Efficient, fair, and strategy-proof allocation of discrete resources under weak preferences and constraints*

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Abstract

We consider the problem of allocating indivisible objects without monetary transfers. Each agent has a unit-demand preference involving indifferences and there is a constraint on possible allocations. Under the assumption that the constraints constitute a discrete structure called an integral polymatroid, our new mechanism is efficient, respects priorities, is strategy-proof, and polynomial-time computable. We discuss applications to the problem of allocating time slots for vaccination.

Keywords: No-transfer allocation, constraints, weak preference relation, efficiency, strategy-proofness, discrete convex analysis

1 Introduction

Governments often need to allocate scarce resources among agents without monetary transfers, as can be widely observed during the Covid-19 pandemic. Real-life allocations possess two key features that are typically precluded from the standard model: weak preferences and constraints. To see their practical relevance, consider the problem of allocating time slots for vaccination thruogh a centralized system: a local government sets up dates on which residents can get vaccinated, each resident submits her list of possible dates in order of preference, and the local government decides who gets vaccinated on which date. In this problem, it is often the case that an agent is available on several dates, i.e., she is indifferent between them.[1] Alongside indifferences, constraints on possible allocations also need to be

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[1] Indifferences of preferences also emanate from other sources such as lack of information. For concrete examples, see, e.g., Erdil and Ergin (2017).

taken into account. Each date has its quota, i.e., the maximum number of residents who can be vaccinated on that day, and the sum of vaccinated people throughout the dates cannot exceeds the number of available vaccine doses.[2] Determining vaccination venues as well as dates further complicates the constraints.

The purpose of the present paper is to introduce a new mechanism for object allocation problems under weak preferences and constraints. To handle indifferences inherent in weak preferences, a standard approach is to break ties and then apply a mechanism under strict preferences,[3] most notably *serial dictatorship* (Satterthwaite and Sonnenschein, 1981).[4] However, this idea might lead to an inefficient outcome.[5] To see this point, suppose that there are three objects, k_1, k_2, k_3 (each has quota 1), and three agents, $1, 2, 3$, with the following preferences:[6]

- Agent 1: $k_1 \sim k_2 \succ k_3$.
- Agent 2: $k_2 \succ k_3 \succ k_1$.
- Agent 3: $k_3 \succ k_1 \succ k_2$.

Let us break 1's indifference relation as $k_2 \succ k_1$ and run serial dictatorship with order $1 \to 2 \to 3$. Then, 1 gets k_2, 2 gets k_3, and 3 gets k_1. There is a (weakly) Pareto-improving allocation: 1 gets k_1, 2 gets k_2, and 3 gets k_3.

The cause of the inefficiency is to finalize 1's allocation as k_2 although she is happy with k_1 *or* k_2. If we allow agent 1 to receive k_1 or k_2, however, then each agent's final allocation has multiple candidates in general, thus creating a combinatorial problem. This poses a computational challenge, particularly when we deal with constraints. To overcome this problem, we assume that the constraints constitute an *integral polymatroid*, a concept in discrete mathematics. Importantly, the class of integral polymatroids permits *hierarchical constraints* as a special case, which are widely observed in real problems. Applying techniques of discrete convex analysis (Murota, 2003), we show that our new mechanism is polynomial-time computable. Furthermore, the mechanism satisfies desirable properties: it is efficient, respects priorities, and is strategy-proof. Here, the second property of respecting priorities

[2]These constraints appear in real problems because a local government receives a certain amount of vaccine doses from the central government and then distributes them; see the vaccination programs of, e.g., India (National Portal of India, 2022), Japan (Prime Minister's Office of Japan, 2022), or the United States (Washinton State Department of Health, 2022).

[3]Abdulkadiroğlu et al. (2009) deal with the issue of breaking ties of school priorities in a school choice problem.

[4]Svensson (1999) proves that a mechanism is neutral and group strategy-proof if and only if it is a serial dictatorship. Pycia and Ünver (2017) provide a generalization of Svensson's result.

[5]This fact was previously pointed out by Bogomolnaia et al. (2005); see the first paragraph of Section 4 therein.

