A Dynamic Model of Rational "Panic Buying"*

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Abstract

This paper analyzes panic buying of storable consumer goods, using a dynamic inventory-adjustment model featuring search frictions in shopping. Even if consumers are fully rational, an anticipated temporary increase in consumer shopping costs (caused by a disaster itself or a state of emergency) can trigger an upward spiral of hoarding demand and result in serious panic buying and misallocation of storable goods due to a coordination failure. We demonstrate that price controls help in mitigating hoarding if retail prices are rigid in nature. We propose several welfare-enhancing policy options, such as taxes on purchases and direct distribution of basic necessities, and argue that the timing of policy interventions crucially influences their effectiveness.

Key Words: Hoarding; Panic buying; Disaster; COVID-19; Search frictions; Coordination failure: Heterogeneous agents model; Mean-field game

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1 Introduction

Panic buying, hoarding, and the scarcity of basic necessities, such as toilet paper, hygiene products, and canned foods, were widespread global phenomena during the COVID-19 pandemic crisis.[1] Panic buying, defined here as consumers' purchasing of unusually large amounts of storable consumer goods, is not an occurrence peculiar to the pandemic. It has often occurred historically in anticipation of and response to various types of emergencies, including the 1973 oil crisis,[2] the 2008 global rice crisis,[3] the 2011 Christchurch earthquake,[4] the 2011 East Japan earthquake and tsunami,[5] and the 2017 Hurricane Irma.[6] In every case, panic buying makes shopping more time-consuming and costly than usual, distressing all consumers, particularly those with reduced mobility. Therefore, it is an urgent matter for policymakers to understand under what circumstances panic buying is likely to occur and to institute measures to prevent or mitigate it.

Current economic theory cannot provide a satisfactory explanation for the global toilet-paper shortage under the COVID-19 pandemic. Classical market theory does not apply well to the paper product market because the pandemic caused neither a supply disruption nor a surge in need for consumption of the paper products. Game theory gives insight into why consumers' buying decisions exhibit strategic complementarity—if some consumers buy more, other consumers should also buy more before the store runs out of goods; Nevertheless, simple coordination-game-like models have not unveiled what triggers the equilibrium shift and provided little quantitative policy implications.

[1] According to Arafat, Kar, Menon, Kaliamoorthy, Mukherjee, Alradie-Mohamed, Sharma, Marthoenis, and Kabir (2020b), who collected English-language media reports from 20 countries and regions, there were 214 news reports including the keyphrase *panic buying* published until May 22, 2020. They report that the majority of media reporting on panic buying was from the United States (40.7%), the United Kingdom (22%), and India (13.6%). See also Keane and Neal (2021), who develop a data set for measuring consumer panic using Google search data from 54 countries.

[2] For example, Malcolm (1974) documents the experiences of panic buying in the United States and Japan.

[3] See Dawe and Slayton (2010) and Hansman, Hong, de Paula, and Singh (2020).

[4] See Lauder (2011). Forbes (2017) studies the short-term changes in consumer behavior.

[5] See Ozasa and Watanabe (2011).

[6] See Alvarez (2017). Note that, since the arrival of hurricanes is (somewhat) predictable, the extensive stocking up that we have defined as "panic buying" occurred in anticipation of the disaster.

The aim of this study is to provide a welfare-based analysis of panic buying of storable necessities in times of disaster. To this end, we develop a novel dynamic consumer-inventory adjustment model with a continuum of consumers. One of the features of our model is that, in the equilibrium dynamics, optimal purchasing behaviors at the individual level can lead to "panic" at the social level. In our model, while all the consumers form rational expectations about the evolution of the market outcomes, a coordination problem among the consumers results in excessive hoarding, leaving room for welfare improvement via government policies to curb it. In this respect, our model shares the spirit of a coordination game, but it differs from coordination-game models with equilibrium multiplicity in that our framework can uncover fundamental shocks that trigger panic buying and quantitatively evaluate the welfare costs of panic buying.

This study focuses on a change in non-pecuniary costs associated with shopping activities (so-called *shopping costs*) as a fundamental driving force behind panic buying. In the event of a large-scale disaster, it often becomes harder than usual to go out shopping. The COVID-19 pandemic was no exception, especially as movement restrictions imposed by various public health policies (e.g., social distancing, lockdown, and travel restrictions) increased the time and effort required to shop for daily use products in many areas. In fact, a recent empirical study by Keane and Neal (2021) finds that announcements of movement restrictions in response to the pandemic played an important role in amplifying panic buying.[7] This study provides a theoretical explanation for the evidence by exploring how a temporary increase in shopping costs causes panic buying. Most importantly, we demonstrate that, even in the absence of fundamental shocks that affect consumption or production as in the case of the toilet-paper market during the COVID-19 pandemic, there can be serious panic buying of storable consumer goods if consumers change their shopping patterns in response to the increased shopping costs.

[7] Specifically, Keane and Neal (2021) measure the degree of movement restrictions using data on a federal closure of primary and secondary schools, a ban on gatherings, encouragement of working from home, restrictions on the use of public spaces, and the shut down of retail and entertainment businesses.

In our dynamic model of the consumer goods, each consumer adjusts their household inventory of daily necessities that are storable at the expense of holding costs. Consumers are willing to consume the goods at a constant rate. Consumers are able to purchase the goods from the marketplace, but purchase of the goods is not instantaneous; prior to a purchase, consumers need to engage in costly shopping search to find the in-store stock. In the presence of shopping costs and uncertainty about purchasing opportunities, consumers adjust their household inventory infrequently and in a lumpy fashion. Specifically, they start a shopping search once their current stock becomes smaller than a certain threshold, and purchase a larger amount upon finding a seller. A salient feature is that the threshold and purchase quantity are affected by the degree of market congestion: as the market becomes more congested, they start shopping earlier (raise the threshold) and purchase a larger amount, expecting that it will take longer to find the in-store stock.

We show that a temporary increase in shopping costs may produce catastrophic consequences for the economy through an upward spiral of demand for hoarding. In response to the shock, consumers attempt to save the cost of shopping searches by purchasing a larger amount per purchase opportunity. The hoarding demand boosts the market demand and sharply reduces the in-store stock available in the market. Expecting that the goods available in the market will be scarce, more consumers rush to the market to purchase the goods before the scarcity takes place. This consumers' action amplifies the market demand further and exacerbates the scarcity of the goods in the market. As a result, consumers face a higher risk of exhausting the goods, spend more time searching for them, and incur a higher holding cost.

In the spiral, individual consumers escalate hoarding for fear of running out of necessities. The individuals act in their own self-interest and fail to internalize the effect of their hoarding behaviors on the market outcome and other consumers. Hence, the optimal decisions at the individual level can result in excess hoarding at the level of society. In this paper, we develop a new decomposition scheme to isolate the portion of the welfare costs that are attributable to the coordination problem. Using the scheme, we find that (i) the coordination problem becomes drastically more serious when the shock is of a certain magnitude, and (ii) when severe panic buying is occurring, the welfare costs attributable to the coordination problem could be much larger than the direct impact of the underlying shock. With our simulation setting, the effect from the coordination failure is over ten times as large as the direct impact.

We demonstrate that the timing of advance announcements for an emergency crucially influences the severity of panic buying. In most countries, movement restrictions imposed against the COVID-19 pandemic were announced in advance of implementation. We examine the dynamic response to an *anticipated* increase in shopping costs. We find that, unless the increase in shopping costs is announced well in advance of its onset, the anticipated shock triggers much more severe panic buying than the unanticipated shock. This implies that there is a non-monotone relation between the severity of panic buying and the lag between announcement and implementation, and that announcing it a few days prior to the implementation leads to the worst consequence.

We also study whether governments should introduce legal price controls on basic necessities in the wake of disasters and emergencies. As of 2021, more than half of the US states have anti-price-gouging laws that restrict retailers from charging exorbitant prices on consumer good during emergency situations to protect consumers from rising living costs.[8] However, there are considerable opposition to such legal price controls as they would exacerbate shortages. Contrary to these concerns, this study demonstrates that, if retail price adjustments are rigid in nature, an increase in demand or a decrease in supply reminds consumers of future price increases, which further accelerates hoarding, and thus, price controls are rather effective in curbing panic buying by discouraging hoarding for fear of future price increases. It is therefore suggested that whether price controls are beneficial in times of disaters relies on the underlying market's ability of price adjustments.

[8]In the United States, starting with the first state law prohibiting price gouging enacted in New York in 1979 in response to rising winter heating oil prices in 1978-1979, these measures got adopted by other states. During the COVID-19 pandemic, 42 states activated some form of price-gouging regulations. See Bae (2009), Giberson (2011), and Chakraborti and Roberts (2021).

Retail price adjustments of daily necessities in regular times are far from flexible (for microdata evidence, see, e.g., Bils and Klenow, 2004; Nakamura and Steinsson, 2008; Klenow and Malin, 2010; Cavallo and Rigobon, 2016). Furthermore, recent empirical evidence has suggested that retail prices do not rise instantaneously even when disasters cause spikes in demand or disruptions in supply.[9] In particular, Gagnon and López-Salido (2019) and Hansman et al. (2020) show that US retailers were hesitant to raise prices even to the extent not prohibited by anti-price-gouging laws, emphasizing that reputable retailers maintained their prices in times of emergency (see also Cabral and Xu (2021) for evidence on the reputation effect observed in times of the COVID-19 pandemic).[10] In light of these facts, retail prices seem highly rigid in nature even under emergency situations. We, therefore, argue that legal price controls, such as anti-price-gouging laws, play a certain role in preventing consumers from accelerating hoarding.

Finally, we propose several policy options to curb panic buying. First, we consider a short-term sales tax hike to temporarily raise the prices of the basic necessities. We find that such a short-term tax hike is effective only if it is implemented immediately upon the announcement of the restricted movement, and even a few days delay in its implementation could result in exacerbating panic buying. Second, we consider a policy that the government distributes the basic necessities to consumers through non-market rationing mechanisms. We find that the distribution policy performs well in reducing the congestion of the market even if the government is unable to distribute to the whole population; The distribution policy indirectly makes everyone better off, including those who fail to receive the rationed

[9]For example, Cavallo, Cavallo, and Rigobon (2014) find that supermarket prices were relatively stable after a sharp decline in product availability due to the 2010 earthquake in Chile and the 2011 earthquake in Japan. Gagnon and López-Salido (2019) report modest effects on retail prices in United States supermarkets in response to large swings in demand triggered by the labor conflicts in 2003 in St. Louis, MO and Southern California, Hurricane Katrina in 2005, and shopping sprees around major snowstorms and hurricanes. Related to these empirical evidences, Nakamura and Zerom (2010) investigate the sources of the delayed and incomplete pass-through of changes in costs to retail prices.

[10]Akerlof's (1980) theory of social norms is one of the theories that suggests that reputable firms refrain from price gouging for fear of damaging their reputation. Related to this theory, the questionnaire study of Kahneman, Knetsch, and Thaler (1986) suggests that fairness considerations influence the price-setting behaviors. Rotemberg (2005) develops the model of price adjustment that allows for customer's reaction based on fairness considerations.

goods.[11] Third, we show that the purchase-quota policy, which is often implemented in stores, is effective if it is enforced before panic buying arises.

The main technical challenge we tackled in this paper is computing the equilibrium dynamic response to a temporal shopping-costs shock. Our model allows for heterogeneity of consumers in quantity of the consumer goods held in their private inventory. In this paper, we formulate the model in continuous time to employ a framework of mean-field games. We solve the system of partial differential equations by customizing the numerical methods established in Achdou, Han, Lasry, Lions, and Moll (2021), which was originally developed for analyzing income and wealth distribution in dynamic general equilibrium models.

The paper proceeds as follows. Section 2 discusses the literature. Section 3 presents a model of the market for storable consumer goods. Section 4 formally defines the rational-expectations equilibrium of the model. Section 5 illustrates the stationary equilibrium of the model. Section 6 studies the economy's dynamic responses to disasters with various scenarios and explores desirable policy interventions. Section 7 makes concluding remarks.

2 Related Literature

Several economic studies have provided empirical analyses for the markets for storable consumption goods, such as laundry detergent (Hendel and Nevo, 2006) and soft drinks (Hendel and Nevo, 2013), and articulated the practical importance of intertemporal demand effects. Recent studies have emphasized the importance of intertemporal demand effects in explaining panic buying. Using US supermarket scanner data covering the 2008 global rice crisis, Hansman et al. (2020) find that, due to the rigidity in retail prices, a negative supply shock produces an expected price rise, which leads consumers to buy early and stockpile. Using online search data during the COVID-19 pandemic, Keane and Neal (2021) and Prentice,

[11] For example, in Japan, amid the spread of COVID-19 and a shortage of masks, the government distributed two washable masks to each of 50 million households. However, this rationing policy policy drew criticism for its unfairness and slow delivery (Eguchi, Kamizawa, and Okazaki, 2020). Our results suggest that, despite these shortcomings, this mask distribution policy might have mitigated panic buying.

Chen, and Stantic (2020) emphasize that the announcement of government measures for combating the pandemic triggered panic buying. Our study contributes to the literature in providing a theoretical framework to explain how intertemporal demand effects lead to panic buying and a formal welfare analysis of how panic buying harms consumers.

Panic buying is also analyzed in the literature of microeconomic theory. Awaya and Krishna (2021) study a two-stage model of misinformation-driven panic buying. In contrast, this paper studies purchasing behaviors of long-lived consumers in an infinite-horizon and continuous-time environment. This formulation provides quantitative implications on the severity of panic buying and the effectiveness of policies.

This study is also related to a large literature in macroeconomics that studies the role of lumpiness in the propagation of aggregate shocks. Following the pioneering work by Caplin (1985), Grossman and Laroque (1990), Caballero and Engel (1991), and Caballero (1993), a number of studies have employed the (S, s) inventory model in analyzing demand for durable or storable consumer goods. In recent years, several papers (e.g., Berger and Vavra, 2015; Baker, Johnson, and Kueng, 2021; McKay and Wieland, 2019) have developed rich (S, s) frameworks to quantitatively study how micro lumpiness translates into aggregate consumption dynamics. Compared with these previous studies, the nature of the adjustment costs in our model differs from that employed in theirs. Consumers in our model cannot choose directly when to adjust their inventory due to search frictions in product markets. Instead, they choose when to start searching for the opportunity to adjust their stock, taking into account how long they have to spend on costly shopping searches.

Several studies have tried to explain panic buying as a consumer's irrational behavior. In particular, Sterman and Dogan (2015) demonstrate that panic buying may occur even in a lab experiment in which panic buying is never rationalized. In contrast to theirs, our study shows that panic buying of storable consumer goods may arise as a result of collective action by fully rational individuals. We conjecture that the behavioral motivations for hoarding can be additional forces that accelerate panic buying.

3 Model

3.1 Overview

We present a model of the dynamic inventory adjustment of a storable consumer good (e.g., toilet paper). Time is continuous and infinite, $t \in \mathbb{T} := [0, \infty)$. In this economy, there is a unit mass of consumers. Non negative random variables $k_i(t) \geq 0$ denote the inventory of the good held by consumer $i \in [0, 1]$. The cross-sectional distribution function of the consumer's inventory at time t is denoted by $G(t, k) = \int_{i \in [0,1]} \mathbb{1}_{\{k_i(t) \leq k\}} di$ for $k \in \mathbb{K} = \mathbb{R}_+$.

We assume, as in the model in Blanchard (1985), consumers stochastically exit from the economy at a Poisson rate $\theta > 0$ and a mass θ of new consumers enters per unit of time, so that total population size is kept at one. We further assume that the consumers who exit take their stock away. Newly-entered consumers start with initial stock $k_o > 0$, which is drawn from a (time-invariant) distribution function G_{new} that has a density function g_{new}.

There is a marketplace in which a store sells storable goods. The store can hold the good in its warehouse. $S(t) \geq 0$ denotes the store's stock in the warehouse at time t. The good is replenished to the warehouse at an exogenous rate $s \geq 0$ every time.

To purchase the good, the consumers have to travel to the marketplace and find a store. However, due to search friction, they cannot find a store instantly and must search for it for a period of time. These processes incur costs such as travel costs, costs of acquiring product information, and opportunity costs of the time spent shopping. These costs are collectively referred to as "shopping costs."[12] Hence, shopping is costly and time-consuming to consumers.

We denote the unit sales price of the good in time t by $p(t)$. We assume that, in the long-run stationary equilibrium, the market price is established so that supply and demand flow are balanced, but this is not the case for the short-term dynamics after a (disaster-induced) shopping-costs shock. As emphasized in Su (2010), the intertemporal pricing policy for

[12]Note that shopping costs in our model are not one-time fixed costs but flow costs incurred while searching in the market.

storable goods is complex and beyond the scope of this study; thus, we instead treat the price as an exogenous variable. However, our model determines market demand endogenously. The flow demand and supply are not always balanced. In particular, when market demand is extremely high, the store is likely to be in short supply. In such a case, the store will remain open while supply lasts and will shut customers out when its stock is sold out.

3.2 The Consumer's Problem

The individual consumer's inventory $k_i(t)$ evolves over time as a result of consumption and purchases from the store in the marketplace. Since the good is storable, the amount not consumed today is kept as inventory for future consumption. Reselling of the good is not alloa.[13] Depreciation of the good is not explicitly considered, since we focus on the short-term behavior of the economy. At every time t, a consumer chooses (i) the flow consumption $x_i(t) \in \mathbb{R}_+$, (ii) whether to do a shopping search $A_i(t) \in \{0, 1\}$, and (iii) how much to buy upon finding available stock at the store, $q_i(t) \in \mathbb{R}_+$.[14]

To purchase the good, a consumer has to engage in a costly shopping search ($A_i(t) = 1$). We assume that, while searching, a consumer finds the store at a Poisson rate $\alpha > 0$. We describe the matching process by an idiosyncratic store-finding shock $\{N_i(t)\}_{t\in\mathbb{T}}$ with $\text{Prob}(dN_i(t) = 1) = 1 - \mathrm{e}^{-\alpha \cdot dt}$ for all consumers.

At every time t, there is a mass of consumers who find a store. In what follows, we refer to the consumers who find a store at time t as *buyers* at time t. Upon finding a store, the buyers are randomly sorted into a queue for purchase, and then allowed to purchase the desired quantity $q_i(t) \geq 0$ in order of the queue *as long as the store is open.* Here, we emphasize that even if a buyer finds a store, she is not necessarily able to make a purchase from the store—the store may be closed because it has run out of stock before her turn comes. Consequently, only a fraction $R(t) \in (0, 1]$ of the buyers are actually able to make a

[13] Hansman et al. (2020) in their empirical analysis find that hoarding during the 2008 Global Rice Crisis was mostly for the consumer's own use. They argue that this seemed to be the case for hoarding during the COVID-19 pandemic as well, referring to media reports at the time.

[14] We assume that total spending on the good is relatively small compared to total expenditures.

purchase. Hence, the individual buyer who has found a store (but *before* knowing her order in the store's queue) faces an idiosyncratic event of whether or not she is able to make a purchase from the store. With assumptions made above, we can represent the idiosyncratic event as an independent idiosyncratic shock $z_i(t)$ drawn from the Bernoulli distribution $Ber(R(t))$.

In sum, a searching consumer faces two types of idiosyncratic risk: (i) whether she can find a store, and (ii) whether she can make a purchase there after finding the store. Accordingly, the time evolution equation of the consumer's inventory is expressed as

$$dk_i(t) = -x_i(t)dt + A_i(t) \cdot [dN_i(t) \cdot z_i(t)] \cdot q_i(t). \tag{1}$$

In the right-hand side of (1), the first term represents consumption, while the second term represents purchase of the good. Note that $k_i(t)$ is a càdlàg process (right continuous with left limit). In particular, when the consumer makes a purchase, the amount of her private stock jumps to $\bar{k}_i(t) = k_i(t^-) + q_i(t)$, where $k_i(t^-) := \lim_{s \uparrow t} k_i(s)$ is the amount of the good in her inventory she held *just before* purchasing the good.

We turn to the decision making faced by the consumers. Each consumer seeks to maximize the expected present value of her total payoff, discounting the future at a rate of $\rho > 0$. The instantaneous payoff is given by

$$d\pi_i(t) = [u_i(t) - b_i(t) - A_i(t) \cdot c(t)] \cdot dt - (A_i(t) \cdot dN_i(t) \cdot z_i(t)) \cdot p(t) \cdot q_i(t),$$

where $u_i(t)$ is the flow utility from consumption, $b_i(t)$ is the flow holding cost, and $c(t)$ is the flow cost associated with shopping searches. Note that $c(t)$ and $p(t)$ are common to all consumers.

