Financial Stability with Fire Sale Externalities*

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May 8, 2022

Abstract

Do policies that aim to mitigate fire sale externalities actually improve financial stability? We study this question in a model of financial intermediation where banks may sell long-term assets in financial markets subject to cash-in-the-market pricing and bank runs. In the absence of interventions, banks hold more long-term assets than is socially optimal, leading to inefficiently large fire sales in a crisis. Policymakers may regulate banks' choices to mitigate this externality, but lack commitment. We show that, in economies with high market liquidity, such actions have the unintended consequence of increasing fragility and lowering welfare.

Keywords: Fire sale externalities; Macroprudential regulations; Limited commitment; Financial fragility

JEL Classification: G21; E61; G28; E44

*We thank Patrick Bolton, Oriol Carbonell-Nicolau, Todd Keister, Greg Phelan, and Haoxi Yang for their helpful comments. We are also grateful to conference and seminar participants at the Midwest Macroeconomics Meetings, the 52nd Annual Conference of the Money, Macro and Finance Society (MMF), the 2021 Midwest Trade and Theory Conference, Nankai University, Rutgers University, the Shanghai University of Finance and Economics, and Waseda University. Yang Li acknowledges financial support from the National Natural Science Foundation of China (Grant Nos. 72103105 and 72001119), the Fundamental Research Funds for Central Universities (Grant No. 63182062), and the Asia Research Center in Nankai University (Grant No. AS2114).

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1 Introduction

Mitigating fire sale externalities has been a central concern of financial stability policy since the financial crisis of 2007-08. There is now a sizable literature on the topic and a variety of policy tools have since been introduced to mitigate fire sales in future periods of financial stress. It is still not well understood, however, how these policies might affect financial fragility and welfare. One challenge in studying this question is to understand how policy choices affect the probability of a crisis. Financial crises are generally thought to have an important self-fulfilling component: creditors are withdrawing from banks and other financial institutions in part because they fear the withdrawals of other creditors will cause losses. This self-fulfilling property can make it difficult to determine the ex-ante probability of a crisis. In this paper, we endogenize the probability of a self-fulfilling bank run using a robust control approach in the spirit of Hansen and Sargent (2001), and studies whether policies that aim to mitigate fire sale externalities actually decrease financial fragility.

We address this question by constructing a version of the Diamond and Dybvig (1983) model of financial intermediation augmented to include fire sale externalities. In particular, we study an environment with financial markets subject to cash-in-the-market pricing based on Allen and Gale (1998) and with limited commitment as in Ennis and Keister (2009). There are two types of assets: short-term and long-term. Banks can sell long-term assets in financial markets before they mature, and the asset price is endogenously determined. The banks face a non-trivial portfolio choice and behave competitively in the sense that they take asset prices as given. When a run occurs, banks may need to sell long-term assets in the market to pay withdrawing depositors. Outside investors have limited funds and, therefore, fire sale pricing will occur when total asset sales are large enough. Because banks rationally neglect the impact of their choices on the asset price, an externality arises. As result of this externality, banks hold more illiquid assets than is socially optimal and fire sales are inefficiently large.

We use this model to study an intervention that aims to mitigate fire sale externalities. A regulator has the ability to regulate banks' short-term liabilities and portfolio choices, but lacks commitment. The lack of commitment implies that runs may occur in equilibrium.[1] The regulator uses these policy instruments to maximize welfare at all times, given the situation at hand. Specifically, the regulator will choose to reduce banks' investment in the long-term asset, which mitigates the fire sale externality.

We study how such an intervention affects financial fragility, measured by the probability of a self-fulfilling bank run. Endogenizing the probability of a self-fulfilling bank run in this framework is nontrivial because there are often multiple equilibria, as in the canonical Diamond and Dybvig (1983) model. In such cases, we follow a robust-control approach: we calculate the maximum probability of a run consistent with equilibrium and measure welfare under this "worst-case" scenario.[2] We show that the resulting probability of a run is well defined and varies depending on the policy tools available to the regulator.

Our main contribution is a novel mechanism through which policies that aim to mitigate fire sale externalities can increase financial fragility. The policymaker requires banks to hold fewer long-term assets, which does mitigate the externality and raise the market value of these assets in a crisis. By itself, this effect would tend to decrease the probability of a run. However, as banks hold a more liquid asset portfolio, the policymaker also finds it optimal to allow banks to issue more short-term liabilities. Having more short-term liabilities, by itself, tends to *increase* the probability of a run. The impact of policy on the probability of a run thus depends on the relative size of two competing effects. We show that when market liquidity is high, meaning outside investors have significant funds available, the second effect dominates and policy intervention increases the maximum probability of a bank run. This fragility effect undermines the benefit of mitigating fire sales and, in some cases, leads to lower welfare. When outside investors have fewer funds, in contrast, policy intervention tends to lower the probability of a run. In these cases, intervention always raises welfare.

[1] The assumption of non-commitment prevents banks and the regulator from costlessly eliminating bank runs by using an off-equilibrium threat such as suspension of convertibility, which is time-inconsistent. See Ennis and Keister (2009, 2010).

[2] This approach follows Li (2017) and Izumi (2021).

Our results show how a version of the classic time-inconsistency problem can arise with policies that aim to mitigate fire sales. In some cases, it would be optimal for the policymaker to commit not to regulate banks at all. Given the ability to regulate, however, the policymaker will want to intervene in an attempt to mitigate fire sales. However, the policy choices can end up giving depositors a stronger incentive to withdraw early and thereby increase the probability of a bank run under the robust-control approach.

The reason market liquidity plays a crucial role in our results is that it determines whether or not banks hold *excess liquidity* in equilibrium. When market liquidity is high, the fire sale discount on assets will be small in the event of a crisis and banks will choose to hold only enough short-term assets to cover the withdrawals that occur in normal times. In this situation, a shift to holding more short-term assets will naturally lead banks to increase its short-term liabilities, which creates the channel through which regulation can increase fragility. When market liquidity is low, in contrast, the equilibrium fire sale discount will be high. Faced with this possibility, banks will choose to hold a precautionary buffer of short-term assets, above what is needed to cover withdrawals in normal times. When this buffer is present, a required change in banks' asset composition will not necessarily lead to a substantial increase in short-term liabilities. We show that, in theses cases, regulation will always decrease the probability of a run and increase welfare.

Related literature: Our paper contributes to the literature studying time-inconsistency problems of policymakers in the spirit of Kydland and Prescott (1977). In the banking context, bailouts have been studied extensively as a typical example of the time-inconsistent problem since the financial crisis of 2007-08. A number of papers reveal how bailing out banks without commitment can worsen welfare, including Farhi and Tirole (2012), Bianchi (2016), Chari and Kehoe (2016), Keister (2016), and Dávila and Walther (2020). These papers study "moral hazard" problems associated with time-inconsistent bailout policies. Our paper is different: We study pecuniary externalities associated with fire sales, and show how a version of the time-inconsistency problem can also arise in this context.

We study consequences of the time-inconsistency problem in the growing literature on fire sale externalities and policy interventions. Lorenzoni (2008), Gale and Gottardi (2015), He and Kondor (2016) and Dávila and Korinek (2018) show that an equilibrium can have either over-investment or under-investment in the presence of fire sale externalities. Gale and Yorulmazer (2020) and Acharya, Shin and Yorulmazer (2011) show how fire sale externalities distort banks' portfolio choices. To correct such externalities, Perotti and Suarez (2011), Walther (2016) and Kara and Ozsoy (2019) discuss optimal designs of capital and liquidity regulations.

Our model follows the sizable literature that builds on the seminal works of Bryant (1980) and Diamond and Dybvig (1983). More specifically, we contribute to the strand of literature that introduces financial markets into this framework and determines asset prices as part of the equilibrium. See, for example, Allen and Gale (2004), Farhi, Golosov and Tsyvinski (2009), and Eisenbach and Phelan (2021). These models feature inefficient portfolio choices in equilibrium associated with pecuniary externalities in financial markets.

We make two contributions to this literature. First, we provide precise conditions under which a policy that aims to correct fire sale externalities is ineffective when the policymaker lacks commitment. This differs from the existing literature which assumes the commitment of policymakers and concludes that regulations are always effective in improving welfare or, at least, do not worsen the situation. Our paper shows that when the policymaker lacks commitment, their results do not always hold. Specifically, when the policymaker lacks commitment, regulation does improve welfare when market liquidity is low, but actually unintentionally worsens welfare when market liquidity is high.

Our second contribution is to endogenize the probability of a self-fulfilling bank run in this framework by focusing on the maximum probability of a run that is consistent with equilibrium. Cooper and Ross (1998), Ennis and Keister (2006) and Ennis and Keister (2010) introduce a positive probability of bank runs into the Diamond-Dybvig framework. Li (2017) and Izumi (2021) evaluate their equilibrium paths using the robust control approach in the spirit of Hansen and Sargent (2001), and use the maximum equilibrium probability as a measure of financial fragility. The degree of fragility is, thus, determined in equilibrium and depends on parameters and policy interventions. Our measurement of fragility follow this approach, and we show that mitigating the externality can increase fragility.

Another approach to endogenize the probability of a bank run is based on global games, as in Goldstein and Pauzner (2005). Within this framework, Eisenbach (2017) studies general equilibrium determination of liquidation values of investments. Ahnert (2016) shows liquidity regulations mitigate fire sale externalities that can occur in this type of environment. Unlike the global-games approach, our model does not place ad hoc restrictions on the banking contract or on regulators. When it becomes clear that a run is underway, banks and policy makers respond by rescheduling payouts from the banking system in a way that resembles policy responses in practice (Ennis and Keister (2009, 2010)). Our assumptions about commitment are also consistent across policies: regulation is subject to the same limited commitment friction as is commonly assumed for bailout policy. We show how to endogenize the probability of a run within this framework and study how this probability responds to policy choices.

The rest of the paper is organized as follows: Section 2 introduces the model environment. Section 3 derives the equilibrium allocation and asset price to find the equilibrium condition for a bank run. Section 4 studies comparative statics of the level of market liqudity, which will be useful in illustrating the impact of interventions. We study policies that aim to correct the externalities and their impacts on fragility in Section 5.

2 The model

Our analysis is based on a version of Diamond and Dybvig (1983) augmented to include the limited commitment features of Ennis and Keister (2009), and to have financial markets as in Allen and Gale (1998). This section describes the model environment including agents, technologies, and markets, and then defines financial fragility in this environment.

2.1 The environment

We consider an economy with three periods indexed by $t = 0, 1, 2$. The economy is populated by a $[0, 1]$ continuum of ex ante identical depositors, indexed by i. We suppose that each depositor has preferences of the following CRRA form:

$$u(c_1, c_2; \omega_i) = \frac{(c_1 + \omega_i c_2)^{1-\gamma}}{1-\gamma},$$

where c_t represents consumption of a single good in period t and the coefficient of relative risk-aversion γ is assumed to be greater than one. The parameter ω_i is a binomial random variable with support $\Omega \equiv \{0, 1\}$ and represents a depositor's type. If $\omega_i = 1$, depositor i is patient, while she is impatient if $\omega_i = 0$. A depositor's type is revealed in period 1 and privately observed by each depositor. Each depositor is chosen to be impatient with a known probability $\pi \in (0, 1)$, and the fraction of impatient depositors in each location is equal to π.

Technologies: In period 0, depositors are each endowed with one unit of goods that can be used for consumption or investment. There are two kinds of assets, short-term and long-term, each representing a constant-returns-to-scale investment technology. The short-term asset is represented by a storage technology that allows one unit of the good placed in period t to be converted into one unit of the good in period $t + 1$. The long-term asset is represented by an investment technology that allows one unit of the good in period 0 to be converted into $R > 1$ units of the good in period 2.

Financial market: The long-term asset can be traded at price p in a competitive asset market in period 1, where investors may purchase it subject to the cash-in-the-market constraint.[3] The investors receive endowments w in period 1, which represent the cash-in-the-market. They can use the endowments to purchase the long-term asset when banks sell, and they value consumption at $t = 2$ only. An investor also has an outside option that yields

[3]While Allen and Gale (1998) assume that investors are risk-neutral, we can assume any risk-preferences. This is because there is no risk in terms of asset returns and investors will buy all long-term assets sold in the market.

the return of $R^* > 0$ in period 2.

Financial intermediation: The investment technologies are operated at a central location, where depositors pool and invest resources together in period 0 to insure individual preference risk. This intermediation technology can be interpreted as a financial intermediary or *bank*. In period 0, the banks invest a fraction x of its portfolio into long-term assets and a fraction $(1-x)$ of its portfolio into short-term assets. Depositors do not observe the banks' portfolio choice. In period 1, upon learning her preference type, each depositor chooses either to withdraw her funds in period 1 or to wait until period 2. Those depositors who contact the bank in period 1 arrive one at a time in the order given by their index i. This index is private information and the bank only observes that a depositor has arrived to withdraw. Under this sequential service constraint, as in Wallace (1988, 1990), the bank determines the payment to each withdrawing depositor based on the number of withdrawals that have been made so far. There is no restriction on these payments; the bank can freely choose the amount received by each depositor when she withdraws. Depositors do not observe the bank's payments made to other depositors, but they can infer the chosen values in equilibrium. As in Ennis and Keister (2009, 2010), the bank cannot pre-commit to future actions, which implies that the bank will always serve depositors optimally depending on the current situation. The objective of the bank is to maximize welfare measured by the equal-weighted sum of depositors' expected utilities,

$$\mathcal{W} = \int_0^1 E\left[u\left(c_1(i), c_2(i); \omega_i\right)\right] di.$$

This intermediation technology is operated by a large number of identical intermediaries.