[6]$k_1 \succ k_2$ is read "k_1 is more preferred than k_2" and $k_1 \sim k_2$ is read "k_1 is indifferent to k_2".

is a fairness condition; assuming that there is a priority order over agents (which is identical across objects), the property guarantees that no agent envies the outcome of the allocation of an agent with a lower priority.

Related literature

The closest to our study is Svensson (1994), who introduces a new mechanism that satisfies efficiency, fairness, and strategy-proofness under weak preferences. Our contribution is threefold. First, while Svensson (1994) assumes that each object has quota 1,[7] we allow for a wider class of constraints on feasible allocations. Second, although Svensson (1994) does not deal with computational problems, we prove that our mechanism is polynomial-time computabile. Third, we describe our algorithm in terms of eliminating excess demand, thus making it intuitive and clarifying the connection to existing matching/auction mechanisms.

Building upon Svensson's (1994) result, Bogomolnaia et al. (2005) characterize a class of mechanisms that satisfy efficiency, strategy-proofness and other desiderata as a selection from a so-called bi-polar serially dictatorial rule. However, they do not offer a computationally tractable selection. Fairness and consitraints are not considered either. Jaramillo and Manjunath (2012) analyze object allocation under weak preferences while allowing some objects to be intially owned by an agent. Our result is distinguished from theirs in that constraints and fairness are taken into account. Erdil and Ergin (2017) develop a general model of two-sided matching under weak preferences. Their model allows both sides of the market to have weak preferences but does not handle constraints. Erdil and Ergin (2017) propose a new algorithm that finds a stable and efficient outcome in polynomial time. The key difference from our result is that their algorithm does not satisfy strategy-proofness.

A notable feature of our analysis is to utilize the notion of a matroid. Recently, matroid and its variations have been integrated into the notion of *M-convexity* in discrete convex analysis (Murota, 2003). Prior work has revealed that M-convexity is fundamental for running the DA/TTC algorithms under constraints; see Hafalir et al. (2022) and the literature review therein.

The remainder is organized as follows. Section 2 introduces our model. Section 3 defines our new mechanism and presents the main theorem about its properties. Section 4 concludes. The proof of the main theorem is relegated to Section 5.

2 Model

Our notation partly follows that of Kojima and Manea (2010).

[7] Under this assumption on quotas, our mechanism coincides with Svensson's mechanism.

Let $N = \{1, \ldots, n\}$ be a set of **agents** and let K be a set of **objects** (more precisely, object *types*). There is a special object, called the **null object**, denoted ϕ; let $\bar{K} := K \cup \{\phi\}$. An **allocation** is a vector $\mu := (\mu_i)_{i \in N}$ that assigns object $\mu_i \in \bar{K}$ to agent i. For an allocation μ, we define $x^\mu \in \mathbb{Z}^K_{\geq 0}$ by

$$x^\mu_k = |\{i \in N : \mu_i = k\}| \text{ for all } k \in K,$$

representing the vector of the number of agents who receive each object (except the null object). We assume that there is a set of **feasible vectors** $\mathcal{F} \subseteq \mathbb{Z}^K_{\geq 0}$ with $\mathcal{F} \neq \emptyset$. An allocation μ is said to be **feasible** if $x^\mu \in \mathcal{F}$. Let $\mathcal{A}$ denote the **set of feasible allocations**.

Remark 1. We assume $\mathcal{F} \subseteq \mathbb{Z}^K_{\geq 0}$ rather than $\mathcal{F} \subseteq \mathbb{Z}^{\bar{K}}_{\geq 0}$, thus imposing no restriction on the number of the null object allocated to the agents. The underlying assumption is that the null object is not scarce. ▮

We illustrate feasible vectors in the example of allocating time slots for vaccination. Suppose that there are two dates, $K = \{k, \ell\}$, on which residents can get vaccinated. Up to 100 residents can be accommodated on either day, but there are only 150 vaccine doses in total. Then,

$$\mathcal{F} = \left\{x \in \mathbb{Z}^K_{\geq 0} : 0 \leq x_k \leq 100,\ 0 \leq x_\ell \leq 100,\ 0 \leq x_k + x_\ell \leq 150\right\}. \tag{1}$$

Each agent i has a weak (complete and transitive) **preference relation** over $\bar{K}$, denoted $\succsim_i$; let $\succ_i$ and $\sim_i$ denote the strict and indifference relations induced from $\succsim_i$, respectively. We denote by $\mathcal{R}$ the set of all weak preference relations. Let $\succsim := (\succsim_i)_{i \in N} \in \mathcal{R}^N$ denote the **preference profile** of all agents. For $j \in N$, we use the notation $\succsim_{-j} := (\succsim_i)_{i \in N \setminus \{j\}}$.

