The Measurement of Population Ageing

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Abstract

Population ageing is one of the most serious problems in numerous developed countries. The level of population ageing is often measured by "usual" measures such as the share of the older population, mean age, median age, and the dependency ratio. However, these measures violate elementary properties for measuring population ageing. We propose a new measure of population ageing that overcomes drawbacks of the measures currently in use. We introduce a new condition called the *working age principle*, which is a sensitivity condition to thickness of the working age population. Our measure is the only measure that satisfies *monotonicity*, the *working age principle*, and the other standard axioms.

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1 Introduction

Population ageing is one of the most serious problems in numerous developed countries. According to the United Nations Population Division Reports, it is unprecedented, pervasive, and enduring problem (United Nations 2002). The level of population ageing is often measured by the ratio of the older population among the entire population (e.g., the share of people aged over 65 years), the so-called *head-count ratio*. For example, in Japan, the head-count ratio is 4.8% in 1950, 26.6% in 2015, and is projected to reach 33.4% by 2035.[1] Although the head-count ratio is well-known and widely applied, it violates at least two elementary properties for measuring population ageing.

First, the head-count ratio violates a monotonicity property with respect to ages. An increase in the age of an individual may not affect the head-count ratio. For example, even though all older individuals become increasingly old, as long as the number of them remains the same, the head-count ratio remains the same. Moreover, the head-count ratio cannot distinguish a distribution with "individuals mostly aged 75–84" from another distribution with "individuals mostly aged 65–74," although these two distributions are quite different.

Second, the head-count ratio fails to take the thickness of the working age population into account (e.g., people aged between 20 and 64 years). One non-negligible facet of population ageing is that the supply of the labor force by the working age population becomes relatively scarce. In particular, the government of Japan is confronted with a substantial budget imbalance due to large increases in public expenditures for social security systems and a decrease in tax-revenue caused by shrinking labor force, e.g., the total expenditures for social security systems exceed 23% of the gross domestic product in 2014 in Japan (Kitao 2015). However, since the head-count ratio simply counts the number of older people and calculates the proportion of these people among the entire population, it

[1] The population structure data and projections in Japan are sourced from the National Institute of Population and Social Security Research 2017.

fails to reflect the thickness of the working age population.

The *mean age* and *median age* are other prevalent measures of population ageing. However, these measures are determined independent of the retirement age and the requirement age for starting work, and thus, they ignore the thickness of the working age population as well as the head-count ratio. The *total dependency ratio*, the proportion of the working age population among the non-working age population, is used to measure the thickness of the working age population, but it violates monotonicity. Similarly, all the measures used in United Nations reports have at least one of these two drawbacks.

The choice of population ageing measures is important because it is difficult to capture, understand, and explore the complex phenomenon of population ageing without quantitative measures, and such measures shape our perceptions of demographic trends. Therefore, if a government uses a "bad" measure, it may misperceive demographic trends and fail to plan appropriate policies against population ageing.

In this paper, we propose a new measure of population ageing that overcomes the two shortcomings of the measures currently in use. We characterize the new measure by *monotonicity*, the *working age principle*, and other standard axioms. The *working age principle*, which is introduced in this paper, is a sensitivity condition to the thickness of the working age population. This axiom focuses on the *residual terms* or *remaining years* for working. The concept of this axiom conforms with the recent influential studies by Sanderson and Scherbov (e.g., Sanderson and Scherbov 2005 and 2010). They cast doubt on conventional measures based on years since birth and make new measures based on age in terms of years left until death or remaining years for working. Thus, our paper gives an axiomatic foundation of such measures focusing on the rest of lifetimes. We also compute our measure using population data for China and Japan; this illustrates the differences between our proposed measure and the head-count ratio.

To the best of our knowledge, this study is the first to axiomatically analyze the measurement of population ageing. Our study is inspired by Sen (1976)'s criticism of the head-count ratio vis-a-vis measuring poverty. The literature on poverty measurement focuses on the left-tail of income distributions (low incomes) and require monotonicity and sensitivity to inequality within those left-tail distributions (e.g., Sen 1976; Foster and Shorrocks 1991). Differing from poverty, population ageing is a trend on the entire distribution; thus, using only the information of right-tail distributions and discarding residual information is an inadequate approach to measuring the level of population ageing. Therefore, we do not simply focus on the right-tail of age distributions (older populations). This is a departure from Chu (1997) who, inspired by the literature on the measurement of poverty, proposed a new measure of population ageing, focusing only on the right-tail of age distributions without providing an axiomatization of his measure.

