Stability and Substitutability in Dynamic Matching Markets*

Keisuke Bando†

Ryo Kawasaki‡

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Abstract

We analyze a dynamic matching market where matching between agents is decided for each time period. To analyze this situation, we embed the situation into the framework of many-to-many matching with contracts where the contract includes the time period at which the matching occurs. While a general stability concept is already defined for the matching with contracts framework, in a dynamic matching model, a stable outcome may not exist when contracts exhibit complementarities across time periods. Thus, we define a stability concept called temporal stability that is more suitable to the dynamic nature of the model. We provide sufficient conditions for the existence of a temporally stable outcome, including a corresponding substitutability condition, ordered substitutability, for the dynamic matching model.

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†Shinshu University, Matsumoto City, Japan; E-mail: k_bando@shinshu-u.ac.jp

‡Tokyo Institute of Technology, Tokyo, Japan; E-mail: kawasaki.r.aa@m.titech.ac.jp

1 Introduction

The classic problem of the two-sided matching problem, first introduced by Gale and Shapley (1962), involves forming pairs between agents from two disjoint groups. Real-life applications of the theory of two-sided matching include matching medical interns and hospitals, matching schools and students. The key property for matchings that is highlighted in the theory is the concept of stability. A matching is stable if no pair finds it profitable to matching with each other than their current partners. We refer the reader to Roth and Sotomayor (1990) for more details on the theory of two-sided matching markets.

One of the assumptions in the early matching literature is that a matching for a given set of agents is decided once and never reconsidered thereafter. However, there are important matching markets where a matching has to be made across multiple time periods. For example, professional athletes in team sports sign possibly multiple contracts throughout their career. Workers and firms may sign different contracts through wages over time. More specific matching markets have been analyzed in the literature. For example, Kennes et al. (2014) consider a dynamic matching model based on the children's daycare system in Denmark and analyze the performance of the allocation mechanism in Aarhus. Pereyra (2013) constructs a model based on the assignments of overlapping generations of teachers to schools in Mexico. In another market, Dimakopoulos and Heller (2019) consider the matching of entry-level lawyers to regional courts in Germany and analyze the performance of the Berlin matching mechanism.

Aside from research on dynamic matching markets that are primarily motivated by real-life applications, there has been a growing literature on the general theory of dynamic matching markets. The first theoretical paper on dynamic matching is Damiano and Lam (2005), where many core concepts are defined as candidates for a stability concept. Subsequent papers such as Kadam and Kotowski (2018a,b), Kotowski (2019), Kurino (2020), and Doval (2021) also analyze a theoretical model of dynamic matching with each paper introducing a stability concept. These papers consider a one-to-one dynamic matching model, where an agent on one side is matched to at most one agent from the other side for each period. The focus of these papers is on defining a suitable stability concept for their models.

In this paper, we provide a theoretical analysis of many-to-many dynamic matching markets using the matching with contracts model, formalized by Hatfield and Milgrom

(2005) and Hatfield and Kominers (2017).[1] These models incorporate additional contractual terms, such as salary or position, in describing a matching not just based on who matches with whom, but under what terms these agents are matched. Thus, we can embed dynamic matching markets into these models by including the time period in the contract terms. Dimakopoulos and Heller (2019) also utilize the matching with contracts model to include the time period in which a match is made. Because the objective of their research is to investigate a specific matching market, their analysis mainly focuses on a particular matching market, which is a many-to-one matching model, and not on building a general theory of a dynamic matching market using the matching with contracts model.

In the model of matching with contracts, substitutability of preferences plays a crucial role in guaranteeing the existence of a stable outcome as defined in Hatfield and Kominers (2017). On the other hand, the preferences of agents in a dynamic matching market may often exhibit complementarities across time periods. For example, a worker may want to work for a foreign firm only after he has learned a skill in a domestic firm. Such preferences may also cause the nonexistence of a stable outcome in our model.

In this paper, we define a new stability concept called *temporal stability*. Our concept weakens the existing stability concept by imposing the condition that blocking agents cannot change their past contracts. We also define a weaker version of the substitutability condition called *ordered substitutability*, which requires that when a contract x is chosen from some available contracts, it is still chosen even if contracts involving time periods that are later than that of x become unavailable. This condition can include some plausible preferences that admit complementarities across time periods as described above. We show that ordered substitutability guarantees the existence of a temporal stable outcome for many-to-one dynamic matching markets. We also provide another sufficient condition that is applicable to many-to-many dynamic matching markets. Our analysis adds to the dynamic matching literature in proposing new stability and substitutability conditions for the many-to-many version of the model. Our result also adds to the growing literature on dynamic matching markets in that we show how tools from static matching markets can be incorporated to analyze a many-to-many dynamic matching market. Our result also exemplifies a new application of the many-to-many matching with contracts model of Hatfield and Kominers (2017).

[1] Prototypes of the matching with contracts models can be found in Roth (1984) and Fleiner (2003).

The rest of the paper is organized as follows. Section 2 surveys the related literature and also explains the relationship between this paper and those in the literature. Section 3 defines the model of dynamic matching markets of this paper and other several concepts. Section 4 contains our main result. Section 5 contains concluding remarks.

2 Literature

There now is a significant amount of theoretical research on the topic of dynamic matching problems. The first paper that explicitly analyzes a dynamic model of the matching problem is Damiano and Lam (2005). They define several core concepts to the dynamic matching market model, some of which are adapted from other models. However, they assume that preferences of agents are time separable in that they can be represented by a utility function that is additively separable in time. Also, they only consider deviations by a set of agents such that in the new matching that they induce, they are matched together throughout. Under the same preference domain, Kurino (2020) relaxes this matching constraint for a set of deviating agents, but instead imposes a credibility condition on a deviation by a set of agents in defining a new stability concept called credible stability. Doval (2021) considers a dynamic matching model in which matching agreements are irreversible, so that agents essentially choose when to be made available to match. When deciding whether to form a match or not, agents have to make predictions as to who will be made available in later periods so that they do not want to forego their opportunities of possibly matching with a more favorable partner.

The papers in the literature that are most closely related are those by Kadam and Kotowski (2018a,b). Both of these papers do not assume time separable preferences for the agents. Moreover, they define a stability that re-evaluates the stability of a matching at each time period. Similarly, our stability also considers the stability of a matching period-by-period, but there are some differences. Aside from the fact that we allow many-to-many matchings for each period instead of the one-to-one matching model of Kadam and Kotowski (2018a,b) and allow for arbitrary sets of agents to deviate instead of just one pair, the blocking conditions in the stability concepts differ slightly. Thus, there is no logical dependence between their stability condition and our stability condition so that one cannot necessarily say which one is stronger. In addition, Kadam and Kotowski (2018a) investigate the existence problem of their stability concept by looking at preferences in which agents are relatively reluctant to change their partners. We, instead, investigate substitutability conditions, which are closely related to the static matching literature. On a slightly different line of research, Kotowski (2019) defines a new robust concept of stability, called perfect α-stability, which is defined by a backwards induction method. In the final period, the stability concept coincides with the usual stability concept for the static matching problem. Supposing that perfect α-stability is defined for time periods later than period t, a pair of agents deviates at period t only when it is worthwhile even at the worst possible matching among the recursively defined stable matchings for periods $t+1$ and beyond. Kotowski (2019) establishes the existence of a perfectly α-stable matching without imposing any restrictions on the preferences. While universal existence is quite desirable, the assumption of the deviating agents' pessimistic outlook does play a large role. We, instead, define our stability concept without agents having to conjecture the worst possible outcome so that it is more closely related to the traditional stability concept used in the literature as our starting point.

Our study also contributes the literature on matching with contracts. We propose a weak substitutability condition, ordered substitutability, that is defined using the dynamic nature of the model. For the matching with contracts model, the substitutability condition is sufficient for the existence of a stable matching outcome but not necessary. Based on this observation, several papers propose weaker substitutability conditions for the model of many-to-one matching with contracts (See Hatfield and Kojima (2010), Hatfield and Kominers (2016), and Hatfield et al. (2021)). These studies show that stable outcomes are still guaranteed to exist under their weak substitutability conditions. Yenmez (2018) and Bando et al. (2021) apply these conditions to the model of many-to-many matching with contracts and provide sufficient conditions for the existence of stable outcomes. Unlike their studies, we consider a weak stability concept since a stable outcome may not exist in the domain of our problem. Ordered substitutability is introduced to guarantee the existence of a temporal stable outcome and thus is a different concept from the existing weak substitutability conditions. Thus, we offer a new approach to analyzing a matching problem that may not satisfy substitutability.

3 Model

We define our model of dynamic matching using the many-to-many matching with contracts framework, formalized in Hatfield and Kominers (2017). Let D be a finite set of doctors and H be a finite set of hospitals where $D \cap H = \emptyset$. We refer to $I \equiv D \cup H$ as the set of *agents*. Let $T = \{1, \cdots, L\}$ be a finite set of periods. Let X be a finite set of *contracts* where each contract is associated with a doctor $x_D \in D$, a hospital $x_H \in H$, and a period $x_T \in T$. Typically, X is given by $D \times H \times T$ where $(d, h, t) \in D \times H \times T$ means that doctor d matches with hospital h at period t.

For each $Y \subseteq X$, let $Y_D = \{y_D \mid y \in Y\}$ be the set of doctors that are associated with some contract in Y, $Y_H = \{y_H \mid y \in Y\}$ be the set of hospitals that are associated with some contract in Y, and $Y_I = Y_D \cup Y_H$. For each $Y \subseteq X$, let $Y_i = \{y \in Y \mid i \in \{y_D, y_H\}\}$ be the set of contracts that are associated with an agent $i \in I$.

For each $Y \subseteq X$ and $t \in T$, let $Y^t = \{y \in Y \mid y_T = t\}$ be the set of contracts at period t in Y, $Y^{\leq t} = \{y \in Y \mid y_T \leq t\}$ be the set of contracts up to period t in Y, and $Y^{\geq t} = \{y \in Y \mid y_T \geq t\}$ be the set of contracts after period t in Y. We denote $Y^0 \equiv \emptyset$.