Following Svensson (1994) and Pathak et al. (2021), we assume that there is a **baseline priority order** π, which is a linear order over N. To quote Pathak et al. (2021): "This priority order captures the ethical values guiding the allocation of the scarce medical resources." In the context of time slot allocation for vaccination, if $j \in N$ is an elderly person or essential personnel and $h \in N$ is a young healthy person, then j is given a higher priority than h, which is represented as $j \, \pi \, h$.[8] Without loss of generality, we assume that

$$j \, \pi \, h \iff j < h.$$

Namely, an agent with a smaller index has a higher priority.

A **mechanism** $\varphi : \mathcal{R}^N \to \mathcal{A}$ maps preference profiles to feasible allocations. At $\succsim \in \mathcal{R}^N$, agent i is assigned object $\varphi_i(\succsim)$. We focus on the following three properties:

[8] The order π could represent other fairness considerations such as needs or waiting time.

- φ is **efficient** if, for any $\succsim \in \mathcal{R}^N$, $\varphi(\succsim)$ is efficient at $\succsim$, i.e., there exists no $\mu \in \mathcal{A}$ such that

$$\big[\mu_i \succsim \varphi_i(\succsim) \text{ for all } i \in N\big] \text{ and } \big[\mu_j \succ \varphi_j(\succsim) \text{ for some } j \in N\big].$$

- φ **respects priorities** if, for any $\succsim \in \mathcal{R}^N$, there exist no $j, h \in N$ with $h > j$ such that

$$\varphi_h(\succsim) \succ_j \varphi_j(\succsim).$$

- φ is **strategy-proof** if, for any $\succsim \in \mathcal{R}^N$, there exists no $j \in N$ and $\succsim'_j \in \mathcal{R}$ such that

$$\varphi_j(\succsim'_j, \succsim_{-j}) \succ_j \varphi_j(\succsim).$$

The first and third properties are standard in the mechanism design literature. The second property was introduced by Svensson (1994) under the name of "weak fairness"; Pathak et al. (2021) and Aziz and Brandl (2021) introduce a related property in the context of medical rationing. It states that an agent j never envies the outcome of another agent h who has a lower priority than j (recall that $j \, \pi \, h$ whenever $h > j$). In the context of time slot allocation, if an elderly person j cannot get vaccinated on any of her possible dates, then a young healthy person $h > j$ cannot get vaccinated on any of j's possible dates either.

3 New mechanism

This section consists of four subsections. Sections 3.1 and 3.2 deal with preliminaries, which are used to define our new mechanism in Section 3.3. Section 3.4 presents our main result about the properties of the new mechanism.

We introduce two pieces of notation. For $k \in K$, let $\mathbb{1}^k \in \mathbb{Z}^K_{\geq 0}$ denote the ***k*-th unit vector**, i.e., $\mathbb{1}^k_k = 1$ and $\mathbb{1}^k_\ell = 0$ for all $\ell \neq k$. For $L \subseteq K$ and $x \in \mathbb{Z}^K_{\geq 0}$, let $x(L) := \sum_{k \in L} x_k$.

3.1 Integral polymatroid

To establish a positive result on efficient computation, we borrow a concept in discrete mathematics. We say that $\mathcal{F} \subseteq \mathbb{Z}^K_{\geq 0}$ with $\mathcal{F} \neq \emptyset$ is an **integral polymatroid** (Welsh, 1976) if it satisfies the following two conditions:

(M1) For any $x \in \mathcal{F}$ and $y \in \mathbb{Z}^K_{\geq 0}$ with $y \leq x$, it holds that $y \in \mathcal{F}$.

(M2) For any $x, y \in \mathcal{F}$ with $x(K) < y(K)$, there exists $k \in K$ with $x_k < y_k$ such that $x + \mathbb{1}^k \in \mathcal{F}$.