The rest of this paper is organized as follows. Section 2 introduces our model. Section 3 presents our axioms. Section 4 gives an axiomatic characterization of our new measure. Section 5 concludes this paper. All omitted proofs are relegated to the Appendix.

2 The Model

Let $\mathbb{N}$ be the set of population sizes. An *age distribution* is a vector $\boldsymbol{y} \in \bigcup_{n=1}^{\infty}[0, \bar{y}]^n$, where $\bar{y} \in \mathbb{R}_{++}$ is an upper bound of individuals' ages. For notational simplicity, we let $\mathscr{D} \equiv \bigcup_{n=1}^{\infty}[0, \bar{y}]^n$. The population size corresponding to $\boldsymbol{y} \in \mathscr{D}$ is denoted by $n(\boldsymbol{y}) \in \mathbb{N}$. For each $\boldsymbol{y} \in \mathscr{D}$ and each $i, j \in \{1, 2, \ldots, n(\boldsymbol{y})\}$, let $\boldsymbol{y}_{-i,j} \equiv (y_k)_{k \in \{1,2,\ldots,n(\boldsymbol{y})\} \setminus \{i,j\}}$.

An age $y_i \in [0, \bar{y}]$ is a *working age* if $y_i \in [x, z]$, where $x, z \in (0, \bar{y}]$ and $x < z$. Age x indicates the requirement age for starting work, and z is the retirement age. The *residual term for working* at $y_i \in [0, \bar{y}]$ is

$$a(y_i) = \begin{cases} z - x & \text{if } y_i < x, \\ z - y_i & \text{if } x \leq y_i \leq z, \\ 0 & \text{if } z < y_i. \end{cases}$$

That is, at age $y_i < x$, individual i has the residual term $z - x$ for working since he has not attained working age; at age $x \leq y_i \leq z$, individual i has the residual term $z - y_i$ for working since he has worked for $y_i - x$ years; and at age $y_i > z$, individual i has no residual term for working since he has already passed the retirement age. An *index function* is a function $I : \mathscr{D} \to \mathbb{R}$ that maps each age distribution $\boldsymbol{y} \in \mathscr{D}$ to a real number $I(\boldsymbol{y}) \in \mathbb{R}$. For example, an index function $H : \mathscr{D} \to \mathbb{R}$ defined by

$$H(\boldsymbol{y}) = \frac{\left|\left\{i \in \{1, 2, \ldots, n(\boldsymbol{y})\} : y_i > z\right\}\right|}{n(\boldsymbol{y})}$$

is called the *head-count ratio*.

3 Axioms

Continuity requires that an index function be robust to small data misspecifications.

Continuity. An index function $I : \mathscr{D} \to \mathbb{R}$ is continuous.

Monotonicity requires that if the ages of individuals increase weakly and that of some individuals increase strictly, then the index strictly increases.

Monotonicity. For each $\boldsymbol{y}, \boldsymbol{y}' \in \mathscr{D}$ with $n(\boldsymbol{y}) = n(\boldsymbol{y}')$, if $y_i = y'_i$ for all $i \neq j$ and $y_j > y'_j$ for some j, then $I(\boldsymbol{y}) > I(\boldsymbol{y}')$.

The next axiom is due to Foster and Shorrocks (1991) and is widely used in the literature on poverty measurement. Consider a situation in which the population is partitioned into two fixed size subgroups. *Subgroup consistency* requires that if population ageing increases in one and stays the other, then the overall level of population ageing increases.

Subgroup Consistency. For each $\boldsymbol{y}, \boldsymbol{y}', \boldsymbol{w}, \boldsymbol{w}' \in \mathscr{D}$ for which $n(\boldsymbol{y}) = n(\boldsymbol{y}')$ and $n(\boldsymbol{w}) = n(\boldsymbol{w}')$, if

$$I(\boldsymbol{y}) > I(\boldsymbol{y}') \text{ and } I(\boldsymbol{w}) = I(\boldsymbol{w}')$$

then

$$I\big(\boldsymbol{y}, \boldsymbol{w}) > I\big(\boldsymbol{y}', \boldsymbol{w}').$$

Replication invariance requires that the index view populaion ageing in per-capita terms.