Each agent $i \in I$ has a choice function $C^i : 2^{X_i} \to 2^{X_i}$ where $C^i(A') \subseteq A'$ for all $A' \subseteq X_i$. We define $R^i(A') = A' \setminus C^i(A')$ for all $A' \subseteq X_i$, which is called the *rejection function*. For each $A \subseteq X$, we denote $C^i(A) = C^i(A_i)$ and $R^i(A) = R^i(A_i)$. When an agent i has a strict preference ordering $\succ_i$ over 2^{X_i}, $C^i(A')$ is defined as the most preferred contacts in $A' \subseteq X_i$, that is, $C^i(A')$ is a subset of A' that satisfies $C^i(A') \succeq_i \hat{A}$ for all $\hat{A} \subseteq A'$. In this case, C^i is called the *choice function induced from* $\succ_i$. We do not assume a unitarity condition of Kominers (2012) on the choice sets where each doctor-hospital pair can appear in at most one contract in a choice set.[2] In fact, because time is explicitly included as a contract, a doctor-hospital pair would be included in multiple contracts if a doctor is to be hired by the hospital for more than one period of time.

We say that $A \subseteq X$ is an *outcome*. An outcome A is *individually rational* if $C^i(A) = A_i$ for all $i \in I$. An outcome A is *blocked* if there exists a nonempty $Z \subseteq X \setminus A$ such that $Z_i \subseteq C^i(A \cup Z)$ for all $i \in Z_I$. When $|Z| = 1$, we say that A is *pairwise blocked*. We say that A is *stable* if it is individually rational and not blocked. We say that A is *pairwise stable* if it is individually rational and not pairwise blocked.

[2]The unitarity condition is also assumed in Klaus and Walzl (2009) and Bando et al. (2021) for example.

We introduce the following standard conditions in the literature that guarantees the existence of stable outcomes. The first of these conditions, irrelevance of rejected contracts, is introduced by Aygün and Sönmez (2013) and is satisfied by choice functions that are generated by preferences.

Definition 1. *An agent i's choice function C^i satisfies irrelevance of rejected contracts (IRC) if for all $A \subseteq X$ and all $x \in X$, if $x \notin C^i(A)$ and $x \in A_i$, then $C^i(A) = C^i(A \setminus \{x\})$.*[3]

Definition 2. *An agent i's choice function C^i satisfies substitutability (SUB) if for any $A \subseteq X$ and any distinct $x, x' \in C^i(A)$, $x \in C^i(A \setminus \{x'\})$.*

If every agent's choice function satisfies IRC and SUB, then a stable outcome exists as shown by Hatfield and Kominers (2017).[4] Moreover, under the same assumption, a doctor-optimal stable outcome A and hospital optimal-outcome B exist in the sense that for any stable outcome A', $C^d(A \cup A') = A$ for all $d \in D$ and $C^h(B \cup A') = B$ for all $h \in H$.

3.1 Temporal stability

While the results in the literature show that substitutability is the essential condition for the existence of a stable outcome, the condition of substitutability is quite strong, as it prohibits any form of complementary between contracts. Indeed, substitutability may be violated in certain situations including those motivated by real-life situations, and in those situations, a stable outcome may not exist. In the following, we consider a weakened version of stability which we call temporal stability that still satisfies meaningful stability properties and exists even in matching problems with some complementarity.

To illustrate the concepts, consider the following example.

[3] IRC is equivalent to the following condition: for all $A, A' \subseteq X$, $C^i(A) \subseteq A' \subseteq A$ implies $C^i(A) = C^i(A')$. This condition is called consistency by Alkan (2002), which is first introduced by Blair (1988) for a matching model.

[4] To be precise, Hatfield and Kominers (2017) assume that each agent has strict preferences over contracts so that the choice function based on these preferences automatically satisfies IRC. In a framework where choice functions are defined as a primitive, we can state their result equivalently by instead assuming IRC on the choice functions. See Chambers and Yenmez (2017) for example.

Example 1. Let $D = \{d_1, d_2\}$ be the set of two consultants and $H = \{h_1, h_2\}$ be the set of two firms where h_1 is a domestic firm and h_2 is a foreign firm. The set of contracts is given by $D \times H \times T$ where $T = \{1, 2\}$. A domestic firm wants to hire a consultant at period 1 and a foreign firm wants to hire one at period 2. Each firm has a strict preference ordering defined as follows:

$$\succ_{h_1}: \{(d_2, 1)\}, \{(d_1, 1)\}, \emptyset, \qquad \succ_{h_2}: \{(d_1, 2)\}, \{(d_2, 2)\}, \emptyset.$$

The above list denotes, for example, that h_1 ranks $\{(d_2, h_1, 1)\}$ first, $\{(d_1, h_1, 1)\}$ second, and $\emptyset$ third, where h_1 is omitted from contracts in the list. Note also that the contracts ranked below $\emptyset$ are also omitted.

Consultant d_1 wants to work at the foreign firm only after he has worked at the domestic firm and learned a skill. Such preferences can be represented by the following ordering:

$$\succ_{d_1}: \{(h_1, 1), (h_2, 2)\}, \{(h_1, 1)\}, \emptyset.$$

Consultant d_2 wants to work at only one period since he wants to spend one period on a vacation. The preferences of d_2 are given by

$$\succ_{d_2}: \{(h_2, 2)\}, \{(h_1, 1)\}, \emptyset.$$

There is no stable outcome in this example. Specifically, $\{(d_1, h_1, 1), (d_1, h_2, 2)\}$ is blocked via $\{(d_2, h_1, 1)\}$, $\{(d_2, h_1, 1), (d_1, h_2, 2)\}$ is not individually rational for d_1, $\{(d_2, h_1, 1)\}$ is blocked via $\{(d_2, h_2, 2)\}$, $\{(d_2, h_2, 2)\}$ is blocked via $\{(d_1, h_1, 1)\}$, and $\{(d_1, h_1, 1), (d_2, h_2, 1)\}$ is blocked via $\{(d_1, h_2, 2)\}$. The remaining individually rational outcomes ($\emptyset$ and $\{(d_1, h_1, 1)\}$) are also blocked.

In the above example, the source of the nonexistence of a stable outcome is that the preferences of consultant d_1 admit complementarities while those of the other agents satisfy substitutability by the assumption of unit demand, that is, each choice set is a singleton. Indeed, consultant d_1 wants to work at the foreign firm at period 2 only if he has worked at the domestic firm at period 1, violating substitutability. In general, the choice of an agent at certain period may depend on his/her past choice. In such a case, substitutability is often violated, which may cause the nonexistence of a stable outcome. To deal with this problem, we introduce a weaker concept of stability as follows.

Definition 3. *Let $A \subseteq X$ be an outcome.*

- *We say that A is blocked at period $t \in T$ if there exists a nonempty $Z \subseteq X^{\geq t} \setminus A$ such that (i) $Z^t \neq \emptyset$, (ii) $Z_i \subseteq C^i(A \cup Z)$ for all $i \in Z_I$, and (iii) $A_i^{\leq t-1} = C^i(A \cup Z)^{\leq t-1}$ for all $i \in Z_I$.*
- *We say that A is temporally stable if it is individually rational and not blocked at any period.*

First, outcomes from period $t-1$ or earlier cannot be used in blocking that outcome, which is contained in the condition $Z \subseteq X^{\geq t} \setminus A$. For an outcome to be blocked at a period t, it must include at least one contract that pertains to period t, which is condition (i). Moreover, outcomes from period $t-1$ or earlier cannot be reneged, which is what condition (iii) states. Condition (iii) includes a restriction where contracts from period $t-1$ and before cannot be used to block in period t, which reflects the idea that past contracts are bygones and cannot be changed. Condition (i) is also important, as without this condition, temporal stability turns to be equivalent to stability.

We illustrate these differences in our stability concept using Example 1, where the outcome $A = \{(d_2, h_2, 1)\}$, while not stable, can be shown to be temporally stable. Recall that this outcome was blocked via $\{(d_2, h_2, 2)\}$ in the definition of stability and this set is the only set Z satisfying $Z_i \subseteq C^i(A \cup Z)$ for all $i \in Z_I$. This set Z does not satisfy the conditions for blocking at period 1, since $Z^1 = \emptyset$, that is, Z does not contain a contract for period 1, and condition (i) is violated. This set Z also does not satisfy the conditions for blocking at period 2, since $C^{d_2}(A \cup Z)^1 = \emptyset$, while $A_{d_2}^1 = \{(d_1, h_2, 1)\}$, thus violating condition (iii).

The concept of temporal stability is similar in spirit of Kadam and Kotowski (2018a,b) in that the stability of an outcome is evaluated at each time period. However, there are subtle differences to the blocking conditions. First, in the stability concept of Kadam and Kotowski (2018a,b), only pairs of agents can block a matching, while in our temporal stability, any group of agents can block a matching outcome. Second, according to the stability concept in Kadam and Kotowski (2018a,b), the blocking pair at period t has to either be matched to each other throughout all periods from t onwards or be unmatched, while for temporal stability, we allow for deviations where a pair may be matched to each other at period t but retain other matches from the original outcome A involving matches with other agents in condition (ii). Third, in a period t block in the stability concept of Kadam and Kotowski (2018a,b), the blocking pair can choose not to be matched in period t, thus violating condition (i) of temporal stability. Because of these differences, the concept defined by Kadam and Kotowski (2018a,b) is neither weaker nor stronger than temporal stability.

3.2 Ordered substitutability

We have noted that complementarity in the dynamic matching market is not unusual. However, we would still like to find a suitable substitutability condition which guarantees the existence of a temporally stable outcome. We define a weaker substitutability concept, called ordered substitutability, in the following way. Recall that SUB requires the substitutability between any two contracts x and x' regardless of the time periods that these contracts are associated with. In contrast, our substitutability condition treats past contracts as bygones so that when re-considering whether x belongs to the choice set $C^i(A \setminus \{x'\})$, this hypothetical situation only makes sense when x is a contract of a time period either earlier than that of x' or in the same time period. If x' is a contract that is earlier than x, then when re-evaluating whether to include x, it is not meaningful to consider a situation where x' is removed, since the contract involving x' has already been enacted and cannot be removed by the time x is reconsidered. Below we give a formal definition of ordered substitutability.