The former condition implies that we deal with an upper bound constraint. The latter condition is the key property of a matroid, stating that a vector x with a smaller coordinate sum than y can move "one step close" to y while staying inside $\mathcal{F}$.

Remark 2. A mechanism φ is said to satisfy **individual rationality** if, for any $\succsim \in \mathcal{R}^N$ and any $i \in N$, it holds that $\varphi_i(\succsim) \succsim_i \phi$. One easily verifies that, under (M1), individual rationality is implied by efficiency. ∎

To see a concrete example of an integral polymatroid, we introduce an additional definition. We say that $\mathcal{F}$ is **hierarchical** if:

- there exists a family $\mathcal{K} \subseteq 2^K$ such that, for any $L, L' \in \mathcal{K}$, either $L \cap L' = \emptyset$ or $L \subseteq L'$ or $L' \subseteq L$ holds; and
- for each $L \in \mathcal{K}$, thete exists $q_L \in \mathbb{Z}_{\geq 0}$ such that

$$\mathcal{F} = \left\{ x \in \mathbb{Z}^K_{\geq 0} : x(L) \leq q_L \text{ for all } L \in \mathcal{K} \right\}. \tag{2}$$

It is known that $\mathcal{F}$ given by (2) is an integral polymatroid. This type of constraints naturally appear in real problems. One such example is provided in (1). Another example is when x vaccine doses are available in January and additional y doses are available in February. In this case, the sum of vaccinated residents in January is no greater than x and the total number of vaccinated residents in January and February is no greater than $x + y$. This case also can be accommodated by hierarchical feasible vectors.

3.2 Requirement function and existence of feasible allocation

Throughout this section, we fix $\succsim \in \mathcal{R}^N$. For $i \in N$, take an integer $r_i \in \{1, \ldots, |\bar{K}|\}$, which we call a **rank**. We define the set of **top r_i ranked objects** (at $\succsim_i$), denoted $\bar{K}_i(r_i; \succsim_i)$, inductively as follows:

$$\bar{K}_i(1; \succsim_i) = \{k \in \bar{K} : k \succsim_i \ell \text{ for all } \ell \in \bar{K}\},$$
$$\bar{K}_i(r_i; \succsim_i) = \left\{k \in \bar{K} : k \succsim_i \ell \text{ for all } \ell \in \bar{K} \setminus \cup_{s=1}^{r_i - 1} \bar{K}_i(s; \succsim_i)\right\} \text{ for all } r_i = 2, \ldots, |\bar{K}|.$$

For example, if $\bar{K} = \{k, \ell, \phi\}$ and 1's preference is $k \sim_1 \ell \succ_1 \phi$, then

$$\bar{K}_1(1; \succsim_1) = \{k, \ell\},\ \bar{K}_1(2; \succsim_1) = \{k, \ell, \phi\},\ \bar{K}_1(3; \succsim_1) = \{k, \ell, \phi\}.$$

We often write $\bar{K}_i(r_i)$ rather than $\bar{K}_i(r_i; \succsim_i)$ when the preference relation is clear from the context. Note that $\bar{K}_i(r_i) \subseteq \bar{K}_i(r'_i)$ whenever $r_i \leq r'_i$.

For $i \in N$, we define i's **requirement function** $\rho_i : 2^K \times \{1, \ldots, |\bar{K}|\} \to \{0, 1\}$ as follows:

$$\rho_i(L, r_i; \succsim_i) = \begin{cases} 1 & \text{if } \bar{K}_i(r_i) \subseteq L, \\ 0 & \text{otherwise.} \end{cases}$$

We often write $\rho_i(L, r_i)$ rather than $\rho_i(L, r_i; \succsim_i)$. In words, $\rho_i(L, r_i) = 1$ means that i requires at least one object in L in order to receive an object ranked r_i or higher. Since L is chosen not from $2^{\bar{K}}$ but from 2^K, the following implication holds:

$$\text{for any } L \in 2^K,\ \phi \in \bar{K}_i(r_i) \Longrightarrow \rho_i(L, r_i; \succsim_i) = 0. \tag{3}$$