Replication Invariance. For each $\boldsymbol{y}, \boldsymbol{y}' \in \mathscr{D}$, if there exists $k \in \mathbb{N}$ such that $n(\boldsymbol{y}) = k \cdot n(\boldsymbol{y}')$ and $\boldsymbol{y} = (\underbrace{\boldsymbol{y}', \boldsymbol{y}', \ldots, \boldsymbol{y}'}_{k \text{ times}})$, then $I(\boldsymbol{y}) = I(\boldsymbol{y}')$.

The next axiom is proposed in this paper, which requires sensitivity to thickness of the working age population. The *working age principle* requires that for any age distribution, if any two individuals' ages are replaced by others while preserving the sum, then an index function weakly decreases whenever the sum of the residual terms for working among these two individuals weakly increases.

The Working Age Principle. For each $\boldsymbol{y} \in \mathscr{D}$, and each $y_i', y_j' \in [0, \bar{y}]$ with $y_i' + y_j' = y_i + y_j$,

$$a(y_i) + a(y_j) \leq a(y_i') + a(y_j') \implies I\big(\boldsymbol{y}\big) \geq I\big(y_i', y_j', \boldsymbol{y}_{-i,j}\big).$$

This axiom is inspired by the *Pigou-Dalton transfer principle* (Dalton 1920). The *working age principle* can be interpreted as follows. A virtual transfer of age from an individual to another individual weakly reduces the value of the index if the sum of their residual terms for working weakly increases. For example, consider the following three age distributions in which $n = 3$:

$$\boldsymbol{y} = (25, 45, 75),$$
$$\boldsymbol{y}' = (25, 50, 70),$$
$$\boldsymbol{y}'' = (25, 40, 80).$$

Suppose that $x = 20$ and $z = 65$ in this society. Since

$$a(y_2) + a(y_3) = (65 - 45) + 0 = 20 > 15 = (65 - 50) + 0 = a(y_2') + a(y_3'),$$

a transfer of five years from individual 3 to individual 2 causes a decrease in the sum of their residual terms for working. Therefore, it follows that $I(\boldsymbol{y}) \leq I(\boldsymbol{y}')$. Similarly, since $a(y_2) + a(y_3) < a(y_2'') + a(y_3'')$, a transfer of five years from individual 2 to individual 3 causes an increase in the sum of their residual terms for working. Thus, it follows that $I(\boldsymbol{y}) \geq I(\boldsymbol{y}'')$. Overall, the working age principle requires $I(\boldsymbol{y}') \geq I(\boldsymbol{y}) \geq I(\boldsymbol{y}'')$.

Finally, *normalization* requires that for any age distribution, if all the individuals are of the same age, then its index takes the value of the age over the maximal age $\bar{y}$.

Normalization. For each $n \in \mathbb{N}$ and each $y \in [0, \bar{y}]$,

$$I(\underbrace{y, \ldots, y}_{\text{n times}}) = \frac{y}{\bar{y}} \in [0, 1].$$

4 A New Measure of Population Ageing

Our purpose is to identify a measure of population ageing that will satisfy the elementary properties stated in the previous section. First, we introduce an index function that represents an ordering on age distributions. For each $\alpha \geq 0$, let $I_\alpha : \mathscr{D} \to \mathbb{R}$ be such that

$$I_\alpha(\boldsymbol{y}) = \frac{1}{n(\boldsymbol{y})} \left(\sum_{j=1}^{n(\boldsymbol{y})} y_j + \alpha \sum_{j=1}^{n(\boldsymbol{y})} \big(z - x - a(y_j)\big) \right).$$

We show that an index function I satisfies *continuity*, *monotonicity*, *subgroup consistency*, *replication invariance* and the *working age principle* if and only if I is given by a monotonic transformation of I_α for some $\alpha \geq 0$.