Definition 4. *An agent i's choice function C^i satisfies ordered substitutability if for any $A \subseteq X$ and any distinct $x, x' \in C^i(A)$ with $x_T \leq x'_T$, we have that $x \in C^i(A \setminus \{x'\})$.*

Ordered substitutability serves as a starting point for our search for a suitable sufficient condition for the existence of a temporally stable outcome. To consider further conditions, we first decompose ordered substitutability into the following two conditions:

- for any $A \subseteq X$ and any distinct $x, x' \in C^i(A)$ with $x_T = x'_T$, $x \in C^i(A \setminus \{x'\})$.
- for any $A \subseteq X$ and $x' \in C^i(A)$, $C^i(A)^t \subseteq C^i(A \setminus \{x'\})^t$ for any $t \in T$ with $t < x'_T$.

The first condition states the substitutability of contracts from the same time period. The second condition considers the substitutability of contracts from different time periods. We first coin the term period-wise substitutability (PS) for the first condition, which is formally given below.

Definition 5. *An agent i's choice function C^i satisfies period-wise substitutability (PS) if for any $A \subseteq X$ and any distinct $x, x' \in C^i(A)$ with $x_T = x'_T$, $x \in C^i(A \setminus \{x'\})$.*

Next, we strengthen the second condition in the decomposition to require the inclusion relation to hold with equality. We call this condition future-invariance (FI), as the choice set for a particular period t is invariant to deletions of contracts of future time periods.

Definition 6. *An agent i's choice function C^i satisfies future-invariance (FI) if for any $A \subseteq X$ and any $x' \in C^i(A)$, $C^i(A)^t = C^i(A \setminus \{x'\})^t$ for any $t \in T$ with $t < x'_T$.*

We provide examples of choice functions that satisfy ordered substitutability. For simplicity, we assume that there are only two periods in the following examples.

Example 2. We introduce a slightly generalized version of time slot specific choice functions introduced by Dimakopoulos and Heller (2019), which is a subclass of slot specific choice functions introduced by Kominers and Sönmez (2016). Let H be the set of departments in a hospital where medical students rotate on departments within the hospital for two periods. We assume that $X_h = \{h\} \times D \times \{1, 2\}$ for all $h \in H$ where D is the set of medical students. Each department h has choice functions $\hat{C}^h$ and $\tilde{C}^h$ over $2^{X_h^1}$ and $2^{X_h^2}$, respectively, and needs to choose different doctors for each period. In such a situation, we can consider a choice function in which h first chooses the most preferred doctors from available doctors at period 1, and then chooses the most preferred doctors from available doctors at period 2 who are not chosen at period 1. That is, for each $Y \subseteq X_h$, $C^h(Y) = \hat{C}^h(Y^1) \cup \tilde{C}^h(\{y \in Y^2 \mid y_D \notin \hat{C}^h(Y^1)_D\})$. Then, C^h satisfies IRC, PS, and FI if $\hat{C}^h$ and $\tilde{C}^h$ satisfy IRC and SUB.[5]

Example 3. We introduce a more general class of choice functions than that in Example 2. Consider once again the setup in Example 1 where D represented the set of consultants and H represented the set of firms. One of the consultants wanted to work for the foreign firm after acquiring the necessary skills to work there. We can generalize this idea in the following way.

Let X_d be the set of possible contracts for d. Suppose that the available contracts in period 2 for d depend on what d has chosen in period 1. This constraint can be described

[5]In Dimakopoulos and Heller (2019) and Kominers and Sönmez (2016), $\hat{C}^h$ and $\tilde{C}^h$ are given by unit demand choice functions or responsive choice functions.

by a function $f : 2^{X_d^1} \to 2^{X_d^2}$ where $f(Y) \subseteq X_d^2$ is a set of available contracts at period 2 provided that d has chosen $Y \subseteq X_d^1$ at period 1. Let d have choice functions $\hat{C}^d$ and $\tilde{C}^d$ over $2^{X_d^1}$ and $2^{X_d^2}$, respectively. Given f, an overall choice function C^d is defined so that d first chooses the most preferred contracts that are available in period 1, and then d chooses the most preferred contracts from available contracts at period 2 given the choice at period 1. That is, for each $Y \subseteq X_d$, $C^d(Y) = \hat{C}^d(Y^1) \cup \tilde{C}^d(Y^2 \cap f(\hat{C}^d(Y^1)))$. Then, C^d satisfies IRC, PS, and FI if $\hat{C}^d$ and $\tilde{C}^d$ satisfy IRC and SUB.

The following condition is a weaker condition than ordered substitutability which is a useful concept for our analysis.

Definition 7. *An agent i's choice function C^i satisfies weak ordered substitutability if for any $A \subseteq X$ and any distinct $x, x' \in C^i(A)$ with $x' \in \arg\max\{y_T \mid y \in A\}$, we have that $x \in C^i(A \setminus \{x'\})$.*

It is well-known that substitutability is equivalent to the monotonicity of the rejection function; that is, a choice function C^i satisfies substitutability if and only if $R^i(A) \subseteq R^i(B)$ for any $A, B \subseteq X$ with $A \subseteq B$, provided that C^i satisfies IRC. It is also well-known that a choice function C^i satisfies substitutability and IRC if and only if it satisfies *path-independence*, that is, $C^i(A \cup B) = C^i(C^i(A) \cup B)$ for any $A, B \subseteq X$. The following lemma shows that weak ordered substitutability is characterized by weaker versions of the monotonicity and path-independence where the proof is given in Appendix A.

Lemma 1. *Let C^i be a choice function that satisfies IRC. Then, the following three statements are equivalent.*

(a) C^i satisfies weak ordered substitutability.

(b) $R^i(A) \subseteq R^i(B)$ for any $A, B \subseteq X$ with $A \subseteq B$ such that $x_T \geq y_T$ for all $x \in B \setminus A$ and all $y \in A$.

(c) $C^i(A \cup B) = C^i(C^i(A) \cup B))$ for any $A, B \subseteq X$ such that $x_T \geq y_T$ for all $x \in B \setminus A$ and all $y \in A$.

It is well-known that pairwise stability and stability are equivalent under substitutability. An analogous result holds under ordered substitutability. We say that A is *pairwise blocked at period $t \in T$* if there exists $z \in X^t \setminus A$ such that (i) $z \in C^i(A \cup \{z\})$ for all

$i \in \{z_D, z_H\}$ and (ii) $A_i^{\leq t-1} = C^i(A \cup \{z\})^{\leq t-1}$ for all $i \in \{z_D, z_H\}$, and A is *pairwise temporally stable* if it is individually rational and not pairwise blocked at any period. Indeed, if every agent's choice function satisfies ordered substitutability, we have that (i) when an outcome A is blocked via Z, then it is blocked via a contract $z \in \arg\min\{z_T \mid z \in Z\}$, and (ii) when an outcome A is blocked via Z at period t, then it is blocked at period t via a contract $z \in Z^t$. Thus, the following proposition holds.

Proposition 1. *Suppose that every agent's choice function satisfies ordered substitutability.*

(a) An outcome is stable if and only if it is pairwise stable.

(b) An outcome is temporally stable if and only if it is pairwise temporally stable.

Note that this result does not hold under weak ordered substitutability. The following example illustrates this fact.

Example 4. Let $X = \{x, y, z\}$ with $x_D = y_D = z_D = d$, $x_H = y_H = z_H = h$, and $x_T = 1, y_T = 2, z_T = 3$. We assume that $C^d(\{x, y, z\}) = \{x, y, z\}$, $C^d(\{x, y\}) = \{x, y\}$, $C^d(\{x, z\}) = \{z\}$, $C^d(\{y, z\}) = \{z\}$, and $C^d(\{x'\}) = \{x'\}$ for all $x' \in \{x, y, z\}$. This choice function does not satisfy ordered substitutability since $x, y \in C^d(\{x, y, z\}) = \{x, y, z\}$ but $x \notin C^d(\{x, z\}) = \{z\}$ while it satisfies weak ordered substitutability. Moreover, C^d satisfies IRC. Then, it is straightforward to see that $\{z\}$ is not stable (or temporally stable) but it is pairwise stable (or pairwise temporally stable) when $C^h(Y) = Y$ for all $Y \subseteq \{x, y, z\}$.

4 Existence

We first provide a sufficient condition for the existence of stable outcomes.

Proposition 2. *If every agent's choice function satisfies IRC, PS, and FI, then a stable outcome exists.*

This proposition does not follow from the existence result of Hatfield and Kominers (2017) since the combination of PS and FI is independent from SUB. However, we can simply construct a stable outcome from period 1 inductively under the assumption of

Proposition 2 by using their result. The formal proof is given in Appendix B. Note that the assumption that every agent's choice function satisfies IRC, PS, and FI is crucial to obtain our result. Indeed, Example 1 shows that there may not exist a stable outcome when the choice function of one agent satisfies IRC, PS, and FI, and the other choice functions satisfy IRC and SUB. Note also that there may not exist a side-optimal stable outcome under the assumption of Proposition 2 . Indeed, in Example 6, we will show that there may not exist a hospital-optimal stable outcome even if every agent's choice function satisfies IRC, PS, and FI.

We now provide sufficient conditions for the existence of temporally stable outcomes. We say that an agent i's choice function C^i satisfies *period-wise unit demand* if $|C^i(A)^t| \leq 1$ for all $t \in T$ and all $A \subseteq X$.

Theorem 1. *Suppose that every hospital's choice function satisfies IRC and weak ordered substitutability.*

(a) If every doctor's choice function satisfies IRC, weak ordered substitutability, and period-wise unit demand, then a temporally stable outcome exists.

(b) If every doctor's choice function satisfies IRC, PS, and FI, then a temporally stable outcome exists.

In this theorem, every agent's choice function is assumed to satisfy IRC and weak ordered substitutability while additional assumptions are imposed on the choice functions of doctors. The assumption of Theorem 1(a) requires that a many-to-one matching between doctors and hospitals is formed for each period. On the other hand, Theorem 1(b) shows that a many-to-many matching between doctors and hospitals is allowed for each period under the assumption that every doctor's choice function satisfies IRC, PS, and FI. It should be remarked that the additional assumptions on the choice functions of doctors are crucial to guarantee the existence of temporally stable outcomes. Indeed, in Example 7, we will show that a temporally stable outcome may not exist even if every agent's choice function satisfies IRC and ordered substitutability. We also note that a side-optimal temporally stable outcome may not exist under the assumption of Theorem 1 (See Example 6).