Take an agent set $\{1, \ldots, m\} \subseteq N$ ($1 \leq m \leq n$) and a profile $r := (r_i)_{i \in N}$ (called a **rank profile**). We say that **excess demand occurs** at $(\{1, \ldots, m\}, r)$ if there exists $L \in 2^K$ such that

$$\sum_{i=1}^{m} \rho_i(L, r_i) > \max_{x \in \mathcal{F}} x(L). \tag{4}$$

The objects in L are in short supply at r in the sense that we cannot give all the agents in $\{1, \ldots, m\}$ a top r_i ranked object. We say that **excess demand does not occur** at $(\{1, \ldots, m\}, r)$ if there exists no $L \in 2^K$ that satisfies (4).

Remark 3. The requirement function was previously introduced in an auction setting; see Demange et al. (1986) or Gul and Stacchetti (2000). Their auction algorithms proceed by increasing the prices of the commodities in excess demand. Our novelty is to convey the technique to a setting without monetary transfers; we adjust ranks, not prices. ∎

The following proposition asserts that, at a given rank profile, excess demand does not occur if and only if there exists a feasible allocation.

Proposition 1. *Fix $\succsim \in \mathcal{R}^N$. Suppose that $\mathcal{F}$ is an integral polymatroid. Let r be a rank profile. Then, the following are equivalent:*

(i) There exists $\mu \in \mathcal{A}$ such that $\mu_i \in \bar{K}_i(r_i)$ for all $i \in N$.

(ii) Excess demand does not occur at (N, r).

Proof. An integral polymatroid is known to satisfy a notion of discrete convexity called

$M^\natural$*-convexity*.[9] Thus, the claim follows from the so-called *discrete separation theorem* for $\mathrm{M}^\natural$-convex sets; see Yokote (2020).[10] □

Remark 4. In practice, $\mathcal{F}$ is bounded; in the example of time slot allocation for vaccination, the number of people who can get vaccinated is always below a certain number. Under this assumption, it is computationally easy to check whether excess demand occurs or not. To see this point, we define the **excess demand function** at $(\{1,\ldots,m\}, r)$, denoted $ED : 2^K \to \mathbb{Z}$, as follows:

$$ED(L) = \sum_{i=1}^{m} \rho_i(L, r_i) - \max_{x\in\mathcal{F}} x(L) \text{ for all } L \in 2^K.$$

One easily verifies from (4) that excess demand occurs at $(\{1,\ldots,m\}, r)$ if and only if $\max_{L\in 2^K} ED(L) > 0$. According to existing results in discrete convex analysis, $ED(\cdot)$ satisfies a condition called *supermodularity* and its maximum value can be computed in time polynomial in the number of agents and objects.[11] It is also computationally easy to find an allocation in Proposition 1(i) (if any) thanks to algorithms for submodular flow problems.[12] ▮

3.3 Generalized Svensson mechanism

Throughout this section, we fix $\succsim \in \mathcal{R}^N$. We define our new algorithm, the **generalized Svensson mechanism**, as follows:

- Step 0: Let $i^0 = 1$ and $r^0 = (1,\ldots,1)$.
- Step $t \geq 1$:
  - (a) If excess demand occurs at $(\{1,\ldots,i^{t-1}\}, r^{t-1})$, then define

    $$i^t = i^{t-1};\ r^t_{i^{t-1}} = r^{t-1}_{i^{t-1}} + 1 \text{ and } r^t_i = r^{t-1}_i \text{ for all } i \neq i^{t-1}.$$

    Go to step $t+1$.
  - (b) Otherwise, define $i^t = i^{t-1} + 1$ and $r^t = r^{t-1}$.
    - ∗ If $i^t \leq n$, then go to step $t+1$.

[9] See, e.g., Section 4.7 of Murota (2003).

[10] As noted by Yokote (2020), the characterization here is a generalization of Hall's (1935) theorem.

[11] Supermodularity follows from the discrete conjugacy theorem; see Theorem 8.12 of Murota (2003). For polynomial-time computability, see Section 10.2 of Murota (2003).

[12] See Section 10.4 of Murota (2003).