2.2 Financial crises

We follow Peck and Shell (2003) and many others in introducing the probability of bank runs through an extrinsic *sunspot* variable. The economy will be in one of two sunspot states, $s \in S \equiv \{\alpha, \beta\}$ with probabilities $\{1-q, q\}$. Depositors observe the realization of the sunspot variable at the beginning of period 1 and may condition their withdrawal strategies on the sunspot variable, while the banks do not observe the sunspot state and must infer it based on the observed withdrawal behavior. In period 1, each depositor chooses to withdraw either in period 1 or 2 based on the sunspot variable and her preference type:

$$y_i : \Omega \times S \to \{0, 1\},$$

where $y_i = 0$ corresponds to withdrawing at $t = 1$ and $y_i = 1$ corresponds to withdrawing at $t = 2$. Let y denote a profile of withdrawal strategies for all depositors. An impatient depositor will choose to withdraw at date 1 in both states, since she does not value consumption in period 2. A patient depositor may or may not withdraw in period 1, and we say that a bank run occurs if a positive measure of patient depositors withdraws in period 1. Therefore, if withdrawals continue after the first π withdrawals, the banks infer that a run is underway. We assume that the banks then react in a way that the run stops, as in Ennis and Keister (2009) and many others.[4] In view of this discussion, we consider the following *run strategy profile* for a depositor i:

$$\begin{aligned} y_i(\omega_i, \alpha) &= \omega_i \qquad \text{for all } i, \text{ and} \\ y_i(\omega_i, \beta) &= \left\{ \begin{array}{c} 0 \\ \omega_i \end{array} \right\} \text{ for } \left\{ \begin{array}{c} i \le \pi \\ i > \pi \end{array} \right\}. \end{aligned} \tag{1}$$

Under this profile, state α has no run while state β incurs a run: each patient depositor with $i \le \pi$ chooses to withdraw early in state β. The banks do not know the realization of the sunspot variable and cannot initially infer if a run is underway. After a fraction π of depositors has been served, the banks can infer the sunspot state and respond in a way such that the run stops, and all remaining patient depositors wait to withdraw in period 2. The following definition provides the notion of financial fragility that we use in the paper.

[4] Ennis and Keister (2010) show that runs are necessarily partial in an environment where the banks can react by changing the repayment schedule. This assumption can be generalized by introducing a richer space for the sunspot variable in which runs can occur in multiple waves. Having multiple waves of runs, however, does not change the mechanisms we will present and results will remain unchanged qualitatively.

Definition 1. A banking system is said to be *fragile* if the strategy profile (1) is part of an equilibrium; otherwise the banking system is said to be *stable*.

We will conduct much of our analysis by studying the set of values for q that makes the run strategy profile an equilibrium and hence the banking system fragile. If q is small, the banks may offer early repayments that are large enough to incentivize depositors to run on the banks if they anticipate others will do so. In such a case, the run strategy profile can be an equilibrium. If q is sufficiently large, however, the banks become conservative in choosing the early payments. When q is large enough, depositors will no longer have an incentive to run on the banks, and the economy is stable. The next section establishes the set of q that is consistent with fragile banking systems.

2.3 Timeline

The timing of events is summarized in Figure 1. In period 0, depositors place their endowments in the banks, the banks choose the portfolio, and the period ends. This choice of portfolio is unknown to both depositors and investors. At the beginning of period 1, depositor i learns her type ω_i and the realized sunspot state, and can choose either to withdraw in period 1 or wait until period 2. When a depositor makes a choice, she privately knows her position in the order of withdrawals. At the same time, the banks choose a repayment plan and begin redeeming deposits withdrawn by depositors from the front of the line sequentially. To make these early repayments, the banks can use the proceeds from matured short-term assets. After π withdrawals have been made, the banks can infer the realization of the sunspot state by whether an additional withdrawal occurs or not. If a run is underway, the banks' reaction halts further withdrawals by patient depositors. In making these additional repayments, the banks may exhaust short-term assets and sell long-term assets in the financial market. The remaining impatient depositors receive repayments in period 1. In period 2, the long-term asset matures and the banks repay the remaining patient depositors.

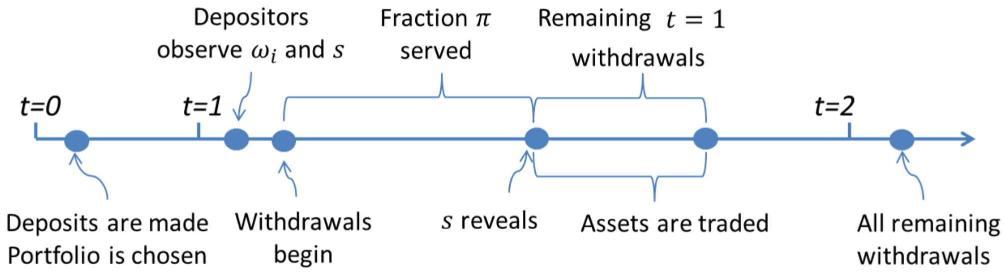


Figure 1: Timeline of the events

2.4 Discussion

We assume depositors choose their withdrawal strategies without directly observing the banks' portfolio choices. In equilibrium, depositors will be able to perfectly infer their bank's choice and, hence, their withdrawal strategies will be a best response to their bank's chosen portfolio. However, our assumptions imply that any deviation from the equilibrium portfolio by the banks (and, later on, the regulator) will not be observed by depositors and thus cannot affect their choices. We could change the model to allow depositors to directly observe their bank's portfolio at the cost of making the model more complex. Assuming they do not simplifies the analysis and creates symmetry between the choice of portfolio and early payments. Moreover, it may be difficult in practice for banks to credibly communicate information about the liquidity profile of their asset portfolio. Dang et al. (2017) point out that the banking industry is inherently opaque, for instance. In such a situation, depositors may be unable to precisely observe changes in the portfolio composition of their financial intermediaries. In addition, Kacperczyk, Sialm and Zheng (2008) report that mutual fund creditors do not observe all actions taken by fund managers despite extensive disclosure requirements.

3 Equilibrium and financial fragility

We begin the analysis by studying equilibrium with no policy intervention as the benchmark, and we will introduce policy interventions in Section 5. Our interest is a withdrawal game in which the banks choose a portfolio and a repayment schedule and depositors choose withdrawal strategies. In this simultaneous-move game, a depositor chooses y_i to maximize her expected utility, and the banks choose a portfolio x and a repayment strategy to maximize the expected utility of depositors. An equilibrium is characterized by the following: each depositor best-responds to the banks' strategies, the banks best-respond to depositors' strategies, and the financial market clears. In this section, we first derive the banks' best-response in their portfolio choice and repayment schedule to the profile of withdrawal strategies in (1), taking the price p as given. We then calculate the market-clearing price to determine the allocation associated with profile (1). We finally verify whether the withdrawal strategy profile is part of an equilibrium and hence whether the banking system is fragile. Throughout the paper, we assume the outside return available to investors, R^*, is equal to the return on the banks' technology, R, which implies that $p \leq 1$ always holds in equilibrium. This assumption does not drive our main results, but allows us to focus on fire sales of assets and simplify the presentation of the mechanism.

3.1 The best-response allocation

Each bank takes one unit of the goods from each of its depositors in period 0 and invests it in a portfolio consisting of long-term assets (x) and short-term assets ($1 - x$). When withdrawals begin at $t = 1$, the banks are initially unable to make any inference about the realization of the sunspot variable and choose to give the same level of consumption c_1 to each withdrawing depositor with $i \leq \pi$. Once π withdrawals have taken place, the banks will be able to infer the sunspot state by observing whether or not withdrawals stop at this point. They will use this information to calculate the fraction of its remaining depositors who are impatient, which we denote as π_s. (Notice that (1) generates $\pi_\alpha = 0$ and $\pi_\beta = \pi$.) All uncertainty has been resolved when π withdrawals have been made, and the banks will give a common amount $c_{1\beta}$ to each of the remaining impatient depositors if a run occurs. In addition, each of the remaining patient depositors will receive a common amount c_{2s} from her bank's remaining resources when she withdraws in period 2. The portfolio and the repayment schedule will be chosen to solve:

$$\max_{\{x,c_1,c_{1\beta},c_{2\alpha},c_{2\beta}\}} \pi u(c_1) + (1-q)(1-\pi)u(c_{2\alpha}) + q(1-\pi)\left[\pi u(c_{1\beta}) + (1-\pi)u(c_{2\beta})\right]. \tag{2}$$

We can simplify the constraint set for this problem by first noting that it will never be optimal for the banks to sell any of the long-term assets in state α. In such a case, the banks could provide more consumption to all depositors by holding more short-term assets and fewer long-term assets. Similarly, the assumption $R > 1$ implies that it will never be optimal for the banks to hold short-term assets over two periods in state β. The banks may, however, hold short-term assets until $t = 2$ in state α, and they may choose to pay additional period-1 withdrawals by selling the long-term assets in state β. Thus, we can write the banks' resource constraints as follows:

$$\pi c_1 \leq (1-x), \tag{3}$$

$$(1-\pi)c_{2\alpha} = Rx + (1-x-\pi c_1), \tag{4}$$

$$(1-x-\pi c_1) \leq (1-\pi)\pi c_{1\beta}, \tag{5}$$

$$(1-\pi)^2 c_{2\beta} = R\left\{x - \tfrac{1}{p}[(1-\pi)\pi c_{1\beta} - (1-x-\pi c_1)]\right\}. \tag{6}$$

The first constraint says that the consumption of the first π depositors to withdraw will always come from the proceeds of the short-term assets. This constraint may or may not hold with equality at the solution. The second constraint says that in state α, the remaining patient depositors will consume all of the remaining resources. The third constraint reflects the fact that additional period-1 payments may come from selling the long-term assets, since all of the short-term assets have already been depleted. The last constraint is the standard pro rata division of remaining resources that determines the payment in period 2. Letting $\mu_1, \mu_{2\alpha}, \mu_{1\beta},$ and $\mu_{2\beta}$ be the Lagrangian multipliers on the above resource constraints, the solution to the problem is characterized by the first order conditions with respect to $(x, c_1, c_{1\beta}, c_{2\alpha}, c_{2\beta})$:

$$-\mu_1 + (R-1)\mu_{2\alpha} + \mu_{1\beta} + (R - R/p)\mu_{2\beta} = 0, \tag{7}$$

$$\pi u'(c_1) - \pi\mu_1 - \pi\mu_{2\alpha} + \pi\mu_{1\beta} - \pi(R/p)\mu_{2\beta} = 0, \tag{8}$$

$$q(1-\pi)\pi u'(c_{1\beta}) + (1-\pi)\pi\mu_{1\beta} - (1-\pi)\pi(R/p)\mu_{2\beta} = 0, \tag{9}$$

$$(1-q)(1-\pi)u'(c_{2\alpha}) - (1-\pi)\mu_{2\alpha} = 0, \tag{10}$$

$$q(1-\pi)^2 u'(c_{2\beta}) - (1-\pi)^2\mu_{2\beta} = 0. \tag{11}$$

These first-order conditions imply that $c_1 < c_{2\alpha}$ and $c_{1\beta} < c_{2\beta}$ always hold.[5]

The solution to the problem characterizes the banks' *best-response* allocation to profile (1) given (p, q). By expressing $(c^*_{2\alpha}, c^*_{1\beta}, c^*_{2\beta})$ as functions (x, c_1), we summarize the best-response allocation by the vector $\mathcal{A}(p, q) \equiv \{x^*, c^*_1\}$.[6] This allocation will involve either (I) no excess liquidity or (II) excess liquidity, where "excess" just means that the banks hold more short-term assets than necessary to pay the first π withdrawals.[7] In Case I, the equality in Equation (3) binds and the banks do not hold more excess liquidity than necessary to pay πc_1. In such a case, the banks will sell the long-term assets to provide additional period-1 payments if a run occurs. In Case II, the inequality in Equation (3) holds and the banks still have liquid assets after paying the first π depositors. If a run occurs, the banks can deplete the proceeds of the remaining short-term assets before it sells long-term assets. If the asset price is low, excess liquidity is a useful option to prepare for runs. For notational convenience, we define the following threshold value of q at which the banks' solution switches from Case I to Case II.

$$q_l = \left\{1 + \frac{R}{R-1}\left(\frac{1}{p} - 1\right)\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{\gamma}\right\}^{-1}.$$

[5]The proof is given in Appendix A

[6]The explicit derivation of this allocation is given in Appendix A.

[7]This statement is not about efficiency. In fact, we will show that, in some economies, the efficient allocation involves excess liquidity. Specifically, when a run is very likely to occur, it can be efficient to hold excess liquidity to prepare for a large redemption.

The best-response allocation, then, has the following property:

Lemma 1. The banks' best-response to profile (1) lies in Case $\begin{Bmatrix} \text{I} \\ \text{II} \end{Bmatrix}$ if $\begin{Bmatrix} 0 < q < q_l \\ q_l \leq q < 1 \end{Bmatrix}$.

The intuition for the above result is as follows: If a crisis is very unlikely, holding excess liquidity is too costly because the banks have to give up some long-term assets and thus opportunities to earn more in state α. In such a case, it is optimal to hold short-term assets only for the purpose of paying πc_1. If a run occurs, the banks sell the long-term assets. As the probability of a crisis increases, the banks will eventually choose to hold excess liquidity. Having more short-term assets lowers the losses from selling long-term assets and thus leaves the banks with more resources in the event of a run.

3.2 Market-clearing price

The price of long-term assets p follows the cash-in-the-market pricing. The demand for long-term assets comes from investors: they purchase long-term assets from the banks subject to the cash-in-the-market as long as $p \leq 1$. The supply of long-term assets comes from the banks that liquidate some of their long-term assets in order to repay additional withdrawal requests in times of crisis. The bank knows exactly how many additional withdrawal requests will come after the π withdrawals, and thus sells the assets all at once. If the banks hold excess liquidity, the banks can pay part of the remaining period-1 payments without liquidating long-term assets. Then, the asset supply measured in units of goods at $t = 1$ is expressed by $L = \pi(1-\pi)c_{1\beta} - (1 - x - \pi c_1)$, where $(1 - x - \pi c_1) = 0$ in Case I and $(1 - x - \pi c_1) > 0$ in Case II. Thus, the market clears all at once, and the market-clearing condition is expressed by

$$\frac{w}{p} = \frac{\pi(1-\pi)c_{1\beta} - (1 - x - \pi c_1)}{p}, \tag{12}$$

where $(x, c_1, c_{1\beta})$ are the functions of the market-clearing price. The patterns of the banks' asset supply are described by the following result and Figure 2.