The flow utility from consumption $u_i(t)$ depends on the flow consumption $x_i(t)$:

$$u_i(t) = u(x_i(t)) = \begin{cases} 0, & x_i(t) \geq 1; \\ -a, & x_i(t) < 1. \end{cases}$$

Considering that the good is one of the daily necessities, we assume that the "need" is highly inelastic because the good is not substitutable: a consumer needs a unit of the good for a unit of time, but she receives a large disutility $a \gg 0$ if she fails to consume it. We assume that a is sufficiently large so that consumers engage in shopping searches at least when they are out of stock (See Assumption 2 presented in Section 5 for the formal condition).

Given this flow utility function u, it is clearly optimal to choose the flow consumption $x_i(t) = 1$ whenever the consumer has some stock of the good (i.e., $k_i(t) > 0$).

$$x_i(t) = x(k_i(t)) = \begin{cases} 1, & k_i(t) > 0; \\ 0, & k_i(t) = 0. \end{cases}$$

Then, the indirect utility $u(x(k_i(t)))$ is a concave function of $k_i(t)$.

The holding cost is the cost associated with storing the good in her storage, which therefore is increasing in $k_i(t)$. We assume that it takes a linear function $b_i(t) = \bar{b} \cdot k_i(t)$ with $\bar{b} > 0$. Then, we define a function $h(k_i(t)) \coloneqq u(x(k_i(t))) - \bar{b} \cdot k_i(t)$, which specifies a net flow utility from holding inventory $k_i(t)$. Accordingly, each consumer uses $k_i(t)$ as a state variable and decides when to start searching and how much to purchase upon finding the store to maximize $\mathbb{E}\left[\int_{s=0}^{\infty} \mathrm{e}^{-rs} d\pi_i(s)\right]$ with $r = \rho + \theta$ being the effective time-discount rate.[15]

3.3 Aggregate Dynamics

Let $dD(t)$ denote the total amount of the goods demanded by the consumers who arrived at the store over the infinitesimal time interval $[t, t + dt]$. With notions introduced above,

[15]Recall that ρ is the subjective time-discount rate, while θ is the exogenous exit rate. Here, we assume that the payoff after exiting from the economy is zero.

$dD(t)$ is given by

$$dD(t) = \int_{i\in[0,1]} A_i(t) \cdot dN_i(t) \cdot q_i(t)di = \alpha \left(\int_{i\in[0,1]} A_i(t) \cdot q_i(t)di \right) dt = d(t)dt,$$

where $d(t) := \alpha \left(\int_{i\in[0,1]} A_i(t) \cdot q_i(t)di \right)$ is the flow rate of demand at time t.

Only the fraction $R(t)$ of such consumers are able to make a purchase at the store. We refer to $R(t)$ as the *availability* (of the goods in the market) at time t. Hence, the total amount of the good actually purchased over the infinitesimal time interval $[t, t + dt]$ is $R(t)dD(t) = R(t)d(t)dt$. In this respect, we refer to $d(t)$ as the *potential demand* flow for the good, as distinguished from the amount purchased.

According to the store's selling rules described above, the availability $R(t)$ is determined by the following rationing rule:

$$R(t) = \begin{cases} 1, & S(t) > 0; \\ \min\left\{ \frac{s}{d(t)}, 1 \right\}, & S(t) = 0. \end{cases} \tag{2}$$

This rule shows that rationing (i.e., $R(t) < 1$) only occurs when the store is out of stock ($S(t) = 0$) and the potential demand flow exceeds the flow of the store's supply ($d(t) > s$). When this occurs, the total amount purchased is limited by the store's supply: $R(t)d(t) = s$.

Finally, we can write the time evolution equation of the store's stock as follows:[16]

$$\dot{S}(t) = s - R(t)d(t), \tag{3}$$

with an initial condition $S(0) = S_o > 0$. That is, the store's stock at time t is the amount of goods left unsold by time t. Note that $S(t) \geq 0$ for all $t \in \mathbb{T}$ since $dS(t) \geq 0$ if $S(t) = 0$.

[16] Below, the dot above a variable denotes the derivative with respect to time.

4 Equilibrium Definition

In this section, we formulate the optimization problem of a consumer and formally define a rational-expectations equilibrium for the economy.

4.1 Consumers' Optimization

Let $\boldsymbol{Y}(t) := (S(t), G(t,k))'$ denote the set of *endogenous aggregate state variables.*[17] Consumers make decisions on when to start shopping searches, making a belief about the future path of availability $\{R(\tau)\}_{\tau \geq t}$. We assume that their belief for $\{R(\tau)\}_{\tau \geq t}$ is rational: the perceived and actual laws of motion for availability is identical. To be more specific, they use the following forecasting rule:

$$\dot{\boldsymbol{Y}}(t) = \Gamma_Y(\boldsymbol{Y}(t)), \qquad \text{and} \qquad R(t) = \Gamma_R(\boldsymbol{Y}(t)). \tag{4}$$

That is, they forecast the evolution of the aggregate state variables $\{\boldsymbol{Y}(\tau)\}_{\tau \geq t}$ recursively, using the current state $\boldsymbol{Y}(t)$ as an initial condition and then apply Γ_R to forecast availability.

The other relevant state variable for the individual consumer is her stock of the good, $k_i(t)$. Let $V(\boldsymbol{Y}(t), k_i(t))$ be the value function for a consumer who has stock $k_i(t)$ at time t, and let $V^*(\boldsymbol{Y}(t), k_i(t))$ be the expected value for her searching in the good market. The consumer's problem can be formulated as the following optimal stopping-time problem:

$$V(\boldsymbol{Y}(t), k_i(t)) = \sup_{T \geq 0} \mathbb{E}\left[\int_t^{t+T} \mathrm{e}^{-r(s-t)} h(k_i(s)) ds + \mathrm{e}^{-r(T-t)} V^*(\boldsymbol{Y}(t+T), k_i(t+T))\right], \tag{5}$$

where V^* satisfies the Hamilton-Jacobi-Bellman (henceforth, HJB) equation:

$$\begin{aligned} rV^*(\boldsymbol{Y}(\tau), k_i(\tau)) =& h(k_i(\tau)) - c(\tau) + \alpha R(\tau)\left[V^A(\boldsymbol{Y}(\tau), k_i(\tau)) - V^*(\boldsymbol{Y}(\tau), k_i(\tau))\right] \\ &+ \frac{\partial V^*(\boldsymbol{Y}(\tau), k_i(\tau))}{\partial \boldsymbol{Y}} \dot{\boldsymbol{Y}}(\tau) - \frac{\partial V^*(\boldsymbol{Y}(\tau), k_i(\tau))}{\partial k} x(k_i(\tau)), \end{aligned} \tag{6}$$

[17] Throughout the analysis below, we do not consider aggregate uncertainty. Thus, take $\boldsymbol{Y}(t)$ to be the deterministic path for a set of endogenous aggregate state variables.

and $V^A(\boldsymbol{Y}(\tau), k)$ is the value right after purchasing at time $\tau \geq t$, which is given by

$$V^A(\boldsymbol{Y}(\tau), k) = \max_{\bar{k} \geq k} V(\boldsymbol{Y}(\tau), \bar{k}) - p(\tau) \cdot (\bar{k} - k), \tag{7}$$

subject to $k_i(\tau) = k_i(t) - \int_t^\tau x(k_i(s))ds$ for $\tau \in [t, t+T)$ and the forecasting rule (4).[18]

The optimal stopping-time problem (5) induces the *optimal stopping-time policy* $T(\boldsymbol{Y}(t), k)$. We define the *action region* as $\mathcal{A}(\boldsymbol{Y}(t)) = \{k \in \mathbb{K} \mid T(\boldsymbol{Y}(t), k) = 0\}$, which is the set of the states k at which the consumer engages in a shopping search. Since consumers have a stronger incentive to go shopping when they have smaller stocks in their inventory, the action region clearly takes an interval structure: $\mathcal{A}(\boldsymbol{Y}(t)) = [0, k^*(\boldsymbol{Y}(t))]$. We refer to k^* as the *go-shopping threshold*.

The maximization problem (7) derives the decision rule on the purchase quantity. Let $\bar{k}(\boldsymbol{Y}(t), k)$ be the solution of (7). It is clear that, if $\bar{k}(\boldsymbol{Y}(t), k) \geq k$, then $\bar{k}(\boldsymbol{Y}(t), k)$ satisfies

$$\frac{\partial V(\boldsymbol{Y}(t), \bar{k}(\boldsymbol{Y}(t), k))}{\partial k} = p(t).$$

Hence, independent of the current stock $k_i(t)$, all searching consumers desire to increase their stock to the same level $\bar{k}(\boldsymbol{Y}(t))$.[19] In what follows, we refer to $\bar{k}(\boldsymbol{Y}(t))$ as the *target stock*.

In the end, the consumers' decision rule can be characterized by two variables: the go-shopping threshold $k^*(\boldsymbol{Y}(t))$ and the target stock $\bar{k}(\boldsymbol{Y}(t))$. As illustrated in Figure 1, they engage in a shopping search if and only if their inventory stock is smaller than $k^*(\boldsymbol{Y}(t))$: once they find an open store, they stock up to $\bar{k}(\boldsymbol{Y}(t))$.

4.2 Law of Motion for the Aggregate State

Given the consumers' decision, we derive the (actual) law of motion for the aggregate variables. First, the consumers' optimal strategy induces a mapping Ψ_d from the aggregate state

[18] See, for example, Stokey (2009) for the formulation of the Bellman equation for optimal stopping-time problems.

[19] The consumers who choose $\bar{k}(\boldsymbol{Y}(t), k) = k$ clearly do not search since there is no gain from searching.

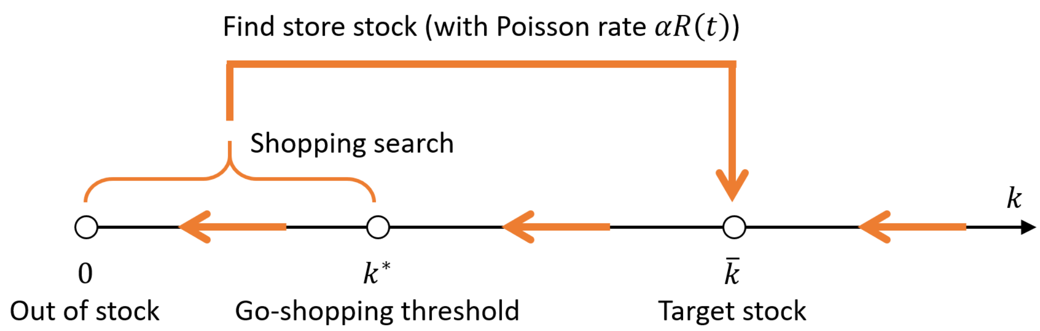


Figure 1: The dynamics of a consumer's stock k, which is characterized by the go-shopping threshold k^* and the target stock $\bar{k}$. For every $k > 0$, the rate of consumption is 1.

$\boldsymbol{Y}(t)$ to the potential demand $d(t)$ as

$$d(t) = \Psi_d(\boldsymbol{Y}(t)) := \alpha \left(\int_{k \in [0, k^*(\boldsymbol{Y}(t))]} q(\boldsymbol{Y}(t), k) g(t, k) dk \right),$$

where $q(\boldsymbol{Y}(t), k)$ is the optimal purchase quantity, defined as $q(\boldsymbol{Y}(t), k) := \max\{\bar{k}(\boldsymbol{Y}(t)) - k, 0\}$, and $g(t, \cdot)$ is a generalized probability density function of the distribution function $G(t, \cdot)$.[20] Recall that the availability $R(t)$ is determined by $d(t)$ and $S(t)$ according to (2). Therefore, $R(t)$ can also be written with a mapping Ψ_R as $R(t) = \Psi_R(\boldsymbol{Y}(t))$.

Then, given the consumer's decisions, the Kolmogorov forward (henceforth, KF) equation for the measure of consumers g can be written as

$$\frac{\partial g(t,k)}{\partial t} = \begin{cases} \frac{\partial g(t,k)}{\partial k} x(k) + \theta \left[g_{new}(k) - g(t,k) \right] - \alpha \Psi_R(\boldsymbol{Y}(t)) g(t,k), & k \in \mathcal{A}(\boldsymbol{Y}(t)), \\ \frac{\partial g(t,k)}{\partial k} x(k) + \theta \left[g_{new}(k) - g(t,k) \right] \\ \qquad + \alpha \Psi_R(\boldsymbol{Y}(t)) G(t, k^*(\boldsymbol{Y}(t))) \delta(k - \bar{k}(\boldsymbol{Y}(t))), & k \notin \mathcal{A}(\boldsymbol{Y}(t)). \end{cases} \tag{8}$$

From (3), the law of motion for $S(t)$ can be written as: $\dot{S}(t) = (s - \Psi_d(\boldsymbol{Y}(t)) \Psi_R(\boldsymbol{Y}(t)))$.

[20] Note that G may have mass points at the boundary ($k = 0$) or in the interior. Thus, we define a generalized probability density function g that satisfies (i) $\int_{k' \in \mathbb{K}} g(t, k') dk' = G(t, k)$ and (ii) $g(t, k) = \hat{g}(t, k) + \sum_{i=1,\ldots,I} m(t, \kappa_i) \delta(k - \kappa_i)$, where $\hat{g}(t, \cdot)$ is a probability density function (a Lebesgue-integrable real valued function), $m(t, \kappa_i)$ is the probability mass at $\kappa_i \in \mathbb{K}$, and $\delta(\cdot)$ is the Dirac delta function.

Therefore, we can write the law of motion for $\boldsymbol{Y}(t)$ as: $\dot{\boldsymbol{Y}}(t) = \Psi_Y(\boldsymbol{Y}(t))$. Accordingly, the consumer's decision rules and the aggregation formulas induce a mapping from the perceived law of motion for the aggregate state variables to an actual law of motion for them.

4.3 Rational-Expectations Equilibrium

In a rational-expectations equilibrium, given the path of exogenous variables $\{c(t), p(t)\}_{t \in \mathbb{T}}$, (i) consumers make optimal decisions based on the perceived law of motion, and (ii) the perceived law of motion is consistent with the actual one.

Definition 1 (Rational-Expectations Equilibrium)**.** A *rational-expectations equilibrium* is defined by a path of the aggregate state variables $\boldsymbol{Y} = (S, G)$, a perceived law of motion Γ_Y, Γ_R, and consumer's decision rules $\{k^*, \bar{k}\}$ with associated value functions $\{V, V^*\}$ such that the following conditions hold:

(i) Consumer's optimization: for every $t \in \mathbb{T}$, $k \in \mathbb{K}$, and $\boldsymbol{Y}(t)$, the decision rules $\{k^*, \bar{k}\}$ and the value functions $\{V, V^*\}$ solve the consumer's optimization problem along with the consumer's beliefs Γ_R and Γ_Y.

(ii) Aggregates are determined by individual actions and the aggregate state variables: $d(t) = \Psi_d(\boldsymbol{Y}(t))$, $R(t) = \Psi_R(\boldsymbol{Y}(t))$, and $\dot{\boldsymbol{Y}}(t) = \Psi_Y(\boldsymbol{Y}(t))$, for all $\boldsymbol{Y}(t)$.

(iii) Consumers' beliefs are rational expectations: $\Gamma_Y = \Psi_Y$ and $\Gamma_R = \Psi_R$.

5 Stationary Equilibrium

As a benchmark of "normal times," we first look at a stationary equilibrium of the economy where all exogenous variables—the flow shopping cost and the price—are constant, i.e., $c(t) = c > 0$ and $p(t) = p$ for all t. We say that an equilibrium is *stationary* if the market demand is constant and the store never runs out of stock. The formal definition is as follows:

Definition 2 (Stationary Equilibrium)**.** A rational-expectations equilibrium is *stationary* if the following conditions are satisfied:

(i) Full availability; Rationing never happens, i.e., $R(t) = 1$.

(ii) The consumer's distribution of the stock is time-invariant; i.e., $G(t, k) = G_o(k)$.

It is clear that in the stationary equilibrium, the value functions and the associated policy functions are all time-invariant, i.e., $V(\boldsymbol{Y}(t), k) = V_o(k)$; $V^*(\boldsymbol{Y}(t), k) = V_o^*(k)$; $k^*(\boldsymbol{Y}(t)) = k_o^*$; $\bar{k}(\boldsymbol{Y}(t)) = \bar{k}_o$. Specifically, the Bellman equations (5) and (6) imply that V_o solves the Hamilton-Jacobi-Bellman variational inequality (HJBVI, henceforth):

$$rV_o(k) = \max\left\{h(k) - V_o^{'}(k)x(k), rV_o^*(k)\right\}, \tag{9}$$

where V_o^* solves the HJB equation:

$$rV_o^*(k) = h(k) - c - V_o^{*'}(k)x(k) + \alpha\left[\left(\max_{q>0} V_o(k+q) - pq\right) - V_o^*(k)\right].$$

The measure of the consumer satisfies the KF equation

$$0 = \begin{cases} g_o'(k)x(k) + \theta\left[g_{new}(k) - g_o(k)\right] - \alpha g_o(k), & k \in \mathcal{A}_o = [0, k_o^*], \\ g_o'(k)x(k) + \theta\left[g_{new}(k) - g_o(k)\right] + \alpha G_o(k_o^*)\delta(k - \bar{k}_o), & k \notin \mathcal{A}_o = [0, k_o^*]. \end{cases} \tag{10}$$

With the notations above, the market demand $d(t)$ in the stationary-equilibrium can be written as $d_o = \alpha \int_{k\in[0,k_o^*]} \max\{\bar{k}_o - k, 0\} dG_o(k)$.

5.1 Characterization of the Stationary Equilibrium

We impose the following assumptions, ensuring that the consumers have a threshold $k_o^* \in (0, \infty)$ to start a shopping search when their stock falls below that level.

Assumption 1 (Sufficient Supply)**.** In the stationary equilibrium, the suppliers of the good adjust their supply so that the flow supply and the flow demand are balanced ($s = d_o$).

Assumption 1 ensures that full availability in the stationary equilibrium (condition (i) of Definition 2).

Assumption 2 (Large Out-of-Stock Disutility)**.** The flow disutility from running out of stock, a, is sufficiently large to satisfy

$$\max_{q \geq 0} \; V^N(q) - pq + \frac{a}{r} > \frac{c}{\alpha}, \tag{11}$$

where $V^N : \mathbb{K} \to \mathbb{R}$ is the value function for the consumers that would be achieved if no control is exercised: $V^N(k) := \int_0^\infty \mathrm{e}^{-rs} h\left(\max\{k - s, 0\}\right) ds$.

Assumption 2 requires that the flow disutility from failing to consume the good is so large that consumers cannot forgo shopping. If the disutility is small (for example, because the good is substitutable), then consumers may optimally choose not to consume it. Since we consider the market of an unsubstitutable necessity good, we exclude such a situation.

Assumption 3 (Large Matching Rate)**.** The matching rate α is sufficiently large such that

$$\alpha p > \bar{b}.$$

Assumption 3 requires that search friction is not too strict. Recalling that the matching rate α captures the easiness of shopping during normal times, it is natural to assume that α is large since shopping is an easy task during normal times.

Assumptions 2 and 3 ensure that a stationary equilibrium exists in which all consumers go shopping periodically. Proposition 1 characterizes such a stationary equilibrium.[21]

Proposition 1. *Suppose Assumptions 1, 2 and 3 hold. Then, the stationary rational-expectations equilibrium satisfies the following properties:*

[21] Proofs are in Online Appendix A.

Table 1: Parameter values

Parameters		Value	Target
ρ	Weekly discount rate	0.01/52	The annual discount rate is 1%
θ	Weekly exit rate	0.04/52	The annual replacement rate is 4%
a	Flow disutility from zero consumption	10,000	Normalization
$\bar{b}$	Scale parameter in the holding cost	1.0	Normalization
p	Market price of the good	10.0	Normalization
α	Matching rate	3.5	2 days/month is spent on shopping.
c	Flow shopping-search cost	25	Shopping frequency: once a month

(i) Consumers engage in shopping periodically; i.e., $0 < k_o^* < \bar{k}_o < +\infty$ *is satisfied.*

(ii) The consumer's go-shopping threshold k_o^* *satisfies* $\alpha\left[V_o^A(k_o^*) - V_o^*(k_o^*)\right] = c$.