* Otherwise, terminate the algorithm and define the outcome as a feasible allocation $\mu \in \mathcal{A}$ such that $\mu_i \in \bar{K}_i(r_i^t)$ for all $i \in N$ (which always exists by Proposition 1).[13]

Note that (a) does not occur indefinitely by (3), and (b) does not occur indefinitely either because the number of agents is finite. This mechanism coincides with Svensson's (1994) mechanism when

$$\mathcal{F} = \Big\{x \in \mathbb{Z}^K_{\geq 0} : x_k \leq 1 \text{ for all } k \in K\Big\}.$$

The algorithm proceeds by sequentially expanding the set of agents and the set of objects $\bar{K}_i(r_i)$ for $i = 1, \dots, n$. The procedure starts as if there is only one agent $\{1\}$, who accepts only her first-ranked objects $\bar{K}_1(1)$. Here, excess demand does not occur if $\phi \in \bar{K}_1(1)$ or $x_k \geq 1$ for some $x \in \mathcal{F}$ and $k \in \bar{K}_1(1)$. Suppose that either of the conditions holds true. Then, we expand the agent set from $\{1\}$ to $\{1, 2\}$ and go to step 2. Both agents accept only their first-ranked objects, namely, those contained in $\bar{K}_1(1)$ (for agent 1) and $\bar{K}_2(1)$ (for agent 2). We check whether, given the acceptable sets of objects, excess demand occurs or not: if not, then we leave agent 2's tentative rank unchanged, i.e., $r_2^2 = r_2^1 = 1$; otherwise, we ask agent 2 to increase her tentative rank, i.e., $r_2^2 = r_2^1 + 1 = 2$. This means that, in step 3, agent 2 accepts the objects in $\bar{K}_2(2)$, the set of first- *and* second-ranked objects. As agent 2 accepts more objects compared to the previous step, there is a higher chance that excess demand does not occur. We continue this process until there is no excess demand. Then, we invite agent 3 to the player set $\{1, 2\}$ and repeat the same procedure.

3.4 Main result

Let φ^{GS} denote the generalized Svensson mechanism.

Theorem 1. *Suppose that $\mathcal{F}$ is an integral polymatroid. Then, φ^{GS} is efficient, respects priorities, and is strategy-proof.*

Proof. See Section 5. □

It is noteworthy that φ^{GS} does not satisfy *group* strategy-proofness, a stronger notion than strategy-proofness.[14] For $M \subseteq N$, let $\succsim_M := (\succsim_i)_{i \in M}$ and $\succsim_{-M} := (\succsim_i)_{i \in N \setminus M}$.

[13] If there are multiple feasible allocations, we choose an arbitrary one.

[14] This observation is consistent with Ehlers's (2002) theorem stating that there exists no efficient and group strategy-proof mechanism under weak preferences.

- φ is **group strategy-proof** if, for any $\succsim \in \mathcal{R}^N$, there exists no $M \subseteq N$ and $\succsim'_M \in \mathcal{R}^M$ such that

$$\big[\varphi_i(\succsim'_M, \succsim_{-M}) \succsim_i \varphi_i(\succsim) \text{ for all } i \in M\big] \text{ and } \big[\varphi_j(\succsim'_M, \succsim_{-M}) \succ_j \varphi_j(\succsim) \text{ for some } j \in M\big].$$

To see that φ^{GS} violates this condition, let $N = \{1, 2, 3\}$, $\bar{K} = \{k, \ell, \phi\}$, and

$$\mathcal{F} = \big\{x \in \mathbb{Z}^K_{\geq 0} : x_k \leq 1, x_\ell \leq 1\big\}.$$

Suppose that the agents have the following true preferences:

- Agent 1: $k \sim \ell \succ \phi$.
- Agent 2: $\ell \succ \phi \succ k$.
- Agent 3: $k \succ \phi \succ \ell$.

Our algorithm proceeds as follows:

- Step 1: $N^0 = \{1\}$, $r^0 = (1, 1, 1)$. Excess demand does not occur. Expand the agent set.
- Step 2: $N^1 = \{1, 2\}$, $r^1 = (1, 1, 1)$. Excess demand does not occur. Expand the agent set.
- Step 3: $N^2 = \{1, 2, 3\}$, $r^2 = (1, 1, 1)$. Excess demand occurs. Increase r^2_3 from 1 to 2.
- Step 4: $N^3 = \{1, 2, 3\}$, $r^3 = (1, 1, 2)$. Excess demand does not occur. The algorithm terminates with final allocation $\mu_1 = k$, $\mu_2 = \ell$, $\mu_3 = \phi$.

