I' \in \mathcal{I}, \frac{1}{|I|}\sum_{i \in I} \alpha(i) = \frac{1}{|I'|}\sum_{i \in I'} \alpha(i).$$

Next, we establish notions of interpersonal fairness. To this end, we introduce a notation to express the relative position of an agent in the affiliated institution. For each $I \in \mathcal{I}$ and each $i \in I$, let r_i^I be the reverse rank of agent i in the institution I, i.e., $r_i^I := |I| - |\{j \in I | j \succ i\}|$. Note that if i is the k-th highest priority agent in I, r_i^I is $|I| - (k-1)$. In the case with no subordination among institutions, each agent may regard a point allocation as fair if the points she gets is similar to that of the agents with similar position in other institutions. For example, supposing that agent i is the 4-th priority agent in an institution with 40 members, she may accept the point assignment similar to that of the 10-th priority agent in a 100 members institution,

[8]The idea of point allocation itself is not new to generate a priority. For example, a point allocation is used to generate a priority order of patients waiting for a deceased donor assignment. Another example is for an allocation of nursery school seats. The novelty of this study lies on the fact that it applies the point allocation system to the market with a partition of agents.

but may not accept the assignment lower to the 10-th priority agent in a 20 members institution. This is because the allocation ignores that agent i's relative position (top 10 %) is better than that of the 10-th agent in 20 members. The following axiom reflects this criterion straightforwardly: A point allocation $\alpha \in \mathcal{A}^P$ is **strongly interpersonally-fair (SIPF)** *if* there exists no pair of institutions $(I, I') \in \mathcal{I} \times \mathcal{I}$ such that for some $(i, i') \in I \times I'$, (1) $\alpha(i') > \alpha(i)$ and (2) $\frac{r_i^I - 1}{|I|} > \frac{r_{i'}^{I'} - 1}{|I'|}$.[9] Although this axiom, which actually singles out a completion except for ties, is seemingly suitable for a reasonable allocation, it could be too restrictive in other aspects. For example, SIPF point allocations always assign the highest points to the top agent in the institution with the greatest members. However, since there is usually little in evaluation to make differences among the top member of each institution, it might be desirable to design a more flexible point allocation rule. The following axiom respects the relative position of agents to some extent, overcoming the inflexibility of SIPF.

Definition 5. A point allocation $\alpha \in \mathcal{A}^P$ is **weakly interpersonally-fair (WIPF)** *if* there exists no pair of institutions $(I, I') \in \mathcal{I} \times \mathcal{I}$ such that for some $(i, i') \in I \times I'$,

(1) $\alpha(i') \geq \alpha(i)$,

(2) $\frac{r_i^I - 1}{|I|} \geq \frac{r_{i'}^{I'}}{|I'|}$, and

(3) at least one of inequalities (1) and (2) is strict.

For an interpretation of the inequality (2) of WIPF, we need to examine the concept of relative position more closely. Let us take a look at the following example. Letting i be a member of an institution I with $|I| = 100$, suppose that i is given the 10-th highest priority in I. Then, the "relative position" of i might be characterized by the two information: "i is a top 10 % agent" and "i is not a top 9 % agent." Note that the endpoints of the interval $\left[\frac{r_i^I - 1}{|I|}, \frac{r_i^I}{|I|}\right]$ clearly represent the information.[10] Thus, the interval could be understood as the set of potentially possible evaluation of agent i's relative positions. Based on this understanding, the inequality (2) indicates that agent i is an agent whose worst-evaluated relative position is not lower than the agent i''s best-evaluated relative position.

[9]The condition (2) is equivalent with $\frac{(|I|+1) - r_i^I}{|I|} < \frac{(|I'|+1) - r_{i'}^{I'}}{|I'|}$.
[10]This is the r_i^I-th interval from the left when we divide $[0, 1]$ into $|I|$ equal parts.

4.3 Midpoint rule

To select an IIF and WIPF point allocation, it is noteworthy that the following is a sufficient condition for a point allocation $\alpha \in \mathcal{A}^P$ to be WIPF.

$$\forall I \in \mathcal{I}, \forall i \in I, \alpha(i) \in \left[\frac{r_i^I - 1}{|I|}, \frac{r_i^I}{|I|}\right].$$ [11]

Specifically, the midpoint of the interval for each agent brings an IIF and WIPF allocation. To see this, let $\alpha^M \in \mathcal{A}^P$ be the point allocation defined as follows: for each $I \in \mathcal{I}$ and each $i \in I$, let

$$\alpha^M(i) := \frac{2r_i^I - 1}{2|I|} \left(= \frac{1}{2} \cdot \frac{r_i^I - 1}{|I|} + \frac{1}{2} \cdot \frac{r_i^I}{|I|} \right).$$

Then, for each $I \in \mathcal{I}$, $\sum_{i \in I} \alpha^M(i) = \frac{|I|}{2}$. Thus, α^M satisfies IIF in addition to WIPF.

A natural question that arises next is: are there other point allocations that satisfy IIF and WIPF? The answer is yes. Furthermore, there are infinitely many IIF and WIPF point allocations. For example, fixing an institution $I \in \mathcal{I}$ with even members, let ϵ be such that $\frac{1}{2|I|} \geq \epsilon > 0$. Let $\beta^+, \beta^- \in \mathcal{A}^P$ be defined as follows:

$$\beta^+(i) = \begin{cases} \alpha^M(i) + \epsilon & \text{if } i \in I \text{ and } r_i^I > \frac{|I|}{2}, \\ \alpha^M(i) - \epsilon & \text{if } i \in I \text{ and } r_i^I \leq \frac{|I|}{2}, \\ \alpha^M(i) & \text{o.w.} \end{cases} \quad \beta^-(i) = \begin{cases} \alpha^M(i) - \epsilon & \text{if } i \in I \text{ and } r_i^I > \frac{|I|}{2}, \\ \alpha^M(i) + \epsilon & \text{if } i \in I \text{ and } r_i^I \leq \frac{|I|}{2}, \\ \alpha^M(i) & \text{o.w.} \end{cases}$$

Obviously, β^+ and β^- satisfy the sufficient condition for WIPF above. Moreover, $\sum_{i \in I} \beta^+(i) = \sum_{i \in I} \beta^-(i) = \sum_{i \in I} \alpha^M(i) = \frac{|I|}{2}$. Thus, β^+ and β^- are IIF and WIPF. Thus, there are infinitely many IIF and WIPF point allocations including β^+ (Resp. β^-), which is favorable to the upper-half (Resp. lower-half) members in I.

Now we would like to reconsider the axiomatic approach to refine the point allocations. We have simply proposed two desirable properties to single out some of point allocations. However, we should be careful for the further selection, especially regarding the selection that might be appropriate to depend on the local information or the policy of institutions, e.g., the selection from β^+ and β^-. In the sequel, we propose a system in which the clearing house selects a set of point allocations (candidates for the final selection). Among the suggested point allocations, the system

[11] Later, it is shown that this is not a necessary condition.

selects one reflecting on the evaluation policy submitted by institutions. Formally,

The midpoint rule (abbreviated as MR).

Step 1. Each institution $I \in \mathcal{I}$ submits a weight vector $w^I \in [-1,1]^I$ with $\sum_{i \in I} w_i^I = 0$ to the clearing house. Set $w := (w^I)_{I \in \mathcal{I}}$.