(iii) The consumer's target stock $\bar{k}_o$ *satisfies*

$$\bar{k}_o = k_o^* + \frac{1}{r}\log\left(1 + \frac{\frac{\alpha p - \bar{b}}{\alpha + r}\left(1 - \mathrm{e}^{-(\alpha+r)k_o^*}\right) + \mathrm{e}^{-(\alpha+r)k_o^*}a - p}{\frac{\bar{b}}{r} + p}\right).$$

(iv) The value function $V_o(k)$ *satisfies*

$$rV_o(k) = \mathbb{1}_{\{k \geq k_o^*\}}\left[\mathrm{e}^{-r(k-k_o^*)}\left(\bar{b}k_o^* - \frac{\bar{b}}{r} + rV_o^*(k_o^*)\right) + \left(\frac{\bar{b}}{r} - \bar{b}k\right)\right] + \mathbb{1}_{\{k<k_o^*\}}rV_o^*(k),$$

where the value of exercising a control $V^*(k)$ *satisfies*

$$V_o^*(k) = \alpha\Lambda(k) + \frac{1}{\alpha + r}\left[\left(1 - \mathrm{e}^{-(\alpha+r)k}\right)\frac{\bar{b}}{\alpha + r} - \bar{b}k - \mathrm{e}^{-(\alpha+r)k}a - c\right],$$

with $\Lambda(k) = \int_0^k \mathrm{e}^{-(\alpha+r)(k-s)}V_o^A(s)ds + \mathrm{e}^{-(\alpha+r)k}Q$, $Q = -[(p+\bar{b}\bar{k}_o)/r + p\bar{k}_o]/(\alpha+r)$, *and* $V^A(k) = \mathbb{1}_{\{k\leq\bar{k}_o\}}[(\alpha+r)Q + pk] + \mathbb{1}_{\{k>\bar{k}_o\}}V_o(k)$.

5.2 Illustration of the Stationary Equilibrium

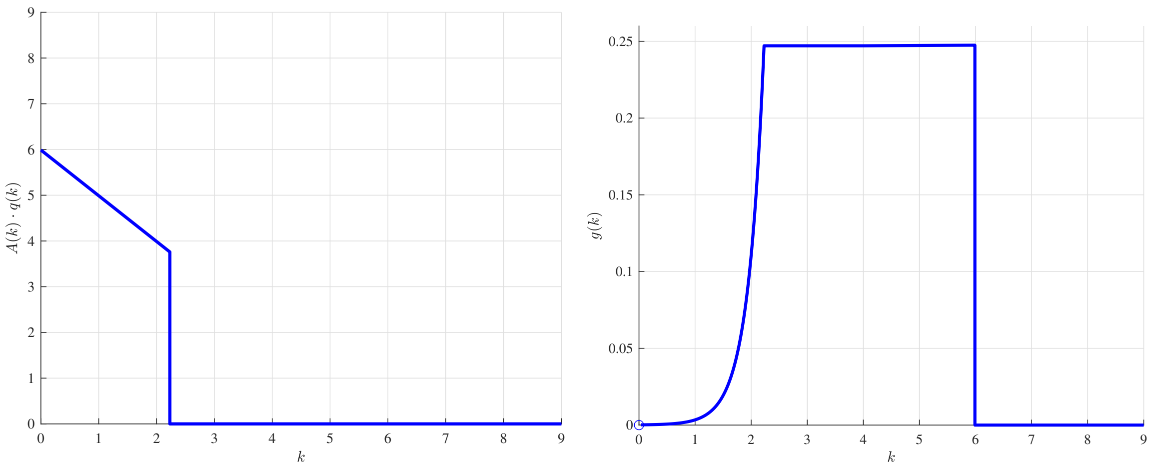
In this subsection, we illustrate a stationary equilibrium using a concrete numerical example. In Table 1, we list the parameter values employed in the numerical exercises. A unit of time is equal to one week. We set the weekly time-discount rate $\rho = 0.01/52$ (the corresponding annual discount rate of 1%) and the weekly exit rate $\theta = 0.04/52$. We consider a market of daily necessities that are hardly substitutable, assuming that the disutility from failing to consume it is large: $a = 10,000$. Note that this specification satisfies Assumption 2. The market price of the good is normalized to $p = 10$. The matching rate α is set to 3.5, implying that it takes two days for searching consumers to find a store. Under these parameter values, in the stationary equilibrium, each consumer goes shopping (roughly) once a month and spends two days per month searching on average. These consumer behaviors are in line with the evidence, drawn from the American Time Use Survey in Petrosky-Nadeau, Wasmer, and Zeng (2016), which documents that the average total shopping time for shoppers in the United States during the years from 2003 to 2012 is 40-50 minutes per day.

In the numerical exercises, we employ the algorithm developed by Achdou et al. (2021). This algorithm is applicable to computing not only the stationary equilibrium but also the equilibrium transition path in response to an unexpected change in model parameters.[22] Specifically, we apply the finite difference method to the HJBVI in (9) and the KF equation in (10) in order to reduce the partial differential equations to a linear complementarity problem, and then solve it iteratively (see Online Appendix B for a detailed description of the computational algorithm we employed).[23]

Figure 2 illustrates a consumer's policy (Panel (a)) and the distribution of the consumer's stock (Panel (b)) in the stationary equilibrium under the parameter values given in Table 1.

[22] The greatest advantage of this algorithm is that it simultaneously solves the HJB equation for the value function and the KF equation for the distribution, using the fact that the HJB operator and the KF operator are adjoints to each other. In recent years, this method has been intensively applied for various continuous-time heterogeneous-agent models in the macroeconomics literature (e.g., Kaplan, Moll, and Violante, 2018; Ahn, Kaplan, Moll, Winberry, and Wolf, 2018; Fernández-Villaverde, Hurtado, and Nuno, 2019).

[23] We employed the algorithm to solve a linear complementarity problem, building on the routines available from Benjamin Moll's personal website `https://benjaminmoll.com/codes/`.



(a) Policy for purchase quantity: $A(k) \cdot q_o(k)$ (b) Consumers' distribution: $g_o(k)$

Note: The horizontal axis represents the amount of the existing consumer's stock. The generalized density function $g_o(k)$ has a mass point at $k = 0$ actually. However, since the mass is very small ($G_o(0) = G_o(k_o^*)\mathrm{e}^{-\alpha k_o^*} \approx 0.3 \times 10^{-5}$), we do not display the math point in Panel (b).

Figure 2: An Illustration of the Stationary Equilibrium

In the stationary equilibrium, as in our daily life, consumers consume the good in their private inventory a constant rate (normalized to 1) and start a shopping search when the stock goes down to $k_o^* \approx 2.3$ (weeks). The searching consumer, upon finding a store, purchases the good to stock up to the target stock, $\bar{k}_o \approx 6$. That is, the amount purchased is $q_o(k) = \bar{k}_o - k \approx 6 - k$. Therefore, no consumer has more than $\bar{k}_o$ in stock, and the fraction $G_o(k_o^*)$ of the consumers engage in shopping searches every time. Since the searching consumer can find an available store with an arrival rate α, the density exponentially increases as k increases for $k \in (0, k_o^*]$. For the inaction region $k \in [k_o^*, \bar{k}_o]$, the density is flat. Although extremely rare, there are consumers who unfortunately continue to fail to find a store and then exhaust the good. With our parameter choice, the share of such stockless consumers ($G_o(k_o^*)\mathrm{e}^{-\alpha k_o^*}$) is less than 0.01%. So, the risk of exhausting the good is very low.

6 Dynamics in an Emergency

In this section, we explore the impacts of an emergency that *temporarily* increases the flow shopping cost $c(t)$ due to various scenarios. Here, we analyze the dynamic response of the economy to an unpredictable (one-time and deterministic) change in $c(t)$, starting from the stationary equilibrium of the model economy. More specifically, until $t = 0$, all the model agents believe that the flow shopping cost is permanently constant, but at time $t = 0$, they are aware of the (future) exogenous change in the flow shopping cost. Below, $X(t)$ denotes the value of a variable X after t time (weeks) after the awareness of the shopping-cost shock.

At time 0, the economy is on the stationary equilibrium: $G(0, k) = G_o(k)$. Our definition of the stationary equilibrium does not determine the initial store stock. In our simulations, we set $S(0) = S_o = 1$, assuming that, in normal times, the store always holds one unit of the goods (which can accommodate the entire population for a week) as a buffer.

In the subsequent sections, we first describe the specification of the shopping-costs shock in Section 6.1 and how we evaluate the welfare impacts in Section 6.2. In Section 6.3, we report the simulation results of various scenarios. In Section 6.4, we investigate the effectiveness of various policy measures, such as increasing the sales tax (Section 6.4.1), nonmarket distribution of basic necessities (Section 6.4.2), and quotas on purchases (Section 6.4.3).

6.1 The Fundamental Shock and the Phases of the Emergency

We specify the path of $c(t)$ using the four parameters $(\bar{c}, T_c^S, T_c^L, T_c^E)$ with $\bar{c} > c$ and $0 \leq T_c^S < T_c^L < T_c^E < \infty$. As illustrated in Figure 3, we consider the following phases of the emergency:

Pre-Disaster Phase ($t < 0$) Prior to time 0, all consumers believe that all the exogenous parameters are stationary, i.e., $(c(t), p(t)) = (c, p)$ for all t. Accordingly, all consumers behave following the stationary-equilibrium strategy, believing that $R(t) = 1$ forever.

Announcement ($t = 0$) At time 0, consumers are aware of unpredictable events that (will) increase flow shopping cost $c(t)$.[24] All consumers are assumed to be fully informed

[24] In some simulation scenarios, the exogenous shift in the sales price $p(t)$ is considered as well.

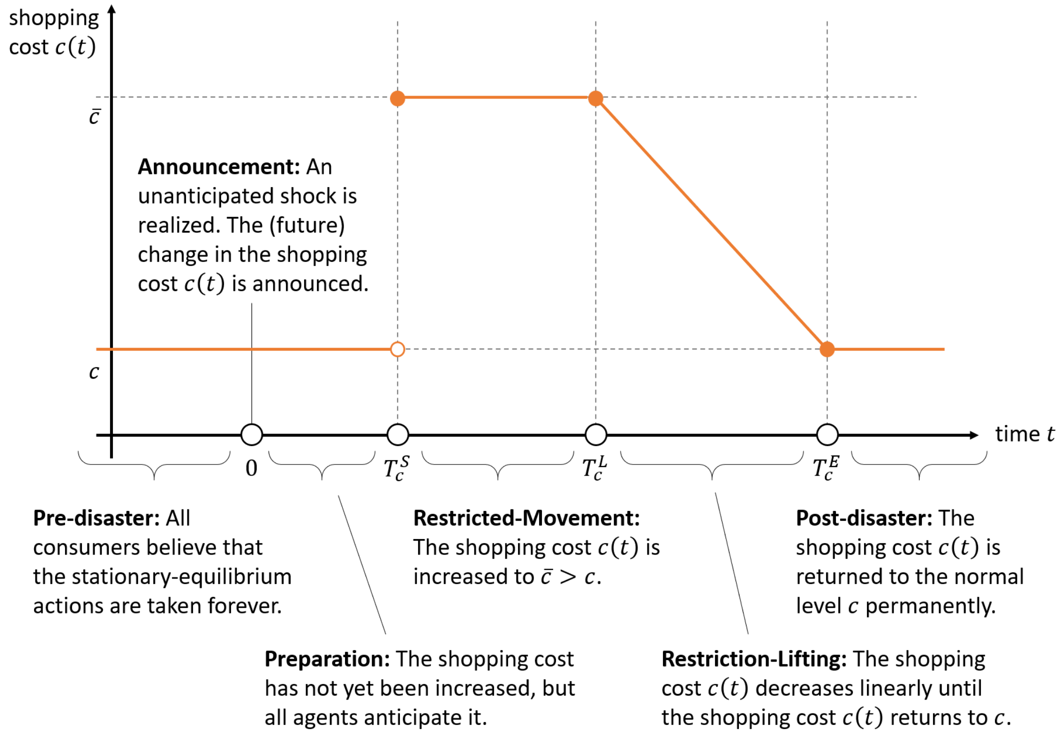


Figure 3: An Illustration of the Phases of the Emergency

about the future path of the economy and thus, at time 0, they immediately react to the change in their beliefs about the path of $\{c(t), p(t)\}$ and endogenous state variables.

Preparation Phase $(0 \leq t < T_c^S)$ Although consumers know that the flow shopping cost $c(t)$ will be increased later, $c(t)$ has not yet increased $(c(t) = c)$. The anticipation of the increase in $c(t)$ could change consumers' behavior even in this phase.

Restricted-Movement Phase $(T_c^S \leq t < T_c^L)$ Movements are restricted due to either the disaster itself or the government's measures. The flow shopping cost $c(t)$ jumps up to $\bar{c}$ at time T_c^S, and it stays at that level until time T_c^L.

Restriction-Lifting Phase $(T_c^L \leq t < T_c^E)$ The restrictions are gradually relaxed. The flow shopping cost $c(t)$ linearly decreases from the maximum level $\bar{c}$ to the normal level c.

Post-Disaster Phase ($T_c^E \leq t$) The "lifting" is completed at time $t = T_c^E$, and then the flow shopping cost $c(t)$ is returned to the normal level ($c(t) = c$) permanently.

To summarize, the dynamics of the flow shopping cost $c(t)$ is given by

$$c(t) = \begin{cases} c, & t < T_c^S, \\ \bar{c}, & T_c^S \leq t < T_c^L, \\ \bar{c}\left(\frac{t - T_c^L}{T_c^E - T_c^L}\right) + c\left(1 - \frac{t - T_c^L}{T_c^E - T_c^L}\right), & T_c^L \leq t < T_c^E, \\ c, & T_c^E \leq t. \end{cases}$$

6.2 Welfare Evaluation

Here, we describe how we evaluate the impact on social welfare. We define the *social welfare* of the economy, denoted by SW, as

$$SW = CS + GR - GE,$$

where CS denotes the (normalized) *consumer surplus*, GR denotes the *government's revenue*, and GE denotes the *government's expenditure*. We define each component as follows.

First, CS measures the average of the changes in the consumers' values at time 0 ($V(0,k) - V_o(k)$) weighted by their measure ($g(0,k)$), i.e.,

$$CS = \int_{k \in \mathbb{K}} [V(0,k) - V_o(k)]\, g(0,k) dk.$$

This captures how the consumers value the surprise at time 0 on average.

Second, GR measures the present value of the government's revenue. In Subsection 6.4.1, we introduce a sales tax as a measure against panic buying. We denote the after-tax price by $\hat{p}(t)$. With this notation, the total government tax revenue at time t is given by $[\hat{p}(t) -$

$p(t)]R(t)d(t)$. Hence, GR is given by

$$GR = \int_0^\infty \mathrm{e}^{-rt} \left[\hat{p}(t) - p(t)\right] R(t)d(t)dt.$$

Third, GE measures the present value of the government's expenditure. In Subsection 6.4.2, we consider the governmental distribution of the goods, that is, a policy according to which the government buys out S_G units of the goods from the market at time t and distributes them directly to consumers immediately. Hence, GE is given by

$$GE = \int_0^\infty \mathrm{e}^{-rt} \left[p(t)S_G\right] dt.$$

In this study, we examine the economy's response against adverse shocks, and thus, because of the increased shopping costs, CS is negative regardless of the efficiency of the equilibrium allocation. We want to isolate the welfare costs attributable to misallocation of the goods due to the coordination failure in consumer's shopping behavior. For that purpose, we consider the following counterfactual. While the shopping cost $c(t)$ is changed by the shock, all consumers (counterfactually) keep taking their stationary-equilibrium strategies. In this case, even after the shock is realized, the availability $R(t)$ would remain one and the same allocation would be achieved as in the stationary equilibrium.[25] This implies that the mass of searching consumers is fixed to $G_o(k_o^*)$ and, therefore, the consumer surplus in the counterfactual becomes

$$LF = G_o(k_o^*) \int_0^\infty \mathrm{e}^{-rt} \left[-\left(c(t) - c\right)\right] dt.$$

The term we labeled LF measures (the negative of) the welfare loss from the fundamental shock. The difference $SW - LF$ captures the welfare costs attributable to misallocation of

[25]The stationary equilibrium allocation is not the first-best one that maximizes the consumer surplus. By distributing the store stock S_o to consumers efficiently, they could be better off. However, we expect that the welfare difference between the first-best allocation and the stationary equilibrium allocation is small. We also conjecture that when $S_o = 0$, the stationary-equilibrium strategy coincides with the first-best one.

Table 2: Summary of Simulation Settings and Results

Shorthand	Benchmark	Magnitude		Announcement		Inflation	
		Large	Small	No Prep	Early	Low	High
Simulation number	1	2	3	4	5	6	7
				Simulation Settings			
$(\bar{c}-c)/c$	100%	120%	80%	100%	100%	100%	100%
T_c^S	1	1	1	0	2	1	1
$T_c^L - T_c^S$	4	4	4	4	4	4	4
$T_c^E - T_c^L$	3	3	3	3	3	3	3
$p(t)$	p	p	p	p	p	Δ2.5%	Δ10%
				Results			
Figure	4	5a	5b	6a	6b	7	7
CS	-130.8	-385.7	-9.0	-8.2	-11.3	-263.1	-744.7
GR	0	0	0	0	0	0	0
GE	0	0	0	0	0	0	0
$SW = CS + GR - GE$	-130.8	-385.7	-9.0	-8.2	-11.3	-263.1	-744.7
LF	-9.7	-11.6	-7.7	-9.7	-9.7	-9.7	-9.7
$SW - LF$	-121.1	-374.1	-1.3	1.5	-1.6	-253.3	-735.0
(Rel. to Benchmark)	(1.0)	(3.09)	(0.01)	(-0.01)	(0.01)	(2.09)	(6.07)

Note: The "No Prep" stands for no preparation phase. See Section 6.1 for the detail descriptions of the four parameters: $\bar{c}$; T_c^S; T_c^L; T_c^E. See Section 6.2 for the definitions of CS, GE, GR, SW, and LF.

the goods due to the coordination failure.

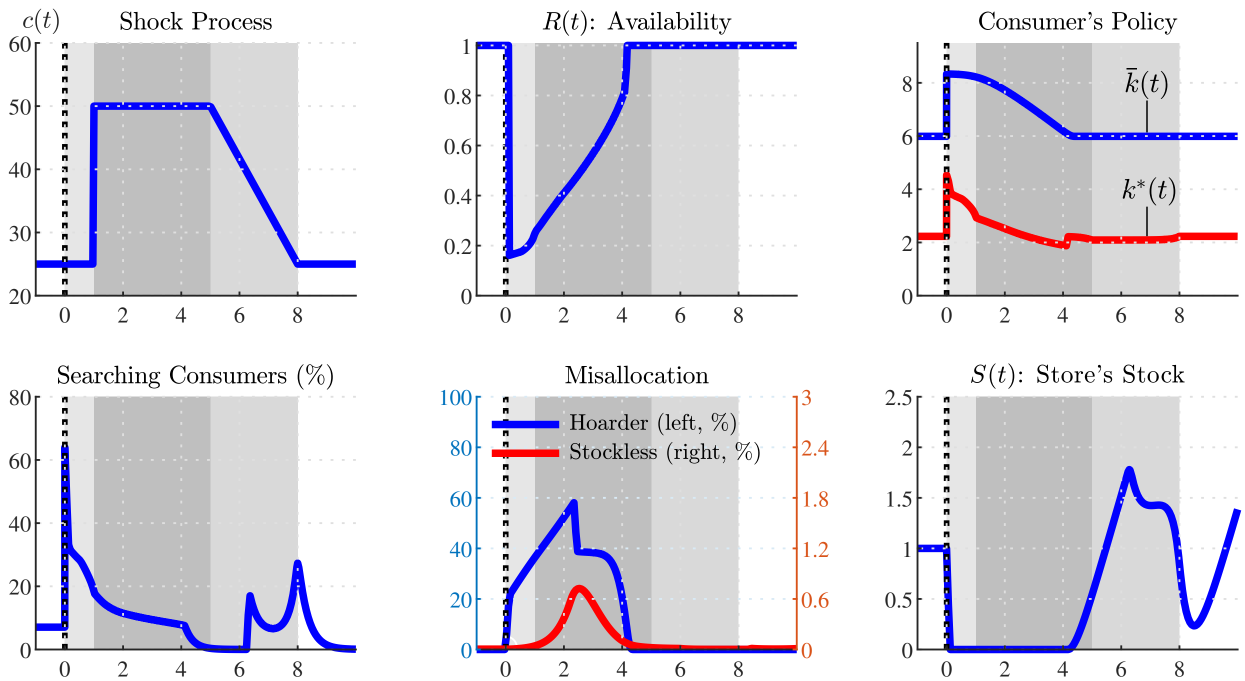
6.3 Simulation Results

In Section 6.3, we report the simulation results for the benchmark scenario (Section 6.3.1), and the alternative scenarios with different magnitudes of the emergency (Section 6.3.2), different duration of the preparation phase (Section 6.3.3), and different underlying price dynamics (Section 6.3.4). Table 2 lists a summary of the settings in each simulation, the corresponding figure, and the results of the welfare analysis.