We explain an outline of the proof of Theorem 1(a) since Theorem 1(b) can be proved in a similar way. The formal proof is given in Appendix C. We will inductively construct a temporally stable outcome from period 1. For any given $t \in T$, we say that an outcome A is *t-temporally stable* if (i) $A \subseteq X^{\leq t}$, and (ii) A is individually rational and not blocked at any period $t' = 1, \cdots, t$. Note that an outcome is L-temporally stable if and only if it is temporally stable where L is the last period. In addition, the existence of a 1-temporally stable outcome follows from the existence result of Hatfield and Kominers (2017).

In the following, we show how to construct a t-temporally stable outcome from a $(t-1)$-temporally stable outcome. Thus, we assume that there exists a $(t-1)$-temporally stable outcome $A \subseteq X^{\leq(t-1)}$ where $t \geq 2$. We construct a t-temporally stable outcome by using the following procedure which is a modification of the DA algorithm.

- Let $Y[0] = X^t$ be the initial set of available contracts for doctors and $B[0] = \emptyset$ be the initial outcome for period t. For each step k, $Y[k]$ represents the set of contracts for period t that have not been rejected, and $B[k]$ represents the set of contracts for period t that are tentatively accepted by some hospital.

  These sets are updated by the following procedure where $Y[k]$ and $B[k]$ are subsets of X^t for each step k.

- Step $k(\geq 0)$: For each $d \in D$, let

  $$O^d[k] = \{x \in Y[k] \setminus B[k] \mid C^d(A \cup \{x\}) = A_d \cup \{x\}\}$$

  be the set of possible proposals for d at step k. Thus, each doctor can propose a new contract only if she wants to keep the past contracts with it. If there exists no $d \in D$ such that $O^d[k] \neq \emptyset$ and $B[k]_d = \emptyset$, then this procedure terminates at this step. Otherwise, pick an arbitrary doctor $d \in D$ such that $O^d[k] \neq \emptyset$ and $B[k]_d = \emptyset$. Note that $C^d(A \cup O^d[k]) \cap O^d[k]$ is a singleton by IRC and period-wise unit demand. Then, doctor d proposes the contract

  $$x(k) \in C^d(A \cup O^d[k]) \cap O^d[k]$$

  to hospital $h = x(k)_H$. Hospital h can accept the new contract only if the past contracts are never dropped by doing so. If $x(k) \in C^h(A \cup B[k] \cup \{x(k)\})$ and $A_h \subseteq C^h(A \cup B[k] \cup \{x(k)\})$, then h accepts $C^h(A \cup B[k] \cup \{x(k)\})$ and rejects $R^h(A \cup B[k] \cup \{x(k)\})$, and define the sets $B[k+1]$ and $Y[k+1]$ by $B[k+1] = (B[k] \cup \{x(k)\}) \setminus R^h(A \cup B[k] \cup \{x(k)\})$ and $Y[k+1] = Y[k] \setminus R^h(A \cup B[k] \cup \{x(k)\})$. Otherwise, $x(k)$ is rejected, and let $B[k+1] = B[k]$ and $Y[k+1] = Y[k] \setminus \{x(k)\}$.

In the following, we explain the procedure for a given time period t. Note that $Y[k+1] \subseteq Y[k]$ holds for each step k. Moreover, either $Y[k+1] \subsetneq Y[k]$ or $B[k+1] \supsetneq B[k]$ holds for each step k, implying that $O^d[k+1] \subsetneq O^d[k]$ where $d = x(k)_D$. Therefore, the procedure for a given time period t must terminate in a finite number of steps k^*. Under the assumption of Theorem 1(a), we can show that $A \cup B[k^*]$ is a t-temporally stable outcome provided that A is a $(t-1)$-temporally stable outcome. Thus, we can find a temporally stable outcome by repeating the above steps for each t until period L.

The most significant difference of this procedure with respect to the DA algorithm is that a hospital h may reject proposal $x(k)$ even if its choice set dictates that it should accept $x(k)$, or equivalently, when $x(k) \in C^h(A \cup B[k] \cup \{x(k)\})$. This happens when its choice set leaves out some contract from A_h, which is precisely when the condition $A_h \subseteq C^h(A \cup B[k] \cup \{x(k)\})$ is violated. However, a hospital in period t cannot cut a contract involving those of periods before t, and therefore, hospital h has no choice but to reject $x(k)$. Because of this difference, the particular choice of the proposing doctor d influences the final outcome that results from this procedure. In other words, the order in which the doctors propose in period t do matter, unlike the DA algorithm.[6]

To illustrate how the procedure depends on the order of proposals, consider the following situation with two doctors d_1 and d_2 and a hospital h with choice functions C^{d_1}, C^{d_2}, and C^h induced from the following preferences:

$$\succ_{d_1}: \{x,y\}, \{x\}, \{y\}, \emptyset \qquad \succ_{d_2}: \{z\}, \emptyset,$$
$$\succ_h: \{y,z\}, \{x,y\}, \{x,z\}, \{y\}, \{z\}, \{x\}, \emptyset,$$

where $\{x,y\}_I = \{d_1, h\}$, $\{z\}_I = \{d_2, h\}$, $x_T = 1$, and $y_T = z_T = 2$. We consider the procedure for period 2 given $\{x\}$ which is a 1-temporally stable outcome. Then, the sets of possible proposals at step 0 are given by $O^{d_1}[0] = \{y\}$ and $O^{d_2}[0] = \{z\}$. If d_1 proposes y to h at step 0, then it is accepted by h since $C^h(\{x,y\}) = \{x,y\}$. In the next step, d_2 proposes z to h, but it is rejected since $x \notin C^h(\{x,y,z\}) = \{y,z\}$. Thus, the above procedure outputs $\{x,y\}$ when d_1 proposes first. On the other hand, it outputs $\{x,z\}$ when d_2 proposes first. Thus, an output of the procedure may depend on the order of

[6]Under IRC and SUB, the standard DA algorithm finds a doctor-optimal stable outcome regardless of the order of proposals. In many-to-one matching with contracts, Hirata and Kasuya (2014) show the order-independence of the cumulative offer process, which is a generalization of the DA algorithm, under a weak substitutability condition.

proposals.

Despite the procedure seemingly forcing hospitals to reject desirable contracts, the resulting matching satisfies our stability condition, regardless of the order of proposals. To see this, consider a step k at which $x(k) \in C^h(A \cup B[k] \cup \{x(k)\})$ but $A_h \nsubseteq C^h(A \cup B[k] \cup \{x(k)\})$. Thus, there exists $x' \in A_h$ such that $x' \notin C^h(A \cup B[k] \cup \{x(k)\})$. By the definition of the procedure, every hospital is weakly better off after each iteration. Together with weak ordered substitutability, we can show that h would also drop x' from $A \cup B[k^*] \cup \{x(k)\}$ where k^* is a step at which the procedure terminates. This property guarantees that an output of the procedure cannot be blocked at period t using any rejected contracts.

A more subtle remark involves why only one doctor proposes in each step. When we allow multiple doctors to make offers simultaneously, it is not unambiguously clear which doctors should be assigned to a hospital when its choice set cuts out a contract from a past period. Without some tie-breaking method, the hospital cannot make a decision on which of the new contracts to keep based only on its choice set. One solution for the hospital in this case is to reject all the contracts it received in that step. In the previous example, if both doctors d_1 and d_2 simultaneously offer y and z, respectively, under this rule, hospital h would have to reject both y and z. However, the resulting matching would violate stability, as either doctor and hospital pair would prefer to be matched in period 2. One way to avoid this would be to introduce tie-breaking rules in each step, but instead of introducing these artificial rules, we define the procedure with only one doctor proposing at a time as a more straightforward procedure.

Period-wise unit demand (or FI) is required due to the constraint that each doctor can make a new proposal only if she wants to keep the past contracts with it. Indeed, doctors may want to withdraw currently accepted contracts in the procedure without period-wise unit demand (or FI). This never happens under the standard DA algorithm when the choice functions of agents satisfy SUB. To see this, consider the following choice function that violates period-wise unit demand:

$$C^d(\{x,y,z,w\}) = \{y,z\}, \quad C^d(\{x,z,w\}) = \{x,z,w\}, \quad C^d(\{x,y,w\}) = \{x,y\},$$
$$C^d(\{x,y\}) = \{x,y\}, \quad C^d(\{x,z\}) = \{x,z\}, \quad C^d(\{x,w\}) = \{x,w\},$$

where $x_T = 1$ and $y_T = z_T = w_T = 2$. Such a choice function is compatible with IRC and weak ordered substitutability (See Example 7). We assume that $\{x\}$ is a 1-temporally stable outcome. Consider the procedure for period 2 given $\{x\}$. Then, the set of possible proposals for d at step 0 is given by $O^d[0] = \{y, z, w\}$ since $C^d(\{x, x'\}) = \{x, x'\}$ for all $x' \in \{y, z, w\}$. We assume that d proposes z and it is accepted at step 0. In the next step, d can propose w but cannot propose y since $C^d(\{x, z, w\}) = \{x, z, w\}$ and $x \notin C^d(\{x, y, z\}) = \{y, z\}$. Thus, we assume that d proposes w and it is accepted at step 1. In addition, we assume that z is rejected by z_H at step 2 since another doctor proposes a contract to z_H. Then, d wants to withdraw the currently accepted contract w by proposing y since $C^d(\{x, y, w\}) = \{x, y\}$. This causes the failure of the procedure. The assumption of period-wise unit demand (or FI) rules out the possibility that doctors may want to withdraw their currently accepted contracts, and thus, the procedure generates a temporally stable outcome.

In the rest of this section, we provide examples that show the limitation of Theorem 1. Example 7, which is given in Appendix D, shows that period-wise unit demand or FI is a crucial assumption in Theorem 1. Specifically, there may not exist a pairwise temporally stable outcome without these conditions even if every agent's choice function satisfies IRC and ordered substitutability. The following example shows that ordered substitutability is a crucial assumption in Theorem 1. Specifically, there may not exist a pairwise temporally stable outcome even if every doctor's choice function satisfies IRC and period-wise unit demand, and every hospital's choice function satisfies IRC, SUB, and period-wise unit demand.