Step 2. The clearing house picks the point allocation $\alpha^w \in \mathcal{A}^P$ defined as follows: for each $I \in \mathcal{I}$ and each $i \in I$,

$$\alpha^w(i) := \alpha^M(i) + \frac{w_i^I}{2|I|}.$$

Step 3. The clearing house picks a completion $\trianglerighteq \in \Gamma(\succeq)$ consistent with α^w.[12]

In the MR, the set of suggested point allocations by the clearing house is $\{\alpha^w \in \mathcal{A}^P \mid w \in [-1,1]^N \text{ and } \forall I \in \mathcal{I}, \sum_{i \in I} w_i = 0\}$. Note that $\alpha^w(i)$ belongs to the interval $\left[\frac{r_i^I - 1}{|I|}, \frac{r_i^I}{|I|}\right]$ since the second term $\frac{w_i^I}{2|I|}$ of $\alpha^w(i)$ represents a small gap from the midpoint of the interval. The weight vector w represents the profile of evaluation policy of institutions. We characterize a class of point allocations, including the range of the MR, with the following axioms. A set of point allocations $S \subseteq \mathcal{A}^P$ satisfies

Interinstitutional fairness (IIF): $\forall \alpha \in S, \alpha$ is IIF.

Weak interpersonal fairness (WIPF): $\forall \alpha \in S, \alpha$ is WIPF.

Independence (IND): $\forall \alpha, \beta \in S, \forall I \in \mathcal{I}, \exists \gamma \in S$ s.t.$\gamma|_I = \alpha|_I$ and $\gamma|_{N \setminus I} = \beta|_{N \setminus I}$.

Continuity (CON): $\forall I \in \mathcal{I}, \forall i, j \in I$ with $r_i^I = r_j^I + 1, \sup_{\alpha \in S} \alpha(j) = \inf_{\alpha \in S} \alpha(i)$.

Existence of the standard evaluation (ESE): $\forall I \in \mathcal{I}, \exists \alpha \in S, \forall i \in I, \alpha(i) = \frac{\sup_{\beta \in S} \beta(i) + \inf_{\beta \in S} \beta(i)}{2}$.

Closedness (CLO): **(i).** $\forall i \in N, \exists \overline{\alpha} \in S$ s.t. $\overline{\alpha}(i) = \sup_{\alpha \in S} \alpha(i)$. In particular, for each $I \in \mathcal{I}$ with $i = \max_{\succeq} I$, there exists $\overline{\alpha} \in S$ such that $\overline{\alpha}(i) = 1$. **(ii).** $\forall i \in N, \exists \underline{\alpha} \in S$ s.t. $\underline{\alpha}(i) = \inf_{\alpha \in S} \alpha(i)$. In particular, for each $I \in \mathcal{I}$ with $i = \min_{\succeq} I$, there exists $\underline{\alpha} \in S$ such that $\underline{\alpha}(i) = 0$.

IIF and WIPF are the main axioms in this section. Other auxiliary axioms are interpretted as follows: IND requires that the point allocation within an institution be

[12] A completion $\trianglerighteq$ is consistent with α^w *if* for any $i, j \in N$, $\alpha^w(i) > \alpha^w(j)$ implies $i \triangleright j$. Ties in α^w are broken arbitrarily.

not affected by the allocation in other institutions. CON requires that the supremum of the potentially possible assignment for an lower-priority agent should not exceed that the infimum of that for an higher-priority agent, i.e., $\sup_{\alpha\in S}\alpha(j) \leq \inf_{\alpha\in S}\alpha(i)$. Moreover, it requires that there be no gap between them. ESE embodies a kind of convexity of the potentially possible evaluations so that the convex combination of the supremum and the infimum is available as an evaluation. CLO is a technical condition that requires that there exist the greatest and the smallest of the potentially possible evaluations. These six axioms almost characterize the range of the midpoint rule. To be precise, we need a definition. The following procedure is a variant of the MR that provides a set of point allocations characterized as the maximum set satisfying the six axioms above.

The generalized midpoint rule (abbreviated as GMR).
Let $f:[0,1]\to[0,1]$ be such that

(GMR-1) f is a continuous strictly increasing function with $f(0)=0$ and $f(1)=1$,

(GMR-2) $\forall I, I' \in \mathcal{I}$, $\frac{1}{|I|}\left[\sum_{i\in I} f\left(\frac{r_i^I}{|I|}\right)-\frac{1}{2}\right]=\frac{1}{|I'|}\left[\sum_{i\in I'} f\left(\frac{r_i^{I'}}{|I'|}\right)-\frac{1}{2}\right]$, and

(GMR-3) $\forall I\in\mathcal{I}, \forall i \in I$, $f\left(\frac{r_i^I}{|I|}\right)-f\left(\frac{r_i^I-1}{|I|}\right)\leq\frac{1}{2}$.

For each $I\in\mathcal{I}$ and each $i\in I$, let $T_i^f := f\left(\frac{r_i^I}{|I|}\right)-f\left(\frac{r_i^I-1}{|I|}\right)$.

Step 1. Each institution $I\in\mathcal{I}$ submits a weight vector $w^I\in[-1,1]^I$ with $\sum_{i\in I} w_i^I T_i^f=0$ to the clearing house. Set $w:=(w^I)_{I\in\mathcal{I}}$.

Step 2. The clearing house picks the point allocation $\alpha^w\in\mathcal{A}^P$ defined as follows: for each $I\in\mathcal{I}$ and each $i\in I$,

$$\alpha^w(i) := \frac{1}{2}\left[f\left(\frac{r_i^I-1}{|I|}\right)+f\left(\frac{r_i^I}{|I|}\right)\right]+\frac{w_i^I T_i^f}{2}.$$

Step 3. The clearing house picks a completion $\trianglerighteq\in\Gamma(\succeq)$ consistent with α^w.

The class of GMRs consists of a variant of the MR parametrized by transformations f satisfying (GMR-1), (GMR-2) and (GMR-3). Under a GMR with a transformation f, the interval that represents the potentially possible relative positions of an agent $i\in I$ is distorted into $\left[f\left(\frac{r_i^I-1}{|I|}\right), f\left(\frac{r_i^I}{|I|}\right)\right]$. Obviously, the GMR with the identity mapping is the MR. (GMR-2) requires that the interinstitutional proportion of the sum of the midpoints (of new potentially possible relative position intervals) should be preserved under the transformation f. (GMR-3) requires that each distorted interval should not exceed the half of the whole interval $[0, 1]$. The main theorem of this section states that a set of point allocations S has a widest variety of options among the sets satisfying the six axioms if and only if S is the range of a GMR.

Theorem 3. *Let $S \subseteq \mathcal{A}^P$. The set of point allocations S is a maximal element of $\mathcal{S} := \{S' \subseteq \mathcal{A}^P \mid S'$ satisfies IIF, WIPF, IND, CON, ESE and CLO$\}$ with respect to $\supseteq$ if and only if there exists $f : [0, 1] \to [0, 1]$ satisfying (GMR-1), (GMR-2) and (GMR-3) such that $S = \mathcal{A}^P_f$.*

Unless there is an unambiguous advantage to adopt a particular distortion f, we advocate the simplest rule in the class of GMRs, the MR, for the selection of a completion.

5 Conclusion

This paper studies the problem of assigning indivisible objects to the agents prioritized within their affiliated institutions. To this end, we formalized the problem as a priority-based indivisible goods allocation problem with incomplete priority. Extending the concept of conditional property right (Balbuzanov and Kotowski, 2019) to a many-to-one setting, we investigate the relationship among several core concepts. Different from the case with an acyclic priority structure, under the assumption of common priority, the weak core coincides with the exclusion core (Theorem 1), and characterized by the range of the priority rule with a priority-completion (Theorem 2).