6.3.1 The Benchmark Case

In this paper, we refer to the following scenario as the *benchmark* case.

Simulation 1 (Benchmark)**.** At time $t = 0$, there is an announcement that movement restriction will be implemented in one week (i.e., $T_c^s = 1$). During the restricted-movement



Note: In all charts, the horizontal axis represents the number of weeks after the announcement (t). The background color of the graph area illustrates the phase of the emergency: the pre-disaster phase (white), the preparation phase (light gray), the restricted-movement phase (dark gray), the restriction-lifting phase (medium gray), and the post-disaster phase (white).

Figure 4: Simulation 1 (Benchmark). $S_0 = 1$, $\bar{c} = 50$, $T_c^S = 1$, $T_c^L - T_c^S = 4$, and $T_c^E - T_c^L = 3$.

phase, the shopping cost is $\bar{c} = 50$; thus, its percentage increase is $(\bar{c} - c)/c = 100\%$. The restricted-movement phase will continue for four weeks ($T_c^L - T_c^S = 4$), and then will be lifted in phases over three weeks ($T_c^E - T_c^L = 3$).

The top-left chart in Figure 4 displays the exogenous path of $c(t)$ used in Simulation 1, where t is the number of weeks after the announcement.[26] The top-middle chart ("$R(t)$: Availability") displays the availability $R(t)$. The top-right chart ("Consumer's Policy") displays the time evolution of the two key variables that characterize the consumers' optimal strategy: the target stock $\bar{k}(t)$ and the go-shopping threshold $k^*(t)$. The lower-left chart ("Searching Consumers (%)") displays the percentage of consumers engaging in a shopping search, $100 \cdot G(t, k^*(t))$. In the lower-middle chart ("Misallocation"), we report the two important moments relevant for the efficiency of the allocation of the goods: (i) the percent-

[26](We use the same layouts for all figures that exhibit the simulation results.

age of consumers who have a larger stock than the maximum level held in the stationary equilibrium, $100(1 - G(t, \bar{k}_o))$, whom we call "*hoarders*"; (ii) the percentage of consumers who run out of stock, $100 \cdot G(t, 0)$, whom we call "*stockless*" consumers. Note that the welfare loss becomes larger as the number of hoarders and stockless consumers increases because hoarders bear unusually high holding costs and stockless consumers suffer large disutility ($a = 10,000$) from not being able to consume.[27] Finally, the lower-right chart ("$S(t)$: Store's Stock") displays in-store stock $S(t)$.

We turn to the result of Simulation 1. Panic buying starts when the announcement is made. Consumers change their purchase policy immediately at $t = 0$: the target stock $\bar{k}(t)$ jumps from 6.0 to 8.3 to avoid shopping during the restricted-movement phase and the go-shopping threshold $k^*(t)$ jumps from 2.2 to 4.5 to start shopping earlier than usual. This sharply increases the fraction of searching consumers from the pre-disaster level of 7% to 64%. The increased demand for the good rapidly reduces and depletes the store's stock. As a result, the availability $R(t)$ decreases to less than 20% at its worst.

What is worse, the low availability persists in the restricted-movement phase because the initial stockpiling demand is so large that many consumers who start searching during the preparation phase cannot finish their shopping by the end of the phase. Such consumers are desperate to shop even during the restricted-movement phase at a higher shopping cost. As a result, there are unusually many consumers who *urgently* need the good—as can be seen from the lower-middle chart, the fraction of stockless consumers reaches about 0.6%, which is more than 100 times the normal level. The increase in stockless consumers is purely due to a coordination failure among the consumers: Since the shock considered here has no impact on neither aggregate consumption nor aggregate supply of the good, the full availability would be maintained if all consumers followed the stationary-equilibrium shopping strategy. Nevertheless, selfish consumers fail to internalize the congestion effect on the market, thereby causing a shortage of the goods in the market and an increased number

[27] Stockless consumers exist even in stationary equilibrium, albeit in very small numbers.

of stockless consumers.

After week 4, the consumer's purchase policies return to their normal level and the availability returns to one. This is natural because, at week 4, consumers know that the shopping cost gets back to the normal level in another four weeks (in week 8). Since consumers in our model go shopping roughly once in four weeks in normal times (i.e., $\bar{k}_o - k_o^* \approx 4$), after week 4, there is no need to excessively hoard.

The bottom parts of Table 2 report the welfare costs of the shopping-costs shock. For the benchmark case, the total welfare costs (SW) are approximately 131, while the welfare costs that are attributable to the fundamental shock (LF) are less than 10—even if the flow shopping cost $c(t)$ is increased by 100% for several weeks, its direct effect is limited since only 7% of consumers are searching in the stationary equilibrium. This result implies that the coordination problem amplifies the disaster damage thirteen times more than the original fundamental shock in the benchmark case.

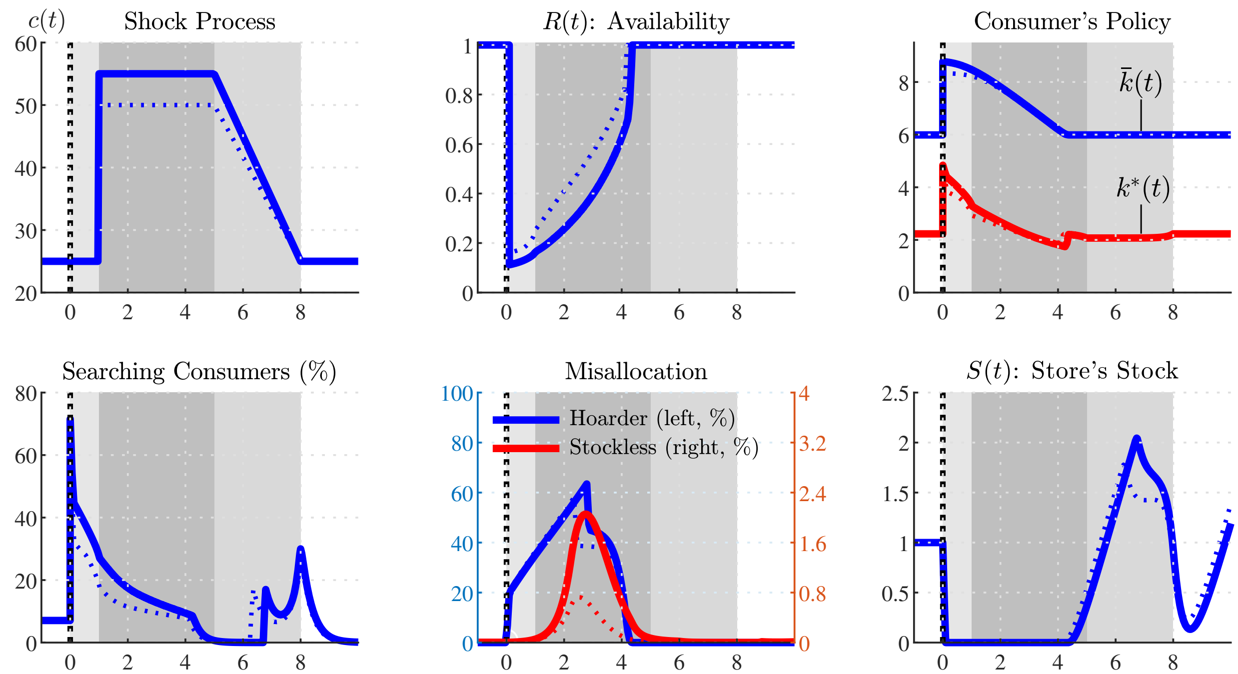
6.3.2 The Magnitude of the Shopping-Cost Shock

In the benchmark case, we considered the case in which the flow shopping cost is increased by 100%. Below, by tuning the parameter $\bar{c}$, we investigate how the tightness of the movement restrictions would affect the market outcomes. Specifically, we consider the cases where the movement restrictions are 20% more strict and 20% looser than in the benchmark case.

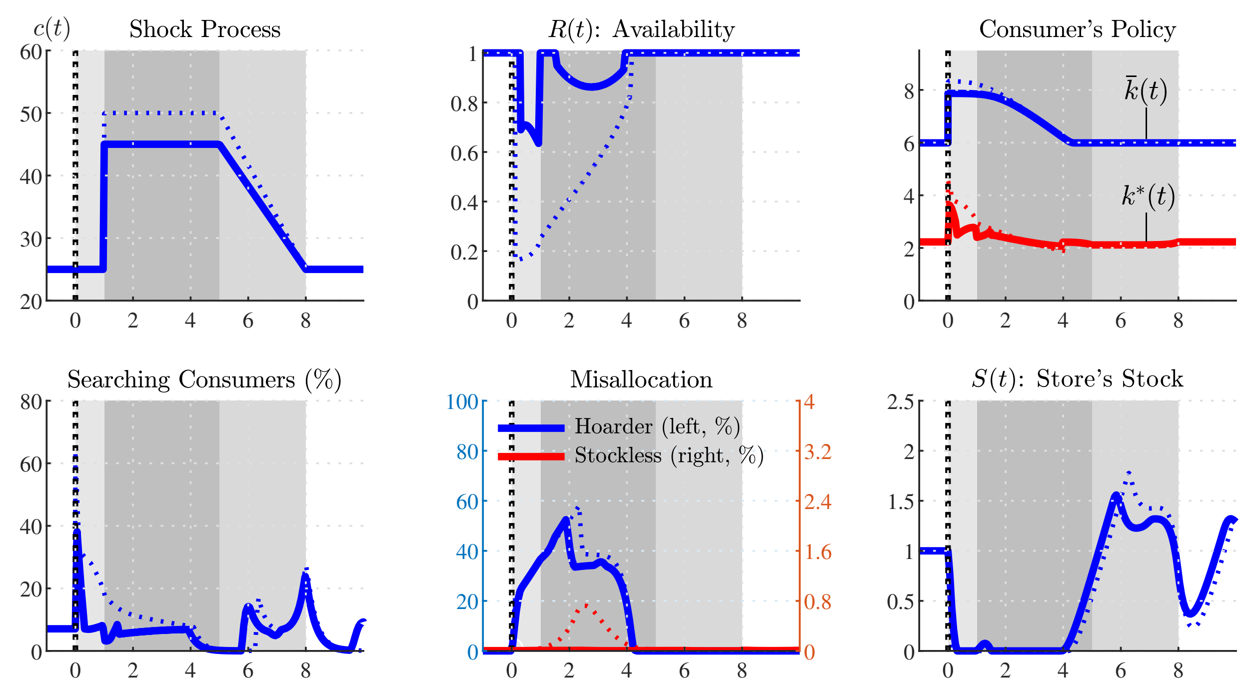
Simulation 2 (Large Shock)**.** During the restricted-movement phase, the shopping cost is increased to $\bar{c} = 55$. Hence, its percentage increase is $(\bar{c} - c)/c = 120\%$.

Simulation 3 (Small Shock)**.** During the restricted-movement phase, the shopping cost is increased to $\bar{c} = 45$. Hence, its percentage increase is $(\bar{c} - c)/c = 80\%$.

Figure 5a displays the result for Simulation 2 (Large Shock). Overall, the behavior of the economy looks qualitatively similar to the benchmark case, but the economic impact is much larger. The availability $R(t)$ declines in the preparation phase about 9 percentage points



(a) Simulation 2: The movement restrictions are 20% more strict than in the benchmark case.



(b) Simulation 3: The movement restrictions are 20% looser than in the benchmark case.

Note: The horizontal axis represents the number of weeks after the announcement. The dotted lines show the results for the benchmark case.

Figure 5: Different Tightness of the Movement Restrictions

more, and the lower availability continues more persistently than in the benchmark case. As a result, at the peak, 2.4% of consumers are out of stock (the numbers of stockless consumers are 0.6% in the benchmark case and less than 0.01% in the stationary equilibrium). This results in substantially large welfare costs: the total welfare costs are 385.7 ($CS = -385.7$), while the welfare costs from the fundamental shock are 11.6 ($LF = -11.6$). This implies that as the magnitude of the disaster (i.e., the size of the shock) is large, the total economic impact becomes sharply larger: compared with the benchmark case, the total welfare costs increase by 295%, while the welfare costs from the fundamental shock increase only by 20%.

We turn to Simulation 3 (Small Shock). As shown in Figure 5b, rationing ($R(t) < 1$) happens during the preparation phase. Nevertheless, the availability returns to the normal level earlier than the benchmark case. Therefore, consumers are hardly concerned with becoming stockless (see the red line in the lower-middle chart) and more consumers are willing to wait for a little while until the availability recovers. This results in a much smaller initial demand surge. Specifically, 40% of the consumers attempt to shop upon hearing the news ($t = 0$) in Simulation 3, while 60% of the consumers do in Simulation 1. As a result, the welfare costs and the degree of misallocation are quite small ($SW = -9$ and $LF = -8$).

The analyses in Section 6.3.2 imply that the impact of the shock size ($\bar{c}$) on social welfare is highly nonlinear. The overall impact of a shopping-costs shock becomes drastically larger when its size is greater than a certain level.

6.3.3 Timing of Announcements

We examine how the duration of the preparation phase T_c^S influences the severity of the panic. In practice, the length of the preparation phase depends on the forecastability of the emergency is considered to vary depending on the types of disasters. For example, landfall of a major hurricane can be forecast in advance, while earthquakes, massive blackouts, and terrorist attacks are virtually unpredictable. Hence, the following simulations are helpful in understanding which disasters are likely to trigger panic buying.

The following simulations also have valuable implications for government decision-making. In some cases, the government can partly control the length of the preparation phase by selecting the timing of announcements and implementation. For example, at the onset of the global spread of COVID-19, many governments placed movement restrictions on their residents after announcing their implementation in advance. Then, in such areas, they suffered from the panic buying that occurred immediately after the announcement of movement restrictions.[28] We believe that the following simulations suggest, in terms of reducing the risk of panic buying, how far in advance it would be desirable for the government to announce such restrictions.

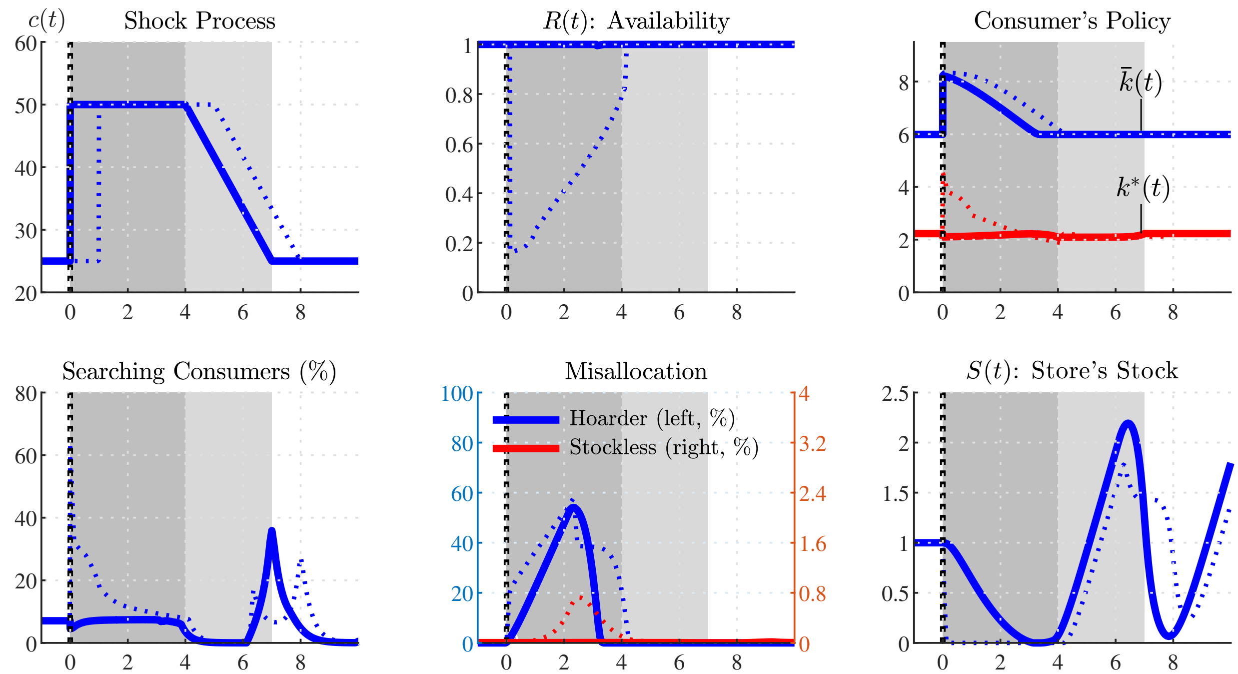
We begin with the case in which the movement restrictions are enforced immediately.

Simulation 4 (Immediate)**.** There is no preparation phase. When the unanticipated shock is announced, the economy immediately shifts to the restricted-movement phase: $T_c^S = 0$.

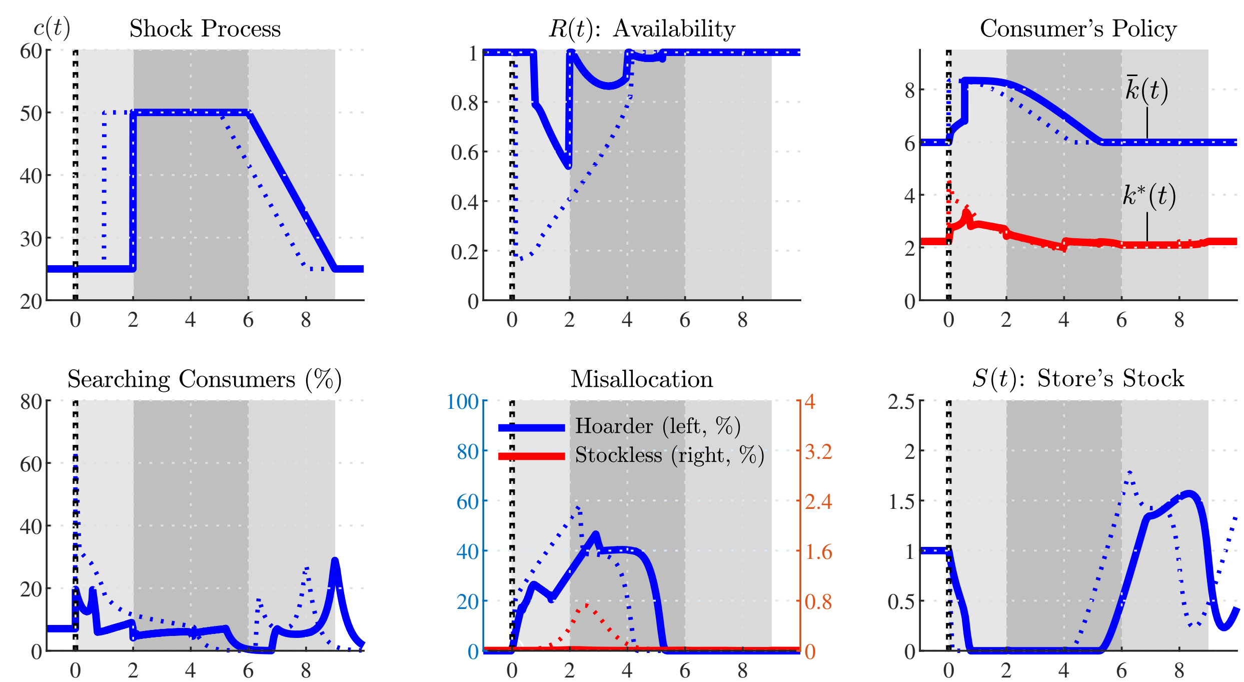
In Figure 6a, we present the simulation result for Simulation 4. In the absence of any preparation phase, the store's stock $S(t)$ declines more slowly than in the benchmark case, and the availability $R(t)$ remains at one, implying that the store is always in stock. Note that consumers increase their target stock $\bar{k}(t)$ for stockpiling but do not increase the go-shopping threshold $k^*(t)$ (see the top-right chart). Therefore, the number of searching consumers does not increase upon announcement (see the bottom-left chart).