Now, suppose that 1 and 3 collude and submit the following preferences:

- Agent 1: $\ell \succ \phi \succ k$.
- Agent 3: $k \succ \phi \succ \ell$ (same as the true preference).

Then, our algorithm proceeds as follows:

- Step 1: $N^0 = \{1\}$, $r^0 = (1, 1, 1)$. Excess demand does not occur. Expand the agent set.
- Step 2: $N^1 = \{1, 2\}$, $r^1 = (1, 1, 1)$. Excess demand occurs. Increase r^1_2 from 1 to 2.
- Step 3: $N^2 = \{1, 2\}$, $r^2 = (1, 2, 1)$. Excess demand does not occur. Expand the agent set.

- Step 4: $N^3 = \{1, 2, 3\}$, $r^3 = (1, 2, 1)$. Excess demand does not occur. The algorithm terminates with final allocation $\mu_1 = \ell$, $\mu_2 = \phi$, $\mu_3 = k$.

Compared to the allocation under true preferences, agent 1 is indifferent and agent 3 becomes strictly better off, showing that group strategy-proofness is violated.

4 Conclusion

In this paper we have developed an efficient, priority-respecting, and strategy-proof mechanism when preferences involve indifferences and constraints are imposed on feasible allocations. The key idea is to reject an agent as being not qualified for object k only if excess demand occurs whenever the agent receives k. Here, indifference relations are taken into account, thus recovering the efficiency loss inherent in serial dictatorship with tie-breaking that ignores indifferences.

5 Proof of Theorem 1

Throughout this section, we abbreviate "generalized Svensson mechanism" as GS.

Proof of efficiency: Fix $\succsim \in \mathcal{R}^N$. Suppose for a contradiction that $\varphi^{GS}(\succsim)$ is not efficient. Then, there exists a feasible allocation μ such that every agent receives a weakly better object than that under $\varphi^{GS}(\succsim)$ and at least one agent receives a strictly better object. For each $i \in N$, we define r_i^* by

$$r_i^* = \min\Big\{r_i \in \{1, \ldots, |\bar{K}|\} : \mu_i \in \bar{K}_i(r_i)\Big\}.$$

Let t be the first step of GS under $\succsim$ at which $r_j^{t-1} = r_j^*$ and $r_j^t > r_j^*$ for some $j \in N$; since there is at least one agent who strictly preferes the object under μ than that under $\varphi^{GS}(\succsim)$, such a step t always exists. Since t is the first step, we have

$$r_i^{t-1} \leq r_i^* \text{ for all } i = 1, \ldots, j-1. \tag{5}$$

By the definition of r^* and the fact that every agent weakly prefers the object under μ than that under $\varphi^{GS}(\succsim)$,

$$r_i^{t-1} \geq r_i^* \text{ for all } i = 1, \ldots, j-1. \tag{6}$$

Combining (5) and (6), together with $r_j^{t-1} = r_j^*$, it holds that

$$r_i^{t-1} = r_i^* \text{ for all } i = 1, \ldots, j.$$

This implies that, for any $L \in 2^K$ and any $i = 1, \ldots, j$,

$$\rho_i(r_i^{t-1}, L; \succsim_i) = 1 \Longrightarrow \mu_i \in L. \tag{7}$$

Then, for any $L \in 2^K$,

$$\max_{x \in \mathcal{F}} x(L) \geq |\{i \in \{1, \ldots, j\} : \mu_i \in L\}| \geq \sum_{i=1}^{j} \rho_i(r_i^*, L; \succsim_i),$$

where the first inequality follows from the fact that μ is a feasible allocation and the second inequality follows from (7). We obtain a contradiction to the fact that excess demand occurs at $(\{1, \ldots, j\}, r^{t-1})$.