To implement the priority rule, we approached the problem of selecting a priority-completion in the framework of the quantified opportunity (point) allocation. We advocate a rule called the midpoint rule, which is reflexive enough for institutions to express their policy for the point allocation. In the main theorem (Theorem 3), we give a characterization of the ranges of generalized midpoint rules based mainly on the two fairness notions: interpersonal and interinstitutional fairness.

Appendix A: Indirect exclusion core under the common priority

In addition to the conditional endowment system in Section 3, Balbuzanov and Kotowski (2019) introduce an extension of the concept. Given an allocation $a \in \mathcal{A}$, **the extended conditional endowment system** $\Omega_a : 2^N \to 2^{\bar{\mathcal{O}}}$ is defined as follows: for each $C \subseteq N$,

$$\Omega_a(C) := \omega_a \left(\cup_{p=0}^{\infty} C_p \right),$$

where $C_0 = C$ and $C_p = C_{p-1} \cup (\bar{a}^{-1} \circ \omega_a)(C_{p-1})$.

The exclusion core in the main text based on the conditional endowment system ω_a is called the **direct exclusion core** in Balbuzanov and Kotowski (2019).[13] On the other hand, the exclusion core defined by replacing ω_a with Ω_a is called the **indirect exclusion core**.[14] Under the assumption of common priority, these two core concepts coincide as shown in the following proposition.

Proposition 1. *Under Assumption 1, for each $a \in \mathcal{A}$ and each $C \in 2^N$, $\omega_a(C) = \Omega_a(C)$.*

Proof. First, we show the following two claims.

Claim 1. $\omega_a(C) = w_a(C_1)$

Proof of Claim 1: By definition, $C \subseteq C_1$. Since ω_a is monotonic, $\omega_a(C) \subseteq \omega_a(C_1)$. In the sequel, we show $\omega_a(C) \supseteq \omega_a(C_1)$ by contradiction. Suppose to the contrary that there exists $o_{k\ell} \in \omega_a(C_1)$ such that $o_{k\ell} \notin \omega_a(C)$. Since $o_{k\ell} \notin \bar{\mathcal{O}} \backslash \bar{a}(N)$, $\bar{a}(i) = o_{k\ell}$ for some $i \in N$. Let $j \in C_1$ be $\succeq_{o_k}$-maximal in C_1 such that $j \succeq_{o_k} i$.

Case 1. $j \notin C$: Since $j \in C_1$ and $j \notin C$, $j \in \left(\bar{a}^{-1} \circ \omega_a\right)(C)$. Thus, there exists a real object $o_{k'\ell'}$ such that $\bar{a}(j) = o_{k'\ell'}$. Since $o_{k'\ell'} \in \omega_a(C) \backslash \left(\bar{\mathcal{O}} \backslash \bar{a}(N)\right)$, there exists $j' \in C$ such that $j' \succ_{o_{k'}} j$. By Assumption 1, $\succeq_{o_k} = \succeq_{o_{k'}}$. Thus, $j' \succ_{o_k} j$. Since $j' \in C \subseteq C_1$, this contradicts the choice of j.

[13] Precisely speaking, they use the terminology only for the setting with initial endowment, i.e., simple economy. Here, we borrow it for the case with priority structure.

[14] Being along with the terminology in Balbuzanov and Kotowski (2019) closer, we should call it the **strong exclusion core** or the **indirect exclusion core based on** ω_a because they define several indirect exclusion core concepts by replacing ω_a with other basic property right formulations. Here, we choose the simpler terminology.

Case 2. $j \in C$: Since $j \succeq_{o_k} i$, $o_{k\ell} \in \omega_a(\{j\})$. Since $\{j\} \subseteq C$, the monotonicity of ω_a implies $o_{k\ell} \in \omega_a(C)$, a contradiction.

Since all cases result in a contradiction, we conclude that $\omega_a(C) \supseteq \omega_a(C_1)$. This completes the proof of Claim 1.

Claim 2. $\forall p \in \mathbb{N}, C_1 = C_{p+1}$

Proof of Claim 2: The induction argument with Claim 1 brings the conclusion. This completes the proof of Claim 2.

Now, we go back to the proof of $\omega_a(C) = \Omega_a(C)$.

$$\begin{aligned}
\Omega_a(C) &= \omega_a\left(\cup_{p=0}^{\infty} C_p\right) \\
&= \omega_a\left(C_0 \cup C_1\right) && (\because \text{Claim 2}) \\
&= \omega_a(C_1) && (\because C_0 \subseteq C_1) \\
&= \omega_a(C) && (\because \text{Claim 1})
\end{aligned}$$

□

By Proposition 1, the direct and the indirect exclusion core coincides under Assumption 1. This is the reason why we introduced only the direct exclusion core in the main text, simply calling it the exclusion core.

Appendix B: Proofs

Lemma 1. *Suppose that an allocation $a \in \mathcal{A}$ is exclusion blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at $R \in \mathcal{R}^N$. Then, for $C' := \{i \in N \mid b_i\ P_i\ a_i\}$,*

$$\forall i \in N, \left[a_i\ P_i\ b_i \Rightarrow \bar{a}(i) \in \omega_a(C')\right].$$

Proof. Note that ω_a is monotonic. Thus, since $C \subseteq C'$, $\bar{a}(i) \in \omega_a(C) \subseteq \omega_a(C')$. □

Before we prove Lemma 2, we introduce a notion of efficiency. The definition is standard. An allocation $a \in \mathcal{A}$ is Pareto efficient at $R \in \mathcal{R}^N$ *if* there exists no allocation $b \in \mathcal{A}$ such that i) $b_i\ R_i\ a_i$ for all $i \in N$, and ii) $b_i\ P_i\ a_i$ for some $i \in N$.

Lemma 2. *Suppose that $a \in \mathcal{A}$ is Pareto efficient at $R \in \mathcal{R}^N$. Then,*

$$\forall b \in \mathcal{A}, \forall i \in N, \left[b_i \; P_i \; a_i \Rightarrow \exists j \in N \; s.t. \; \bar{a}(j) = \bar{b}(i)\right].$$

Proof. Suppose to the contrary that for some $b \in \mathcal{A}$ and $i \in N$, $b_i \; P_i \; a_i$ and

$$\nexists j \in N \text{ s.t. } \bar{a}(j) = \bar{b}(i).$$

Then, $\bar{b}(i) \in \left(\bar{\mathcal{O}} \backslash \bar{a}(N)\right) \cup \{o_0\}$. Let $c \in \mathcal{A}$ be such that $c(i) = b(i)$ and $c|_{N\backslash\{i\}} = a|_{N\backslash\{i\}}$. Then, a is not Pareto efficient at R, a contradiction. □

Lemma 3. *Let $R \in \mathcal{R}^N$ and $a \in \mathcal{WC}(R)$. Then, a is Pareto efficient at R.*

Proof. Suppose to the contrary that for some $b \in \mathcal{A}$, i) $b_i \; R_i \; a_i$ for all $i \in N$, and ii) $b_i \; P_i \; a_i$ for some $i \in N$. Let $C := \{i \in N \mid b_i \; P_i \; a_i\}$. Since $a \in \mathcal{WC}(R)$, $b_i \neq o_0$ for all $i \in C$. Moreover, by the definition of C, $b(i) = a(i)$ for $i \notin C$. Thus, $b|_C$ could be constructed by reallocating

- the real objects disposed at a and
- the real objects assigned to members of C.