The comparison between Simulations 1 and 4 implies that the very existence of a preparation phase plays an important role in amplifying panic buying. If there is a preparation phase (as in Simulation 1), consumers desperately attempt to shop before the restricted-movement phase begins and shopping costs rise. As a result, more consumers attempt to hoard the good during the preparation phase, causing a scarcity of the good at once. By contrast, if there is no preparation phase (as in Simulation 4), when consumers become aware

[28] For example, in New York City, an epicenter of COVID-19 infections, after confirming the state's first case of COVID-19 on March 1, New York Governor Andrew Cuomo declared a state of emergency on March 7 and issued stay-at-home order on March 14. Wallace (2020) reports that panic buying of toilet paper already became serious during the week that ended March 14.



(a) Simulation 4: There is no preparation phase: $T_c^S = 0$.



(b) Simulation 5: There is a two-week preparation phase: $T_c^S = 2$.

Note: The horizontal axis represents the number of weeks after the announcement. The dotted lines show the results for the benchmark case.

Figure 6: Different Timing of the Emergency Announcement

of the increase in their shopping costs, it is too late to rush to the market since movement restrictions are already in effect. Thus, the increase in market demand is mild.

It is somewhat difficult to find historical evidence of a situation with “no panic buying.” Nevertheless, Burney and Jones (2005) report that panic buying was not observed in London after the terrorist bombing incident in 2005, even though (i) this terrorist attack disrupted the transportation system of London, and (ii) there were security alerts at many locations throughout the United Kingdom. Likewise, we found no newspaper article that reports panic buying after the September 11, 2001 terrorist attacks in New York City. Considering that these terrorist attacks restricted the daily lives of the residents there but were not predicted in advance, these experiences are consistent with the result of Simulation 4.

Next, we turn to the case with a longer preparation phase than in the benchmark case.

Simulation 5 (Early Notice)**.** The movement restrictions are announced two weeks before being implemented: $T_c^S = 2$.

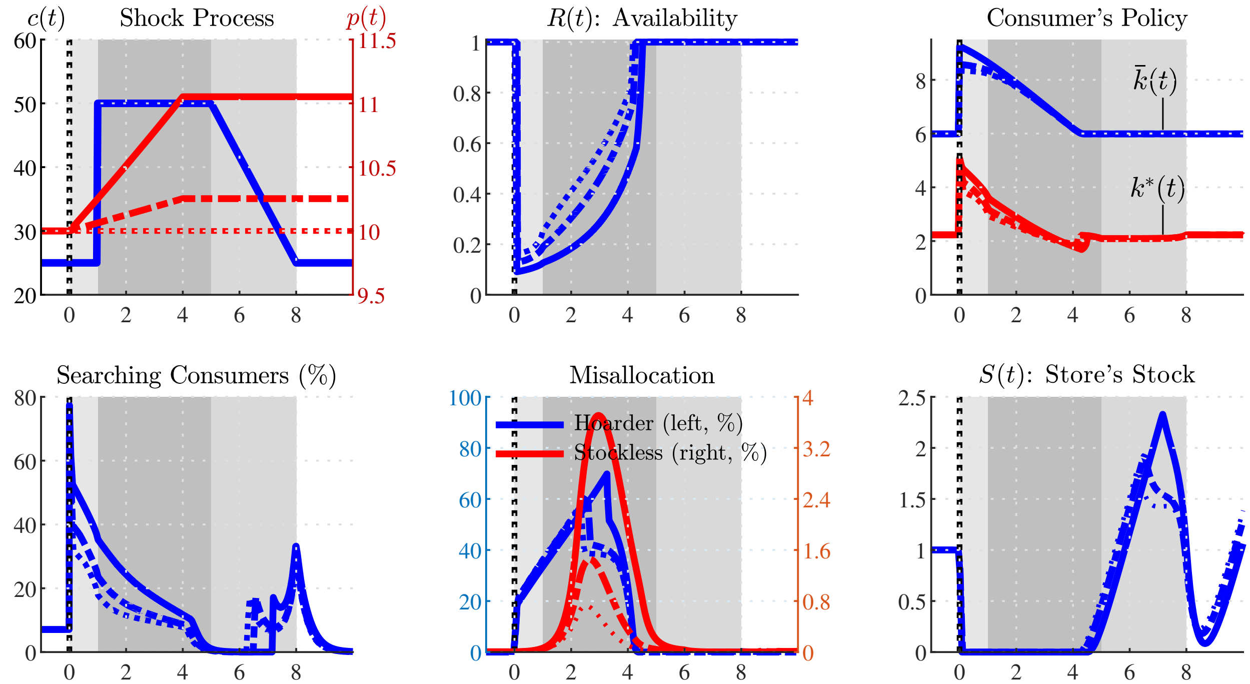
As shown in Figure 6b, in Simulation 5, consumers who visited the store within several days after hearing the news do not hoard so much (see the top-right chart), and rationing does not occur immediately after the news is announced (see top-middle chart). Since consumers do not sharply increase the go-shopping threshold, the number of consumers who attempt to shop right after hearing the news is less than one-third of the number observed in the benchmark case (at most 20% in Simulation 5 versus more than 60% in Simulation 1). On the whole, the shortage of the good is not as serious as in the benchmark case.

The simulation demonstrates that the extended duration of the preparation phase results in diversifying the timing of shopping, thereby mitigating panic buying. Viewed from time $t = 0$, in Simulation 5, the movement restrictions are set to be lifted one week later than in the benchmark case ($T_c^L = 6$ and $T_c^E = 9$ in Simulation 5, while $T_c^L = 5$ and $T_c^E = 8$ in the benchmark case). Thus, consumers are indeed eager to stockpile more than in the benchmark case, but this is not optimal in the light of holding costs. As a result, those who go shopping within the first couple of days after hearing the news purchase their usual quantity, and then conduct further shopping searches one month later when the shortage of the good is nearly eliminated. On the other hand, consumers who visited the store just before the restricted-movement phase begins purchase a larger quantity than usual and do not search during the restricted-movement phase. In this manner, the timing of shopping is not as concentrated as in the benchmark case.

It should be emphasized that the duration of the preparation phase has a non-monotonic effect on the severity of panic buying. Among Simulations 1 (with $T_c^S = 1$), 4 (with $T_c^S = 0$), and 5 (with $T_c^S = 2$), the most severe panic buying occurs in Simulation 1. In Simulation 1, the preparation phase has only one week and therefore the shopping is concentrated during that time. In contrast, in Simulation 5, the preparation phase has two weeks, consumers have relatively more time to choose when to shop.

6.3.4 Price Dynamics: Are Price Controls Necessarily Bad?

Thus far, we fixed the market price $p(t)$ at the stationary-equilibrium level, on the grounds of the anti-price gouging laws. Although price controls on daily necessities during emergency situations have become common, many scholars criticize such measures for encouraging hoarding and exacerbating shortages (e.g., Avoy, 1971; Brewer, 2007; Chakraborti and Roberts, 2020). In the following simulations, in order to study whether legal price controls indeed lead to escalating panic buying, we allow the market price to react in response to the increase in demand. Of course, in the absence of any regulations, it is reasonable to assume that market price would rise in response to the increased demand. However, whether the market mechanism would *flexibly* raise the retail price in the event of a disaster is nontrivial. In fact, there is growing empirical evidence suggesting that stores tend to hesitate to increase the price in order to maintain their reputation particularly in times of emergency situations (e.g., Cavallo, Cavallo, and Rigobon, 2014; Gagnon and López-Salido, 2019; Hansman et al., 2020; Cabral and Xu, 2021). In light of the evidence, it would be realistic to consider that the retail prices in times of emergency situations are highly rigid in nature because of reputation



Note: The horizontal axis represents the number of weeks after the announcement. The solid, dash-dotted, dotted lines show the results for Simulation 7 (10% inflation), 6 (2% inflation), and the benchmark case, respectively.

Figure 7: With and Without Anti-Price Gouging Regulations

effects.

We therefore consider the following two scenarios where the market price gradually rises in response to the spike of demand:

Simulation 6 (Low Inflation)**.** During the first four weeks after time 0, the market price increases at a monthly rate of 2.5%.

Simulation 7 (High Inflation)**.** During the first four weeks after time 0, the market price increases at a monthly rate of 10%.

Figure 7 displays the simulation results. The greater the increase in the market price is, the more severe the shortage is and the more serious the impact on the welfare is (e.g., $SW = -744.7$ in Simulation 7, while $SW = -130.8$ in the benchmark case). The intuition is straightforward: an expectation of a gradual rise in the price encourages consumers to buy storable goods earlier and hoard more. This channel is emphasized by Hansman et al.

Table 3: Summary of Simulation Settings and Results: With Policy Interventions

Shorthand	Benchmark	Sales Tax		Governmental Distribution		Quota
		Immediate	Delayed	All	Half	
Simulation number	1	8	9	10	11	12
				Simulation Settings		
$(\bar{c}-c)/c$	100%	100%	100%	100%	100%	100%
T_c^S	1	1	1	1	1	1
$T_c^L - T_c^S$	4	4	4	4	4	4
$T_c^E - T_c^L$	3	3	3	3	3	3
Sales Tax	0	5%: $t \in [0,1]$	5%: $t \in [0.5, 1.5]$	0	0	0
Rationing	0	0	0	1/2 unit to all ppl	1 unit to half of ppl	0
Purchase quota	0	0	0	0	0	$q \leq 4$
				Results		
Figure	4	8a	8b	9a	9b	10
CS	-130.8	-10.6	-68.2	-9.5	-9.3	-9.4
GR	0	0.1	0.1	0	0	0
GE	0	0	0	5.0	5.0	0
$SW = CS + GR - GE$	-130.8	-10.6	-68.2	-14.5	-14.3	-9.4
LF	-9.7	-9.7	-9.7	-9.7	-9.7	-9.7
$SW - LF$	-121.1	-0.98	-58.5	-4.9	-4.6	0.3
(Rel. to Benchmark)	(1.0)	(0.01)	(0.48)	(0.04)	(0.04)	(-0.00)

Note: Throughout the simulations in Section 6.4, the fundamental shock (the path of $c(t)$ and $p(t)$) is fixed to the one used in the benchmark case (Simulation 1). See Section 6.1 for the detail descriptions of the four parameters: $\bar{c}$; T_c^S; T_c^L; T_c^E. See Section 6.2 for the definitions of CS, GE, GR, SW, and LF.

(2020), who empirically analyze the 2008 global rice crisis and point out that the gradual price increase played a central role in causing the panic buying. In our simulation setting, an introduction of the anti-price-gouging law moderates panic buying by reducing this effect.[29]

6.4 Policy Interventions

In this section, we turn to policy options for curbing panic buying. Our analyses thus far have demonstrated that panic buying is an upward spiral of demand for hoarding. We can naturally infer that breaking this upward spiral is essential to curb panic buying. In the subsequent subsections, we evaluate the performance of the following three types of policy: a short-term sales tax increase, nonmarket distribution of the good, and quotas on purchases. Table 3 shows the simulation settings and the summary of the main results.

[29] Awaya and Krishna (2021) provide a two-period model framework in which the price of a storable good is endogenously determined by the market clearing price. In their model, when consumers purchase a large amount in the first period, the second-period price increases and becomes even higher than the first-period price. This situation resembles Simulations 6 and 7 in the sense that the price does not increase instantaneously in the beginning of the game. Awaya and Krishna (2021) also show that price controls mitigate panic buying and enhance social welfare.

6.4.1 A Short-Term Sales Tax Increase

The first policy option we analyze is a temporary increase in the sales tax. This policy is expected to disincentivize consumers from buying a large quantity. We consider a 5% sales tax increase for one week. We examine the following two scenarios regarding its implementation period. For the first scenario, we assume that the government can increase the tax rate as soon as the news is known:

Simulation 8 (Immediate Sales Tax Increase)**.** The government imposes a special sales tax of 5% during the preparation phase ($t \in [0, T_c^S] = [0, 1]$). The after-tax price is given by[30]

$$\hat{p}(t) = \begin{cases} 1.05 \cdot p(t) & \text{if } t \in [0, T_c^S]; \\ p(t) & \text{otherwise.} \end{cases}$$

However, such a flexible taxation system seems to be unrealistic in practice at least as of this writing.[31] Then, as a more realistic "best-case scenario," we consider a case in which the government increases the tax rate half a week after realizing the shock:

Simulation 9 (Delayed Sales Tax Increase)**.** Half a week after realizing the shock, the government imposes the special sales tax for a week:

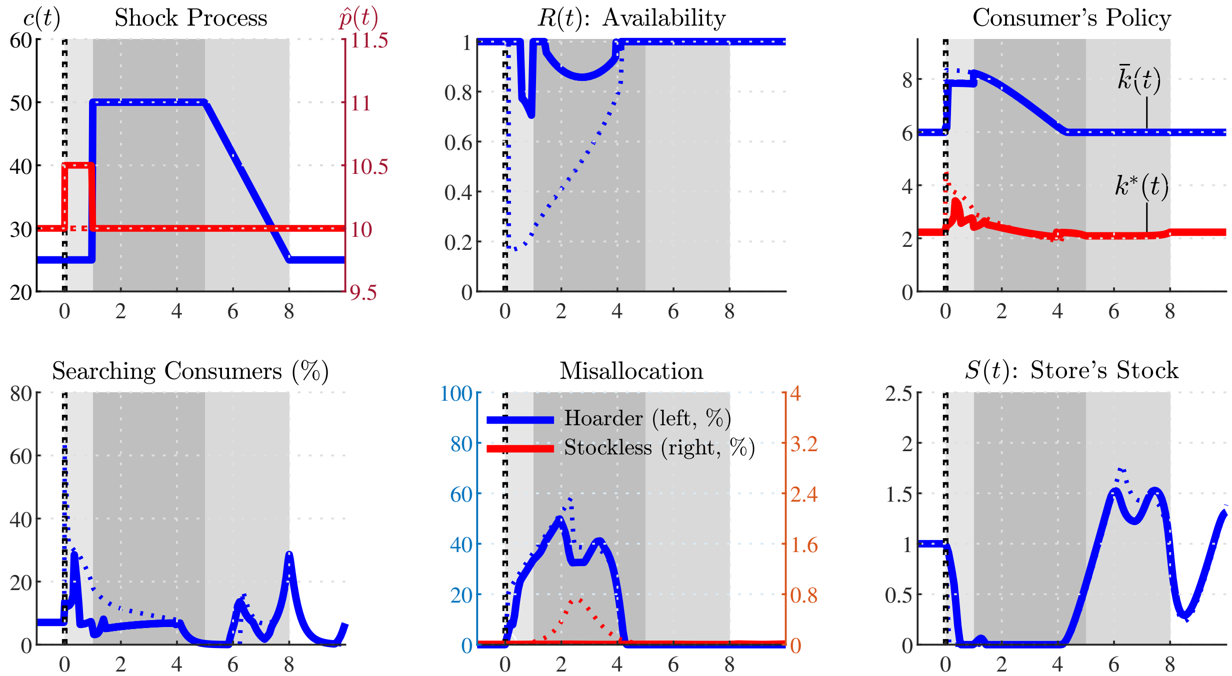
$$\hat{p}(t) = \begin{cases} 1.05 \cdot p(t) & \text{if } t \in [0.5, T_c^S + 0.5]; \\ p(t) & \text{otherwise.} \end{cases}$$

We first look at the result of Simulation 8. As shown in Figure 8a, the (immediate) short-term tax increase effectively mitigates shortages of the good in the market.[32] Even

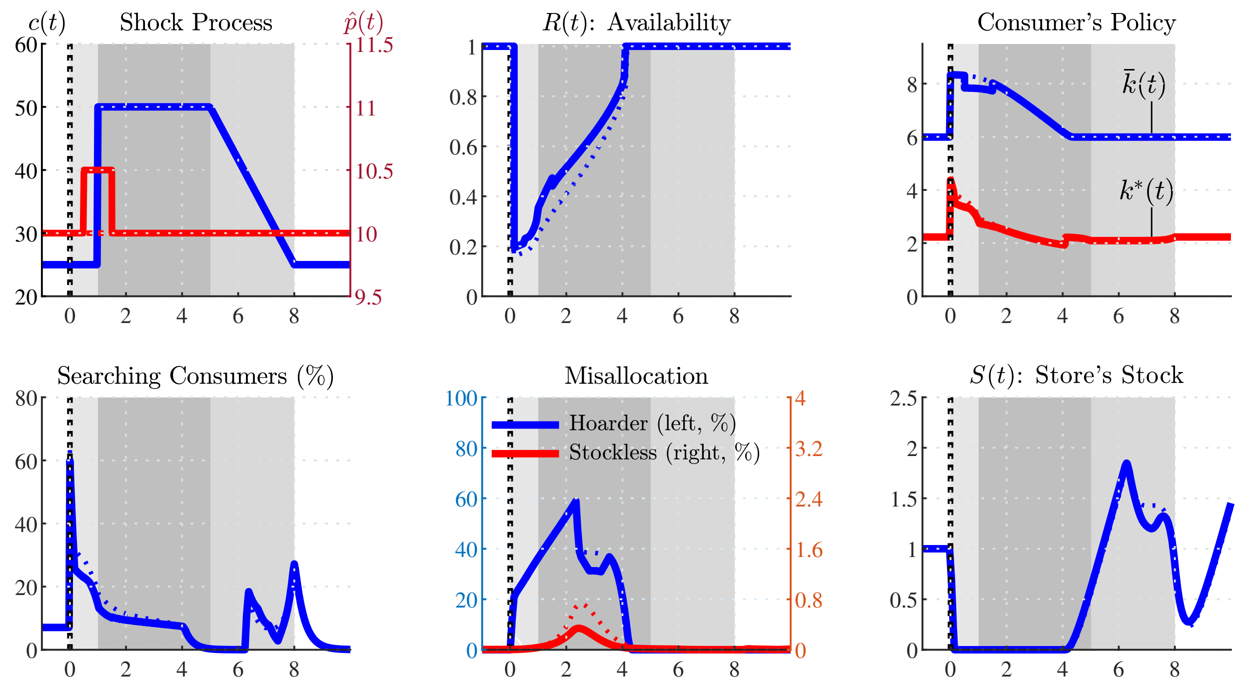
[30] In this simulation, the market price is fixed at $p(t) = p$ for all times.

[31] As pointed out in Nielsen Holdings PLC (2020), the growth of electronic commerce retail sales has been changing consumer's shopping behavior and efficiency of the supply chain, which played some roles in panic buying during the COVID-19 pandemic. Further widespread use of electronic commerce could facilitate flexible adjustments to the sales tax rate in the future.

[32] Of course, we confirm the policy is not very effective if the tax increase is small. See Figure A.1 in Online Appendix A.2 for the simulation result with an immediate sales tax increase of 2%.



(a) Simulation 8: Immediate Sales Tax. The price $p(t)$ is raised by 5% for $t \in [0, T_c^S]$.



(b) Simulation 9: Delayed Sales Tax. The price $p(t)$ is raised by 5% for $t \in [0.5, T_c^S + 0.5]$.

Note: The horizontal axis represents the number of weeks after the announcement. The dotted lines show the results for the benchmark case.

Figure 8: A Short-Term Sales Tax Increase

though the tax increase is not so large compared with the magnitude of the increase in the flow shopping cost, the immediate sales tax increase encourages consumers to do shopping *after* the tax increase ends, thereby mitigating the market congestion during the preparation phase. As a result, this policy is successful in enhancing the social welfare ($SW = -10.6$ in Simulation 8, while $SW = -130.8$ in the benchmark).

In contrast, as can be seen from Figure 8b, if its implementation is delayed even a few days, the short-term tax increase has a much more limited effect. In fact, with the delayed tax increase as in Simulation 9, the availability $R(t)$ declines to about 20% as in the benchmark case. The sharp difference between the effects of immediate and delayed tax increases is due to the intertemporal demand effect. In the case of the delayed tax increase, at $t = 0$, consumers become aware that not only the shopping costs but also the (after-tax) price will increase in the very near term. This further encourages consumers to shop early and amplifies the stockpiling motive.