Example 5. Let $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$, and $T = \{1, 2\}$. The set of contracts is given by $D \times H \times T$. Each agent $i \in I$ has a choice function C^i induced from a strict preference ordering $\succ_i$ over 2^{X_i}, which is defined as follows:

$$\succ_{d_1}: \{(h_1, 2), (h_2, 1)\}, \{(h_1, 2)\}, \emptyset, \quad \succ_{d_2}: \{(h_2, 1)\}, \{(h_1, 2)\}, \emptyset,$$
$$\succ_{h_1}: \{(d_2, 2)\}, \{(d_1, 2)\}, \emptyset, \quad \succ_{h_2}: \{(d_1, 1)\}, \{(d_2, 1)\}, \emptyset.$$

Note that C^{d_1} does not satisfy ordered substitutability while the choice functions of the other agents satisfy SUB. Every agent's choice function satisfies period-wise unit demand and period-wise SUB. We can confirm the nonexistence of a pairwise temporally stable outcome by the same argument as that in Example 1.

The following example shows that a hospital-optimal stable outcome (or a hospital-optimal temporally stable outcome) may not exist even if every agent's choice function satisfies IRC, PS, and FI.

Example 6. Let $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$, and $T = \{1, 2\}$. The set of contracts is given by $D \times H \times T$. We first define choice functions of hospitals. For each $Y \subseteq X$, let $C^{h_1}(Y) = \hat{C}^{h_1}(Y^1_{h_1}) \cup \tilde{C}^{h_1}(Y^2_{h_1})$ where $\hat{C}^{h_1}$ is the choice function over $2^{X^1_{h_1}}$ induced from $\hat{\succ}_{h_1} : \{(d_1, 1)\}, \{(d_2, 1)\}, \emptyset$ and $\tilde{C}^{h_1}$ is the choice function over $2^{X^2_{h_1}}$ induced from $\tilde{\succ}_{h_1} : \{(d_1, 2)\}, \{(d_2, 2)\}, \emptyset$. For each $Y \subseteq X$, let $C^{h_2}(Y) = \hat{C}^{h_2}(Y^1_{h_2}) \cup \tilde{C}^{h_2}(Y^2_{h_2})$ where $\hat{C}^{h_2}$ is the choice function over $2^{X^1_{h_2}}$ induced from $\hat{\succ}_{h_2} : \{(d_2, 1)\}, \{(d_1, 1)\}, \emptyset$ and $\tilde{C}^{h_2}$ is the choice function over $2^{X^2_{h_2}}$ induced from $\tilde{\succ}_{h_2} : \{(d_2, 2)\}, \{(d_1, 2)\}, \emptyset$. Then, it is straightforward to see that C^{h_1} and C^{h_2} satisfy IRC, SUB, and FI.

Each doctor $d \in D$ has a choice function C^d induced from $\succ_d$, which is defined as follows:

$$\succ_{d_1}: \{(h_2, 1)\}, \{(h_1, 1), (h_1, 2)\}, \{(h_1, 1)\}, \emptyset; \qquad \succ_{d_2}: \{(h_1, 1), (h_2, 2)\}, \{(h_1, 1)\}, \{(h_2, 1)\}, \emptyset.$$

Then, C^{d_1} and C^{d_2} satisfy IRC and FI while they do not satisfy SUB.

Let A be a stable outcome. Then, $|A^1_d| = 1$ for all $d \in D$ since otherwise A is not individually rational or A is blocked at period 1. Thus, we have that $\{(d_1, h_1, 1), (d_2, h_2, 1)\} \subseteq A$ or $\{(d_1, h_2, 1), (d_2, h_1, 1)\} \subseteq A$. If $\{(d_1, h_1, 1), (d_2, h_2, 1)\} \subseteq A$, then $A^2_{d_2} = \emptyset$ by individual rationality of A, which implies $A = \{(d_1, h_1, 1), (d_2, h_2, 1), (d_1, h_1, 2)\}$. If $\{(d_1, h_2, 1), (d_2, h_1, 1)\} \subseteq A$, then $A^2_{d_1} = \emptyset$ by individual rationality of A, which implies $A = \{(d_1, h_2, 1), (d_2, h_1, 1), (d_2, h_2, 2)\}$. Moreover, it is straightforward to see that $\{(d_1, h_1, 1), (d_2, h_2, 1), (d_1, h_1, 2)\}$ and $\{(d_1, h_2, 1), (d_2, h_1, 1), (d_2, h_2, 2)\}$ are stable. Thus, only these two outcomes are stable. However,

$$C^{h_2}(\{(d_2, h_2, 1)\} \cup \{(d_2, h_2, 1), (d_1, h_2, 2)\}) = \{(d_2, h_2, 1), (d_2, h_2, 2)\}.$$

Therefore, there exists no hospital-optimal stable outcome. Note that the set of stable outcomes coincides with the set of temporally stable outcomes when every agent's choice function satisfies FI. Thus, there exists no hospital-optimal temporally stable outcome in this example.

5 Conclusion

In this paper, we have formulated the many-to-many dynamic matching market using the many-to-many matching with contracts model of Hatfield and Kominers (2017). The stability concept defined in Hatfield and Kominers (2017) is quite strong in that when substitutability is violated, a stable outcome may not exist. This violation is not uncommon in the dynamic matching market. We have instead defined a weaker stability concept, temporal stability, that is similar in spirit to the stability concept of Kadam and Kotowski (2018a), but with some subtle differences. We have also defined a corresponding substitutability concept as a sufficient condition that guarantees the existence of a temporally stable outcome. We also have provided a procedure that constructs a temporally stable outcome, using a procedure similar to the DA algorithm of Gale and Shapley (1962).

One possible direction of further research is to consider analogues of other stability concepts defined for the dynamic matching model. For example, some stability concepts are defined recursively as justification of the set of outcomes used to block a certain outcome. Other stability concepts, when considering a block, may consider outcomes of other agents to change as well. It may be interesting to consider these additional stability concepts in the many-to-many setting.

Another direction of further research would be to consider choice functions which also depend on the history of previous matching outcomes. In particular, in a model where preferences are given, the preferences of agents may depend on whom they were matched to previous periods. For example, firms may take into account the past experiences of workers. This extension is particularly important in dynamic matching markets as agents may take into account these changes in preferences when considering earlier matching outcomes.

Appendix A. Proof of Lemma 1

We can show this lemma by the same argument as that of Blair (1988). We show (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a) as follows.

(a) $\Rightarrow$ (b): Let $A, B \subseteq X$ with $A \subseteq B$ such that $x_T \geq y_T$ for all $x \in B \setminus A$ and all $y \in A$. Let $x^1 \in \arg\min\{x_T \mid x \in B \setminus A\}$. Then, $R^i(A) \subseteq R^i(A \cup \{x^1\})$ by weak ordered substitutability and IRC. Moreover, $x_T \geq y_T$ for all $x \in B \setminus (A \cup \{x^1\})$ and all $y \in A \cup \{x^1\}$. Thus, by repeating this argument, we have that $R^i(A) \subseteq R^i(B)$.

(b) $\Rightarrow$ (c): Let $A, B \subseteq X$ such that $x_T \geq y_T$ for all $x \in B \setminus A$ and all $y \in A$. Pick any $x \in C^i(A \cup B)$. When $x \notin B$, we have $x \in C^i(A)$ since otherwise $x \notin C^i(A \cup B)$ holds by (b). Thus, $C^i(A \cup B) \subseteq C^i(A) \cup B \subseteq A \cup B$. By IRC, we have that $C^i(A \cup B) = C^i(C^i(A) \cup B)$.

(c) $\Rightarrow$ (a): Let $A \subseteq X$. Consider any distinct $x, x' \in C^i(A)$ with $x' \in \arg\max\{y_T \mid y \in A\}$. By (c), we have $x \in C^i(A) = C^i(C^i(A \setminus \{x'\}) \cup \{x'\})$. Thus, $x \in C^i(A \setminus \{x'\})$.

Appendix B. Proof of Proposition 2

We show Proposition 2 by induction. Given $t \in T$, we say that an outcome A is *t-stable* if $A \subseteq X^{\leq t}$, A is individually rational, and there exists no $Z \subseteq X^{\leq t} \setminus A$ such that $Z \neq \emptyset$ and $Z_i \subseteq C^i(A \cup Z)$ for all $i \in Z_I$. Suppose that every agent's choice function satisfies IRC, PS, and FI. By the existence result of Hatfield and Kominers (2017), there exists a 1-stable outcome. Suppose that there exists a $(t-1)$-stable outcome $A \subseteq X^{\leq (t-1)}$ where $t \geq 2$. For each $i \in I$, we define $\hat{C}^i$ over $2^{X_i^t}$ by $\hat{C}^i(Y) = C^i(A \cup Y) \setminus A$ for all $Y \subseteq X_i^t$. Then, $\hat{C}^i$ satisfies IRC and SUB since C^i satisfies IRC and PS. By the existence result of Hatfield and Kominers (2017), there exists a stable outcome B within X^t, that is, $B \subseteq X^t$ such that $\hat{C}^i(B) = B_i$ for all $i \in I$ and there exists no $Z \subseteq X^t \setminus B$ such that $Z \neq \emptyset$ and $Z_i \subseteq \hat{C}^i(B \cup Z)$ for all $i \in Z_I$. Note that $C^i(A \cup Y)^{\leq (t-1)} = A_i$ for all $i \in I$ and all $Y \subseteq X_i^t$ since C^i satisfies IRC and FI, and $C^i(A) = A_i$ holds. This implies that $A \cup B$ is individually rational. Suppose that $A \cup B$ is not t-stable. Then, there exists a nonempty $Z \subseteq X^{\leq t} \setminus (A \cup B)$ such that $Z_i \subseteq C^i(A \cup B \cup Z)$ for all $i \in Z_I$. When $Z^{\leq (t-1)} \neq \emptyset$, we have that $Z_i^{\leq (t-1)} \subseteq C^i(A \cup Z^{\leq (t-1)})$ for all $i \in Z_I^{\leq (t-1)}$ by FI, contradicting the assumption that A is $(t-1)$-stable. Thus, $Z \subseteq X^t$. Then, $Z_i \subseteq \hat{C}^i(B \cup Z)$ for all $i \in Z_I$ since $Z_i \subseteq C^i(A \cup B \cup Z)$ for all $i \in Z_I$, contradicting the fact that B is a stable outcome within X^t. Thus, $A \cup B$ is t-stable. Therefore, there exists a stable outcome.