Note that $\left(\bar{\mathcal{O}} \backslash \bar{a}(N)\right) \cup \bar{a}(C) \subseteq \omega_a(C)$. Thus, $b|_C$ is achevable on $\omega_a(C)$. This contradicts $a \in \mathcal{WC}(R)$. □

Proof of Theorem 1. Let $R \in \mathcal{R}^N$. Obviously, $\mathcal{SC}(R) \subseteq \mathcal{WC}(R)$. Thus, we only prove $\mathcal{EC}(R) = \mathcal{WC}(R)$.

($\mathcal{EC}(R) \supseteq \mathcal{WC}(R)$) Suppose to the contrary that there exists $a \in \mathcal{WC}(R)$ such that a coalition $C \in 2^N \backslash \{\emptyset\}$ exclusion blocks through an allocation $b \in \mathcal{A}$ at R. By Lemma 1, we may assume, without loss of generality, that (i) $i \in C$ if and only if $b(i) \; P_i \; a(i)$, and (ii) $a(i) \; P_i \; b(i)$ implies $\bar{a}(i) \in \omega_a(C)$. We show that $b|_C$ is achievable on $\omega_a(C)$.

To this end, we first show that $\omega_a(C)$ contains at least one copy of $b(i)$ for each $i \in C$. The following inductive argument brings a chain of agents $\{i_1, \ldots, i_p\}$ that finally hits an owner of a copy of $b(i)$ in $\omega_a(C)$ (under a or b). Note that, by Lemma 2 and 3, there exists $i_1 \in N$ such that $\bar{a}(i_1) = \bar{b}(i)$. For each $p \geq 1$, the search for a chain stops if one of the following three conditions holds.

Stopping Rule (SR) 1. $i_p \in C$.

SR 2. $i_p \notin C$ and $a(i_p)\ P_{i_p}\ b(i_p)$.

SR 3. $i_p \notin C$, $a(i_p) = b(i_p)$ and $\bar{b}(i_p) \notin \bar{a}(N)$.

If one of SR 1, 2, 3 holds, then $\{i_1, \ldots, i_p\}$ is the desired chain with the length p. If none of three rules holds, i.e., $i_p \notin C$, $a(i_p) = b(i_p)$ and $\bar{b}(i_p) \in \bar{a}(N)$, go to the next step. Note that, in the latter case, since $\bar{b}(i_p) \in \bar{a}(N)$, there exists $i_{p+1} \in N$ such that $\bar{a}(i_{p+1}) = \bar{b}(i_p)$. The following two are important features of the chain (at step p in the construction).

$$[\forall q \in \{1, \ldots, p-1\}, a(i_q) = b(i_q)] \text{ and } [\forall q \in \{1, \ldots, p-1\}, \bar{a}(i_{q+1}) = \bar{b}(i_q)]$$

Note that for any $p \geq 2$, $i_p \notin \{i_1, \ldots, i_{p-1}\}$.[15] Since N is finite, one of the stopping rules holds for some i_p with $p \leq n$. Then, under SR 1, the copy of $b(i)$ is $\bar{a}(i_p) \in \omega_a(\{i_p\}) \subseteq \omega_a(C)$. Under SR 2, it is $\bar{a}(i_p) \in \omega_a(C)$ ($\because (ii)$). Under SR 3, it is $\bar{b}(i_p) \in \omega_a(C)$ $\big(\because \big(\bar{\mathcal{O}} \backslash \bar{a}(N)\big) \subseteq \omega_a(C)\big)$.

Next, we show that for any $i, j \in C (i \neq j)$, the copies of $b(i), b(j) \in \omega_a(C)$ found by the above argument are different. Let $\{i_1, \ldots, i_p\}$ and $\{j_1, \ldots, j_{p'}\}$ be the chains of agents to find copies of $b(i)$ and $b(j)$, respectively. We claim that $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_{p'}\} = \emptyset$.[16] Thus, $i_p \neq j_{p'}$. Note that the following three cases exhaust all possibile combinations of i_p and $j_{p'}$. In any case, the copies of $b(i)$ and $b(j)$ found by the corresponding chains are different.

Case 1. Both i_p and $j_{p'}$ satisfy one of SR 1 or 2: For this case, the copies of $b(i)$ and $b(j)$ are $\bar{a}(i_p)$ and $\bar{a}(j_{p'})$, respectively. If they are identical, then $i_p = j_{p'}$, a contradiction.

Case 2. Both i_p and $j_{p'}$ satisfy SR 3: For this case, the copies of $b(i)$ and $b(j)$ are $\bar{b}(i_p)$ and $\bar{b}(j_{p'})$, respectively. If they are identical, then $i_p = j_{p'}$, a contradiction.

Case 3. i_p satisfies one of SR 1 or 2 while $j_{p'}$ satisfies SR 3: For this case, the

[15] *Proof.* Suppose to the contrary that $i_p \in \{i_1, i_2, \ldots, i_{p-1}\}$. Assume, without loss of generality, that p is the smallest among such indices. Let $p' \in \{1, \ldots, p-1\}$ be $i_p = i_{p'}$. Then, $\bar{b}(i_{p'-1}) = \bar{a}(i_{p'}) = \bar{a}(i_p) = \bar{b}(i_{p-1})$, where $i_0 = i$. Thus,$i_{p'-1} = i_{p-1}$, a contradiction to the choice of p.

[16] *Proof.* Suppose to the contrary that for $q \in \{1, \ldots, p\}$ and $q' \in \{1, \ldots, p'\}$, $i_q = j_{q'}$. Assume, without loss of generality, that q is the smallest first coordinate among the pairs of such indices. Then, $\bar{b}(i_{q-1}) = \bar{a}(i_q) = \bar{a}(j_{q'}) = \bar{b}(j_{q'-1})$, where $i_0 = i$ and $j_0 = j$. Thus, $i_{q-1} = j_{q'-1}$, a contradiction to the choice of q.

copies of $b(i)$ and $b(j)$ are $\bar{a}(i_p)$ and $\bar{b}(j_{p'})$, respectively. If they are identical, then $\bar{a}(i_p) = \bar{b}(j_{p'})$. This implies $\bar{b}(j_{p'}) \in \bar{a}(N)$, a contradiction to SR 3.

Summing up, $b|_C$ is achievable on $\omega_a(C)$, a contradiction to $a \in \mathcal{WC}(R)$.

($\mathcal{EC}(R) \subseteq \mathcal{WC}(R)$) Suppose to the contrary that there exists $a \in \mathcal{EC}(R)$ such that a is strongly blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at R, i.e., (i) $b(i)\ P_i\ a(i)$ for all $i \in C$, and (ii) $b|_C$ is achievable on $\omega_a(C)$. Note that , by (ii), there exists an injective function $T : C \to \omega_a(C)$ such that for each $i \in C$, $T(i) \in \{o_{k1}, \ldots, o_{kq_k}\}$, where k satisfies $b(i) = o_k$. Let $b' \in \mathcal{A}$ be such that

$$b'(i) = \begin{cases} b(i) & \text{if } i \in C, \\ o_0 & \text{if } i \notin C \text{ and } \bar{a}(i) \in T(C), \\ a_i & \text{o.w.} \end{cases}$$

Note that for any $i \in N$, $a(i)\ P_i\ b'(i)$ only if $i \notin C$ and $\bar{a}(i) \in T(C)$. Thus, $\bar{a}(i) \in \omega_a(C)$. Summing up, the coalition C exclusion blocks a through b' at R, a contradiction. □

Theorem 4 in Balbuzanov and Kotowski (2019) shows that any indirect exclusion core allocation could be reached by the generalized top-trading cycles (GTTC) algorithm which coincides with the priority rule undet the assumption of common priority.[17] By Proposition 1 in Appendix A and Theorem 1, $\mathcal{WC}(R)$ coincides with the indirect exclusion core in the current setting. Thus, to prove $\mathcal{WC}(R) \subseteq \Phi(R)$, the argument for the proof of Theorem 4 in Balbuzanov and Kotowski (2019) works apart from the fact that the current setting accompanies multiple copies of the objects. We provide a proof for completeness.