We therefore conclude that raising the sales tax is a double-edged sword. It is an effective policy measure if the tax is raised immediately after news of the emergency comes out. However, if it is delayed even a little, raising the sales tax has little effect on curbing hoarding; on the contrary, it could exacerbate hoarding.

6.4.2 Governmental Distribution

The second policy option is government rationing of basic necessities. Concretely, we assume that the government can purchase the good from the market at the market price p and distribute it to consumers instantaneously. However, in the conduct of this policy, the government cannot target specific consumers, for example, consumers who need the goods urgently, since it can observe neither individual consumers' stock level (each consumer's k) nor their behaviors (e.g., whether they are searching or not).[33]

[33] This sort of rationing policy has been implemented in Japan and Taiwan during the COVID-19 pandemic. In Japan, the government distributed reusable cloth masks, dubbed the "Abenomask," in April 2020. In Taiwan, the government began to distribute face masks by allowing each resident to purchase two masks in seven days in February 2020 (see https://www.nhi.gov.tw/english/Content_List.aspx?n=022B9D97EF66C076

In Simulation 10, we first consider a case in which the government distributes the good to all consumers. Since fairness is an important policy concern, the government often wants to accommodate the whole population, when it cannot observe specific consumers' needs.

Simulation 10 (Governmental Distribution to All Consumers)**.** The government distributes *one-half* unit of the good to *all* consumers at $t = 0$: The initial condition is set to $S(0) = S_o - 1/2$ and $G(0, k) = G_o(k - 1/2)$ for all k.

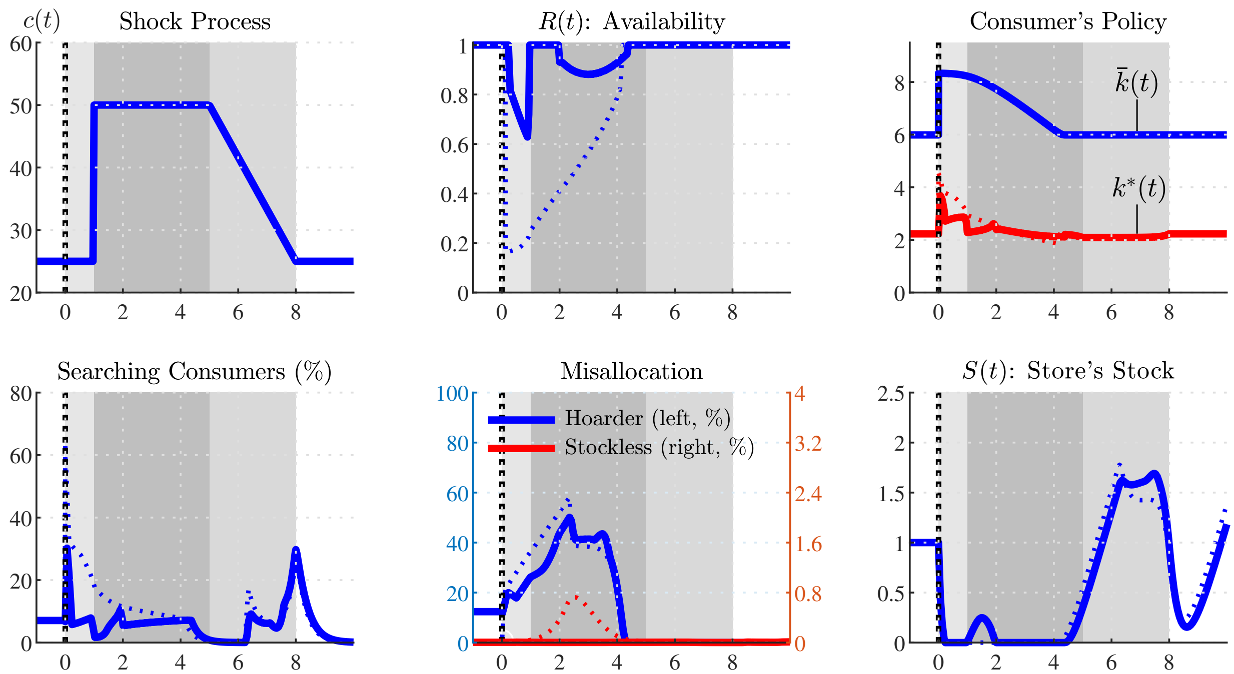
As shown in Figure 9a, the distribution policy considered in Simulation 10 effectively mitigates the risk of the good becoming scarce. Specifically, the number of consumers who attempt to shop upon the news becomes less than half the number that would be seen in the absence of the government intervention. Hence, governmental distribution is helpful in relaxing the market congestion and substantially enhances social welfare.

Next, we consider a different distribution rule. Indeed many real-world governments want to fairly distribute scarce goods, but accommodating the entire population equally is often costly and time-consuming in practice. Simulation 11 analyzes whether the government can make the *entire* population better off by distributing to only *a part of* the population.

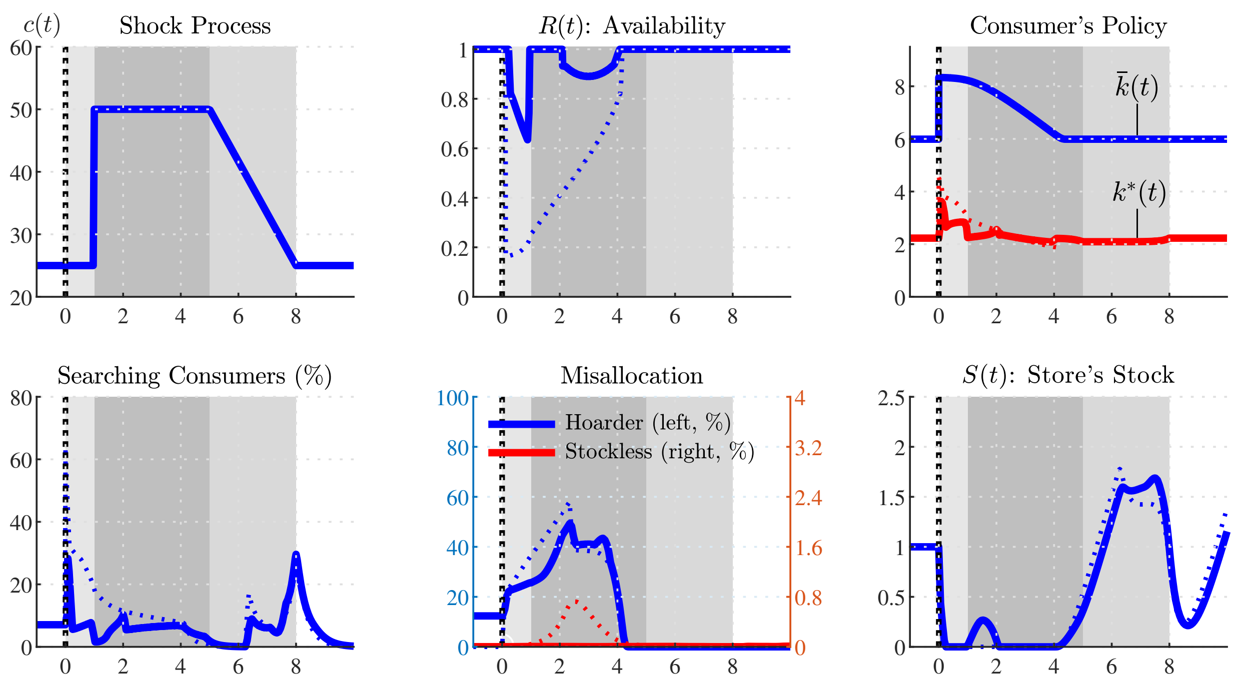
Simulation 11 (Governmental Distribution to One Half of Consumers)**.** The government distributes *one* unit of the good to *one half* of consumers at $t = 0$: the initial condition is set to $S(0) = S_o - 1/2 = 1/2$ and $G(0, k) = 1/2G_o(k) + 1/2G_o(k - 1)$ for all k.

Figure 9b displays the result of Simulation 11. Surprisingly, the economy's dynamics in Simulation 11 is very similar to those in Simulation 10, implying that distributing to half the population performs as well as distributing to the entire population. This is because the distribution policy is able to reduce excessive congestion in the market even when only half of the consumers can receive the rationed good. The consumers who failed to receive the rationed good are also better off because the policy indirectly reduces the shopping cost. This result suggests that the government should not hesitate to support only "easy-to-support people" in implementing the distribution policy. Even when the government can

for the detailed rationing procedure).



(a) Simulation 10: The government distributes $1/2$ unit to all the consumers at time 0: $S(0) = 1/2$ and $G(0, k) = G_o(k - 1/2)$ for all k.



(b) Simulation 11: The government distributes one unit to $1/2$ of consumers at time 0: $S(0) = 1/2$ and $G(0, k) = 1/2G_o(k) + 1/2G_o(k - 1)$ for all k.

Note: The horizontal axis represents the number of weeks after the announcement. The dotted lines show the results for the benchmark case.

Figure 9: Governmental Distribution

neither reach all the consumers nor observe consumers' individual stock, the governmental distribution would improve social welfare during times of disaster.

Remark 1. In Online Appendix A.2, we conduct an additional simulation regarding the governmental distribution (Figure A.3). It indicates that even when the government distributes one unit of the good to *one-fourth* of the consumers at time 0, such policy decreases the searching consumers at $t = 0$ by about 14 percentage points (from 64% to 50%).

6.4.3 Quotas on Purchases

The third option is to impose a restriction on the quantity purchased. When faced with a sudden increase in demand, stores often limit the number of items that can be purchased by each shopper. In this section, we evaluate the performance of the purchase-quota policy, assuming that such a quota is perfectly enforceable (we briefly discuss this issue later).

Here, we look at the case where the government imposes the following restriction:

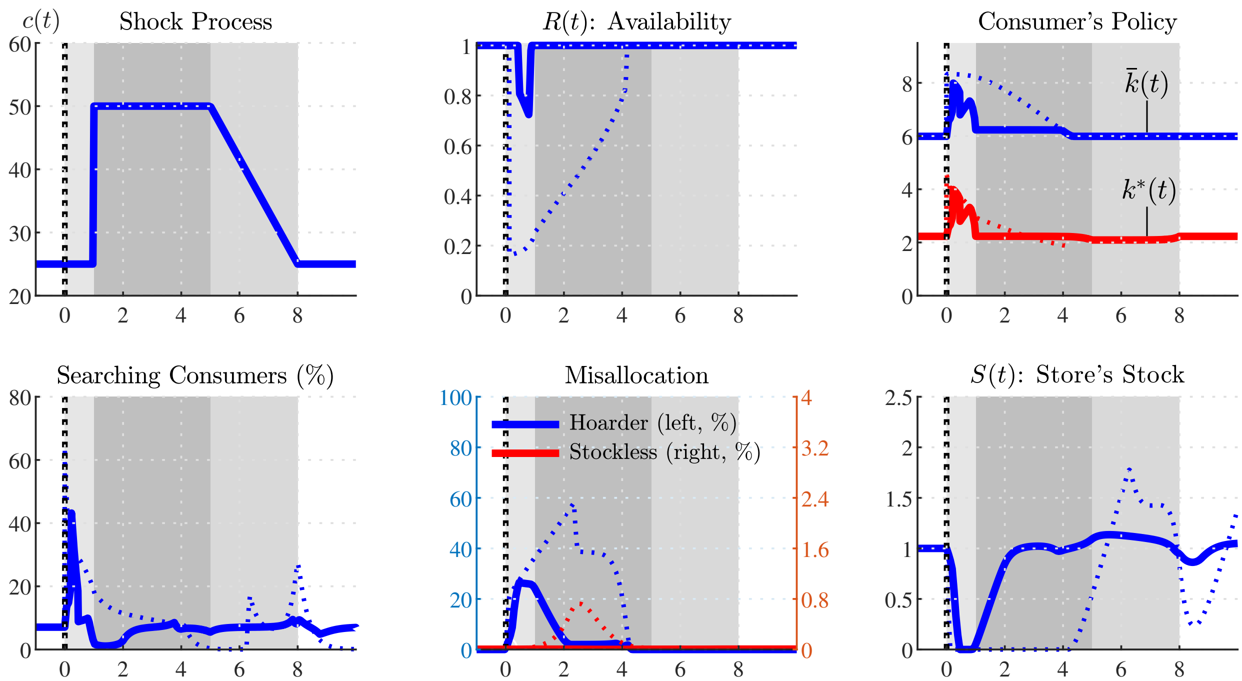
Simulation 12 (Quota)**.** Consumers are not allowed to purchase more than 4 units until the restricted-movement phase ends; i.e., $q_i(t) \leq 4$ for $t \leq T_c^L = 5$.

Under this restriction, consumers are allowed to purchase only up to four units of the goods during the preparation and restricted-movement phases. Those who want to purchase more must restart a shopping search again.

As shown in Figure 10, the purchase-quota policy effectively prevents panic buying.[34] Although the measure of searching consumers increases at time 0, each consumer is not allowed to stock up to the optimal level $\bar{k}(t)$ at a time. As a result, the shortages of the goods do not become serious, and consumers rarely exhaust their individual stock.

Remark 2. It is arguable whether this form of a purchase-quota policy is enforceable in practice. As of this writing, it is difficult for stores (and the government) to track who have

[34] Note that, in a stationary equilibrium, the minimum amount of the good a consumer purchases at a time is (roughly) 4 units: $q_o(k) = \bar{k}_o - k \geq \bar{k}_o - k_o^* \approx 4$. Hence, even when a consumer expects that the good will be always available ($R(t) = 1$ for all t), a consumer would purchase at least 4 units. Accordingly, the purchase-quota $q_i(t) \leq 4$ is almost always binding.



Note: The horizontal axis represents the number of weeks after the announcement. The dotted lines show the results for the benchmark case.

Figure 10: Simulation 12 (Quota). The quota $q_i(t) \leq 4$ is imposed until the end of the restricted-movement phase (for $t \in [0, T_c^L] = [0, 5]$).

already made purchases. If stores cannot track this information, consumers can easily violate the quota policy.[35]

7 Concluding Remarks

This paper has studied the fundamental causes and the welfare costs of the panic buying of storable consumer goods that have repeatedly been observed during times of disaster. We developed a dynamic model of the market for the storable daily necessities, in which a mass of consumers adjusts the stock of their daily necessities by infrequent and lumpy purchases in the presence of search frictions. We highlighted the following features of our model and the implications derived from our simulation analyses:

[35]Nowadays, due to the growth of digital payments, many stores track their customers' purchasing history for their marketing strategies. Nevertheless, consumers can easily create multiple accounts (e.g., store cards) and violate the quota in practice.

1. Panic buying could occur even when (i) all consumers are fully rational and there is no misinformation, and (ii) neither consumption nor production of the goods is affected by the disaster. A shock to the flow shopping cost influences the optimal strategies of selfish consumers, and it could cause a surge in hoarding-driven demand.

2. The welfare costs of panic buying could be large. When panic buying causes a shortage of the goods, consumers are forced to (i) bear a higher holding cost, (ii) engage in longer costly shopping searches, and (iii) face a higher risk of exhausting all stock.

3. The severity of panic buying is nonlinear in the shock size. Due to strategic complementarity, once the shock size exceeds a certain value, shortages become drastically severe.

4. Anticipated shopping-costs shocks produce more severe panic buying than unanticipated ones because consumers stockpile the goods before the shopping cost increases. Hence, an announcement immediately triggers a shortage, and it tends to persist because of strategic complementarity. The government can avoid this by either (i) implementing immediately movement restrictions, or (ii) announcing them well in advance.

5. A temporary sales tax increase discourages consumers from stockpiling and prevents panic buying if it is implemented before panic buying takes place. Governmental distribution of consumer goods can be an effective policy option to lighten the congestion of the market. It is effective even when (i) the government cannot observe consumers' existing inventory, and (ii) the government cannot reach all consumers.

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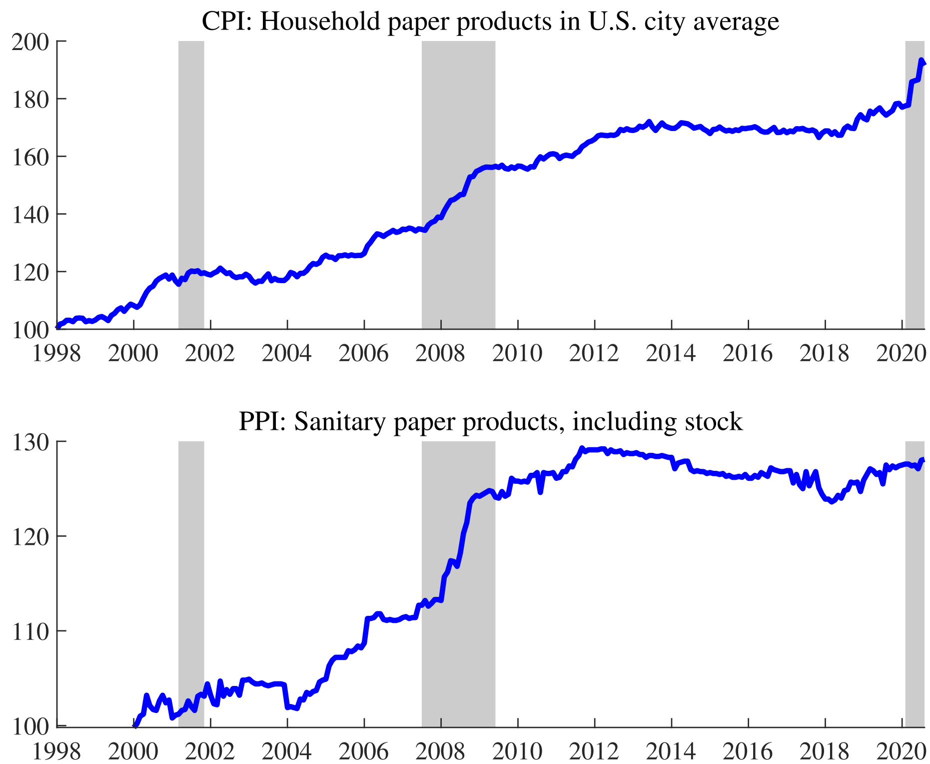
Appendices

APPENDIX A Supplemental Figures

A.1 Figures for Section 1

Figure A.1: Prices of Paper Products



Note: The top chart displays "Household paper products in US city average, all urban consumers (Not seasonally adjusted, Dec 1997=100)." The bottom chart displays "the producer price index for sanitary paper products, including stock (Not Seasonally Adjusted, Index Dec 1999=100)." Both series are collected by the US Bureau of Labor Statistics. Shaded areas correspond to NBER recessions.

A.2 Additional Simulation Results

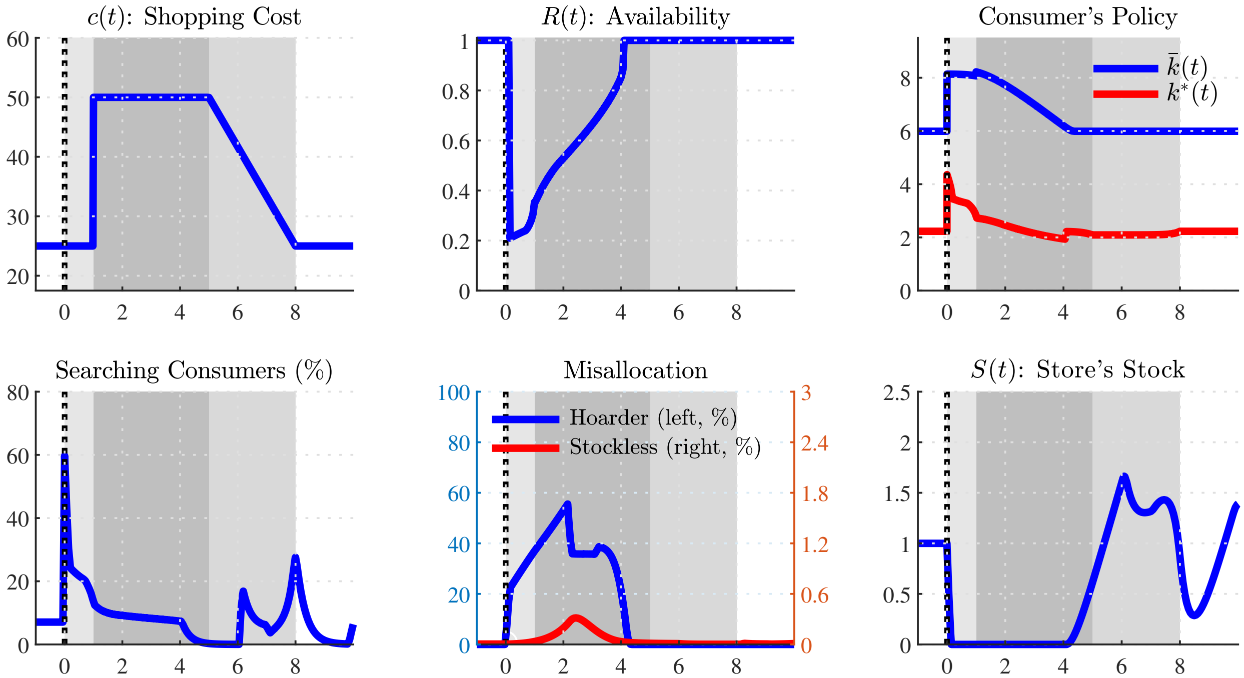


Figure A.1: 2% Tax on Purchase for $t \in [0, 1]$.