Lemma 2. L is strictly increasing in p in both Cases I and II.

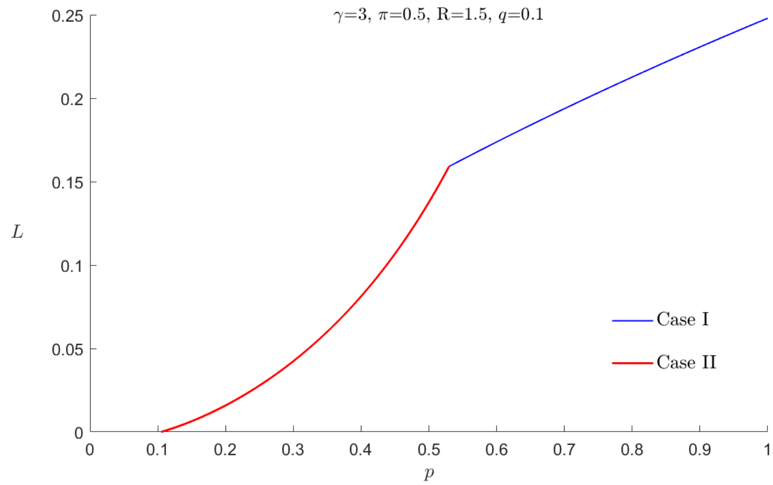


Figure 2: The asset supply measured in units of goods at $t = 1$ over price

The market-clearing condition implies that the equilibrium price is affected by the banks' choice of repayment schedule. As we pointed out in Lemma 1, the best-response of the banks to (1) involves holding excess liquidity and/or fire sales depending on the probability of a bank run. The market-clearing price p^* is, therefore, a function of the probability of a bank run.

Lemma 3. The market-clearing price p^* is strictly $\begin{Bmatrix} \text{decreasing} \\ \text{increasing} \end{Bmatrix}$ in q if $q \begin{Bmatrix} < \\ \geq \end{Bmatrix} q_l$.

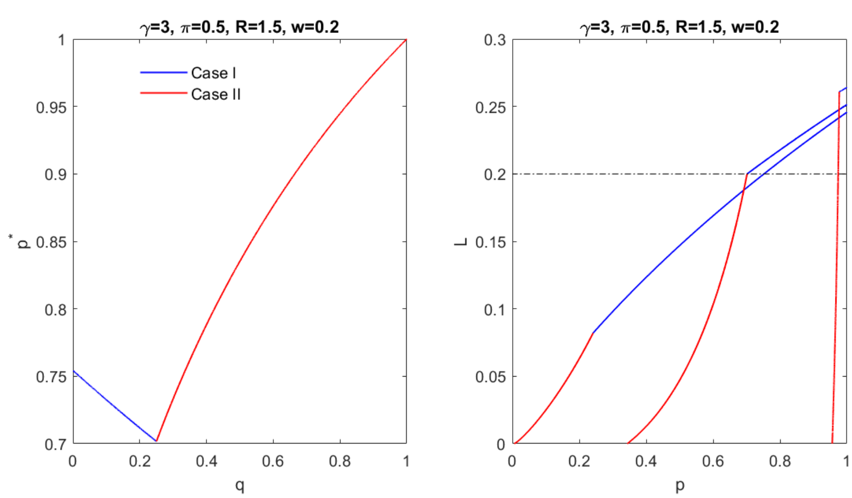


Figure 3: The impact of q on the market-clearing price p^*

Recall that when the solution lies in Case I, the banks have no excess liquidity. In this case, the banks need to sell long-term assets to repay the additional withdrawals in period 1 if a run occurs. Thus, the banks are expected to sell more long-term assets as a crisis becomes more likely. When the solution lies in Case II, the banks become conservative and hold excess liquidity to repay the extra withdrawals in times of a run. In this case, the banks sell fewer long-term assets and hold even more excess liquidity as q increases. As a result, the price is increasing over q.

3.3 Equilibrium bank runs

We now verify whether the strategy profile in (1) is part of an equilibrium, and hence whether the banking system is fragile. The (indirect) expected utility of depositor i can be defined as

$$v_i(x, c_1, y, p) = \mathbb{E}u(c_1, c_2; \omega_i), \tag{13}$$

where the expectation operator is over ω_i. An equilibrium features that depositors and banks are best-responding to each other with the market-clearing price, which is defined as follows:

Definition 2. An *equilibrium without regulation* is profile of strategies (x^*, c_1^*, y^*) and a price p^* such that

1. $v_i(x^*, c_1^*, (y_i^*(s), y_{-i}^*(s)), p^*) \geq v_i(x^*, c^*, (y_i(s), y_{-i}^*(s)), p^*)$ for all s, for all y_i, for all i,
2. $(x^*, c_1^*) \in \mathcal{A}(p, q)$,
3. p^* satisfies the market clearing condition (12).

Recall that an impatient depositor will always strictly prefer to withdraw early, regardless of whatever payment she receives, since she values period-1 consumption only. Therefore, we only need to consider the actions of patient depositors. After π withdrawals, the banks infer if a run is underway, and are able to implement the first-best continuation allocation. Thus, a patient depositor with $i > \pi$ prefers to wait in state β. For patient depositors with $i \leq \pi$, the banks must consider each of the two possible sunspot states separately. In state α, a patient depositor will strictly prefer to wait as specified in 1. In state β, a patient depositor with $i \leq \pi$ receives c_1 if she joins the run and $c_{2\beta}$ if she leaves her funds in the banking system. The discussion above establishes that the profile (1) emerges as part of an equilibrium if and only if the allocation satisfies $c_1^* \geq c_{2\beta}^*$.

We define $\bar{q}$ as the maximum probability in which a run can occur in equilibrium. Following Li (2017) and Izumi (2021), we use this $\bar{q}$ as the measure of financial fragility.

Definition 3. Given (γ, π, w, R), let $\bar{q}$ be the maximum value of q such that $c_1^* \geq c_{2\beta}^*$ holds. If $c_1^* \geq c_{2\beta}^*$ does not hold for any value of q, then define $\bar{q} = 0$.

If the probability of a run exceeds this cutoff value, the banks will become sufficiently cautious that a run is no longer an equilibrium behavior for depositors. This cutoff value provides a natural measure of financial fragility; if a change in parameter values decreases the maximum probability of a run equilibrium, it makes the banking system less fragile and more stable. In the next section, we study comparative statics of the $\bar{q}$ with no policy intervention as the preliminary step to explore how interventions affect the $\bar{q}$.

3.4 Discussion

The financial market is one of the central pieces of our framework, and we made the assumption ($R^* = R$) to focus on the case where the market-clearing price is $p \leq 1$. This could be generalized by allowing the price to be above 1, which requires one to consider the case in which the banks invest only or mostly in long-term assets. Such a generalization does not undermine our findings in $p < 1$, but focusing on this range allows us to demonstrate our mechanisms and results in the clearest way. This type of assumption in which the long-term asset pays less than or equally to the short-term asset in period 1 is a standard approach in the literature.[8]

The equilibrium price involves an externality, which we call a *fire sale externality*. The

[8]See, for example, Diamond and Dybvig (1983), Cooper and Ross (1998) and many others.

asset supplies are characterized by the banks' best-response allocation $\mathcal{A}$. Since we suppose the competitive banking sector, banks choose the allocation taking the price as given. At the equilibrium, all banks make the same choice, and hence the equilibrium price is dependent on this choice. As a result, the banks over-invest in long-term assets and sell more long-term assets in the financial market than the efficient level, and the price drops further. We will discuss this inefficiency further in Section 5.

4 Market liquidity and financial fragility

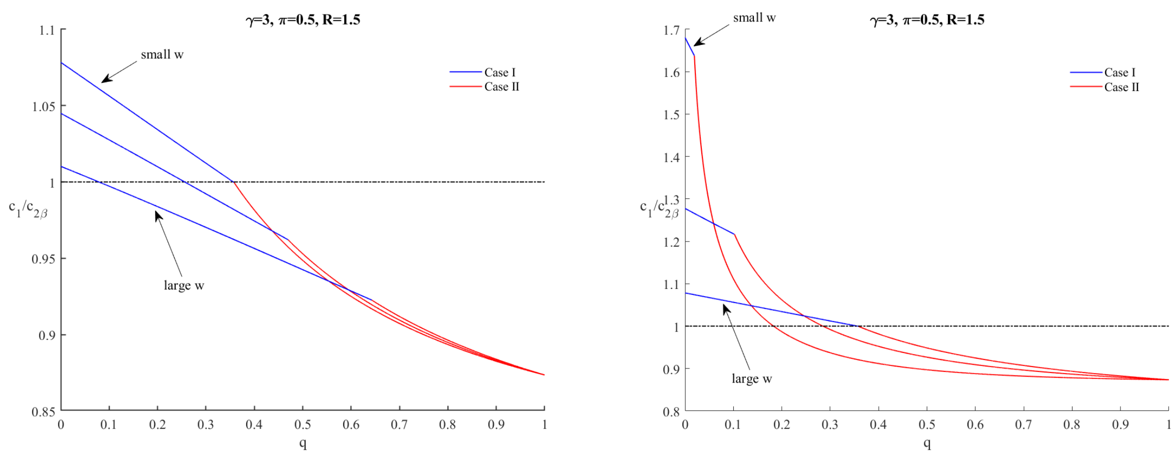
In analyzing how interventions affect fragility, it is first useful to explore fragility in the equilibrium with no policy intervention. Specifically, this section explores the comparative statics of fragility ($\bar{q}$) over market liquidity (w). Later in Section 5, we will study how the introduction of policy interventions affects this result, which is useful in illustrating the effect of interventions.

An important channel linking market liquidity and fragility is the banks' holding of excess liquidity. Lemma 2 implies that the market-clearing price is monotonically increasing over w. When w is small, the market-clearing price is also low, and the banks choose to hold excess liquidity to leave more resources for a time of crisis (Case II). As w increases, the market-clearing price rises, and the banks choose to hold less excess liquidity. When w reaches a particular level, the market-clearing price rises enough and the banks hold no excess liquidity (Case I). Then, an increase in w has no impact on the magnitude of the banks' excess liquidity because the banks hold no excess liquidity in the first place.

The determination of $\bar{q}$ depends on the ratio of $\frac{c_1}{c_{2\beta}}$ as in Definition 3. When w increases, the banks can sell the asset at a higher price, and adjust the repayment schedule. If the banks respond by raising c_1 more than $c_{2\beta}$, the value of q that sustains $c_1^* \geq c_{2\beta}^*$ increases: the $\bar{q}$ rises. That is, when a run is more likely to occur, but also market liquidity is larger, the banks will still pay $c_1^* \geq c_{2\beta}^*$. Similarly, if the banks raise c_1 so that it is less than $c_{2\beta}$, the $\bar{q}$ declines. How the banks change these consumption variables depends on whether the banks hold excess liquidity or not:

Lemma 4. $c_1/c_{2\beta}$ is monotonically $\begin{pmatrix} \text{decreasing} \\ \text{increasing} \end{pmatrix}$ in w if the solution lies in $\begin{pmatrix} \text{Case I} \\ \text{Case II} \end{pmatrix}$.

This result is depicted in Figure 4, which shows how the banks change $\frac{c_1}{c_{2\beta}}$ in response to a change in w and q. In this figure, $\frac{c_1}{c_{2\beta}}$ is monotonically decreasing over q: the banks become conservative and pay less c_1 as q increases. On the other hand, the banks' response to an increase in w depends on whether the banks hold excess liquidity or not.



(a) $\bar{q}$ lies in Case I

(b) $\bar{q}$ lies in Case II

Figure 4: The impact of w on $c_1/c_{2\beta}$

To better understand the banks' response, we decompose the ratio further into two parts as: $c_1/c_{2\beta} = c_1/c_{2\alpha} \times c_{2\alpha}/c_{2\beta}$. Then, the following relationship holds:

Lemma 5. $c_1/c_{2\alpha}$ is monotonically $\begin{pmatrix} \text{increasing} \\ \text{decreasing} \end{pmatrix}$ in w while $c_{2\alpha}/c_{2\beta}$ is monotonically $\begin{pmatrix} \text{decreasing} \\ \text{increasing} \end{pmatrix}$ in w if the solution lies in $\begin{pmatrix} \text{Case I} \\ \text{Case II} \end{pmatrix}$. And then, the effect of $c_{2\alpha}/c_{2\beta}$ is always dominant in determining the change in $c_1/c_{2\beta}$.

In response to an increase in w, the banks raise not only $(c_{1\beta}, c_{2\beta})$ but also c_1 as a substitution over time. The magnitude of this increase depends on whether the banks hold excess liquidity or not. When the solution lies in Case I, the equality in Equation (3) holds: a desire to smooth consumption between c_1 and $(c_{1\beta}, c_{2\beta})$ requires the banks to hold fewer long-term assets, and the banks reduce $c_{2\alpha}$. To avoid further decline in $c_{2\alpha}$, the banks increase c_1 to a lesser extent than $c_{2\beta}$. As a result, $\frac{c_1}{c_{2\beta}}$ is decreasing over w. When the solution lies in Case II, another substitution effect arises: the banks raise x and thus also $c_{2\alpha}$, leaving less excess liquidity. The increase in x increases $c_{2\alpha}$ more than $c_{2\beta}$ because long-term assets are not liquidated in period 1 in state α. This increase in $c_{2\alpha}$ drives the banks to raise c_1 more than $c_{2\beta}$ to better smooth consumption. As a result, $\frac{c_1}{c_{2\beta}}$ is increasing over w. The $\bar{q}$ is, therefore, increasing over w in Case II and decreasing over w in Case I:

Proposition 1. If $\bar{q}$ lies in $\begin{Bmatrix} \text{Case I} \\ \text{Case II} \end{Bmatrix}$, then $\bar{q}$ is monotonically $\begin{Bmatrix} \text{decreasing} \\ \text{increasing} \end{Bmatrix}$ in w.