Proof of respecting priorities: Fix $\succsim \in \mathcal{R}^N$. Suppose for a contradiction that there exist $j, h \in N$ with $h > j$ such that

$$\varphi_h^{GS}(\succsim) \succ_j \varphi_j^{GS}(\succsim).$$

Let $k^* := \varphi_h^{GS}(\succsim)$. By $k^* \succ_j \varphi_j^{GS}(\succsim)$, there exists a step t of GS under $\succsim$ at which $k^* \in \bar{K}_j(r_j^{t-1})$ and excess demand occurs, i.e., there exists $L \in 2^K$ such that

$$\sum_{i=1}^{j} \rho_i(r_i^{t-1}, L) > \max_{x \in \mathcal{F}} x(L). \tag{8}$$

Since excess demand does not occur when the agent set is $\{1, \ldots, j-1\}$,

$$\sum_{i=1}^{j-1} \rho_i(r_i^{t-1}, L) \leq \max_{x \in \mathcal{F}} x(L). \tag{9}$$

By definition,

$$\rho_j(r_j^{t-1}, L) \leq 1. \tag{10}$$

Combining (8)-(10), the inequalities of (9) and (10) reduce to equalities. By (10) (holding as equality),

$$\bar{K}_j(r_j^{t-1}) \subseteq L,$$

which together with $k^* \in \bar{K}_j(r_j^{t-1})$ implies $k^* \in L$. Together with (9) (holding as equality),

$$\sum_{i=1}^{j-1} \rho_i(r_i^{t-1}, L) = \max_{x \in \mathcal{F}} x(L).$$

This implies that, at $\varphi^{GS}(\succsim)$, all the objects in L are allocated exhaustively to the agents in $\{1, \ldots, j-1\}$. Since $k^* \in L$, we obtain a contradiction to $k^* = \varphi_h^{GS}(\succsim)$ and $h > j$.

Proof of strategy-proofness: Fix a true preference profile $\succsim \in \mathcal{R}^N$. Suppose for a contradiction that an agent $j \in N$ becomes stricly better off by submitting a false preference $\succsim_j' \in \mathcal{R}$, i.e.,

$$\varphi_j^{GS}(\succsim_j', \succsim_{-j}) \succ_j \varphi_j^{GS}(\succsim).$$

Let r' denote the rank profile at the end of GS under $(\succsim_j', \succsim_{-j})$.

The proof goes in parallel with that of φ respecting priorities. Let $k^* := \varphi_j^{GS}(\succsim_j', \succsim_{-j})$. By $k^* \succ_j \varphi_j^{GS}(\succsim)$, there exists a step t of GS under $\succsim$ at which $k^* \in \bar{K}_j(r_j^{t-1}; \succsim_j)$ and excess demand occurs, i.e., there exists $L \in 2^K$ such that

$$\sum_{i=1}^{j} \rho_i(r_i^{t-1}, L; \succsim_i) > \max_{x \in \mathcal{F}} x(L). \tag{11}$$

Since excess demand does not occur when the agent set is $\{1, \ldots, j-1\}$,

$$\sum_{i=1}^{j-1} \rho_i(r_i^{t-1}, L; \succsim_i) \leq \max_{x \in \mathcal{F}} x(L). \tag{12}$$

By definition,

$$\rho_j(r_j^{t-1}, L; \succsim_j) \leq 1. \tag{13}$$

Combining (11)-(13), the inequalities of (12) and (13) reduce to equalities. By (13) (holding as equality),

$$\bar{K}_j(r_j^{t-1}; \succsim_j) \subseteq L,$$

which together with $k^* \in \bar{K}_j(r_j^{t-1}; \succsim_j)$ implies $k^* \in L$. Since all the agents in $\{1, \ldots, j-1\}$ submit the same preferences between $\succsim$ and $(\succsim'_j, \succsim_{-j})$, it holds that $r_i^{t-1} = r'_i$ for all $i = 1, \ldots, j-1$. Together with (12) (holding as equality),

$$\sum_{i=1}^{j-1} \rho_i(r'_i, L; \succsim_i) = \max_{x \in \mathcal{F}} x(L).$$

This implies that, at $\varphi^{GS}(\succsim'_j, \succsim_{-j})$, all the objects in L are allocated exhaustively to the agents in $\{1, \ldots, j-1\}$. Since $k^* \in L$, we obtain a contradiction to $k^* = \varphi_j^{GS}(\succsim'_j, \succsim_{-j})$. $\square$

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