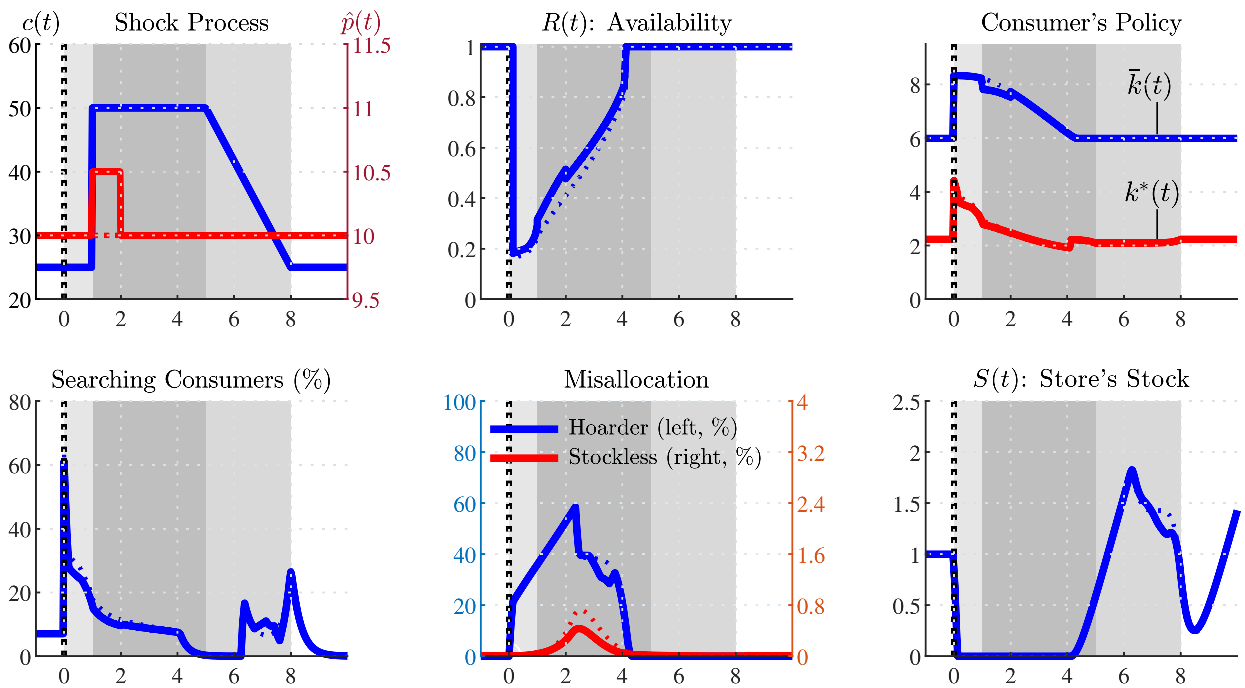


Figure A.2: 5% Tax on Purchase for $t \in [1, 2]$.

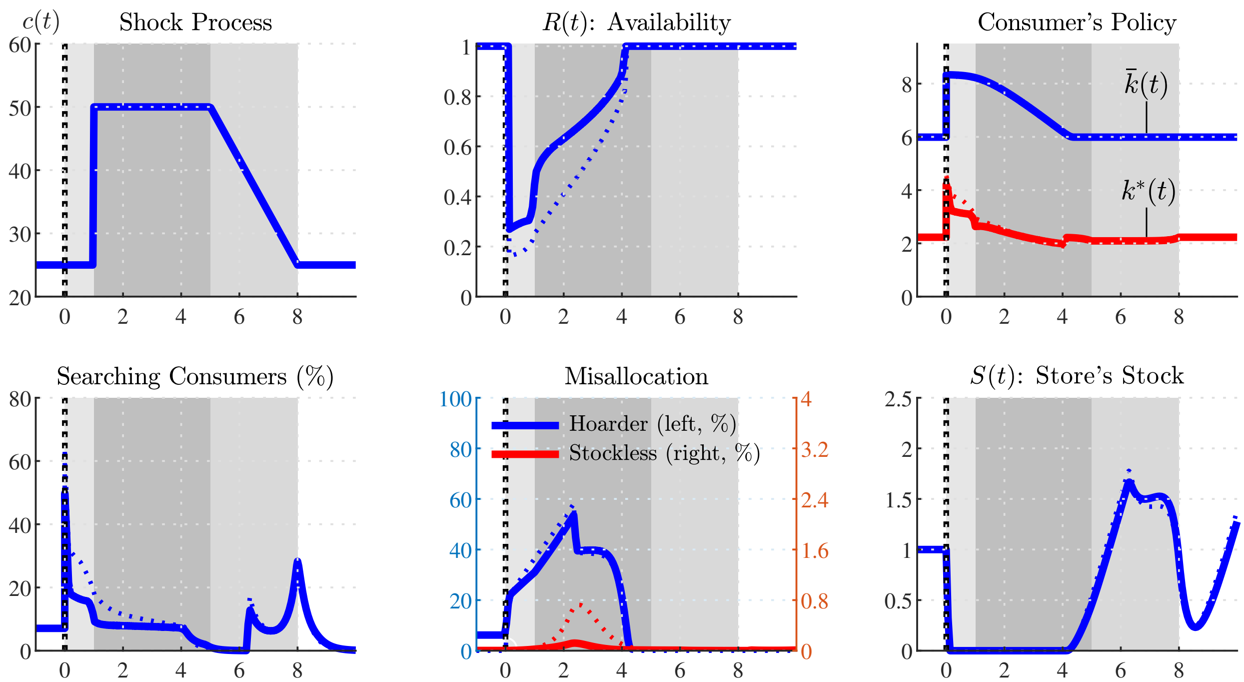


Figure A.3: The Governmental Distribution of One Unit of the Goods to One Fourth of Consumers: $S(0) = 3/4$ and $G(0, k) = G(k - 1/4)$ for all k.

Online Appendix (Not for Publication)

A Proof of Proposition 1

Proof. We prove the proposition in five steps.

Step 1. We first prove $0 \in \mathcal{A}$ and

$$\alpha \left(V^A(0) - V^*(0)\right) - c > 0 \tag{A.1}$$

by contradiction. Suppose $0 \notin \mathcal{A}$, we must have

$$V(0) = -\frac{a}{r} = V^N(0) > V^*(0), \tag{A.2}$$

where

$$V^N(k) := \int_0^\infty \mathrm{e}^{-rs} h\left(\max\{k-s,0\}\right) ds = \frac{1}{r}\left[1-(1+a)\mathrm{e}^{-rk} - \bar{b}\left[\mathrm{e}^{-rk}\left(\frac{1}{r}+k\right) - \frac{1}{r}\right]\right].$$

By definition of V^*,

$$\begin{aligned}
V^*(0) &= -\frac{a+c}{r} + \alpha \frac{V^A(0) - V^*(0)}{r} \\
&> -\frac{a+c}{r} + \alpha \frac{\sup_{q\geq 0}\ V^N(q) - pq - V(0)}{r} \\
&= -\frac{a+c}{r} + \alpha \frac{\sup_{q\geq 0}\ V^N(q) - pq + a/r}{r},
\end{aligned}$$

where the second line used the fact $V^A(k) = \sup_{q\geq 0}\ V(k+q) - pq \geq \sup_{q\geq 0}\ V^N(k+q) - pq$ and the third line used $V^N(0) = -a/r$. Then, using (A.2), we have

$$-\frac{a}{r} > -\frac{a+c}{r} + \alpha \frac{\sup_{q\geq 0}\ V^N(q) - pq + a/r}{r},$$

or

$$c > \alpha \left[\sup_{q \geq 0} \ V^N(k+q) - pq + a/r \right].$$

This clearly contradicts (11). Then, we must have $0 \in \mathcal{A}$, which implies

$$V(0) = V^*(0) = -\frac{a+c}{r} + \alpha \frac{V^A(0) - V^*(0)}{r} > -\frac{a}{r}.$$

This immediately implies (A.1).

Step 2. We next prove $[0, \varepsilon] \in \mathcal{A}$ for sufficiently small $\varepsilon > 0$ by contradiction. Suppose that $\mathcal{A} = \{0\}$, that is $V_o^*(k) < V_o(k)$ for all $k > 0$.

By construction of V_o and V_o^*, we have $V_o(\varepsilon) = \max\{\tilde{V}_o(\varepsilon), V_o^*(\varepsilon)\}$, where

$$\tilde{V}_o(\varepsilon) = h(\varepsilon)dt + (1 - rdt)V_o(\varepsilon - dt) \tag{A.3}$$

and

$$V_o^*(\varepsilon) = \left[h(\varepsilon) - c + \alpha V^A(\varepsilon)\right] dt + (1 - (\alpha + r)dt)V_o^*(\varepsilon - dt) \tag{A.4}$$

for any $\varepsilon > 0$. Take $\varepsilon = dt > 0$. Then, taking difference (A.3) from (A.4), we have

$$V_o^*(\varepsilon) - \tilde{V}_o(\varepsilon) = \left[\alpha \left(V^A(\varepsilon) - V_o^*(0)\right) - c\right] \varepsilon. \tag{A.5}$$

Since

$$V^A(\varepsilon) = \sup_{q \geq 0} V_o(\varepsilon + q) - pq = \sup_{q' \geq \varepsilon} V_o(q') - p(q' - \varepsilon) = \left(\sup_{q' \geq \varepsilon} V_o(q') - pq'\right) + p\varepsilon,$$

we have, for a sufficiently small ε,

$$V^A(\varepsilon) = V^A(0) + p\varepsilon. \tag{A.6}$$

Substituting (A.6) into (A.5), we have

$$\frac{V_o^*(\varepsilon) - \tilde{V}_o(\varepsilon)}{\varepsilon} = \alpha \left(V^A(0) - V^*(0) \right) - c + p\varepsilon.$$

Rearranging the terms yields

$$\alpha \left(V^A(0) - V^*(0) \right) - c = -\frac{\tilde{V}_o(\varepsilon) - V_o^*(\varepsilon)}{\varepsilon} - p\varepsilon < 0. \tag{A.7}$$

where the last inequality comes from the assumption $V_o(\varepsilon) = \tilde{V}_o(\varepsilon) > V_o^*(\varepsilon)$. Here, (A.7) contradicts to (A.1).

Step 3. Using the same arguments as Step 2, we can show that if $[0, \hat{k}] \in \mathcal{A}$ such that $\alpha \left(V^A(\hat{k}) - V_o^*(\hat{k}) \right) - c > 0$, then $[0, \hat{k} + \varepsilon'] \in \mathcal{A}$ for a small $\varepsilon' > 0$. Then, continuity of V_o^* and the instantaneous payoff function $h(k)$ show that $[0, k^*] \in \mathcal{A}$ with $\alpha \left(V^A(k^*) - V_o^*(k^*) \right) = c$.

Step 4. We show that the interval $\mathcal{A}$ is connected. That is, $\mathcal{A} = [0, k^*]$. This is almost obvious. Because $h(k)$ is strictly decreasing for $k \geq k^*$, there is no reason to increase k at the cost of shopping search.

Step 5. Finally, given that optimal policy, we derive V and V^* satisfying:

$$V_o(k) = \mathbb{1}_{\{k \geq k^*\}} \left[\int_0^{k-k^*} \mathrm{e}^{-rs'} h(k - s') ds' + \mathrm{e}^{-r(k-k^*)} V_o^*(k^*) \right] + \mathbb{1}_{\{k < k^*\}} V_o^*(k), \tag{A.8}$$

and

$$V_o^*(k) = \int_0^\infty \mathrm{e}^{-(\alpha+r)s'} \left[h(\max\{k - s', 0\}) + \alpha V^A(\max\{k - s', 0\}) - c \right] ds',$$

where

$$V^A(k) = \max_{q \geq 0} \; V_o(k + q) - pq.$$

It is, therefore, confirmed that

$$rV_o(k) = \mathbb{1}_{\{k \geq k^*\}} \left[h(k) - \frac{\partial V_o(k)}{\partial k} x(k) \right] + \mathbb{1}_{\{k < k^*\}} V_o^*(k) \tag{A.9}$$

and

$$rV^*(k) = h(k) - c - \frac{\partial V^*(k)}{\partial k}x(k) + \alpha\left[V^A(k) - V^*(k)\right]. \tag{A.10}$$

Lemma 1. *$V_o(k)$ and $V_o^*(k)$ are, respectively, expressed as follows:*

$$V_o(k) = \mathbb{1}_{\{k\geq k^*\}}\left[\frac{1}{r}\mathrm{e}^{-r(k-k^*)}\left(b(k^*) - \frac{\bar{b}}{r} + rV_o^*(k^*)\right) + \frac{1}{r}\left(\frac{\bar{b}}{r} - b(k)\right)\right] + \mathbb{1}_{\{k<k^*\}}V_o^*(k), \tag{A.11}$$

and

$$V_o^*(k) = \alpha\Lambda(k) + \frac{1}{\alpha+r}\left[\left(1 - \mathrm{e}^{-(\alpha+r)k}\right)\frac{\bar{b}}{\alpha+r} - b(k) - \mathrm{e}^{-(\alpha+r)k}a - c\right], \tag{A.12}$$

where

$$\Lambda(k) \coloneqq \int_0^k \mathrm{e}^{-(\alpha+r)(k-s)}V^A(s)ds + \mathrm{e}^{-(\alpha+r)k}\frac{V^A(0)}{\alpha+r}.$$

They satisfy the value matching condition

$$V_o(k^*) = V_o^*(k^*) = \lim_{k\uparrow k^*} V_o^*(k) = \lim_{k\uparrow k^*} V_o(k), \tag{A.13}$$

and the smooth pasting condition

$$V_o'(k^*) = V_o^{*\prime}(k^*) = \lim_{k\uparrow k^*} V_o^{*\prime}(k) = \lim_{k\uparrow k^*} V_o^{\prime}(k). \tag{A.14}$$

Proof of Lemma 1. First, we derive (A.11). The first term of the right-hand side of (A.8) is

$$\begin{aligned}\int_0^{k-k^*} \mathrm{e}^{-rs'}h(k-s')ds' + \mathrm{e}^{-r(k-k^*)}V_o^*(k^*) &= -\int_k^{k^*} \mathrm{e}^{-r(k-s)}h(s)ds + \mathrm{e}^{-r(k-k^*)}V_o^*(k^*)\\ &= \int_k^{k^*} \mathrm{e}^{-r(k-s)}b(s)ds + \mathrm{e}^{-r(k-k^*)}V_o^*(k^*).\end{aligned}$$

Then, use $b(k) = \bar{b}k$ and then apply integration by part to obtain

$$\begin{aligned}\int_k^{k^*} \mathrm{e}^{-r(k-s)} b(s) ds =& \bar{b} \int_k^{k^*} \mathrm{e}^{-r(k-s)} s ds \\ =& \bar{b} \left[\frac{1}{r} \left[\mathrm{e}^{-r(k-s)} s \right]_k^{k^*} - \frac{1}{r} \int_k^{k^*} \mathrm{e}^{-r(k-s)} ds \right] \\ =& \frac{\bar{b}}{r} \left[\mathrm{e}^{-r(k-s)} \left(s - \frac{1}{r} \right) \right]_k^{k^*} \\ =& \frac{1}{r} \left[\mathrm{e}^{-r(k-k^*)} \left(b(k^*) - \frac{\bar{b}}{r} \right) + \frac{\bar{b}}{r} - b(k) \right].\end{aligned}$$

Then, we derive (A.13) and (A.14). Given (A.8), it is immediate to derive the value matching condition (A.13). Then, (A.10) and the fact $\alpha \left(V^A(k^*) - V_o^*(k^*) \right) = c$ implies

$$rV^*(k^*) = -b(k^*) - V^{*'}(k^*). \tag{A.15}$$

Then, the value matching condition and (A.9) yield the smooth pasting condition (A.14).

Finally, we derive (A.12).

$$\begin{aligned}V^*(k) &= \int_0^\infty \mathrm{e}^{-(\alpha+r)s'} \left[h(\max\{k - s', 0\}) + \alpha V^A((\max\{k - s', 0\}) - c \right] ds' \\ &= \int_0^k \mathrm{e}^{-(\alpha+r)(k-s)} \left[h(s) + \alpha V^A(s) \right] ds + \frac{1}{\alpha + r} \left[\mathrm{e}^{-(\alpha+r)k} \left(h(0) + \alpha V^A(0) \right) - c \right] \\ &= \alpha \Lambda(k) + \frac{1}{\alpha + r} \left[\left(1 - \mathrm{e}^{-(\alpha+r)k} \right) \frac{\bar{b}}{\alpha + r} - b(k) \right] - \frac{1}{\alpha + r} \left(\mathrm{e}^{-(\alpha+r)k} a + c \right) \\ &= \alpha \Lambda(k) + \frac{1}{\alpha + r} \left[\left(1 - \mathrm{e}^{-(\alpha+r)k} \right) \frac{\bar{b}}{\alpha + r} - b(k) - \mathrm{e}^{-(\alpha+r)k} a - c \right],\end{aligned}$$

where

$$\Lambda(k) = \int_0^k \mathrm{e}^{-(\alpha+r)(k-s)} V^A(s) ds + \mathrm{e}^{-(\alpha+r)k} \frac{V^A(0)}{\alpha + r}.$$

□

Note that Lemma 1 implies that, for $k \geq k^*$,

$$V_o^{'}(k) = -\mathrm{e}^{-r(k-k^*)}\left[b(k^*) - \frac{\bar{b}}{r} + rV_o^*(k^*)\right] - \frac{\bar{b}}{r},$$

and

$$\begin{aligned} V_o^{''}(k) =& r\mathrm{e}^{-r(k-k^*)}\left[b(k^*) - \frac{\bar{b}}{r} + rV^*(k^*)\right] \\ =& -r\mathrm{e}^{-r(k-k^*)}\left[\frac{\bar{b}}{r} + V_o^{*'}(k^*)\right], \end{aligned}$$

where the second line used (A.15) and (A.14).