This result is illustrated in Figure 5, which depicts $\bar{q}$ over w. When w is small, the banks' solution lies in Case II and the $\bar{q}$ is increasing over w. Once w reaches a particular level, the banks' solution lies in Case I and an increase in market liquidity reduces $\bar{q}$. When w is substantially high such that $p = 1$ holds, the $\bar{q}$ becomes inelastic to w. In such a situation, an increase in market liquidity has no effect on the banks' repayment schedule.

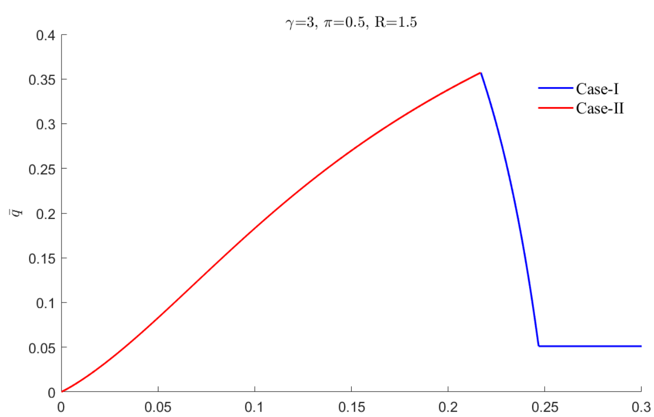


Figure 5: The impact of w on the $\bar{q}$

In the next section, we study how interventions affect the $\bar{q}$ and explore the comparative statics of the $\bar{q}$ with interventions.

5 Policy intervention

We will now consider government interventions that aim to mitigate fire sale externalities. Specifically, we study two types of policy instruments: intervention in short-term payments and intervention in portfolio choice. The regulator maximizes the expected utility of depositors and is able to control (x, c_1) subject to the same financial market as the banks. We suppose that the regulator cannot direct depositors' withdrawal decisions and cannot control the actions of banks after π withdrawals. The banks, therefore, still decide $(c_{2\alpha}, c_{1\beta}, c_{2\beta})$ taking p as given. However, unlike banks, the regulator takes into account the effect of choosing (x, c_1) on the asset price. One example of such an intervention is the liquidity requirements in the revised Basel III accord. These requirements aim to ensure that banks' liquidity holdings are sufficient to meet short-term cash outflows, and this measure corresponds closely to the choices on (x, c_1).[9]

We also suppose that the regulator is subject to the same limited commitment friction as the banks: The regulator cannot credibly promise to use contracts that rule out runs,[10] and the choices of (x, c_1) will be made to maximize welfare taking as given depositors' choice of withdrawal strategy. As in the previous section, depositors do not directly observe the regulators' actions, but correctly anticipate them in equilibrium. The withdrawal game will be, therefore, played by the regulator and depositors, and the difference from the previous section is that (x, c_1) are now chosen by internalizing their effects on the price such that $(x^*, c_1^*) \in \mathcal{A}(p^*, q)$. Our interest is in how such an intervention affects financial fragility $\bar{q}$, which, in turn, affects welfare.

[9] This interpretation is commonly used in the literature to study liquidity regulations. See, for example, Keister (2016) and Li (2020).

[10] The assumption in which policymakers lack commitment is standard in the literature. The classic reference is Kydland and Prescott (1977). In the bank runs context, see Ennis and Keister (2009, 2010) and Keister (2016).

5.1 A regulator's problem

We begin our analysis by studying banks' choices. Given (x, c_1) and p, the banks choose $(c_{2\alpha}, c_{1\beta}, c_{2\beta})$ to solve:

$$V(x, c_1) = \max_{\{c_{2\alpha}, c_{1\beta}, c_{2\beta}\}} (1-q)(1-\pi)u(c_{2\alpha}) + q(1-\pi)\left[\pi u(c_{1\beta}) + (1-\pi)u(c_{2\beta})\right], \tag{14}$$

subject to (4-6). The solution to this problem depends on (x, c_1), and the regulator takes the banks' actions and the price into account while choosing (x, c_1). The regulator solves

$$\max_{\{x, c_1\}} \pi u(c_1) + V(x, c_1), \tag{15}$$

subject to (3). The asset price p will be determined at the market, and the regulator internalizes the effect of choosing (x, c_1) on p. Letting μ_1 be the Lagrangian multiplier for the constraint, the first-order conditions are characterized by

$$\pi\left[u'(c_1) - \mu_1 - (1-q)u'(c_{2\alpha}) - q\frac{R}{p}u'(c_{2\beta})\right] + q\frac{w}{p}\frac{R}{p}u'(c_{2\beta})\frac{\partial p}{\partial c_1} = 0, \tag{16}$$

$$\left[(1-q)(R-1)u'(c_{2\alpha}) - \mu_1 - q\left(\frac{R}{p} - R\right)u'(c_{2\beta})\right] + q\frac{w}{p}\frac{R}{p}u'(c_{2\beta})\frac{\partial p}{\partial x} = 0, \tag{17}$$

with respect to c_1 and x respectively. This regulator's solution (x,c_1) has the following patterns:

Lemma 6. $c_1^* \leq c_{1R}$ and $x^* \geq x_R$ in Case I,

where the subscript R indicates the regulator's solution. The regulator invests less in long-term assets, which indicates that the banks always over-invest in long-term assets in Case I. The regulator knows that holding more long-term assets may decrease the asset price in period 1, and this extra marginal cost of holding more long-term assets drives $x_R \leq x^*$. The regulator, instead, holds more short-term assets and allocates more resources for c_1. As the equality in Equation 3 holds, the regulator chooses $c_{1R} \leq c_1^*$ in Case I. As the first-order conditions include marginal effects on the price, holding long-term assets affects the relative returns $\frac{R}{p}$, on which the marginal rate of substitutions between c_1 and other consumption variables depend, in the regulator's problem. The marginal rate of substitution changes in a way that the banks' c_1 is inefficiently low, and therefore, the regulator moves resources to early payments in internalizing the fire sale externalities. We derive the market-clearing asset price based on the regulator's solution:

Proposition 2. $p^* \leq p_R$ in Case I,

where p_R is the market-clearing price associated with the regulator's solution. Since the regulator leaves less long-term assets, the asset supply decreases and the market-clearing price rises. These mechanisms are common both in Cases I and II and appear clearly in Case I as we have shown, while, in Case II, the holding of excess liquidity creates a complication. When c_1 reaches some level, the regulator desires to leave some resources for later consumption by holding excess liquidity and thus the solution switches to Case II. In such a situation, the regulator moves resources not only to c_1 but also for later consumption in decreasing the holdings of long-term assets. Figure 6 illustrates a numerical example of these results and shows that the regulator's solution involves a higher asset price and higher welfare both in Case I and Case II.

5.2 Mitigating externalities

We can see the sources of the inefficiency in the first-order conditions (7-11). Rewriting these conditions, we can derive

$$\pi \left[u'(c_1) - \mu_1 - (1-q)u'(c_{2\alpha}) - q\frac{R}{p}u'(c_{2\beta}) \right] = 0, \tag{18}$$

$$\left[(1-q)(R-1)u'(c_{2\alpha}) - \mu_1 - q\left(\frac{R}{p} - R\right)u'(c_{2\beta}) \right] = 0. \tag{19}$$

Notice that the only differences from (16-17) are the second terms in each equation. These

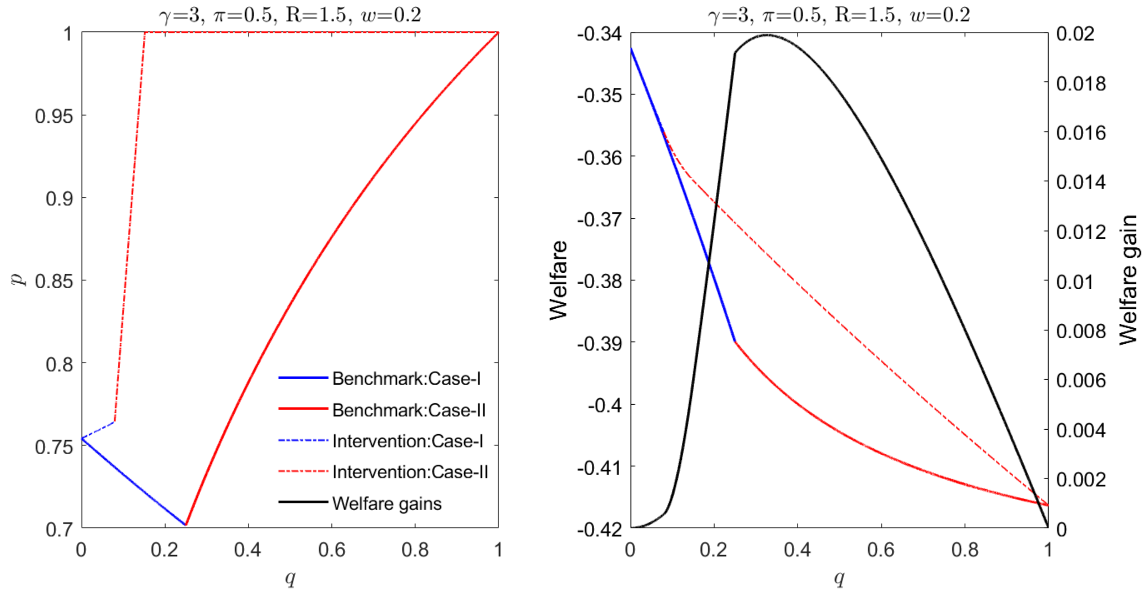


Figure 6: Market-clearing prices, welfare, and welfare gains over q

second terms capture the incentives of internalizing the asset prices. The regulator chooses (x, c_1) by taking into account that these choices change the asset prices. The banks ignore this effect and leave fewer resources for a crisis time. Once a crisis occurs, the banks fire-sell assets and the price drops substantially, affecting other banks as a fire sale externality. The asset price is inefficiently low in the equilibrium without interventions, and the intervention mitigates the fire sale externalities.

A numerical example of welfare gain of policy intervention is captured in Figure 6. When q is zero, there is no benefit of the intervention because fire sales are not anticipated. As q increases, fire sales are more likely to occur and hence the benefit of intervention starts rising. It is perhaps worth noting that the welfare gain diminishes when q is sufficiently large because we assume $p \leq 1$ and the asset prices remain at 1 as q becomes sufficiently large.

5.3 Competing effects on fragility

A bank run can still occur in the regulator's problem because we assume that the regulator cannot direct a depositor's withdrawal decision. Thus, the regulator's choices also depend on q. We study the set of q that satisfies $c_{1,R} \geq c_{2\beta,R}$ and let $\bar{q}_R$ denote the maximum value of q

in this set, as in Section 3. We find that

Proposition 3. $\bar{q}_R \geq \bar{q}$ holds:

- if $\bar{q}_R$ and $\bar{q}$ are both in Case I, and
- for some parameter values with $\bar{q}_R$ and/or $\bar{q}$ in Case II.

This result indicates that the intervention can be associated with a higher degree of financial fragility. When both the banks and the regulator hold no excess liquidity (Case I), the regulator always chooses a higher c_1 and smaller x than the banks. The banks choose c_1 subject to the marginal rates of substitution: (i) c_1 and $c_{2\alpha}$ and (ii) c_1 and $(c_{1\beta}, c_{2\beta})$. Behind this decision, the banks are choosing x but assume that choosing (x, c_1) does not affect the asset price. The regulator considers the same factors as the banks, and additionally, considers the effect of choosing (x, c_1) on the asset price. By reducing the asset supply, the regulator knows that the asset price will go up. This benefit of reducing the asset supply encourages the regulator to hold less x and more c_1 than the banks. This intervention raises the price of long-term assets in period 1 but leaves fewer long-term assets for $(c_{1\beta}, c_{2\beta})$. The increased price partially offsets the fewer remaining resources: it reduces the cost of paying $c_{1\beta}$ but not $c_{2\beta}$ as the return of long-term assets in period 2 is fixed at R. The net effect on $c_{2\beta}$ is a decline. By decreasing x, the regulator instead holds more short-term assets and allocates more resources for c_1. While $c_{2\beta}$ decreases, c_1 rises. As a result, the $\bar{q}$ always increases.

If both the banks and the regulator, or only the regulator, choose to hold excess liquidity, the impact on $\bar{q}$ will depend on parameters.[11] The regulator still chooses a higher c_1 and smaller x than the banks, but the regulator also holds excess liquidity ($\pi c_1 < 1 - x$). Holding excess liquidity mitigates the negative effect on fragility: It provides additional resources to pay $c_{1\beta}$ and reduces the asset supply, which raises the price of long-term assets furthermore. This action leaves more resources for $c_{2\beta}$, and the net effect on $c_{2\beta}$ is now ambiguous. Holding

[11] If both of them choose to hold excess liquidity, the regulator always holds more excess liquidity in internalizing the asset price.

excess liquidity, therefore, makes the change in the $\bar{q}$ ambiguous, and whether the $\bar{q}$ increases or not depends on parameter values.

This result is illustrated in the middle sub-figure in Figure 9, which shows that an intervention involves the higher fragility when w is near 0.2. The banks do not hold any excess liquidity because they think the asset price will be high enough and rather want to use funds for profitable long-term investments. The regulator takes it differently: the equilibrium asset price will be still low enough to hold excess liquidity. This difference comes from the fact that the regulator internalizes the effects of the choices on asset prices. The regulator decreases the asset supply by holding fewer long-term assets and instead holding more short-term assets. The regulator uses some of these extra short-assets for paying more c_1 as the opportunity cost of paying out funds at $t = 1$ is lower when there is excess liquidity. However, the regulator uses the rest of short-term assets to pay out $c_{1\beta}$ in case of runs, which reduces the asset supply and also raises the asset prices, leaving more long-term assets for $c_{2\beta}$. Therefore, the net effect on $c_{2\beta}$ is ambiguous. When w is near 0.2 in this numerical example, excess liquidity is not enough for $c_{2\beta}$ to rise as much as c_1. As a result, the intervention makes the banks more fragile: $\bar{q}_R > \bar{q}$.