Proof of Theorem 2. Let $R \in \mathcal{R}^N$.

($\mathcal{WC}(R) \supseteq \Phi(R)$) Suppose to the contrary that there exists $\trianglerighteq\ \in \Gamma(\succeq)$ such that $a := \varphi^{\trianglerighteq}(R) \notin \mathcal{WC}(R)$. Let $C \in 2^N \backslash \{\emptyset\}$ be a coalition that strongly blocks a through an allocation $b \in \mathcal{A}$ at R, i.e., (i) $b(i)\ P_i\ a(i)$ for all $i \in C$, and (ii) $b|_C$ is achievable on $\omega_a(C)$. Let $i_0 \in C$ be the highest-priority agent in C with respect to $\trianglerighteq$. First, we show the following Claim.

[17] See Appendix A for the definition of the indirect exclusion core.

Claim. $\forall i \in N, [a(i) = b(i_0) \Rightarrow i \rhd i_0]$.

Proof of Claim: Suppose to the contrary that for some $i \in N$ with $a(i) = b(i_0)$, $i_0 \rhd i$. This implies that at least one unit of $b(i_0)(= a(i))$ remains at the step of priority rule with $\trianglerighteq$ in which $\varphi^{\trianglerighteq}_{i_0}(R)(= a(i_0))$ is determined. Thus, $\varphi^{\trianglerighteq}_{i_0}(R) \neq a(i_0)$, a contradiction. This completes the proof of Claim.

By (ii), $\omega_a(C)$ contains at least one unit of $b(i_0)$. Let $o_{k\ell} \in \omega_a(C)$ be a copy of $b(i_0)$. Note that $o_{k\ell} \in \bar{a}(N)$.[18] By the definition of $\omega_a(C)$,

$$\exists i \in C \text{ s.t. } i \succeq_{o_k} \bar{a}^{-1}(o_{k\ell}).$$

Let $j_0 := \bar{a}^{-1}(o_{k\ell})$. Since $\trianglerighteq$ is a completion of $\succeq_{o_k}$, $i \trianglerighteq j_0$. Since i_0 is the $\trianglerighteq$-greatest in C, $i_0 \trianglerighteq i$. Thus, $i_0 \trianglerighteq j_0$. However, $a(j_0) = o_k = b(i_0)$, a contradiction to Claim.

($\mathcal{WC}(R) \subseteq \Phi(R)$) By Theorem 1, $\mathcal{EC}(R) = \mathcal{WC}(R)$. Thus, we show $\mathcal{EC}(R) \subseteq \Phi(R)$. Let $a \in \mathcal{EC}(R)$ be arbitrary. Let $G_1 := \{i \in N \mid i \text{ is } \succeq\text{-maximal}\}$. We first prove

$$(*) \exists i \in G_1 \text{ s.t. } \bar{a}(i) \text{ is a copy of } i\text{'s most favorite object with respect to } R_i$$

by contradiction. Suppose the contrary. We show that there exists a cycle of agents $(i_1, j_1, \ldots, i_p, j_p, i_{p+1})$ such that

(i) $\{i_1, \ldots, i_p\} \subseteq G_1$,

(ii) $\forall p' \in \{1, \ldots, p\}$, $\bar{a}(j_{p'})$ is a copy of $i_{p'}$'s most favorite object with respect to $R_{i_{p'}}$,

(iii) $\forall p' \in \{1, \ldots, p\}, i_{p'+1} \succeq j_{p'}$,

(iv) $|\{i_1, \ldots, i_p\}| = p = |\{j_1, \ldots, j_p\}|$, and

(v) $i_{p+1} = i_1$.

The following inductive procedure, by definition, finds such a cycle.

<u>Step 1.</u> Fix $i_1 \in G_1$ arbitrarily. Let $\alpha_{i_1} := \max_{R_{i_1}} \mathcal{O}$. By the contradiction hypothesis, $a(i_1) \neq \alpha_{i_1}$. Since a is Pareto efficent at R ($\because$ Lemma 3), $\bar{\mathcal{O}} \backslash \bar{a}(N)$ does not contain a copy of α_{i_1}. Thus, there exists $j_1 \in N$ such that $a(j_1) = \alpha_{i_1}$.

[18] *Proof.* Suppose to the contrary that $o_{k\ell} \in \bar{\mathcal{O}} \backslash \bar{a}(N)$. Then, at least one unit of o_k remains at the step of priority rule in which $\varphi^{\trianglerighteq}_{i_0}(R)$ is determined. Thus, $\varphi^{\trianglerighteq}_{i_0}(R) \neq a(i_0)$, a contradiction.

By the definition of G_1, there exists $i_2 \in G_1$ such that $i_2 \succeq j_1$. If $i_2 \in \{i_1\}$, (i_1, j_1, i_2) is the desired cycle. Otherwise, i.e., $i_2 \notin \{i_1\}$, go to Step 2.

Induction hypothesis. Let $p \geq 2$. Suppose that a sequence $(i_1, j_1, \ldots, i_{p-1}, j_{p-1}, i_p)$ satisfies the conditions (i) - (iv). Suppose also that $i_p \notin \{i_1, \ldots, i_{p-1}\}$.

Step p. Let $\alpha_{i_p} := \max_{R_{i_p}} \mathcal{O}$. By the contradiction hypothesis, $a(i_p) \neq \alpha_{i_p}$. Since a is Pareto efficent at R ($\because$ Lemma 3), $\bar{\mathcal{O}} \backslash \bar{a}(N)$ does not contain a copy of α_{i_p}. Thus, there exists $j_p \in N$ such that $a(j_p) = \alpha_{i_p}$. If $j_p \in \{j_1, \ldots, j_{p-1}\}$, $(i_{p'}, j_{p'}, \ldots, i_p, j_p, i_{p+1})$ is the desired cycle, where $j_{p'} = j_p$ and $i_{p+1} := i_{p'}$. Otherwise, i.e., $j_p \notin \{j_1, \ldots, j_{p-1}\}$, then by the definition of G_1, there exists $i_{p+1} \in G_1$ such that $i_{p+1} \succeq j_p$. If $i_{p+1} \in \{i_1, \ldots, i_p\}$, $(i_{p'}, j_{p'}, \ldots, i_p, j_p, i_{p+1})$ is the desired cycle, where $i_{p'} = i_{p+1}$. Otherwise, i.e., $i_{p+1} \notin \{i_1, \ldots, i_p\}$, go to Step $p+1$.

Since N is finite, the procedure stops in a finite steps. This completes the description of a procedure that finds a cycle $(i_1, j_1, \ldots, i_p, j_p, i_{p+1})$ satisfying the properties (i) - (v).

Now, we go back to the proof of $(*)$. Let $C := \{i_1, \ldots, i_p\}$. Define $b \in \mathcal{A}$ as follows.

$$b(i) = \begin{cases} \alpha_{i_{p'}} & \text{if } i \in C \text{ and } i = i_{p'}, \\ o_0 & \text{if } i \in \{j_1, \ldots, j_p\} \backslash C, \\ a(i) & \text{o.w.} \end{cases}$$

Obviously, $b(i)\ P_i\ a(i)$ for all $i \in C$. By definition, for any $i \in N$ with $a(i)\ P_i\ b(i)$, $i \in \{j_1, \ldots, j_p\} \backslash C$. By (iii), for such i, $\bar{a}(i) \in \omega_a(C)$. However, this contradicts $a \in \mathcal{EC}(R)$. This completes the proof of $(*)$.