Appendix C. Proof of Theorem 1

We provide proofs of Theorem 1 (a) and (b) as follows. We assume that every agent's choice function satisfies IRC and weak ordered substitutability. We first show that there exists a 1-temporally stable outcome. By the existence result of Hatfield and Kominers (2017), there exists a stable outcome within X^1; that is, $A \subseteq X^1$ such that A is individually rational and there exists no $Z \subseteq X^1 \setminus A$ satisfying $Z \neq \emptyset$ and $Z_i \subseteq C^i(A \cup Z)$ for all $i \in Z_I$. By weak ordered substitutability, this implies that A is a 1-temporally stable outcome. We now prove Theorem 1(a) and (b) below.

Proof of Theorem 1(a). We assume that every doctor's choice function satisfies IRC, weak ordered substitutability, and period-wise unit demand. We show Theorem 1 by induction. We have already shown that there exists a 1-temporally stable outcome. Suppose that there exists a $(t-1)$-temporally stable outcome $A \subseteq X^{\leq(t-1)}$ where $t \geq 2$. We assume that the procedure described in Section 4 terminates at step k^*. We show that $A \cup B[k^*]$ is a t-temporally stable outcome.

We first show that $A \cup B[k^*]$ is individually rational by induction. Clearly, $A \cup B[0] = A$ is individually rational. Suppose that $A \cup B[k]$ is individually rational for some step $k = 0, \cdots, k^* - 1$. Let $d = x(k)_D$ and $h = x(k)_H$. If $B[k] = B[k+1]$, then the proof is complete. Thus, we assume that $B[k] \neq B[k+1]$, that is, $x(k)$ is accepted by h. Then, $B[k+1] = (B[k] \cup \{x(k)\}) \setminus R^h(A \cup B[k] \cup \{x(k)\})$. By $B[k]_d = \emptyset$, we have that $(A \cup B[k+1])_d = A_d \cup \{x(k)\}$. By $x(k) \in O^d[k]$, $C^d(A \cup \{x(k)\}) = A_d \cup \{x(k)\}$. Thus, $A \cup B[k+1]$ is individually rational for d. Since $x(k)$ is accepted by h, $A_h \subseteq C^h(A \cup B[k] \cup \{x(k)\})$. This implies that $(A \cup B[k+1])_h = C^h(A \cup B[k] \cup \{x(k)\})$. Thus, $C^h(A \cup B[k+1]) = (A \cup B[k+1])_h$ by IRC. Thus, $A \cup B[k+1]$ is individually rational for h. Note that $A \cup B[k+1]$ is individually rational for all $h' \in H \setminus \{h\}$ by $B[k+1]_{h'} = B[k]_{h'}$ and the induction hypothesis. Note also that $B[k+1]_{d'} \subseteq B[k]_{d'}$ for all $d' \in D \setminus \{d\}$. This implies that $C^{d'}(A \cup B[k+1]) = (A \cup B[k+1])_{d'}$ for all $d' \in D \setminus \{d\}$ by weak ordered substitutability and the induction hypothesis. Thus, $A \cup B[k+1]$ is individually rational.

We next show that that for all $h \in H$,

$$C^h(A \cup B[k] \cup B[k^*]) = C^h(A \cup B[k^*]) \text{ for all } k = 0, \cdots, k^*. \tag{1}$$

Let $h \in H$. By the definition of the procedure and IRC, we have that $C^h(A \cup B[k-1] \cup$

$B[k]) = C^h(A \cup B[k])$ for all $k = 1, \cdots, k^*$. By Lemma 1(c), we have that

$$\begin{aligned} C^h(A \cup B[k^*-2] \cup B[k^*]) &= C^h(C^h(A \cup B[k^*]) \cup B[k^*-2]) \\ &= C^h(A \cup B[k^*-1] \cup B[k^*] \cup B[k^*-2])) \\ &= C^h(C^h(A \cup B[k^*-2] \cup B[k^*-1]) \cup B[k^*])) \\ &= C^h(A \cup B[k^*-1] \cup B[k^*])) \\ &= C^h(A \cup B[k^*]). \end{aligned}$$

Thus, $C^h(A \cup B[k^*-2] \cup B[k^*]) = C^h(A \cup B[k^*])$. Together with $C^h(A \cup B[k^*-3] \cup B[k^*-2]) = C^h(A \cup B[k^*-2])$, we have that $C^h(A \cup B[k^*-3] \cup B[k^*]) = C^h(A \cup B[k^*])$ by the same argument above. By repeating this argument, we obtain (1).

We now show that $A \cup B[k^*]$ is a t-temporally stable outcome. Suppose that $A \cup B[k^*]$ is not t-temporally stable. Then, $A \cup B[k^*]$ is blocked at some period $\hat{t} = 1, \cdots, t$ since we have already shown that $A \cup B[k^*]$ is individually rational. Thus, there exists a nonempty $Z \subseteq X^{\geq \hat{t}} \setminus (A \cup B[k^*])$ such that (i) $Z^{\hat{t}} \neq \emptyset$, (ii) $Z_i \subseteq C^i(A \cup B[k^*] \cup Z)$ for all $i \in Z_I$, and (iii) $A_i^{\leq(\hat{t}-1)} = C^i(A \cup B[k^*] \cup Z)^{\leq(\hat{t}-1)}$ for all $i \in Z_I$.

We first consider the case with $\hat{t} < t$. Note that $Z^{\leq(t-1)}$ is nonempty by $Z^{\hat{t}} \subseteq Z^{\leq(t-1)}$. By weak ordered substitutability, we have that $Z_i^{\leq(t-1)} \subseteq C^i(A \cup Z^{\leq(t-1)})$ and $A_i^{\leq(\hat{t}-1)} = C^i(A \cup Z^{\leq(t-1)})^{\leq(\hat{t}-1)}$ for all $i \in Z_I^{\leq(t-1)}$. This means that A is blocked at period $\hat{t}(\leq t-1)$ contradicting the assumption that A is $(t-1)$-temporally stable.

It remains to consider the case with $\hat{t} = t$. Then, $Z \subseteq X^{\geq t}$ and Z^t is nonempty. Pick any $z \in Z^t$. By weak ordered substitutability, we have that $z \in C^i(A \cup B[k^*] \cup \{z\})$ and $A_i = C^i(A \cup B[k^*] \cup \{z\})^{\leq(t-1)}$ for all $i \in \{z_D, z_H\}$. Hence, $A \cup B[k^*]$ is blocked at period t via $\{z\}$. Let $d = z_D$ and $h = z_H$.

We claim that $z \notin Y[k^*]$. Suppose that $z \in Y[k^*]$. Note that $C^d(A \cup \{z\}) = A_d \cup \{z\}$ by weak ordered substitutability since $z \in C^d(A \cup B[k^*] \cup \{z\})$ and $A_d = C^d(A \cup B[k^*] \cup \{z\})^{\leq(t-1)}$. If $B[k^*]_d = \emptyset$, then $z \in O^d[k^*] = \{x \in Y[k^*] \setminus B[k^*] \mid C^d(A \cup \{x\}) = A_d \cup \{x\}\}$, contradicting that the procedure terminates at step k^*. Thus, $B[k^*]_d \neq \emptyset$. By period-wise unit demand, $|B[k^*]_d| = 1$. By the definition of the procedure, $B[k^*]_d = \{x(k')\}$ for some $k' \leq k^*$. Thus, d has proposed $x(k')$ at step k'. This means that $x(k') \in O^d[k']$ and $B[k']_d = \emptyset$. Moreover, $z \in O^d[k'] = \{x \in Y[k'] \setminus B[k'] \mid C^d(A \cup \{x\}) = A_d \cup \{x\}\}$ since $C^d(A \cup \{z\}) = A_d \cup \{z\}$ and $z \in Y[k^*] \subseteq Y[k']$. Recall that $z \in C^i(A \cup B[k^*] \cup \{z\}) = C^i(A \cup \{x(k'), z\})$. By period-wise unit demand, $x(k') \notin C^i(A \cup \{x(k'), z\})$. Thus, $x(k') \notin$

$C^i(A \cup O^d[k'])$ by weak ordered substitutability, contradicting $x(k') \in C^i(A \cup O^d[k'])$. Therefore, $z \notin Y[k^*]$.

By $z \notin Y[k^*]$, there exists some step $\hat{k} < k^*$ at which z is rejected; that is, $z \in Y[\hat{k}]$ but $z \notin Y[\hat{k}+1]$. We consider the following two cases.

Case 1. Suppose that $z \neq x[\hat{k}]$. Then, $x(\hat{k})$ is accepted at step $\hat{k}$ and $z \in R^h(A \cup B[\hat{k}] \cup \{x(\hat{k})\})$. This implies that $z \notin C^h(A \cup B[\hat{k}+1] \cup \{z\})$ by the definition of the procedure and IRC. By weak ordered substitutability, $z \notin C^h(A \cup B[\hat{k}+1] \cup B[k^*] \cup \{z\})$. By Lemma 1(c) and (1), $C^h(A \cup B[\hat{k}+1] \cup B[k^*] \cup \{z\}) = C^h(C^h(A \cup B[\hat{k}+1] \cup B[k^*]) \cup \{z\}) = C^h(A \cup B[k^*] \cup \{z\})$. Thus, $z \notin C^h(A \cup B[k^*] \cup \{z\})$, contradicting the fact that $A \cup B[k^*]$ is blocked at period t via $\{z\}$.

Case 2. Suppose that $z = x[\hat{k}]$. Then, there are two possibilities.

Case 2-1. Suppose that $z \notin C^h(A \cup B[\hat{k}] \cup \{z\})$. By weak ordered substitutability, $z \notin C^h(A \cup B[\hat{k}] \cup B[k^*] \cup \{z\})$. By Lemma 1(c) and (1), this implies that $z \notin C^h(A \cup B[k^*] \cup \{z\})$, contradicting the fact that $A \cup B[k^*]$ is blocked at period t via $\{z\}$.