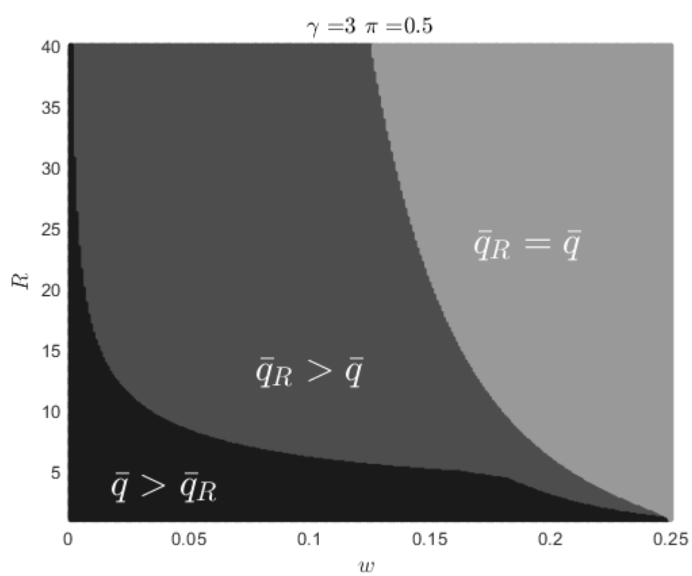


Figure 7: Comparing the sets where $\bar{q} > \bar{q}_R$, $\bar{q}_R > \bar{q}$, and $\bar{q} = \bar{q}_R$

In comparing outcomes across different market liquidity, it will be useful to study the set of economies that are more fragile with or without the intervention. We calculate $(\bar{q}, \bar{q}_R)$ for combinations of the parameters, and identify whether the intervention increases or decreases fragility. Figure 7 depicts the results of this numerical exercise over (w, R). This numerical exercise shows that, when w is small or R is low, the intervention is likely to decrease fragility: $\bar{q}_R < \bar{q}$. It is worth noting that, in such a situation, the intervention involves with holding excess liquidity: The asset price is too low and holding excess liquidity will be useful in paying $c_{1\beta}$. As a result, the regulator raises $c_{2\beta}$ more than raises c_1, rendering $\bar{q}_R < \bar{q}$. However, when w is large or R is high, the intervention involves with a higher fragility: $\bar{q}_R > \bar{q}$. In this region, both the asset price is anticipated to be high and the marginal benefit of holding excess liquidity is small. The intervention accompanies few excess liquidity or even no excess liquidity, decreasing fragility. When w is sufficiently high, the asset price will be at the upper bound of 1 both in the banks' and regulator's solution. The externality, therefore, does not arise and the intervention does not change allocation: $\bar{q}_R = \bar{q}$.

The next proposition characterizes the sufficient condition for $\bar{q}_R > \bar{q}$:

Proposition 4. There exists $\{w_{s,j}\}_{j=1}^{4}$ such that $\bar{q}_R > \bar{q}$ when both $w_{s1} \leq w \leq w_{s2}$ and $w_{s3} < w < w_{s4}$ hold,

where the formulas for the thresholds can be found in Appendix B. Similarly to Lemma 1, we define the threshold $q_{l,R}$ such that the regulator's solution always lies in Case I if $q < q_{l,R}$. In such a case, the regulator views that the probability of runs is small enough to hold no excess liquidity. It is straightforward to show that $q_{l,R} < q_l$ because the regulator internalizes the price and is more likely to hold excess liquidity. We focus on the economies where $p_R = 1$ when $q = q_{l,R}$ for tractability of the analysis, and then the sufficient condition guarantees $\bar{q}_R > \bar{q}$. The conditions correspond to (i) When $w \geq w_{s1}$, $p_R = 1$ always holds at $q = q_{l,R}$, (ii) $w \leq w_{s2}$ establishes $\bar{q}_R > q_{l,R}$, (iii) $w < w_{s3}$ leads to $\bar{q} < q_{l,R}$, and (iv) $w > w_{s4}$ is required for $p^* < 1$ when $q = q_{l,R}$. Perhaps, it is worth emphasizing that, when $w > w_{s4}$, there will be no fire sale externality as $p^* = p_R = 1$, and hence $\bar{q} = \bar{q}_R$. We illustrate these conditions

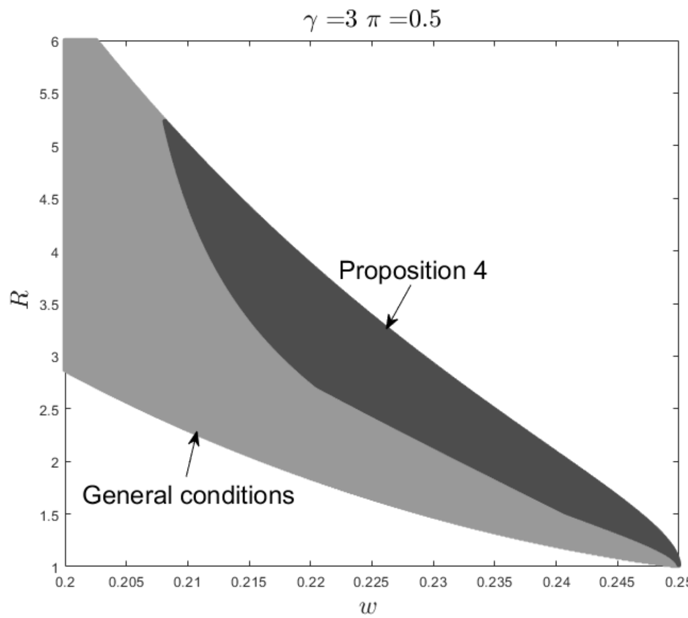


Figure 8: The sufficient conditions for $\bar{q}_R > \bar{q}$

in Figure 8. This figure shows that the economies that satisfy these conditions are a strict subset of the general case in which we examined the set of economies that satisfy $\bar{q}_R > \bar{q}$.

5.4 Welfare implications

This unintended effect on fragility negatively affects welfare evaluated at $\bar{q}_R$. The increased fragility sometimes undermines the benefit of mitigating the fire sale externality, which decreases welfare evaluated at $\bar{q}$.

Proposition 5. Let $\mathcal{W}_R$ denote welfare with the intervention. Then $\mathcal{W}_R(\bar{q}_R) \leq \mathcal{W}(\bar{q})$ holds:

- if $\bar{q}_R$ and $\bar{q}$ are both in Case I, and
- for some parameter values with $\bar{q}$ and/or $\bar{q}_R$ in Case II.

The intervention, thus, does not always improve welfare. In some cases, the intervention mitigates the fire sale externality and also decreases fragility. In other cases, however, the intervention has the unintended consequence of increasing fragility and lowering welfare. This result can be interpreted as a version of the classic time-inconsistency problem as in Kydland and Prescott (1977). If the regulator has commitment, intervention cannot lower welfare. However, it is well known that an intervention without commitment can worsen welfare. Figure 7 and Proposition 4 suggest that the intervention is likely to lower welfare when w is greater. In such a situation, it would be desirable to commit to *not* correcting fire sale externalities. The implication of this analysis is that, in some cases, policymakers should not be given macroprudential tools to mitigate fire sale externalities. If intervention is allowed, the policymaker has discretion in setting the policy tools, which can result in increasing fragility and lowering welfare.

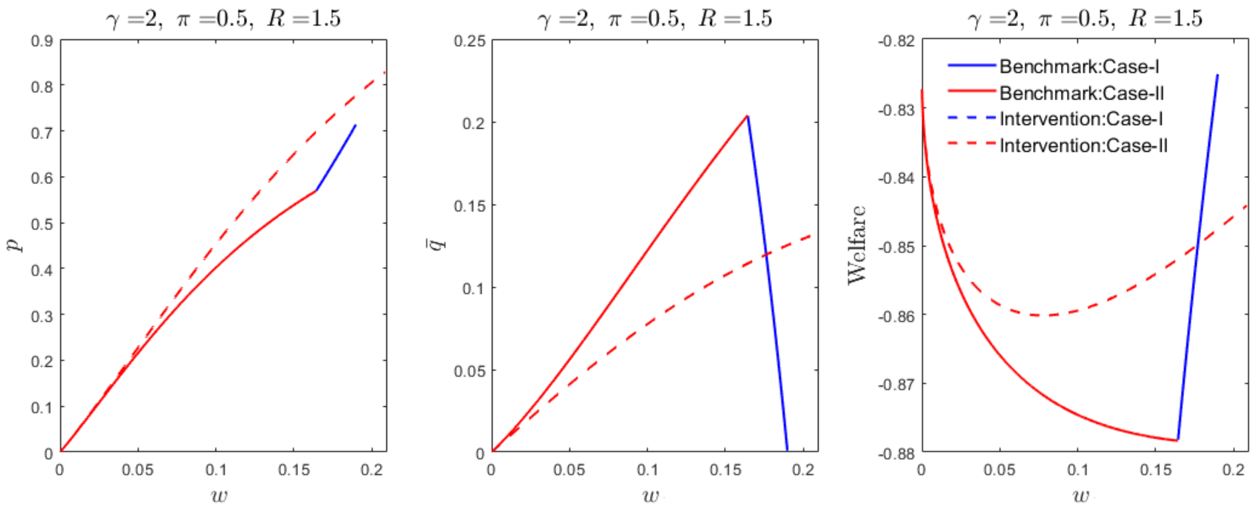


Figure 9: Market-clearing prices, $\bar{q}$, and welfare over w

These results are illustrated in Figure 9, which depicts (i) p^* and p_R^*, (ii) $\bar{q}$ and $\bar{q}_R$, and (iii) $\mathcal{W}(\bar{q})$ and $\mathcal{W}_R(\bar{q_R})$ as functions of w. The equilibrium with no intervention and the equilibrium with intervention have the same pattern: the market-clearing prices increase in both Cases I and II, and the fragility increases in Case II. Welfare evaluated at $\bar{q}$ initially declines as w increases because financial fragility increases. Eventually, welfare begins to increase as an increase in p^* becomes substantial.

5.5 Discussion

The feature that the regulator cannot pre-commit to (x, c_1) is reminiscent of the financial crisis of 2007-8, in which banks were exposed to distress regarding financial assets related to housing for over a year. This made depositors nervous about their funds at banks, and the crisis reached a climax in the fall of 2008 with runs.[12] During such periods of distress, banks may be able to gradually adjust the propotion of liquid assets in their portfolios, and regulators may be able to adjust the requirements they impose on banks in light of the situation. Allowing the regulator to choose both (x, c_1) at the same time when depositors make decisions is suitable for capturing this type of situation.

Some policymakers, including Stein (2012), consider applying these liquidity regulations to some shadow banking arrangements. However, our analysis shows that such an intervention may actually make the financial system fragile. While introducing a restriction on the composition of banks' assets and liabilities indeed reduces asset fire sales, it creates competing effects on depositors' incentive to run. Specifically, requiring banks to hold fewer long-term assets tends to make the banks more stable, but as banks are more liquid, the policymaker also finds it optimal to allow banks to issue more short-term liabilities. This action tends to increase the incentive for depositors to run, which in turn undermines welfare. We show that whether or not the intervention successfully raises welfare depends on the level of market liquidity. Our results suggest that liquidity requirements may mitigate the externality but end up with unintended consequences of increasing fragility and lowering welfare when the market liquidity is high.

6 Conclusion

There has been much discussion of fire-sale externalities since 2008, and policies have been adopted that aim to mitigate fire sales in future periods of financial stress. However, how these policies might affect financial fragility and welfare is not yet well understood. The challenge in studying this question is to determine the ex-ante probability of a crisis. We study how policies that aim to mitigate fire sale externalities affect the probability of a crisis and welfare using a robust-control approach.

Our benchmark model has the following three key features: (i) banks choose its portfolio

[12] For example, Gorton and Metrick (2012) discuss runs on repo in 2007-8.

endogenously, (ii) the price of long-term assets depends on the banks' choices and cash-in-the-market, and (iii) the banks cannot credibly promise to use contracts that rule out runs. The banks choose the portfolio anticipating the possibility of runs and fire sales. The banks do not, however, internalize the effects of their choices on asset prices. As a result, the banks hold more long-term assets than is socially optimal, leading to inefficiently large fire sales in a crisis.

We then studied an intervention that contains two policy instruments in mitigating fire sale externalities: intervention in the short-term payments and intervention in the banks' portfolio. The regulator has the ability to regulate these choices, but lacks commitment. We have found that the impact of such an intervention on fragility is mixed: While the intervention does raise the market value of each long-term asset, it leaves fewer long-term assets for long-term payments. These competing effects render the impact of the intervention on long-term payments ambiguous. In decreasing the holdings of long-term assets, the regulator also allocates more resources toward short-term payments. These effects may or may not raise short-term payments relative to payments in a crisis time. When market liquidity is high, the net effect increases fragility and lowers welfare. On the other hand, when market liquidity is low, the regulator chooses to hold excess liquidity and leave more resources for later payments in the form of excess liquidity. Holding excess liquidity can alleviate a depositor's incentive to run on the banks, and hence, the intervention can decrease fragility when market liquidity is sufficiently low.

The primary takeaway from our analysis is that there may be situations in which it would be desirable to prevent policymakers from intervening to correct fire sale externalities. While mitigating the externality itself is desirable, a policymaker with discretion may end up increasing the incentive to depositors to join in a run, which can have the unintended consequence of increasing fragility and lowering welfare. In such cases, a commitment to not intervene, if feasible, would lead to better outcomes.