Finally, we construct $\trianglerighteq \in \Gamma(\succeq)$ such that $a = \varphi^{\trianglerighteq}(R)$. By $(*)$,

$$\exists i'_1 \in G_1 \text{ s.t. } \bar{a}(i'_1) \text{ is a copy of the } i'_1\text{'s most favorite object with respect to } R_{i'_1}.$$

Define a subproblem by removing i'_1 with one unit of $a(i'_1)$ if $a(i'_1)$ is a real object. Remove only i'_1 if $a(i'_1) = o_0$. Let G_2 be the set of $\succeq$-maximal agents in the subproblem.

By $(*)$,

$$\exists i_2' \in G_2 \text{ s.t. } \bar{a}(i_2') \text{ is a copy of the } i_2'\text{'s most favorite object with respect to } R_{i_2'}.$$[19]

Repeating this procedure n times, we obtain a sequence of agents $\{i_p'\}_{p=1}^{n}$. Let $\trianglerighteq$ be $i_1' \trianglerighteq i_2' \trianglerighteq \ldots \trianglerighteq i_n'$. Obviously, $\trianglerighteq \in \Gamma(\succeq)$ and $a = \varphi^{\trianglerighteq}(R)$. □

Lemma 4 and Lemma 5 below are used in the proof of Theorem 3. In the sequel, as is defined in the statement of Theorem 3, $\mathcal{S}$ denotes the set of subsets of $\mathcal{A}^P$ satisfying IIF, WIPF, IND, CON, ESE and CLO.

Lemma 4. *Let $S \in \mathcal{S}$. Then, there exists $f : [0,1] \to [0,1]$ satisfying (GMR-1), (GMR-2) and (GMR-3) such that $S \subseteq \mathcal{A}_f^P$.*

Proof. For each $I \in \mathcal{I}$ and each $i \in I$, let $\overline{x}_i^I := \sup_{\alpha \in S} \alpha(i)$ and $\underline{x}_i^I := \inf_{\alpha \in S} \alpha(i)$. We first prove the following claims.

Claim 1. $\forall I, I' \in \mathcal{I}, \forall i \in I, \forall i' \in I', \frac{r_{i'}^{I'}}{|I'|} \leq \frac{r_i^I}{|I|} \Rightarrow \overline{x}_{i'}^{I'} \leq \overline{x}_i^I$.

Proof of Claim 1: Suppose to the contrary that $\frac{r_{i'}^{I'}}{|I'|} \leq \frac{r_i^I}{|I|}$ and $\overline{x}_{i'}^{I'} > \overline{x}_i^I$. Note that $r_i^I < |I|$.[20] Thus, we have $j \in I$ such that $r_j^I = r_i^I + 1$. By CON, $\overline{x}_i^I = \inf_{\alpha \in S} \alpha(j)$. Thus, by CLO(ii), there exists $\alpha \in S$ such that $\alpha(j) = \overline{x}_i^I$. Moreover, by CLO(i), there exists $\beta \in S$ such that $\beta(i') = \overline{x}_{i'}^{I'}$. Note that $I \neq I'$.[21] By IND, there exists $\gamma \in S$ such that $\gamma|_I = \alpha|_I$ and $\gamma|_{N \setminus I} = \beta|_{N \setminus I}$. Since $\gamma(j) < \gamma(i')$ and $\frac{r_{i'}^{I'}}{|I'|} \leq \frac{r_i^I}{|I|} = \frac{r_j^I - 1}{|I|}$, γ is not WIPF, a contradiction. This completes the proof of Claim 1.

Claim 2. $\forall I, I' \in \mathcal{I}, \forall i \in I, \forall i' \in I', \frac{r_{i'}^{I'}}{|I'|} < \frac{r_i^I}{|I|} \Rightarrow \overline{x}_{i'}^{I'} < \overline{x}_i^I$.

Proof of Claim 2: Suppose to the contrary that $\frac{r_{i'}^{I'}}{|I'|} < \frac{r_i^I}{|I|}$ and $\overline{x}_{i'}^{I'} \geq \overline{x}_i^I$. By Claim 1, $\overline{x}_{i'}^{I'} = \overline{x}_i^I$. Note that $r_i^I < |I|$.[22] Thus, there exists $j \in I$ such that $r_j^I = r_i^I + 1$. By CON, $\inf_{\alpha \in S} \alpha(j) = \overline{x}_i^I \left(= \overline{x}_{i'}^{I'}\right)$. By CLO(ii), there exists $\alpha \in S$ such that $\alpha(j) = \overline{x}_i^I$. Moreover, by CLO(i), there exists $\beta \in S$ such that $\beta(i') = \overline{x}_{i'}^{I'}$. Note

[19] Here, "the most favorite object" means the one remaining in the subproblem.

[20] *Proof of $r_i^I < |I|$*: If $r_i^I = |I|$, $1 = \overline{x}_i^I < \overline{x}_{i'}^{I'}$ by CLO(ii). This contradicts the definition of $\overline{x}_{i'}^{I'}$.

[21] *Proof of $I \neq I'$*: If $I = I'$, $\overline{x}_{i'}^{I'} > \overline{x}_i^I$. Thus, by the definition of $\mathcal{A}^P$, $r_{i'}^I > r_i^I$, a contradiction.

[22] *Proof of $r_i^I < |I|$*: Suppose to the contrary that $r_i^I = |I|$. Then, by CLO(i), $1 = \overline{x}_i^I = \overline{x}_{i'}^{I'}$. Since $\frac{r_{i'}^{I'}}{|I'|} < 1$, there exists $j' \in I'$ such that $r_{j'}^{I'} = r_{i'}^{I'} + 1$. By CLO(i), there exists $\delta \in S$ such that $\delta(i') = \overline{x}_{i'}^{I'} = 1$. For this δ, $\delta(j') > \delta(i') = 1$. However, this contradicts the definition of $\mathcal{A}^P$.

that $I \neq I'$.[23] By IND, there exists $\gamma \in S$ such that $\gamma|_I = \alpha|_I$ and $\gamma|_{N\setminus I} = \beta|_{N\setminus I}$. Thus, $\gamma(j) = \gamma(i')$ and $\frac{r_{i'}^{I'}}{|I'|} < \frac{r_i^I}{|I|} = \frac{r_j^I - 1}{|I|}$. Thus, γ is not WIPF, a contradiction. This completes the proof of Claim 2.

By Claim 1 and 2, there exists a function $f : [0,1] \to [0,1]$ satisfying (GMR-1) such that

$$\forall I \in \mathcal{I}, \forall i \in I, f\left(\frac{r_i^I}{|I|}\right) = \overline{x}_i^I.$$

In the sequel, we show that f satisfies (GMR-2), (GMR-3) and $S \subseteq \mathcal{A}_f^P$.

(f satisfies (GMR-2)): By ESE, there exists $\alpha_f^M \in S$ such that $\alpha_f^M(i) = \frac{\overline{x}_i^I + \underline{x}_i^I}{2}$ for each $I \in \mathcal{I}$ and each $i \in I$. For each $I \in \mathcal{I}$ with $I = \{i_1, \ldots, i_K\}$ and $i_K \succ \ldots \succ i_1$,

$$\begin{aligned}
\sum_{i \in I} \alpha_f^M(i) &= \frac{\overline{x}_{i_1}^I + \underline{x}_{i_1}^I}{2} + \ldots + \frac{\overline{x}_{i_K}^I + \underline{x}_{i_K}^I}{2} \\
&= \frac{0 + \overline{x}_{i_1}^I}{2} + \frac{\overline{x}_{i_1}^I + \overline{x}_{i_2}^I}{2} + \ldots + \frac{\overline{x}_{i_{K-1}}^I + 1}{2} \quad (\because \text{ CON and CLO }) \\
&= \sum_{k=1}^{K-1} \overline{x}_{i_k}^I + \frac{1}{2} \\
&= \sum_{i \in I} \overline{x}_i^I - \frac{1}{2}. \qquad \text{(A)}
\end{aligned}$$

Thus, combined with IIF, f satisfies (GMR-2).