Case 2-2. Suppose that $A_h \nsubseteq C^h(A \cup B[\hat{k}] \cup \{z\})$. Then, there exists $y \in A_h$ such that $y \notin C^h(A \cup B[\hat{k}] \cup \{z\})$. By weak ordered substitutability, we have $y \notin C^h(A \cup B[\hat{k}] \cup B[k^*] \cup \{z\})$. By Lemma 1(c) and (1), this implies that $y \notin C^h(A \cup B[k^*] \cup \{z\})$ and thus $A_h \neq C^h(A \cup B[k^*] \cup \{z\})^{\leq (t-1)}$, contradicting the fact that $A \cup B[k^*]$ is blocked at period t via $\{z\}$.

Therefore, every case yields a contradiction. Thus, $A \cup B[k^*]$ is t-temporally stable.

Proof of Theorem 1(b). We assume that every doctor's choice function satisfies IRC, PS, and FI. We show Theorem 1(b) by induction. We have already shown that there exists a 1-temporally stable outcome. Suppose that there exists a $(t-1)$-temporally stable outcome $A \subseteq X^{\leq (t-1)}$ where $t \geq 2$. For each $d \in D$, we define a choice function $\hat{C}^d$ over $2^{X^t_d}$ by $\hat{C}^d(Y) = C^d(A \cup Y) \setminus A$ for all $Y \subseteq X^t_d$. Note that $\hat{C}^d$ satisfies IRC and SUB for all $d \in D$. Consider the following procedure which is a variation of the one in Section 4.

- Let $Y[0] = X^t$ be the initial set of available contracts for doctors and $B[0] = \emptyset$ be the initial outcome for period t. For each step k, $Y[k]$ represents the set of contracts for period t that have not been rejected, and $B[k]$ represents the set of contracts for period t that are tentatively accepted by some hospital. These sets are updated by the following procedure where $Y[k]$ and $B[k]$ are subsets of X^t for each step k.

- Step $k(\geq 0)$: For each $d \in D$, $\hat{C}^d(Y[k]) \setminus B[k]_d$ is the set of possible proposals for d at step k. If $\hat{C}^d(Y[k]) \setminus B[k]_d = \emptyset$ for all $d \in D$, then this procedure terminates at this step. Otherwise, pick an arbitrary $d \in D$ such that $\hat{C}^d(Y[k]) \setminus B[k]_d \neq \emptyset$. Then, doctor d proposes a contract

$$x(k) \in \hat{C}^d(Y[k]) \setminus B[k]_d$$

to hospital $h = x(k)_H$. For hospitals, this procedure works in the same way as that of Section 4. If $x(k) \in C^h(A \cup B[k] \cup \{x(k)\})$ and $A_h \subseteq C^h(A \cup B[k] \cup \{x(k)\})$, then h accepts $C^h(A \cup B[k] \cup \{x(k)\})$ and rejects $R^h(A \cup B[k] \cup \{x(k)\})$, and define the sets $B[k+1]$ and $Y[k+1]$ by $B[k+1] = (B[k] \cup \{x(k)\}) \setminus R^h(A \cup B[k] \cup \{x(k)\})$ and $Y[k+1] = Y[k] \setminus R^h(A \cup B[k] \cup \{x(k)\})$. Otherwise, $x(k)$ is rejected, and let $B[k+1] = B[k]$ and $Y[k+1] = Y[k] \setminus \{x(k)\}$.

This procedure also terminates at a finite step. Let k^* be a step at which the above procedure terminates. Note that for all $h \in H$, $C^h(A \cup B[k] \cup B[k^*]) = C^h(A \cup B[k^*])$ holds for all $k = 0, \cdots, k^*$ by the same argument as the proof of Theorem 1(a).

For doctors, the procedure works in the same way as in the standard DA algorithm. This implies that $B[k^*]_d = \hat{C}^d(Y[k^*])$ for all $d \in D$. To see this, let $d \in D$ and $x \in B[k^*]_d$. Then, $x = x(k')$ for some $k' \leq k^*$ and $x(k') \in \hat{C}^d(Y[k'])$. Note that $B[k] \subseteq Y[k]$ for any $k = 0, \cdots, k^*$ by the definition of the procedure. Thus, $x \in Y[k^*] \subseteq Y[k']$. By SUB of $\hat{C}^d$, $x \in \hat{C}^d(Y[k^*])$. Thus, $B[k^*]_d \subseteq \hat{C}^d(Y[k^*])$. Since the procedure terminates at step k^*, $\hat{C}^d(Y[k^*]) \subseteq B[k^*]_d$ and hence $B[k^*]_d = \hat{C}^d(Y[k^*])$.

We next show that $A \cup B[k^*]$ is individually rational. For all $d \in D$, $\hat{C}^d(B[k^*]_d) = B[k^*]_d$ by $B[k^*]_d = \hat{C}^d(Y[k^*])$ and IRC. Since the choice functions of doctors satisfy FI, this implies that $C^d(A \cup B[k^*]) = (A \cup B[k^*])_d$ for all $d \in D$. By the same argument as the proof of Theorem 1(a), $A \cup B[k^*]$ is individually rational for all hospitals. Therefore, $A \cup B[k^*]$ is individually rational. Moreover, for any $z \in X^t \setminus B[k^*]$, $z \in \hat{C}^{z_D}(B[k^*] \cup \{z\})$ implies $z \notin Y[k^*]$ by IRC and $B[k^*]_d = \hat{C}^d(Y[k^*])$. Then, we can show that $A \cup B[k^*]$ is a t-temporally stable outcome by the same argument as the proof of Theorem 1(a).

Appendix D. Example 7

Example 7. Let $D = \{d_1, d_2, d_3\}$, $H = \{h_1, h_2, h_3\}$, and $T = \{1, 2\}$. The set of contracts at period 1 is given by $\{(d_1, h_1, 1), (d_1, h_2, 1)\}$. The set of contracts at period 2 is given by

$$(D \setminus \{d_1\}) \times H \times \{2\} \cup \{(d_1, h_1, 2), (d_1, h_2, 2, \alpha), (d_1, h_2, 2, \beta), (d_1, h_3, 2)\}.$$

Note that there are two contracts between d_1 and h_2 at period 2.

We first construct d_1's choice function. Note that $X^1_{d_1} = \{(d_1, h_1, 1), (d_1, h_2, 1)\}$ and $X^2_{d_1} = \{(d_1, h_1, 2), (d_1, h_2, 2, \alpha), (d_1, h_2, 2, \beta), (d_1, h_3, 2)\}$. If d_1 matches with h_2 at period 1, then d_1's preferences at period 2 (strict preferences over $2^{X^2_{d_1}}$) are given as follows:

$$\begin{aligned}\hat{\succ}_{d_1} :& \{(h_1, 2), (h_3, 2)\}, \{(h_1, 2), (h_2, 2, \alpha)\}, \{(h_3, 2), (h_2, 2, \beta)\},\\ & \{(h_2, 2, \alpha), (h_2, 2, \beta)\}, \{(h_1, 2)\}, \{(h_3, 2)\}, \{(h_2, 2, \alpha)\}, \{(h_2, 2, \beta)\}, \emptyset.\end{aligned}$$

If d_1 does not matches with h_2 at period 1, then d_1 never wants to sign $(d_1, h_2, 2, \alpha)$ and $(d_1, h_2, 2, \beta)$, and d_1's preferences at period 2 (strict preferences over $2^{X^2_{d_1}}$) are given as follows:

$$\tilde{\succ}_{d_1} : \{(h_1, 2), (h_3, 2)\}, \{(h_1, 2)\}, \{(h_3, 2)\}, \emptyset.$$

Let $\hat{C}^{d_1}$ and $\tilde{C}^{d_1}$ be the choice functions over $2^{X^2_{d_1}}$ that are induced from $\hat{\succ}_{d_1}$ and $\tilde{\succ}_{d_1}$, respectively. Then, it can be confirmed that $\hat{C}^{d_1}$ and $\tilde{C}^{d_1}$ satisfy IRC and SUB. We define choice functions $\hat{\mathcal{C}}^{d_1}$ and $\tilde{\mathcal{C}}^{d_1}$ over $2^{\{(d_1,h_1,1)\} \cup X^2_{d_1}}$ as follows: for all $Y \subseteq \{(d_1, h_1, 1)\} \cup X^2_{d_1}$

$$\hat{\mathcal{C}}^{d_1}(Y) = \begin{cases} \{(d_1, h_1, 2), (d_1, h_3, 2)\} (= \hat{C}^{d_1}(Y \setminus \{(d_1, h_1, 1)\})) & \text{if } \{(d_1, h_1, 2), (d_1, h_3, 2)\} \subseteq Y, \\ \hat{C}^{d_1}(Y \setminus \{(d_1, h_1, 1)\}) \cup [Y \cap \{(d_1, h_1, 1)\}] & \text{if } \{(d_1, h_1, 2), (d_1, h_3, 2)\} \not\subseteq Y, \end{cases}$$

$$\tilde{\mathcal{C}}^{d_1}(Y) = \begin{cases} \{(d_1, h_1, 2), (d_1, h_3, 2)\} (= \tilde{C}^{d_1}(Y \setminus \{(d_1, h_1, 1)\})) & \text{if } \{(d_1, h_1, 2), (d_1, h_3, 2)\} \subseteq Y, \\ \tilde{C}^{d_1}(Y \setminus \{(d_1, h_1, 1)\}) \cup [Y \cap \{(d_1, h_1, 1)\}] & \text{if } \{(d_1, h_1, 2), (d_1, h_3, 2)\} \not\subseteq Y. \end{cases}$$

Then, it is straightforward to see that $\hat{\mathcal{C}}^{d_1}$ and $\tilde{\mathcal{C}}^{d_1}$ satisfy IRC and SUB since $\hat{C}^{d_1}$ and $\tilde{C}^{d_1}$ satisfy them. We now define a choice function C^{d_1} over $2^{X_{d_1}}$ as follows: for all $Y \subseteq X_{d_1} = \{(d_1, h_1, 1), (d_1, h_2, 1)\} \cup X^2_{d_1}$,

$$C^{d_1}(Y) = \begin{cases} \hat{\mathcal{C}}^{d_1}(Y \setminus \{(d_1, h_2, 1)\}) \cup \{(d_1, h_2, 1)\} & \text{if } (d_1, h_2, 1) \in Y, \\ \tilde{\mathcal{C}}^{d_1}(Y) & \text{if } (d_1, h_2, 1) \notin Y. \end{cases}$$