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Appendix A Equilibrium preliminaries

In this appendix, we first derive the best response of the banks to the strategy profile (1) given the asset price p and then verify whether the asset market is clear under the competitive equilibrium. Secondly, we drive those conditions under different policy regimes. The expressions derived here are used in the proofs of the propositions given in Appendix B.

A.1 The best-response allocation of the banks under the competitive equilibrium

Given the asset price p, the banks choose $(x, c_1, c_{1\beta}, c_{2\alpha}, c_{2\beta})$ to solve the problem (2) following the timeline described in Figure 1.

$$\pi u(c_1) + (1-q)(1-\pi)u(c_{2\alpha}) + q(1-\pi)[\pi u(c_{1\beta}) + (1-\pi)u(C_{2\beta})]$$

subject to the following resources constraints

$$\begin{aligned}
\pi c_1 &\leq 1 - x, \\
(1-\pi)c_{2\alpha} &= Rx + 1 - x - \pi c_1, \\
(1-\pi)\pi c_{1\beta} &\geq 1 - x - \pi c_1, \\
(1-\pi)^2 c_{2\beta} &= R\{x - \tfrac{1}{p}[(1-\pi)\pi c_{1\beta} - (1 - x - \pi c_1)]\}.
\end{aligned}$$

Let $\mu_1, \mu_{2\alpha}, \mu_{1\beta}, \mu_{2\beta}$ be the Lagrangian multipliers on the above resource constraints, the first order conditions with respect to $(x, c_1, c_{1\beta}, c_{2\alpha}, c_{2\beta})$ are:

$$-\mu_1 + (R-1)\mu_{2\alpha} + \mu_{1\beta} + (R - R/p)\mu_{2\beta} = 0,$$

$$\pi u'(c_1) - \pi\mu_1 - \pi\mu_{2\alpha} + \pi\mu_{1\beta} - \pi(R/p)\mu_{2\beta} = 0,$$

$$q(1-\pi)\pi u'(c_{1\beta}) + (1-\pi)\pi\mu_{1\beta} - (1-\pi)\pi(R/p)\mu_{2\beta} = 0,$$

$$(1-q)(1-\pi)u'(c_{2\alpha}) - (1-\pi)\mu_{2\alpha} = 0,$$

$$q(1-\pi)^2 u'(c_{2\beta}) - (1-\pi)^2\mu_{2\beta} = 0.$$

The allocation will lie in different cases, depending on the value of q given the other parameters (γ, π, R, p). For notational convenience, we define the following threshold value of q at which the banks' solution switches from Case I to Case II.

$$q_l = \left\{1 + \frac{R}{R-1}\left(\frac{1}{p} - 1\right)\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{\gamma}\right\}^{-1}.$$

Case I: If $0 < q < q_l$, then there is no excess liquidity (i.e. $\mu_1 > 0$), fire sale occurs (i.e.

$\mu_{1\beta} = 0$), and the solution is given by:

$$\pi c_1 = 1 - x, \tag{20}$$

$$(1-\pi)c_{2\alpha} = Rx, \tag{21}$$

$$c_{1\beta}/c_{2\beta} = (R/p)^{-1/\gamma} < 1, \tag{22}$$

$$c_{1\beta} = px/[(1-\pi)\pi + (1-\pi)^2(R/p)^{1/\gamma-1}], \tag{23}$$

$$u'(c_1) = \mu_1 + \mu_{2\alpha} + (R/p)\mu_{2\beta} = (1-q)Ru'(c_{2\alpha}) + qRu'(c_{2\beta}) = R\mu_{2\alpha} + R\mu_{2\beta}, \tag{24}$$

$$c_1 = \left(\pi + (1-\pi)\{(1-q)R^{1-\gamma} + q[\pi p^{1/\gamma-1} + (1-\pi)R^{1/\gamma-1}]^{\gamma}\}^{1/\gamma}\right)^{-1}, \tag{25}$$

$$x = \left(1 + \pi/(1-\pi)R\{(1-q)R + qR[\pi(R/p)^{1-1/\gamma} + (1-\pi)]^{\gamma}\}^{-1/\gamma}\right)^{-1}, \tag{26}$$

$$c_1/c_{2\alpha} = \{(1-q)R + qR[\pi(R/p)^{1-1/\gamma} + (1-\pi)]^{\gamma}\}^{-1/\gamma} < 1, \tag{27}$$

$$c_{2\alpha}/c_{2\beta} = \pi(R/p)^{1-1/\gamma} + (1-\pi), \tag{28}$$

$$c_1/c_{2\beta} = \{(1-q)R[\pi(R/p)^{1-1/\gamma} + (1-\pi)]^{-\gamma} + qR\}^{-1/\gamma}. \tag{29}$$

Note that $(1-\pi)\pi c_{1\beta} > 0$ (i.e., $\mu_{1\beta} = 0$) always holds and that $0 < q < q_l$ implies that $\mu_1 > 0$ holds in this case. It is worth emphasizing that $c_{1\beta} < c_{2\beta}$ and $c_1 < c_{2\alpha}$ always hold if the solution lies in Case I. As a result, the equilibrium allocation will be in Case I if $c_1^I \geq c_{2\beta}^I$ holds.

As we discussed in the text, there is a representative investor who purchases the long-term asset from the banks and the banks sell it for obtaining liquidity to meet early withdrawal demand. In equilibrium, the asset market is clear, that is, $L = w$. When the solution lies in Case I, the liquidity obtained by the banks is $L^I = \pi(1-\pi)c_{1\beta}$. Using the equations we derived above, L^I is given by:

$$L^I = \pi(1-\pi)\cdot(p/R)^{1/\gamma}\cdot\{(1-q)R[\pi(R/p)^{1-1/\gamma}+(1-\pi)]^{-\gamma}+qR\}^{1/\gamma}\cdot c_1, \text{ where } c_1 \text{ is given by (9).}$$

It is straightforward to show c_1 is strictly increasing in p, and both the second and third term in the right-hand side of the above equation are strictly increasing in p. Thus, L^I is strictly increasing in p when the solution lies in Case I.

Case II: If $q_l \leq q < 1$, then there is excess liquidity (i.e. $\mu_1 = 0$), fire sale occurs (i.e. $\mu_{1\beta} = 0$), and the solution is given by:

$$(1-\pi)c_{2\alpha} = Rx + 1 - x - \pi c_1, \tag{30}$$

$$c_{1\beta}/c_{2\beta} = (R/p)^{-1/\gamma} < 1, \tag{31}$$

$$c_{1\beta} = px/[(1-\pi)\pi + (1-\pi)^2(R/p)^{1/\gamma-1}], \tag{32}$$

$$u'(c_1) = (1-q)u'(c_{2\alpha}) + qu'(c_{1\beta}) = (1-q)Ru'(c_{2\alpha}) + qRu'(c_{2\beta}), \tag{33}$$

$$\begin{aligned} c_1 = \{&\pi + (1-\pi)[(R-p)/(1-p)]^{1/\gamma-1}(1-q)^{1/\gamma} \\ &+ (1-\pi)[\pi + (1-\pi)(R/p)^{1/\gamma-1}][(R-p)/(R-1)]^{1/\gamma-1}q^{1/\gamma}\}^{-1}, \end{aligned} \tag{34}$$

$$x = \{[\pi + (1-\pi)[(R-p)/(1-p)]^{1/\gamma}(1-q)^{1/\gamma}]c_1 - 1\}/(R-1), \tag{35}$$

$$c_1/c_{2\alpha} = [(1-q)(R-p)/(1-p)]^{-1/\gamma} < 1 \text{ as long as } p \geq p_l, \tag{36}$$

$$c_{2\alpha}/c_{2\beta} = [q/(1-q)\cdot(R/p)\cdot(1-p)/(R-1)]^{-1/\gamma}, \tag{37}$$

$$c_1/c_{2\beta} = [q\cdot(R/p)\cdot(R-p)/(R-1)]^{-1/\gamma}. \tag{38}$$

Note that if $q_l < q < 1$ which implies that $\pi c_1 \leq 1-x$ and $(1-\pi)\pi c_{1\beta} > 1 - x - \pi c_1$ hold, which in turn implies $\mu_1 = 0$ and $\mu_{1\beta} = 0$ (i.e. the banks will sell the long-term asset and hold excess liquidity). It is worth emphasizing that $c_{1\beta} < c_{2\beta}$ and $c_1 < c_{2\alpha}$ always hold if the solution lies in Case II. As a result, the equilibrium allocation will be in Case II if $c_1^{II} \geq c_{2\beta}^{II}$ holds.

When the solution lies in Case II, the liquidity obtained by the banks is $L^{II} = \pi(1-\pi)c_{1\beta} - (1 - x - \pi c_1)$. Using the equations we derived above, L^{II} is given by:

$$L^{II} = \Delta \cdot c_1 - R/(R-1), \text{ where } c_1 \text{ is given by (33) and}$$

$$\Delta = \pi + \pi(1-\pi)[(R-p)/(R-1)]^{1/\gamma}q^{1/\gamma} + [\pi + (1-\pi)[(R-p)/(1-p)]^{1/\gamma}(1-q)^{1/\gamma}]/(R-1).$$

Differentiating c_1 and Δ with respect to p, and the derivative of these expressions are given by:

$$\partial c_1/\partial p \propto (1/q-1)-(1-p)/(R-1)[\pi+(1-\pi)(R/p)^{1/\gamma}]^{\gamma},$$
$$\partial\Delta/\partial p \propto \{1+[\pi+(1-p)^{1+1/\gamma}/(R-1)^{1/\gamma}]^{\gamma}\}^{-1}-q.$$

It is straightforward to show that both $\partial c_1/\partial p$ and $\partial\Delta/\partial p$ are positive as long as $q_l<q<1$. Thus, L^{II} is strictly increasing in p when the solution lies in Case II.

A.2 Policy intervention

We will now assume that the regulator chooses c_1 and x while the banks still choose the remaining payments to its depositors. In this subsection, we derive the best-response of the banks and the policymaker backwards.

Bank's problem. After π withdrawals served, in the good state, the banks choose $c_{2\alpha}$ to maximize

$$\max_{\{c_{2\alpha}\}}(1-q)(1-\pi)u(c_{2\alpha}),$$

subject to

$$(1-\pi)c_{2\alpha}=Rx+1-x-\pi c_1.$$

Combining them, we obtain

$$V_{\alpha}(x,c_1)=(1-q)(1-\pi)u(c_{2\alpha})\text{ where }c_{2\alpha}=\frac{Rx+1-x-\pi c_1}{1-\pi}.$$

In the bad state, the banks choose $c_{1\beta}$ and $c_{2\beta}$ to maximize

$$\max_{\{c_{1\beta},c_{2\beta}\}}q(1-\pi)\left[\pi u(c_{1\beta})+(1-\pi)u(c_{2\beta})\right]$$

subject to

$$(1-\pi)\pi c_{1\beta} \geq 1 - x - \pi c_1$$
$$(1-\pi)^2 c_{2\beta} = R\left\{x - \frac{1}{p}[(1-\pi)\pi c_{1\beta} - (1 - x - \pi c_1)]\right\}$$

Let $\mu_{1\beta}$ and $\mu_{2\beta}$ be the Lagrangian multipliers for the above two constraints, respectively, differentiating with respect to $c_{1\beta}$ and $c_{2\beta}$ yields

$$qu'(c_{1\beta}) + \mu_{1\beta} = q\frac{R}{p}u'(c_{2\beta}).$$

It is straightforward to show that $(1-\pi)\pi c_{1\beta} > 1 - x - \pi c_1$ or $\mu_{1\beta} = 0$ holds, then we have

$$qu'(c_{1\beta}) = q\frac{R}{p}u'(c_{2\beta}) \text{ or } c_{1\beta} = (R/p)^{-1/\gamma}c_{2\beta}.$$

Plugging the above equation into the second constraint, we can obtain

$$(1-\pi)^2 c_{2\beta} = R\left\{x - \frac{1}{p}[(1-\pi)\pi c_{1\beta} - (1 - x - \pi c_1)],\right\}$$

which yields

$$c_{1\beta} = \frac{px + (1 - x - \pi c_1)}{(1-\pi)[\pi + (1-\pi)(R/p)^{1/\gamma - 1}]}. \tag{39}$$

We substitute this formula into the market-clearing condition:

$$\frac{w}{p} = \frac{(1-\pi)\pi c_{1\beta} - (1 - x - \pi c_1)}{p},$$

and then, the price level p is determined by

$$w = \frac{\pi p x - (1-\pi)(R/p)^{1/\gamma - 1}(1 - x - \pi c_1)}{\pi + (1-\pi)(R/p)^{1/\gamma - 1}}. \tag{40}$$

In addition, the utility level is given by

$$V_\beta(x, c_1) = q(1-\pi)[\pi + (1-\pi)(R/p)^{1/\gamma - 1)}]u(c_{1\beta}),$$

where $c_{1\beta}$ and p are determined by (39) and (40) respectively.