(f satisfies (GMR-3)): Suppose to the contrary that there exist $I \in \mathcal{I}$ and $i \in I$ such that $\overline{x}_i^I - \underline{x}_i^I > \frac{1}{2}$. Note that $\overline{x}_i^I - \underline{x}_i^I = f\left(\frac{r_i^I}{|I|}\right) - f\left(\frac{r_i^I - 1}{|I|}\right)$ ($\because$ CON and CLO(ii)). By CLO, there exist $\alpha, \beta \in S$ such that $\alpha(i) = \overline{x}_i^I$ and $\beta(i) = \underline{x}_i^I$. Since S satisfies IIF and IND, the point assignment to institution $I \in \mathcal{I}$ under any point allocation in S is equal to (A). Thus, $\overline{x}_i^I + \sum_{j \in I\setminus\{i\}} \alpha(j) = \underline{x}_i^I + \sum_{j \in I\setminus\{i\}} \beta(j) < \left(\overline{x}_i^I - \frac{1}{2}\right) + \sum_{j \in I\setminus\{i\}} \beta(j)$. Thus, $\sum_{j \in I\setminus\{i\}} (\beta(j) - \alpha(j)) > \frac{1}{2}$. Note that for each $j \in I\setminus\{i\}, \overline{x}_j^I - \underline{x}_j^I \geq \beta(j) - \alpha(j)$ because $\alpha(j), \beta(j) \in [\underline{x}_j^I, \overline{x}_j^I]$. Thus, $\sum_{j \in I\setminus\{i\}} \left(\overline{x}_j^I - \underline{x}_j^I\right) > \frac{1}{2}$. Summing up,

$$\left(\overline{x}_i^I - \underline{x}_i^I\right) + \sum_{j \in I\setminus\{i\}} \left(\overline{x}_j^I - \underline{x}_j^I\right) > \frac{1}{2} + \frac{1}{2} = 1.$$

[23] *Proof of $I \neq I'$*: Suppose to the contrary that $I = I'$. By CLO(i), there exists $\delta \in S$ such that $\delta(i') = \overline{x}_{i'}^{I'}$. Since $\frac{r_{i'}^{I'}}{|I'|} < \frac{r_i^I}{|I|}$, $r_{i'}^{I'} < r_i^I$. Thus, $\delta(i') < \delta(i)$ ($\because$ the definition of $\mathcal{A}^P$). This contradicts $\overline{x}_i^I = \overline{x}_{i'}^{I'}$.

However, the left-hand side of the inequality is 1, a contradiction.

($S \subseteq \mathcal{A}_f^P$): First, we introduce a notation. For each $I \in \mathcal{I}$ and each $i \in I$, let $T_i^f := \overline{x}_i^I - \underline{x}_i^I$. Let $\alpha \in S$ be arbitrary. For each $I \in \mathcal{I}$ and each $i \in I$, there exists $w_i \in [-1, 1]$ such that $\alpha(i) = \frac{\underline{x}_i^I + \overline{x}_i^I}{2} + w_i \frac{T_i^f}{2}$ ($\because \alpha(i) \in [\underline{x}_i^I, \overline{x}_i^I]$). Since S satisfies IIF and IND, the point assignment to institution $I \in \mathcal{I}$ under any point allocation in S is equal to (A), i.e., $\sum_{i \in I} \overline{x}_i^I - \frac{1}{2} = \sum_{i \in I} \alpha(i)$. Thus,

$$\begin{aligned}\sum_{i \in I} \overline{x}_i^I - \frac{1}{2} &= \sum_{i \in I} \left(\frac{\underline{x}_i^I + \overline{x}_i^I}{2} + w_i \frac{T_i^f}{2} \right) \\ &= \sum_{i \in I} \alpha_f^M(i) + \sum_{i \in I} w_i \frac{T_i^f}{2} \\ &= \left(\sum_{i \in I} \overline{x}_i^I - \frac{1}{2} \right) + \sum_{i \in I} w_i \frac{T_i^f}{2}.\end{aligned}$$

Thus, $\sum_{i \in I} w_i T_i^f = 0$. Thus, $\alpha \in \mathcal{A}_f^P$. □

Lemma 5. *Let $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ be such that*

- *both f and g satisfy (GMR-1), (GMR-2) and (GMR-3), and*
- *$\exists I \in \mathcal{I}, \exists i \in I$ s.t. $f\left(\frac{r_i^I}{|I|}\right) \neq g\left(\frac{r_i^I}{|I|}\right)$.*

Then, $\mathcal{A}_f^P \not\subseteq \mathcal{A}_g^P$.

Proof. We show the conclusion in the following two cases separately.

Case 1. $f\left(\frac{r_i^I}{|I|}\right) < g\left(\frac{r_i^I}{|I|}\right)$.

Since f and g satisfy (GMR-1), $f(1) = 1 = g(1)$. Thus, $r_i^I < |I|$. Thus, there exists $j \in I$ such that $r_j^I = r_i^I + 1$. Thus, by the definition of GMR, there exists $\alpha \in \mathcal{A}_f^P$ such that $\alpha(j) = f\left(\frac{r_i^I}{|I|}\right)$. Since $\min_{\beta \in \mathcal{A}_g^P} \beta(j) = g\left(\frac{r_i^I}{|I|}\right)$, $\alpha \notin \mathcal{A}_g^P$.

Case 2. $f\left(\frac{r_i^I}{|I|}\right) > g\left(\frac{r_i^I}{|I|}\right)$.

By the definition of GMR, there exists $\alpha \in \mathcal{A}_f^P$ such that $\alpha(i) = f\left(\frac{r_i^I}{|I|}\right)$. Since $\max_{\beta \in \mathcal{A}_g^P} \beta(i) = g\left(\frac{r_i^I}{|I|}\right)$, $\alpha \notin \mathcal{A}_g^P$. □

Proof of Theorem 3. ($\Leftarrow$) For each $I \in \mathcal{I}$ and each $i \in I$, let $\overline{x}_i^I := f\left(\frac{r_i^I}{|I|}\right)$, $\underline{x}_i^I := f\left(\frac{r_i^I - 1}{|I|}\right)$ and $T_i^f := \overline{x}_i^I - \underline{x}_i^I$.

($\mathcal{A}_f^P$ satisfies WIPF): Let $\alpha \in \mathcal{A}_f^P$ be arbitrary. Suppose that $\alpha(i') \geq \alpha(i)$ and $\frac{r_{i'}^{I'}}{|I'|} \leq \frac{r_i^I - 1}{|I|}$ for $I, I' \in \mathcal{I}, i \in I$ and $i' \in I'$.