Then, C^{d_1} satisfies IRC since $\hat{\mathcal{C}}^{d_1}$ and $\tilde{\mathcal{C}}^{d_1}$ satisfy it. In addition, C^{d_1} satisfies ordered substitutability. To see this, consider any $Y \subseteq X_{d_1}$ and any distinct $x, x' \in C^{d_1}(Y)$ with $x_T \leq x'_T$. When $x = (d_1, h_2, 1)$, clearly $x \in C^{d_1}(Y \setminus \{x'\})$. Suppose that $x = (d_1, h_1, 1)$. By $(d_1, h_1, 1) \in C^{d_1}(Y)$, $\{(d_1, h_1, 2), (d_1, h_3, 2)\} \nsubseteq Y$. Thus, $\{(d_1, h_1, 2), (d_1, h_3, 2)\} \nsubseteq Y \setminus \{x'\}$. This implies that $(d_1, h_1, 1) \in C^{d_1}(Y \setminus \{x'\})$ by the constructions of $\hat{\mathcal{C}}^{d_1}$ and $\tilde{\mathcal{C}}^{d_1}$. Hence, we assume that $x_T = 2$. By $x_T \leq x'_T$, we must have $x'_T = 2$. Then, it is straightforward to see that $x \in C^{d_1}(Y \setminus \{x'\})$ since $\hat{\mathcal{C}}^{d_1}$ and $\tilde{\mathcal{C}}^{d_1}$ satisfy SUB. Therefore, C^{d_1} satisfies ordered substitutability. Note that C^{d_1} does not satisfy SUB since $(d_1, h_2, 2, \alpha) \in C^{d_1}(\{(d_1, h_2, 2, \alpha), (d_1, h_2, 1)\})$ but $(d_1, h_2, 2, \alpha) \notin C^{d_1}(\{(d_1, h_2, 2, \alpha)\})$ for example.

We next construct h_2's choice function. Note that $X^1_{h_2} = \{(d_1, h_2, 1)\}$ and $X^2_{h_2} = \{(d_1, h_2, 2, \alpha), (d_1, h_2, 2, \beta), (d_2, h_2, 2), (d_3, h_2, 2)\}$. We assume that h_2 has a strict preference ordering over $2^{X^2_{h_2}}$ defined by

$$\begin{aligned}\hat{\succ}_{h_2} :&\{(d_1, 2, \alpha), (d_1, 2, \beta)\}, \{(d_1, 2, \alpha), (d_3, 2)\}, \{(d_1, 2, \beta), (d_2, 2)\},\\ &\{(d_3, 2), (d_2, 2)\}, \{(d_1, 2, \alpha)\}, \{(d_1, 2, \beta)\}, \{(d_3, 2)\}, \{(d_2, 2)\}, \emptyset.\end{aligned}$$

Let $\hat{C}^{h_2}$ be the choice function over $2^{X^2_{h_2}}$ induced from $\hat{\succ}_{h_2}$. Then, $\hat{C}^{h_2}$ satisfies IRC and SUB. We define a choice function C^{h_2} over $2^{X_{h_2}}$ as follows: for any $Y \subseteq \{(d_1, h_2, 1)\} \cup X^2_{h_2}$,

$$C^{h_2}(Y) = \begin{cases} \{(d_1, h_2, 2, \alpha), (d_1, h_2, 2, \beta)\} (= \hat{C}^{h_2}(Y \setminus \{(d_1, h_2, 1)\})) & \text{if } \{(d_1, h_2, 2, \alpha), (d_1, h_2, 2, \beta)\} \subseteq Y, \\ \hat{C}^{h_2}(Y \setminus \{(d_1, h_2, 1)\}) \cup [Y \cap \{(d_1, h_2, 1)\}] & \text{if } \{(d_1, h_2, 2, \alpha), (d_1, h_2, 2, \beta)\} \nsubseteq Y. \end{cases}$$

Then, C^{h_2} satisfies IRC and SUB since $\hat{C}^{h_2}$ satisfies them.

Every agent $i \in I \setminus \{d_1, h_2\}$ has a choice function C^i over 2^{X_i} that is induced from a strict preference ordering $\succ_i$ over 2^{X_i} which is defined as follows:

$$\begin{aligned}&\succ_{d_2}: \{(h_2, 2)\}, \{(h_1, 2)\}, \emptyset,\\ &\succ_{d_3}: \{(h_2, 2)\}, \{(h_3, 2)\}, \emptyset,\\ &\succ_{h_1}: \{(d_1, 1), (d_2, 2)\}, \{(d_1, 1), (d_1, 2)\}, \{(d_1, 1)\}, \emptyset,\\ &\succ_{h_3}: \{(d_3, 2)\}, \{(d_1, 2)\}, \emptyset.\end{aligned}$$

Note that C^{h_1} does not satisfy SUB while it satisfies PS and FI. The choice functions of the other agents satisfy IRC and SUB.

We show that there exists no pairwise temporally stable outcome. Let A be an individually rational outcome. Suppose that $(d_1, h_1, 1) \notin A$. Then, $(d_1, h_1, 2) \notin A$ since A is individually rational for h_1. Thus, $\{(d_1, h_1, 2), (d_1, h_3, 2)\} \nsubseteq A$. Thus, $(d_1, h_1, 1) \in C^{d_1}(A \cup$

$\{(d_1,h_1,1)\}$) by the definition of C^{d_1}. Clearly, $(d_1,h_1,1) \in C^{h_1}(A \cup \{(d_1,h_1,1)\})$. Thus A is blocked at period 1 via $\{(d_1,h_1,1)\}$. Suppose that $(d_1,h_2,1) \notin A$. Then, $(d_1,h_2,2,\alpha) \notin A$ since A is individually rational for d_1. Thus, $\{(d_1,h_2,2,\alpha),(d_1,h_2,2,\beta)\} \nsubseteq A$. Thus, $(d_1,h_2,1) \in C^{h_2}(A \cup \{(d_1,h_2,1)\})$ by the definition of C^{h_2}. Clearly, $(d_1,h_2,1) \in C^{d_1}(A \cup \{(d_1,h_2,1)\})$. Thus, A is blocked at period 1 via $\{(d_1,h_2,1)\}$. Therefore, if $(d_1,h_1,1) \notin A$ or $(d_1,h_2,1) \notin A$, then A is not temporally stable.

We assume that $\{(d_1,h_1,1),(d_1,h_2,1)\} \subseteq A$ and show that A is not temporally stable. If $A^2_{h_2} = \emptyset$, then A is blocked at period 2 by $\{(d_2,h_2,2)\}$ since $A^1_{d_2} = \emptyset$, d_2 ranks $\{(d_2,h_1,2)\}$ first, and $C^{h_2}(\{(d_1,h_2,1),(d_2,h_2,2)\}) = \{(d_1,h_2,1),(d_2,h_2,2)\}$. When $|A^2_{h_2}| = 1$, then A is blocked at period 2 (if $A^2_{h_2} = \{(d_1,h_2,2,\alpha)\}$, A is blocked at period 2 via $\{(d_3,h_2,2)\}$, if $A^2_{h_2} = \{(d_1,h_2,2,\beta)\}$, A is blocked at period 2 via $\{(d_2,h_2,2\}$, if $A^2_{h_2} = \{(d_2,h_2,2)\}$, A is blocked at period 2 via $\{(d_3,h_2,2)\}$, and if $A^2_{h_2} = \{(d_3,h_2,2)\}$, A is blocked at period 2 via $\{(d_2,h_2,2)\}$). Thus, we assume that $|A^2_{h_2}| = 2$. Note that individual rationality of A implies that $A^2_{h_2} \neq \{(d_1,h_2,2,\alpha),(d_1,h_2,2,\beta)\}$ by $(d_1,h_2,1) \in A$ and $A^2_{d_1} \neq \{(d_1,h_1,2),(d_1,h_3,2)\}$ by $(d_1,h_1,1) \in A$. By $A^2_{h_2} \neq \{(d_1,h_2,2,\alpha),(d_1,h_2,2,\beta)\}$, it remains to consider the following three cases.

(1) Suppose that $A^2_{h_2} = \{(d_2,h_2,2),(d_3,h_2,2)\}$. By individual rationality of A, A^2 is either

$$\begin{aligned}&\{(d_2,h_2,2),(d_3,h_2,2)\},\\&\{(d_1,h_1,2),(d_2,h_2,2),(d_3,h_2,2)\},\text{or}\\&\{(d_1,h_3,2),(d_2,h_2,2),(d_3,h_2,2)\}.\end{aligned}$$

The former two are blocked at period 2 via $\{(d_1,h_2,2,\alpha)\}$. The last one is blocked at period 2 via $\{(d_1,h_2,2,\beta)\}$.

(2) Suppose that $A^2_{h_2} = \{(d_1,h_2,2,\alpha),(d_3,h_2,2)\}$. By individual rationality of A, A^2 is either

$$\begin{aligned}&\{(d_1,h_2,2,\alpha),(d_3,h_2,2)\},\\&\{(d_1,h_1,2),(d_1,h_2,2,\alpha),(d_3,h_2,2)\},\text{or}\\&\{(d_2,h_1,2),(d_1,h_2,2,\alpha),(d_3,h_2,2)\}.\end{aligned}$$

The former two are blocked at period 2 via $\{(d_2,h_1,2)\}$. The last one is blocked at period 2 via $\{(d_1,h_3,2)\}$.

(3) Suppose that $A^2_{h_2} = \{(d_1, h_2, 2, \beta), (d_2, h_2, 2)\}$. By individual rationality of A, A^2 is either

$$\{(d_1, h_2, 2, \beta), (d_2, h_2, 2)\},$$
$$\{(d_1, h_3, 2), (d_1, h_2, 2, \beta), (d_2, h_2, 2)\}, \text{or}$$
$$\{(d_3, h_3, 2), (d_1, h_2, 2, \beta), (d_2, h_2, 2)\}.$$

The former two are blocked at period 2 via $\{(d_3, h_3, 2)\}$. The last one is blocked at period 2 via $\{(d_1, h_1, 2)\}$.

The above argument shows that every individually rational outcome is pairwise blocked at period 1 or 2. Thus, there exists no pairwise temporally stable outcome in this example.

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