The regulator's problem. The policymaker chooses x and c_1 to maximize

$$\max_{\{x,c_1\}} \pi u(c_1) + V_\alpha + V_\beta$$

subject to

$$\pi c_1 \leq 1 - x$$

Let μ_1 be the Lagrangian multiplier for the constraint, we have the following F.O.Cs:

wrt c_1,

$$\pi\left[u'(c_1) - \mu_1 - (1-q)u'(c_{2\alpha}) - q\frac{R}{p}u'(c_{2\beta})\right] + q\frac{w}{p}\frac{R}{p}u'(c_{2\beta})\frac{\partial p}{\partial c_1} = 0,$$

wrt x,

$$\left[(1-q)(R-1)u'(c_{2\alpha}) - \mu_1 - q\left(\frac{R}{p} - R\right)u'(c_{2\beta})\right] + q\frac{w}{p}\frac{R}{p}u'(c_{2\beta})\frac{\partial p}{\partial x} = 0.$$

Differentiating equation (40) with respect to x and c_1 yields

$$\frac{\partial p}{\partial x} = \frac{\left[\pi p + (1-\pi)(R/p)^{1/\gamma-1}\right] p \left[\pi + (1-\pi)(R/p)^{1/\gamma-1}\right]}{(1-1/\gamma)\pi(1-\pi)(R/p)^{1/\gamma-1}(1-x-\pi c_1) - \left[\pi + 1/\gamma(1-\pi)(R/p)^{1/\gamma-1}\right]\pi p x}$$
$$\frac{\partial p}{\partial c_1} = \frac{\pi(1-\pi)(R/p)^{1/\gamma-1} p \left[\pi + (1-\pi)(R/p)^{1/\gamma-1}\right]}{(1-1/\gamma)\pi(1-\pi)(R/p)^{1/\gamma-1}(1-x-\pi c_1) - \left[\pi + 1/\gamma(1-\pi)(R/p)^{1/\gamma-1}\right]\pi p x}$$

Plugging $\partial p/\partial x$ and $\partial p/\partial c_1$ into the above two F.O.Cs, we have

$$u'(c_1) = \mu_1 + (1-q)u'(c_{2\alpha}) + q\frac{R}{p}u'(c_{2\beta}) \tag{41}$$
$$\cdot \frac{\left[\left(1-\frac{1}{\gamma}\right)\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right](1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}(1-x-\pi c_1) - \left[\pi + \frac{1+\gamma}{\gamma}(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]\pi p x}{\left(1-\frac{1}{\gamma}\right)\pi(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}(1-x-\pi c_1) - \left[\pi + \frac{1}{\gamma}(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]\pi p x}$$

$$\mu_1 = (1-q)(R-1)u'(c_{2\alpha}) + q\frac{R}{p}u'(c_{2\beta}) \tag{42}$$
$$\cdot \frac{\left[\pi + \left(1+\frac{1-p}{\gamma}\right)(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]\pi p x - \left[\left(1-\frac{1-p}{\gamma}\right)\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right](1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}(1-x-\pi c_1)}{\left(1-\frac{1}{\gamma}\right)\pi(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}(1-x-\pi c_1) - \left[\pi + \frac{1}{\gamma}(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]\pi p x}$$

Note that there exist two possible cases depending on whether the constraint $\pi c_1 \leq 1-x$ is binding or not.

Case I. If $\pi c_1 = 1-x$ or $\mu_1 > 0$, then the above two F.O.Cs lead to the following equation that determines the equilibrium price:

$$\begin{aligned}\frac{u'(c_1)}{u'(c_{2\beta})} &= \left(\frac{1-\pi}{\pi R}\left[\pi\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1} + (1-\pi)\right]\left\{\frac{\pi p}{\left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w} - 1\right\}\right)^{-\gamma} \\ &= (1-q)R\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{-\gamma} + qR\frac{(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}{\gamma\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}.\end{aligned} \tag{43}$$

In addition, the solution lies in the case where $(1-\pi)\pi c_{1\beta} > 1-x-\pi c_1 \equiv 0$ and $\mu_1 > 0$ hold, if the following condition is satisfied

$$\frac{u'(c_{2\alpha})}{u'(c_{2\beta})} = \left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{-\gamma} > \frac{q\frac{R}{p}}{(1-q)(R-1)}\frac{\gamma\pi + (\gamma+1-p)(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}{\gamma\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}} \tag{44}$$

In other words, if the above condition (44) holds, then the solution under the regime regulating

x and c_1 lies in Case I: no excess liquidity holds in state α (i.e., $\mu_1 > 0$) and selling long-term assets in state β (i.e., $(1-\pi)\pi c_{1\beta} > 1 - x - \pi c_1 \equiv 0$). Moreover, it is easy to show that $\mu_1 > 0$ implies that

$$\frac{u'(c_1)}{u'(c_{2\alpha})} = (1-q)R + qR\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{\gamma} \frac{(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}{\gamma\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}} > 1$$

always holds, or $c_1 < c_{2\alpha}$ is satisfied. Thus, there exists a run equilibrium when the solution lies in Case I under intervention if

$$\frac{u'(c_1)}{u'(c_{2\beta})} = (1-q)R\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{-\gamma} + qR\frac{(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}{\gamma\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}} \leq 1 \tag{45}$$

Case II. If $\pi c_1 < 1 - x$ or $\mu_1 = 0$, then the above two F.O.Cs characterizes the solution pair of (x, p) as follows:

$$\begin{cases} \dfrac{u'(c_1)}{u'(c_{2\alpha})} = (1-q)\dfrac{(R-p)\pi x + \left[\left(1+(\gamma-1)\frac{R}{p}\right)\pi + \gamma(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}}\right]w}{(1-p)\pi x + \left[\left(1+(\gamma-1)\frac{1}{p}\right)\pi + \gamma\frac{1}{p}(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w} \\[2ex] \dfrac{u'(c_1)}{u'(c_{2\beta})} = \dfrac{qR}{(R-1)}\dfrac{(R-p)\pi x + \left[\left(1+(\gamma-1)\frac{R}{p}\right)\pi + \gamma(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}}\right]w}{\pi p x + (\gamma-1)\pi w} \end{cases} \tag{46}$$

where

$$\frac{c_1}{c_{2\alpha}} = \frac{1-\pi}{\pi}\frac{(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1} + \left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w - \left[\pi p + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]x}{\left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}}\right]px - \left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w}$$

$$\frac{c_1}{c_{2\beta}} = \frac{1-\pi}{\pi}\frac{(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1} + \left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w - \left[\pi p + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]x}{\left(Rx - \frac{R}{p}w\right)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}$$

In addition, the solution lies in the case where $(1-\pi)\pi c_{1\beta} > 1-x-\pi c_1$ and $\pi c_1 < 1-x$ hold, if the following condition is satisfied

$$\pi\left(\frac{R}{p}\right)^{-\frac{1}{\gamma}} + (1-\pi) < \frac{c_{2\alpha}}{c_{2\beta}} < \pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)$$

where

$$\frac{c_{2\alpha}}{c_{2\beta}} = \frac{\left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}}\right]px - \left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w}{\left(Rx - \frac{R}{p}w\right)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}$$

$$\text{or} \tag{47}$$

$$\frac{u'(c_{2\alpha})}{u'(c_{2\beta})} = \frac{qR}{(1-q)(R-1)} \frac{(1-p)\pi x + \left[\left(1+(\gamma-1)\frac{1}{p}\right)\pi + \gamma\frac{1}{p}(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w}{\pi p x + (\gamma-1)\pi w}$$

Notice that as the solution pair (x, p) is feasible the ration $c_{2\alpha}/c_{2\beta}$ given by (47) is always strictly larger than $\pi(R/p)^{-\frac{1}{\gamma}} + (1-\pi)$. Thus, the solution under the regime regulating x and c_1 characterized by (46) lies in Case II: excess liquidity holds in state α (i.e., $\pi c_1 < 1-x$) and selling long-term assets in state β (i.e., $(1-\pi)\pi c_{1\beta} > 1-x-\pi c_1$) if the following condition is satisfied

$$\frac{c_{2\alpha}}{c_{2\beta}} < \pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi) \text{ or } \frac{u'(c_{2\alpha})}{u'(c_{2\beta})} > \left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{-\gamma} \tag{48}$$

It is worth emphasizing that $c_1 < c_{2\alpha}$ always hold if the solution lies in Case II. As a result, the equilibrium allocation will be in Case II if

$$\frac{u'(c_1)}{u'(c_{2\beta})} = \frac{qR}{(R-1)} \frac{(R-p)\pi x + \left[\left(1+(\gamma-1)\frac{R}{p}\right)\pi + \gamma(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}}\right]w}{\pi p x + (\gamma-1)\pi w} \leq 1. \tag{49}$$

Appendix B The thresholds in Proposition 4

The explicit formulas for $\{w_{s,j}\}_{j=1}^{4}$ in the proposition are summarized below:

$$w_{s1} = \frac{\pi(1-\pi)R^{1-\frac{1}{\gamma}}}{\pi R^{1-\frac{1}{\gamma}} \left\{ \frac{\gamma \frac{R}{R-1}\left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]+(1-\pi)}{\gamma \frac{R}{R-1}\left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]^{\gamma+1}+\left[\gamma\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]} \right\}^{-\frac{1}{\gamma}} + (1-\pi)\left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]}$$

$$w_{s2} = \frac{\pi(1-\pi)R^{1-\frac{1}{\gamma}}}{\pi R + (1-\pi)\left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]}$$

$$w_{s3} = \frac{\pi(1-\pi)R^{1-\frac{1}{\gamma}} \left(\frac{1}{\pi}\left\{ \frac{1}{R} - \frac{1}{\gamma}\left(\frac{R-1}{R}\right)^2 \left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]^{-\gamma-1} \left[\gamma\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right] \right\}^{-\frac{1}{\gamma}} - \frac{1-\pi}{\pi} \right)^{\frac{1}{1-\gamma}}}{\pi R^{1-\frac{1}{\gamma}} + (1-\pi)R^{-\frac{1}{\gamma}} \left\{ \frac{1}{R} - \frac{1}{\gamma}\left(\frac{R-1}{R}\right)^2 \left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]^{-\gamma-1} \left[\gamma\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right] \right\}^{-\frac{1}{\gamma}}}$$

$$w_{s4} = \frac{\pi(1-\pi)R^{1-\frac{1}{\gamma}}}{\pi R^{1-\frac{1}{\gamma}} \left\{ \frac{\gamma \frac{R}{R-1}\left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]+\left[\gamma\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]}{\gamma \frac{R}{R-1}\left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]^{\gamma+1}+\left[\gamma\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]} \right\}^{-\frac{1}{\gamma}} + (1-\pi)\left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]}$$

Appendix C Proofs of Propositions

Lemma 1. The banks' best response to profile (1) lies in Case $\begin{Bmatrix} \text{I} \\ \text{II} \end{Bmatrix}$ if $\begin{Bmatrix} 0 < q < q_l \\ q_l \leq q < 1 \end{Bmatrix}$.

Proof. According to Appendix A.1, we have the desired result. ■

Lemma 2. The amount of liquidity obtained by the banks L is strictly increasing in p in both Cases I and II.

Proof. According to Appendix A.1, we have the desired result. ■

Lemma 3. The market-clearing price p^* is strictly $\begin{Bmatrix} \text{decreasing} \\ \text{increasing} \end{Bmatrix}$ in q if $q \begin{Bmatrix} < \\ \geq \end{Bmatrix} q_l$.

Proof. According to Appendix A.1, the price is determined by $w = L^I$ when the solution lies in Case I, that is,

$$w = L^I = \frac{\pi p}{[\pi + (1-\pi)(R/p)^{\frac{1}{\gamma}-1}]} \cdot \frac{1}{1 + \frac{\pi}{1-\pi} R^{1-\frac{1}{\gamma}} \{(1-q) + q[\pi(R/p)^{1-\frac{1}{\gamma}} + (1-\pi)]^{\gamma}\}^{-\frac{1}{\gamma}}}.$$

It is easy to show that L^I is strictly increasing in q since $[\pi(R/p)^{1-\frac{1}{\gamma}} + (1-\pi)]^{\gamma} > 1$ always holds. Combined with the fact that L^I is strictly increasing in p, we have p^* is strictly decreasing in q given the market liquidity w fixed when the solution lies in Case I.

We next focus on Case II, in which the price is determined by

$$w = L^{II} = \frac{(1-\pi)\frac{p}{R-p}\left\{[(1-q)(R-p)/(1-p)]^{\frac{1}{\gamma}} - [\pi + (1-\pi)(R/p)^{\frac{1}{\gamma}}][q(R-p)/(R-1)]^{\frac{1}{\gamma}}\right\}}{\pi + (1-\pi)\left\{[\pi + (1-\pi)(R/p)^{\frac{1}{\gamma}-1}]q^{\frac{1}{\gamma}}[(R-p)/(R-1)]^{\frac{1}{\gamma}-1} + (1-q)^{\frac{1}{\gamma}}[(R-p)/(1-p)]^{\frac{1}{\gamma}-1}\right\}}.$$

After some algebra, we have L^{II} is strictly decreasing in q. Combined with the fact that L^{II} is strictly increasing in p, we have p^* is strictly increasing in q given the market liquidity w fixed when the solution lies in Case II.

■

Proposition 1. If $\bar{q}$ lies in $\begin{Bmatrix} \text{Case I} \\ \text{Case II} \end{Bmatrix}$, then $\bar{q}$ is strictly $\begin{Bmatrix} \text{decreasing} \\ \text{increasing} \end{Bmatrix}$ in w.

Proof. This proof follows Lemma 4.

■

Lemma 4. $c_1/c_{2\beta}$ is monotonically $\begin{pmatrix} \text{decreasing} \\ \text{increasing} \end{pmatrix}$ in w if the solution lies in $\begin{pmatrix} \text{Case I} \\ \text{Case II} \end{pmatrix}$.

Proof. According to Appendix A.1, when the equilibrium is in Case I, differentiating the expression of $c_1/c_{2\beta}$ with respect to p, we have $c_1/c_{2\beta}$ is strictly decreasing in p given other parameters. Similarly, we have $c_1/c_{2\beta}$ is strictly increasing in p when the equilibrium lies in Case II. In addition, it is straightforward to show that the market-clearing price p^* is strictly increasing in w. Recall the definition of $\bar{q}$ that is the maximum probability q such that $c_1/c_{2\beta}$ crosses 1, we then have Lemma 4 as desired.

$\blacksquare$

Lemma 5. $c_1/c_{2\alpha}$ is monotonically $\begin{pmatrix}\text{increasing}\\ \text{decreasing}\end{pmatrix}$ in w while $c_{2\alpha}/c_{2\beta}$ is monotonically $\begin{pmatrix}\text{decreasing}\\ \text{increasing}\end{pmatrix}$ in w if the solution lies in $\begin{pmatrix}\text{Case I}\\ \text{Case II}\end{pmatrix}$. And then, the effect of $c_{2\alpha}/c_{2\beta}$ is always dominant in determining the change in $c_1/c_{2\beta}$.