First, we show that $\frac{r_{i'}^{I'}}{|I'|} = \frac{r_i^I - 1}{|I|}$. Suppose to the contrary that $\frac{r_{i'}^{I'}}{|I'|} < \frac{r_i^I - 1}{|I|}$. Since f is strictly increasing ($\because$ GMR-1), $f\left(\frac{r_{i'}^{I'}}{|I'|}\right) < f\left(\frac{r_i^I - 1}{|I|}\right)$. By the definition of GMR,

$$\alpha(i') \leq f\left(\frac{r_{i'}^{I'}}{|I'|}\right) < f\left(\frac{r_i^I - 1}{|I|}\right) \leq \alpha(i),$$

a contradiction. Thus, $\frac{r_{i'}^{I'}}{|I'|} = \frac{r_i^I - 1}{|I|}$.

Next, we show that $\alpha(i') = \alpha(i)$. Since $\frac{r_{i'}^{I'}}{|I'|} = \frac{r_i^I - 1}{|I|}$, by the definition of GMR, $\alpha(i') \leq f\left(\frac{r_{i'}^{I'}}{|I'|}\right) = f\left(\frac{r_i^I - 1}{|I|}\right) \leq \alpha(i)$. Thus, $\alpha(i') = \alpha(i)$.

Summing up the previous two paragraphs, α satisfies WIPF. Thus, $\mathcal{A}_f^P$ satisfies WIPF.

($\mathcal{A}_f^P$ satisfies IIF): Let $\alpha \in \mathcal{A}_f^P$ be arbitrary. Suppose that for $w \in [-1, 1]^N$, $\alpha(i) = \frac{\underline{x}_i^I + \overline{x}_i^I}{2} + w_i \frac{T_i^f}{2}$. Let $I, I' \in \mathcal{I}$ be arbitrary. By (GMR-2), $\frac{1}{|I|}\left(\sum_{i \in I} \overline{x}_i^I - \frac{1}{2}\right) = \frac{1}{|I'|}\left(\sum_{i' \in I'} \overline{x}_{i'}^{I'} - \frac{1}{2}\right)$. Thus,

$$\begin{aligned}
\sum_{i \in I} \alpha(i) &= \sum_{i \in I} \left(\frac{\underline{x}_i^I + \overline{x}_i^I}{2} + w_i \frac{T_i^f}{2}\right) \\
&= \sum_{i \in I} \frac{\underline{x}_i^I + \overline{x}_i^I}{2} \qquad \left(\because \sum_{i \in I} w_i T_i^f = 0\right) \\
&= \sum_{i \in I} \overline{x}_i^I - \frac{1}{2}.
\end{aligned}$$

The same calculation brings $\sum_{i' \in I'} \alpha(i') = \sum_{i' \in I'} \overline{x}_{i'}^{I'} - \frac{1}{2}$. Thus, $\frac{1}{|I|} \sum_{i \in I} \alpha(i) = \frac{1}{|I'|} \sum_{i' \in I'} \alpha(i')$. Thus, α satisfies IIF. Thus, $\mathcal{A}_f^P$ satisfies IIF.

($\mathcal{A}_f^P$ satisfies IND): Obvious.

($\mathcal{A}_f^P$ satisfies CLO): We only show CLO(i) since the proof of CLO(ii) is similar. Let $I \in \mathcal{I}$ and $i \in I$ be arbitrary. We construct a point allocation in $\mathcal{A}_f^P$ under which the point assignment for i is $\overline{x}_i^I \left(= \frac{\overline{x}_i^I + \underline{x}_i^I}{2} + \frac{\overline{x}_i^I - \underline{x}_i^I}{2}\right)$. By (GMR-3), $\sum_{j \in I \setminus \{i\}} \left(\overline{x}_j^I - \underline{x}_j^I\right) \geq \overline{x}_i^I - \underline{x}_i^I$. Thus, $\sum_{j \in I \setminus \{i\}} \frac{\overline{x}_j^I - \underline{x}_j^I}{2} \geq \frac{\overline{x}_i^I - \underline{x}_i^I}{2}$. Note that the right-hand side of the inequality respresents the extra points needed for i's assignment to be $\overline{x}_i^I$ beyond the midpoin of $[\underline{x}_i^I, \overline{x}_i^I]$. On the other hand, the left-hand side of the inequality represents the sum of the points maximally took away from other agents $j \in I\backslash\{i\}$ beyond the midpoint of $[\underline{x}_j^I, \overline{x}_j^I]$. Let $w \in [-1,1]^N$ be

$$w_k := \begin{cases} 1 & \text{if } k = i, \\ -\frac{\overline{x}_i^I - \underline{x}_i^I}{\sum_{j \in I\backslash\{i\}} \left(\overline{x}_j^I - \underline{x}_j^I\right)} & \text{if } k \in I\backslash\{i\}, \\ 0 & \text{if } k \in N\backslash I. \end{cases}$$

Since $\sum_{k\in I} w_k T_k^f = (\overline{x}_i^I - \underline{x}_i^I) + \sum_{k \in I\backslash\{i\}} \left(-\frac{\overline{x}_i^I - \underline{x}_i^I}{\sum_{j\in I\backslash\{i\}}\left(\overline{x}_j^I - \underline{x}_j^I\right)}\right)(\overline{x}_k^I - \underline{x}_k^I) = 0$, $\alpha^w \in \mathcal{A}_f^P$. Note that $\alpha^w(i) = \overline{x}_i^I$. Thus, $\mathcal{A}_f^P$ satisfies CLO(i).

($\mathcal{A}_f^P$ satisfies ESE): Since $\mathcal{A}_f^P$ satisfies CLO, by the definition of GMR, for each $I \in \mathcal{I}$ and each $i \in I$, $\inf_{\alpha\in\mathcal{A}_f^P} \alpha(i) = f\left(\frac{r_i^I - 1}{|I|}\right)$ and $\sup_{\alpha\in\mathcal{A}_f^P} \alpha(i) = f\left(\frac{r_i^I}{|I|}\right)$. Let $w = (0,\ldots,0) \in [-1,1]^N$. Obviously, $\alpha^w \in \mathcal{A}_f^P$ satisfies $\alpha^w(i) = \frac{\inf_{\alpha\in\mathcal{A}_f^P}\alpha(i) + \sup_{\alpha\in\mathcal{A}_f^P}\alpha(i)}{2}$ for all $i \in I$.

($\mathcal{A}_f^P$ satisfies CON): Let $I \in \mathcal{I}$. Suppose that $i, j \in I$ are such that $r_i^I = r_j^I + 1$. Since $\mathcal{A}_f^P$ satisfies CLO, by the definition of GMR, $\sup_{\alpha\in\mathcal{A}_f^P} \alpha(j) = f\left(\frac{r_j^I}{|I|}\right) = f\left(\frac{r_i^I - 1}{|I|}\right) = \inf_{\alpha\in\mathcal{A}_f^P} \alpha(i)$.

($\mathcal{A}_f^P$ is $\supseteq$-maximal in $\mathcal{S}$): Suppose to the contrary that there exists $T \in \mathcal{S}$ such that $\mathcal{A}_f^P \subsetneq T$. By Lemma 4, there exists $g : [0,1] \to [0,1]$ satisfying (GMR-1), (GMR-2) and (GMR-3) such that $T \subseteq \mathcal{A}_g^P$. Thus, $\mathcal{A}_f^P \subsetneq \mathcal{A}_g^P$. However, this contradicts Lemma 5.

($\Rightarrow$) By Lemma 4, there exists $f : [0,1] \to [0,1]$ satisfying (GMR-1), (GMR-2) and (GMR-3) such that $S \subseteq \mathcal{A}_f^P$. Note that $\mathcal{A}_f^P \in \mathcal{S}$ as shown in the sufficiency part of the proof of Theorem 3. Since S is a $\supseteq$-maximal element in $\mathcal{S}$, $S = \mathcal{A}_f^P$. □

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