Here, we postulate $V_o^{*'}(k^*) > 0$ and therefore $V_o^{''}(k) < 0$ for $k \geq k^*$, implying that $V_o(k)$ is strictly concave for $k \geq k^*$. Under this, $V^A(0) = \max_{q \geq 0} V_o(q) - pq$ has a unique solution. We denote the solution by $\bar{k}$, which must be (i) $\bar{k} = k^*$ if $V_o'(k^*) \leq p$ or (ii) $\bar{k} > k^*$ if $V_o'(k^*) > p$. But it is clear that the case (i) contradicts to the fact that $k^* \in \mathcal{A}$. Hence, it must be true that $V_o'(k^*) > p$. So, the case (ii) must be held, and therefore $\bar{k}$ solves

$$V_o'(\bar{k}) = -\mathrm{e}^{-r(\bar{k}-k^*)}\left(b(k^*) - \frac{\bar{b}}{r} + rV^*(k^*)\right) - \frac{\bar{b}}{r} = p,$$

or

$$\bar{k} = k^* - \frac{1}{r}\log\left(-\frac{\bar{b}/r + p}{b(k^*) - \bar{b}/r + rV_o^*(k^*)}\right) = k^* + \frac{1}{r}\log\underbrace{\left(1 + \frac{V_o'(k^*) - p}{\bar{b}/r + p}\right)}_{>1}.$$

As a consequence, (when postulating $V_o^{*'}(k^*) > 0$), we must have

$$V^A(k) = \max_{q \geq 0} V_o(k+q) - pq = \begin{cases} V_o(\bar{k}) - p(\bar{k} - k), & \text{for } k \in [0, \bar{k}], \\ V_o(k), & \text{for } k \in (\bar{k}, \infty). \end{cases} \tag{A.16}$$

Furthermore, use (A.9) to derive the following

$$V_o(\bar{k}) = -\frac{b(\bar{k}) + V_o'(\bar{k})}{r} = -\frac{b(\bar{k}) + p}{r}.$$

Plugging this into (A.16) yields

$$V^A(k) = \begin{cases} -\dfrac{p + b(\bar{k})}{r} - p(\bar{k} - k) & \text{for} \quad k \in [0, \bar{k}], \\ \dfrac{1}{r}\left[\mathrm{e}^{-r(k-k^*)}\left(b(k^*) - \dfrac{\bar{b}}{r} + rV^*(k^*)\right) + \left(\dfrac{\bar{b}}{r} - b(k)\right)\right] & \text{for} \quad k \in (\bar{k}, \infty). \end{cases}$$

Finally, we verify that our postulation was true. Using (A.12), we have

$$V^{*'}(k) = \alpha\Lambda'(k) - \left(1 - \mathrm{e}^{-(\alpha+r)k}\right)\frac{\bar{b}}{\alpha + r} + \mathrm{e}^{-(\alpha+r)k}a.$$

We then show that, for $k \in [0, \bar{k}]$

$$\Lambda'(k) = -(\alpha + r)\Lambda(k) + V^A(k) = \left(1 - \mathrm{e}^{-(\alpha+r)k}\right)\frac{p}{\alpha + r}.$$

since

$$\begin{aligned} \Lambda'(k) &= -(\alpha + r)\left[\int_0^k \mathrm{e}^{-(\alpha+r)(k-s)}\left(V^A(0) + ps\right)ds + \mathrm{e}^{-(\alpha+r)k}\frac{V^A(0)}{\alpha + r}\right] + (V^A(0) + pk) \\ &= -\left(1 - \mathrm{e}^{-(\alpha+r)k}\right)V^A(0) - \mathrm{e}^{-(\alpha+r)k}V^A(0) + (V^A(0) + pk) - p(\alpha + r)\int_0^k \mathrm{e}^{-(\alpha+r)(k-s)}s ds \\ &= pk - pk + \left(1 - \mathrm{e}^{-(\alpha+r)k}\right)\frac{p}{\alpha + r} \end{aligned}$$

for $k \in [0, \bar{k}]$. Hence, we have

$$V^{*'}(k) = \left(1 - \mathrm{e}^{-(\alpha+r)k}\right)\frac{\alpha p - \bar{b}}{\alpha + r} + \mathrm{e}^{-(\alpha+r)k}a.$$

for $k < \bar{k}$. Assumption 3 ensures $\alpha p > \bar{b}$ and thus $V^{*'}(k) > 0$ for all $k < \bar{k}$. Since $k^* < \bar{k}$, we have shown $V^{*'}(k^*) > 0$.

□

B Algorithm description

We define a differential operator (or an infinitesimal generator of the process) $\mathscr{K}$ as

$$(\mathscr{K}V)(t,k) = -\partial_k V(t,k)x(k) + \partial_t V(t,k).$$

Then, the value function $V(t,k)$ can be written as a viscosity of the solution of the Hamilton-Jacobi-Bellman variational inequality (HJBVI, henceforth):

$$\min\{rV(t,k) - h(k) - (\mathscr{K}V)(t,k), V^*(t,k)\} = 0, \tag{B.1}$$

where $V^*(t,k)$ is the function that satisfies the HJB equation:

$$(r + \alpha R(t))V^*(t,k) = h(k) - c(t) + (\mathscr{K}V^*)(t,k) + \alpha R(t)V^A(t,k).$$

Here, solving the HJB variational inequality is equivalent to finding the function $V(t,k)$ that satisfies complementary slackness:

$$\begin{aligned} V(t,k) &\geq V^*(t,k) \quad \text{if} \quad rV(t,k) = h(k) + (\mathscr{K}V)(t,k) \\ V(t,k) &= V^*(t,k) \quad \text{if} \quad rV(t,k) \geq h(k) + (\mathscr{K}V)(t,k) \end{aligned}.$$

We will find an approximated solutions of (B.1) in a discretized space. We begin with the description of our notations. Set an equidistant grid over the consumer's stock level, $k_1, k_2, \dots k_L$ with $\Delta_k = k_\ell - k_{\ell-1}$ for all $\ell = 2, \dots, L$. Below, we use the bold letters to denote vectors, such as $\boldsymbol{h} = [h(k_1), \dots, h(k_L)]'$, $\boldsymbol{v}(t) = [V(t,k_1), \dots, V(t,k_L)]'$, and $\boldsymbol{v}^*(t) = [V^*(t,k_1), \dots, V^*(t,k_L)]'$. We use the subscript ℓ to denote the ℓ-th element of a vector, for example u_ℓ and $v_\ell(t)$ denote the ℓ-th element of $\boldsymbol{u}$ and $\boldsymbol{v}(t)$, respectively.

We discretize the differential operator $\mathscr{K}$. Recall that an operator is the infinite-dimensional analogue of a matrix, so we approximate the operator by a matrix $\boldsymbol{K}$. To this end, we approximate the partial derivative based on the following finite difference scheme:

$$\partial_k V(t, k_\ell) = \frac{V(t, k_\ell) - V(t, k_{\ell-1})}{\Delta_k}.$$

Using the above scheme along with the boundary condition, we can write

$$-\partial_k V(t, k_\ell) x(k_\ell) = \begin{cases} 0, & \ell = 1 \\ -\dfrac{V(t, k_\ell) - V(t, k_{\ell-1})}{\Delta_k}\mu = v_{\ell-1}(t)\omega_+ + v_\ell(t)\omega_-, & \ell = 2, \ldots, L \end{cases}$$

where $\omega_+ = \mu/\Delta_k$ and $\omega_- = -\mu/\Delta_k$. Then, we can build a $L \times L$ sparse matrix $\boldsymbol{K}$ such that

$$\boldsymbol{K}\boldsymbol{v}(t) = [0, v_1(t)\omega_+ + v_2(t)\omega_-, \ldots, v_{L-1}(t)\omega_+ + v_L(t)\omega_-]',$$

that is

$$\boldsymbol{K} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \omega_+ & \omega_- & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \omega_+ & \omega_- & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \omega_+ & \omega_- & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \omega_+ & \omega_- \end{pmatrix}. \tag{B.2}$$

Then, the approximation of (B.1) in the discretized space is given by

$$\min\left\{ r\boldsymbol{v}(t) - \boldsymbol{h} - \boldsymbol{K}\boldsymbol{v}(t) - \frac{\boldsymbol{v}(t+dt) - \boldsymbol{v}(t)}{dt}, \boldsymbol{v}(t) - \boldsymbol{v}^*(t) \right\} = 0,$$

In the similar way, we can find the expression for $\boldsymbol{v}^*(t)$ in the discretized space as follows:

$$(\alpha + rR(t))\boldsymbol{v}^*(t) = \boldsymbol{h} - c(t)\mathbf{1}_L + \boldsymbol{K}\boldsymbol{v}^*(t) + \left(\frac{\boldsymbol{v}^*(t+dt) - \boldsymbol{v}^*(t)}{dt} \right) + \alpha R(t)\boldsymbol{v}^A(t). \tag{B.3}$$

where $\boldsymbol{v}^A(t)$ is the approximation of $V^A(t, k)$ in the discretized space.

For the later use, we define a $L \times L$ sparse matrix $\boldsymbol{M}(t)$ that captures the rate of transition of the consumer's stock associated with the market activity.[36] The (ℓ, n) elements are given by

$$\boldsymbol{M}_{\ell,n}(t) = \begin{cases} -\alpha R(t), & \text{for} \quad n = \ell \quad \text{if} \quad k_\ell \in \mathcal{A}(t) \\ \alpha R(t), & \text{for} \quad n = \bar{k}(t) \quad \text{if} \quad k_\ell \in \mathcal{A}(t) \\ 0, & \text{otherwise} \end{cases}$$

The sum of each row equals to zero. Furthermore, we define a $L \times L$ diagonal matrix $\boldsymbol{D}$ all of whose on-diagonal elements are $-\theta$, which captures the rate of transition of the consumer's stock associated with exit.

We turn to the time evolution of the cross-sectional distribution of the stock level. We denote $\boldsymbol{g}(t) = [g(t, k_1), \ldots, g(t, k_L)]'$ and $\boldsymbol{g}_{new} = [g_{new}(k_1), \ldots, g_{new}(k_L)]'$. Since the KF operator is the adjoint operator of the HJB operator, in the discretized space, the KF equation (8) can be written as

$$\dot{\boldsymbol{g}}(t) = \left(\boldsymbol{K}^T + \boldsymbol{M}(t)^T + \boldsymbol{D}^T\right) \boldsymbol{g}(t) + \theta \boldsymbol{g}_{new}.$$

where $\dot{\boldsymbol{g}}(t) = [\partial g(t, k_1)/\partial t, \ldots, \partial g(t, k_L)/\partial t]'$ and $\boldsymbol{A}^T$, $\boldsymbol{M}(t)^T$, and $\boldsymbol{D}^T$ are the transpose of the intensity matrices $\boldsymbol{A}$, $\boldsymbol{M}(t)$, and $\boldsymbol{D}$, respectively.

B.1 Stationary distribution

1. Set a concave function $\boldsymbol{v}^0$ as an initial guess for the value function. Here, we use $\boldsymbol{v}^0$ such that $r\boldsymbol{v}^0 = \boldsymbol{h} + \boldsymbol{K}\boldsymbol{v}^0$.

2. Given $\boldsymbol{v}^n$, find $\boldsymbol{v}^{n+1}$ by solving

$$\min \left\{ \frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^n}{\Delta} + r\boldsymbol{v}^{n+1} - \boldsymbol{h} - \boldsymbol{K}\boldsymbol{v}^{n+1}, \boldsymbol{v}^{n+1} - \boldsymbol{v}^*(\boldsymbol{v}^n) \right\} = 0, \tag{B.4}$$

[36] The matrix $\boldsymbol{K}$, which is given by (B.2), can be interpreted as the rate of transition of the consumer's stock associated with consumption.

where

$$\boldsymbol{v}^*(\boldsymbol{v}^n) = \boldsymbol{B_a}^{-1}\left(\boldsymbol{h} + \alpha \boldsymbol{v}^{\boldsymbol{A}}(\boldsymbol{v}^n)\right)$$

with $\boldsymbol{B_a} = (\alpha + r)\boldsymbol{I}_L - \boldsymbol{K}$.

2-1. Define matrix $\boldsymbol{B}$ as

$$\boldsymbol{B} = \left(r + \frac{1}{\Delta}\right)\boldsymbol{I}_L - \boldsymbol{K}.$$

Then, rewrite (B.4) into

$$\min\left\{\boldsymbol{B}\boldsymbol{v}^{n+1} - \frac{1}{\Delta}\boldsymbol{v}^n - \boldsymbol{h}, \boldsymbol{v}^{n+1} - \boldsymbol{v}^*\left(\boldsymbol{v}^n\right)\right\} = 0. \tag{B.5}$$

Now, find that that solving (B.5) is equivalent to solving the following problem:

$$\begin{aligned} \left(\boldsymbol{v}^{n+1} - \boldsymbol{v}^*(\boldsymbol{v}^n)\right)'\left(\boldsymbol{B}\boldsymbol{v}^{n+1} - \frac{1}{\Delta}\boldsymbol{v}^n - \boldsymbol{h}\right) &= 0 \\ \boldsymbol{v}^{n+1} - \boldsymbol{v}^*(\boldsymbol{v}^n) &\geq 0 \\ \boldsymbol{B}\boldsymbol{v}^{n+1} - \frac{1}{\Delta}\boldsymbol{v}^n - \boldsymbol{h} &\geq 0 \end{aligned} \tag{B.6}$$

2-2. Define

$$\boldsymbol{z}^{n+1} = \boldsymbol{v}^{n+1} - \boldsymbol{v}^*(\boldsymbol{v}^n) \quad \text{and} \quad \boldsymbol{y}^n = \boldsymbol{B}\left(\boldsymbol{v}^*(\boldsymbol{v}^n) - c\boldsymbol{1}\right) - \boldsymbol{v}^n/\Delta - \left(\boldsymbol{u} - \boldsymbol{h}\right).$$

Then, (B.6) is reduced to the following Linear Complementarity Problem (LCP):

$$\begin{aligned} (\boldsymbol{z}^{n+1})'(\boldsymbol{B}\boldsymbol{z}^{n+1} + \boldsymbol{y}^n) &= 0 \\ \boldsymbol{z}^{n+1} &\geq 0 \\ \boldsymbol{B}\boldsymbol{z}^{n+1} + \boldsymbol{y}^n &\geq 0 \end{aligned}$$

Then, given $\boldsymbol{v}^n$ (equivalently $\boldsymbol{y}^n$), the above problem solves $\boldsymbol{z}^{n+1}$ and therefore $\boldsymbol{v}^{n+1}$.

3. Repeat the step 2 until $\boldsymbol{v}^{n+1}$ is sufficiently close to $\boldsymbol{v}^n$.

4. Find $\boldsymbol{g}$

4-1. Set $\boldsymbol{M}$. The (ℓ, n) elements are given by

$$\boldsymbol{M}_{\ell,n} = \begin{cases} -\alpha, & \text{for} \quad n = \ell \quad \text{if} \quad k_\ell \in \mathcal{A} \\ \alpha, & \text{for} \quad n = \bar{k} \quad \text{if} \quad k_\ell \in \mathcal{A} \\ 0, & \text{otherwise} \end{cases}$$

4-2. Find $\boldsymbol{g}$ such that

$$\boldsymbol{0} = \left(\boldsymbol{K}^{\mathrm{T}} + \boldsymbol{M}^{\mathrm{T}} + \boldsymbol{D}^{\mathrm{T}}\right) \boldsymbol{g} + \theta \boldsymbol{g}_{new},$$

or

$$\boldsymbol{g} = -\left(\boldsymbol{K}^{\mathrm{T}} + \boldsymbol{M}^{\mathrm{T}} + \boldsymbol{D}^{\mathrm{T}}\right)^{-1} \theta \boldsymbol{g}_{new}.$$

B.2 Transitional dynamics

Here, we describe the algorithm to find the transitional dynamics over a time period $\boldsymbol{T} = \{t_0, \ldots, t_\tau, \ldots, t_T\}$ for $\tau = 0, \ldots, T$ with a large integer T. We take equi-distance grid points for time with $\Delta_t = \Delta_k$ (i.e., $\Delta_t = t_\tau - t_{\tau-1}$ for all $\tau = 1, \ldots, T$). Below, let $\boldsymbol{v}$, $\boldsymbol{v}^*$ and $\boldsymbol{g}$ denote the value functions and the density function for the consumer in the stationary equilibrium (in the discretized space), respectively. We use the following notation: $\boldsymbol{x}(t_\tau) = \boldsymbol{x}_\tau$.

1. Set $\boldsymbol{v}_T = \boldsymbol{v}$, $\boldsymbol{v}_T^* = \boldsymbol{v}^*$, $\boldsymbol{g}_0 = \boldsymbol{g}$, and $S_0 = S_o > 0$.

2. Set initial guess $\{\widetilde{R}_\tau\}_{\tau=0}^T$ for $\{R(t_\tau)\}_{\tau=0}^T$.

3. Given $\{\widetilde{R}_\tau\}_{\tau=0}^T$, find the paths $\{\boldsymbol{v}_\tau\}_{\tau=0}^T$ and $\{\boldsymbol{M}_\tau\}_{\tau=0}^T$ backward.

3-A. Set $\tau = T$.

3-B. Given $\boldsymbol{v}_\tau$, find $\tilde{\boldsymbol{v}}_{\tau-1}$ such that

$$r\tilde{\boldsymbol{v}}_{\tau-1} = \boldsymbol{h} + \boldsymbol{K}\tilde{\boldsymbol{v}}_{\tau-1} + \frac{\boldsymbol{v}_\tau - \tilde{\boldsymbol{v}}_{\tau-1}}{\Delta_t}.$$

3-C. Given $\boldsymbol{v}^*_\tau$ and $\tilde{\boldsymbol{v}}_{\tau-1}$, find $\boldsymbol{v}^*_{\tau-1}$ such that

$$(\alpha + r\widetilde{R}_{\tau-1})\boldsymbol{v}^*_{\tau-1} = \boldsymbol{h} - c_{\tau-1}\mathbf{1}_L + \boldsymbol{K}\boldsymbol{v}^*_{\tau-1} + \frac{\boldsymbol{v}^*_\tau - \boldsymbol{v}^*_{\tau-1}}{\Delta_t} + \alpha\widetilde{R}_{\tau-1}\boldsymbol{v}^A(\tilde{\boldsymbol{v}}_{\tau-1}).$$

3-D. Find $\boldsymbol{v}_{\tau-1}$ such that

$$\boldsymbol{v}_{\tau-1} = \max\{\tilde{\boldsymbol{v}}_{\tau-1}, \boldsymbol{v}^*_{\tau-1}\}$$

that is, $\boldsymbol{v}_{\tau-1}$ is the element-wise maximum of $\tilde{\boldsymbol{v}}_{\tau-1}$ and $\boldsymbol{v}^*_{\tau-1}$.

3-E. Set the transition intensity matrix $\boldsymbol{M}_\tau$ as in (B.3)

3-F. The optimal policy is denoted by k^*_τ and $\bar{k}_\tau$

3-G. Repeat until $\tau = 1$

4. Given $\{\boldsymbol{M}_\tau\}_{\tau=0}^T$, find the paths $\{\boldsymbol{g}_\tau\}_{\tau=0}^T$ and $\{R_\tau\}_{\tau=0}^T$ forward.

4-A. Set $\tau = 0$

4-B. Given $\boldsymbol{g}_\tau$ and the optimal policy, find D_τ as follows:

$$D_\tau = \boldsymbol{g}_\tau^T \left[\mathbb{1}_{k^*} \odot \left(\bar{\boldsymbol{k}}_\tau - \boldsymbol{k}\right)\right]$$

where $\boldsymbol{k} = \{k_1, \ldots, k_\ell, \ldots, k_L\}'$, $\odot$ represents the element-wise product of vectors, and $\mathbb{1}_{k^*}$ is a $L \times 1$ vector whose the ℓ-th element $\mathbb{1}_{k^*}(\ell)$ satisfies

$$\mathbb{1}_{k^*}(\ell) = \begin{cases} 1 & \text{if} \quad k_\ell \leq k^*_\tau \\ 0 & \text{otherwise} \end{cases}$$

4-C. Given $\boldsymbol{g}_\tau$ and S_τ, find R_τ using the following rule:

$$R_\tau = \min\left\{\frac{S_\tau + s \cdot \Delta_t}{D_\tau}, 1\right\}.$$

4-D. Given $\boldsymbol{g}_\tau$, find $\boldsymbol{g}_{\tau+1}$ using an implicit method:

$$\frac{\boldsymbol{g}_{\tau+1} - \boldsymbol{g}_\tau}{\Delta_t} = \left(\boldsymbol{A} + \boldsymbol{M}_\tau + \boldsymbol{D}\right)^{\mathrm{T}} \boldsymbol{g}_{\tau+1} + \theta \boldsymbol{g}_{new}.$$

4-E. Given R_τ and S_τ, find $S_{\tau+1}$ using the following rule:

$$S_{\tau+1} = S_\tau + (s \cdot \Delta_t - R_\tau D_\tau).$$

4-F. Repeat until $\tau = T - 1$

5. Update the guess $\{\widetilde{R}_\tau\}_{\tau=0}^T$ until $\{\widetilde{R}_\tau\}_{\tau=0}^T$ and $\{R_\tau\}_{\tau=0}^T$ become close enough, based on the following rule:

$$\widetilde{R}_\tau = \lambda \widetilde{R}_\tau + (1 - \lambda) R_\tau$$