Proof. Using the best-response allocation from Appendix A.1, we have

$$\begin{aligned}
(c_1/c_{2\alpha})_I &= \{(1-q)R + qR[\pi(R/p)^{1-1/\gamma} + (1-\pi)]^\gamma\}^{-1/\gamma},\\
(c_{2\alpha}/c_{2\beta})_I &= \pi(R/p)^{1-1/\gamma} + (1-\pi),\\
(c_1/c_{2\alpha})_{II} &= [(1-q)(R-p)/(1-p)]^{-1/\gamma},\\
(c_{2\alpha}/c_{2\beta})_{II} &= [q/(1-q)\cdot(R/p)\cdot(1-p)/(R-1)]^{-1/\gamma}.
\end{aligned}$$

Differentiating these expressions with respect to p, we have this proposition as desired.

$\blacksquare$

Lemma 6. $c_1^* \leq c_{1R}^*$ and $x^* \geq x_R^*$ in Case I.

Proof. Based on the derivation in Appendix A, we have $\pi c_1 = 1 - x$ and $\pi p x = [\pi + (1-\pi)(R/p)^{\frac{1}{\gamma}-1}]w$, which implies that c_1 is increasing in p while x is decreasing in p when the solution lies in Case I. Thus, we have the desired result since $p^* \leq p_R^*$ in this case as shown later in Proposition 2.

$\blacksquare$

Proposition 2. $p^* \leq p_R^*$ in Case I.

Proof. According to Appendix A, the solution in Case I under the competitive equilibrium is characterized by

$$\left(\frac{1-\pi}{\pi R}\left\{\frac{\pi p}{\left[\pi+(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w}-1\right\}\right)^{-\gamma}=(1-q)R+qR\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}}+(1-\pi)\right]^{\gamma}$$

The solution of the regulator's problem in Case I is determined by

$$\left(\frac{1-\pi}{\pi R}\left\{\frac{\pi p}{\left[\pi+(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w}-1\right\}\right)^{-\gamma}=(1-q)R+qR\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}}+(1-\pi)\right]^{\gamma}\cdot\frac{(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}{\gamma\pi+(1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}}$$

It is straightforward to show that $p_I^* \leq p_{R,I}^*$ since the right-hand side of the solution equation in the competitive equilibrium RHS_I is higher than that in the regulator's problem $RHS_{R,I}$, and thus we have the desired result.

■

Proposition 3. $\bar{q}_R \geq \bar{q}$ holds:

- if $\bar{q}_R$ and $\bar{q}$ are both in Case I, and
- for some parameter values with $\bar{q}_R$ and/or $\bar{q}$ in Case II.

Proof. Recall that we define $\bar{q}$ is the maximum crisis probability beyond that there is a run equilibrium, i.e., $c_1 \geq c_{2\beta}$, which implies that $\bar{q}_R \geq \bar{q}$ as long as $(c_1/c_{2\beta})_R \geq (c_1/c_{2\beta})$ when the equilibrium lies in Case I under both regimes. In this case, according to Appendix A, the ratio $c_1/c_{2\beta}$ under both regimes is given by

$$\frac{c_1}{c_{2\beta}} = \frac{1-\pi}{\pi R}\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]\left\{\frac{\pi p}{\left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w} - 1\right\}.$$

It is easy to show that this ratio is increasing in p, which in turn yields the first part of the proposition since $p^*_{R,I} \geq p^*_I$ in this case. Finally, the second part of this result can be illustrated in the figure in the text.

■

Proposition 4. There exists $\{w_{s,j}\}_{j=1}^4$ such that $\bar{q}_R > \bar{q}$ when both $w_{s1} \leq w \leq w_{s2}$ and $w_{s3} < w < w_{s4}$ hold.

Proof. Recall that $L_{I,R}(q = q_{l,R}) = w$ determines the price $p_{I,R}(q = q_{l,R})$ at $q = q_{l,R}$ and we have $p_{I,R}(q = q_{l,R}) = 1$ as long as $w \geq w_{s1}$. Recall also that $(c_1/c_{2\beta})_{I,R}$ at $q = q_{l,R}$ with $p_R = 1$ is given by

$$(c_1/c_{2\beta})_{I,R}|_{q=q_{l,R} \text{ and } p_R=1} = \frac{1-\pi}{\pi R}\left[\pi R^{1-\frac{1}{\gamma}} + (1-\pi)\right]\left\{\frac{\pi R^{1-\frac{1}{\gamma}}}{\left[\pi R^{1-\frac{1}{\gamma}} + (1-\pi)\right]w} - 1\right\},$$

which yields $(c_1/c_{2\beta})_{I,R} \geq 1$ at $q = q_{l,R}$ with $p_R = 1$ as long as $w \leq w_{s2}$. Thus, $\bar{q}_R$ is no less than $\hat{q} \equiv \left\{1 + \gamma\frac{R}{R-1}\frac{\left[\pi R^{1-\frac{1}{\gamma}}+(1-\pi)\right]^{\gamma+1}}{\gamma\pi R^{1-\frac{1}{\gamma}}+(1-\pi)}\right\}^{-1}$ according to our definition of $\bar{q}$.

Recall next that $q_{l,R} < q_l$ always holds, which implies that the solution lies in Case-I in the competitive equilibrium at $q = \hat{q}$ defined above. Recall also that the ratio of $(c_1/c_{2\beta})_I$ is given by

$$(u'(c_1)/u'(c_{2\beta}))_I = (1-q)R\left[\pi(R/p)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{-\gamma} + qR,$$

which implies that $(c_1/c_{2\beta})_I < 1$ as long as $(1-q)R\left[\pi(R/p)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{-\gamma} + qR > 1$ at $q = \hat{q}$ with $p_I < 1$. Note that here we focus on the case with $p_I < 1$; otherwise there is no need of regulation (i.e., the regulation regime yields the same result as that in the competitive equilibrium). The above inequality will be satisfied if

$$R\left[\frac{1}{\pi}\left(\frac{\frac{1}{R}-\hat{q}}{1-\hat{q}}\right)^{-\frac{1}{\gamma}}-\frac{1-\pi}{\pi}\right]^{\frac{\gamma}{1-\gamma}}<p_I|_{q=\hat{q}}<1.$$

Using the market-clearing condition $w = L_I|_{q=\hat{q}}$, it is straightforward to show that $c_1 < c_{2\beta}$ at $q = \hat{q}$ with $p_I < 1$ when $w_{s3} < w < w_{s4}$ holds. Thus, $\bar{q}$ will be less than $\hat{q}$ if $c_1 < c_{2\beta}$ still holds when $q > \hat{q}$.

Now, we are going to prove that $c_1 < c_{2\beta}$ holds when $q > \hat{q}$. Note that when the solution lies in Case-II in the competitive equilibrium, the price is determined by

$$w=\frac{\left(\frac{q}{1-q}\frac{\frac{R}{p}-R}{R-1}\right)^{-\frac{1}{\gamma}}-\left[\pi\left(\frac{R}{p}\right)^{-\frac{1}{\gamma}}+(1-\pi)\right]}{\left(\frac{1}{p}-1\right)\left(\frac{q}{1-q}\frac{\frac{R}{p}-R}{R-1}\right)^{-\frac{1}{\gamma}}+\frac{\pi}{1-\pi}\left(\frac{R}{p}-1\right)\left(q\frac{R}{p}\frac{R-p}{R-1}\right)^{-\frac{1}{\gamma}}+\left(1-\frac{1}{R}\right)\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}}+(1-\pi)\right]}\equiv\frac{A}{B}$$

Differentiating both sides with respect to q, we have $(dA/dq)B - (dB/dq)A = 0$, where

$$\begin{aligned}
\frac{dA}{dq} &= -\frac{1}{\gamma}\frac{1}{q(1-q)}\left(\frac{q}{1-q}\frac{\frac{R}{p}-R}{R-1}\right)^{-\frac{1}{\gamma}}+\frac{1}{\gamma}\frac{1}{p(1-p)}\left(\frac{q}{1-q}\frac{\frac{R}{p}-R}{R-1}\right)^{-\frac{1}{\gamma}}\frac{dp}{dq}\\
\frac{dB}{dq} &= -\frac{1}{\gamma}\left[\left(\frac{1}{p}-1\right)\frac{1}{q(1-q)}\left(\frac{q}{1-q}\frac{\frac{R}{p}-R}{R-1}\right)^{-\frac{1}{\gamma}}+\frac{\pi}{1-\pi}\left(\frac{R}{p}-1\right)\frac{1}{q}\left(q\frac{R}{p}\frac{R-p}{R-1}\right)^{-\frac{1}{\gamma}}\right]\\
&\quad-\left(1-\frac{1}{\gamma}\right)\left[\frac{1}{p^2}\left(\frac{q}{1-q}\frac{\frac{R}{p}-R}{R-1}\right)^{-\frac{1}{\gamma}}+\frac{\pi}{1-\pi}\frac{R}{p^2}\left(q\frac{R}{p}\frac{R-p}{R-1}\right)^{-\frac{1}{\gamma}}+\left(1-\frac{1}{R}\right)\frac{1}{p}\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}}\right]\frac{dp}{dq}
\end{aligned}$$

It is easy to show that $(dp/dq) > 0$, which implies that $(dB/dq) < 0$, combined with $(dA/dq)B = (dB/dq)A$, we have $(dA/dq) < 0$. This inequality can be expressed as

$$\frac{dp}{dq}<\frac{p(1-p)}{q(1-q)}.$$

Recall next that $(u'(c_1)/u'(c_{2\beta})) = q(R/p)[(R-p)/(R-1)]$ when the solution lies in Case-II in the competitive equilibrium. Differentiating both sides with respect to q, yields

$$\frac{d}{dq}\left(\frac{u'(c_1)}{u'(c_{2\beta})}\right) = \frac{R}{R-1}\left[\left(\frac{R}{p}-1\right) - q\frac{R}{p^2}\frac{dp}{dq}\right]$$

Combined with $(dp/dq) < [p(1-p)]/[q(1-q)]$, we have

$$\frac{d}{dq}\left(\frac{u'(c_1)}{u'(c_{2\beta})}\right) > \frac{R}{R-1}\left[\left(\frac{R}{p}-1\right) - \left(\frac{R}{p}-R\right)\frac{1}{1-q}\right]$$

In addition, suppose $c_1 \geq c_{2\beta}$ holds when the solution lies in Case-II, which implies that

$$\frac{u'(c_1)}{u'(c_{2\beta})} = q\frac{R}{p}\frac{R-p}{R-1} \text{ or } q \leq \frac{p(R-1)}{R(R-p)}$$

Thus, we have

$$\left(\frac{R}{p}-1\right) - \left(\frac{R}{p}-R\right)\frac{1}{1-q} \geq \frac{(R-1)^2(R-p)}{(R-p)^2+p(1-p)} > 0,$$

which implies that $(c_1/c_{2\beta})$ is strictly decreasing in q when $c_1 \geq c_{2\beta}$ in Case-II.

Finally, when the solution lies in Case-I in the competitive equilibrium, the ratio of $c_1/c_{2\beta}$ is strictly decreasing in q in Case-I. Therefore, the fact that when the solution lies in Case-II $c_1/c_{2\beta}$ is strictly decreasing in q as long as $c_1 \geq c_{2\beta}$ holds in this case, which implies that $c_1/c_{2\beta}$ is expected to across 1 once. This fact implies that once $c_1 < c_{2\beta}$ holds when the solution lies in Case-I then the economy is always stable as the solution lies in Case-II.

In sum, when both $w_{s1} \leq w \leq w_{s2}$ and $w_{s3} < w < w_{s4}$ hold, we have $\bar{q}_R \geq \hat{q}$ and $\bar{q} < \hat{q}$, which gives us the desired result.

■

Proposition 5. Let $\mathcal{W}_R$ denote welfare with the intervention. Then $\mathcal{W}_R(\bar{q}_R) \leq \mathcal{W}(\bar{q})$ holds:

- if $\bar{q}_R$ and $\bar{q}$ are both in Case I, and
- for some parameter values with $\bar{q}$ and/or $\bar{q}_R$ in Case II.

Proof. When the $\bar{q}$ lies in Case I under both regimes, according to Appendix A, the ratio $c_1/c_{2\beta}$ in Case I is given by

$$\frac{c_1}{c_{2\beta}} = \frac{1-\pi}{\pi R}\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]\left\{\frac{\pi p}{\left[\pi + (1-\pi)\left(\frac{R}{p}\right)^{\frac{1}{\gamma}-1}\right]w} - 1\right\}.$$

Recall that we have $c_1/c_{2\beta} = 1$ at $q = \bar{q}$, and thus $c_1/c_{2\beta} = 1$ yields the same price level evaluated at $\bar{q}$ in Case-I under both regimes. This fact implies that the early payment c_1 is the same as well.

Recall also that the welfare level in Case I is given by

$$\begin{aligned} W &= \pi u(c_1) + (1-q)(1-\pi)u(c_2) + q(1-\pi)\left[\pi u(c_{1\beta}) + (1-\pi)u(c_{2\beta})\right] \\ &= \pi u(c_1) + (1-\pi)\left\{(1-q)\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]^{1-\gamma} + q\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]\right\}u(c_{2\beta}). \end{aligned}$$

After some algebra, it is easy to show that the welfare level evaluated at $\bar{q}$ is determined by

$$W = \left(\pi + (1-\pi)\left\{(1-q)\left(\frac{q}{1-q}\frac{R/p - R}{R-1}\right)^{1-\gamma} + q\left[\pi\left(\frac{R}{p}\right)^{1-\frac{1}{\gamma}} + (1-\pi)\right]\right\}\right)u(c_1)$$

which is decreasing in q given p. This fact in turn yields the first part of the proposition since $\bar{q}_R \geq \bar{q}$ holds if both $\bar{q}$ are in Case I. Finally, the second part of this result can be illustrated in the figure in the text.

■