Theorem 1. For each index function $I : \mathscr{D} \to \mathbb{R}$, the following statements (i) and (ii) are equivalent:

(i) $I : \mathscr{D} \to \mathbb{R}$ satisfies *continuity*, *monotonicity*, *subgroup consistency*, *replication invariance*, and the *working age principle*;

(ii) there exist $\alpha \geq 0$ and a *continuous* and *strictly increasing* function $F : \mathbb{R} \to \mathbb{R}$ such that for each $\boldsymbol{y} \in \mathscr{D}$,

$$I(\boldsymbol{y}) = F\left[I_\alpha(\boldsymbol{y})\right].$$

The first term of I_α is the mean age and the second term is the mean period that individuals have worked for; I_α is a linear combination of both the terms. Parameter α is the degree of sensitivity to the thickness of the working age population. It could coincide with the mean age when $\alpha = 0$. This is caused by the weakness of the *working age principle*. Indeed, it permits that whenever $y_i' + y_j' = y_i + y_j$ holds,

$$I\big(\boldsymbol{y}\big) = I\big(y_i', y_j', \boldsymbol{y}_{-i,j}\big).$$

We introduce the *strict working age principle* to exclude the mean age, which fails to respect the thickness of the working age population.

The Strict Working Age Principle. For each $\boldsymbol{y} \in \mathscr{D}$ and each $y_i', y_j' \in [0, \bar{y}]$ with $y_i' + y_j' = y_i + y_j$,

$$a(y_i) + a(y_j) = a(y_i') + a(y_j') \implies I\big(\boldsymbol{y}\big) = I\big(y_i', y_j', \boldsymbol{y}_{-i,j}\big),$$
$$a(y_i) + a(y_j) < a(y_i') + a(y_j') \implies I\big(\boldsymbol{y}\big) > I\big(y_i', y_j', \boldsymbol{y}_{-i,j}\big).$$

Note that mean age and head-count ratio violate strict working age principle. Replacing the *working age principle* with the strict version, we have the following corollary.

Corollary 1. For each index function $I : \mathscr{D} \to \mathbb{R}$, the following statements (i) and (ii) are equivalent:

(i) $I : \mathscr{D} \to \mathbb{R}$ satisfies *continuity*, *monotonicity*, *subgroup consistency*, *replication invariance*, and the *strict working age principle*;

(ii) there exist $\alpha > 0$ and a *continuous* and *strictly increasing* function $F : \mathbb{R} \to \mathbb{R}$ such that for each $\boldsymbol{y} \in \mathscr{D}$,

$$I(\boldsymbol{y}) = F\left[I_\alpha(\boldsymbol{y})\right].$$

In the Appendix, we show the tightness of the axioms in Theorem 1 and Corollary 1. Finally, we characterize an index function that satisfies *continuity*, *monotonicity*, *subgroup consistency*, *replication invariance*, the *strict working age principle*, and *normalization*.

For each $\alpha > 0$, let $f_\alpha : [0, \bar{y}] \to \mathbb{R}$ be such that for each $y_i \in [0, \bar{y}]$,

$$f_\alpha(y_i) = \begin{cases} y_i & \text{if } y_i < x, \\ (1+\alpha)y_i - \alpha x & \text{if } x \leq y_i \leq z, \\ y_i + \alpha(z - x) & \text{if } z < y_i. \end{cases}$$

By definition of I_α, we have

$$I_\alpha(\boldsymbol{y}) = \frac{1}{n(\boldsymbol{y})} \sum_{j=1}^{n(\boldsymbol{y})} f_\alpha(y_j).$$

Corollary 2. For each index function $I : \mathscr{D} \to \mathbb{R}$, the following statements (i) and (ii) are equivalent:

(i) $I : \mathscr{D} \to \mathbb{R}$ satisfies continuity, monotonicity, subgroup consistency, replication invariance, the strict working age principle, and *normalization*;

(ii) there exists $\alpha > 0$ such that

$$I(\boldsymbol{y}) = \frac{1}{\bar{y}} \cdot f_\alpha^{-1}\left[I_\alpha(\boldsymbol{y})\right].$$

We illustrate the differences between the head-count ratio and our new measure, using population data for China and Japan.[2] Figure 1 shows computation

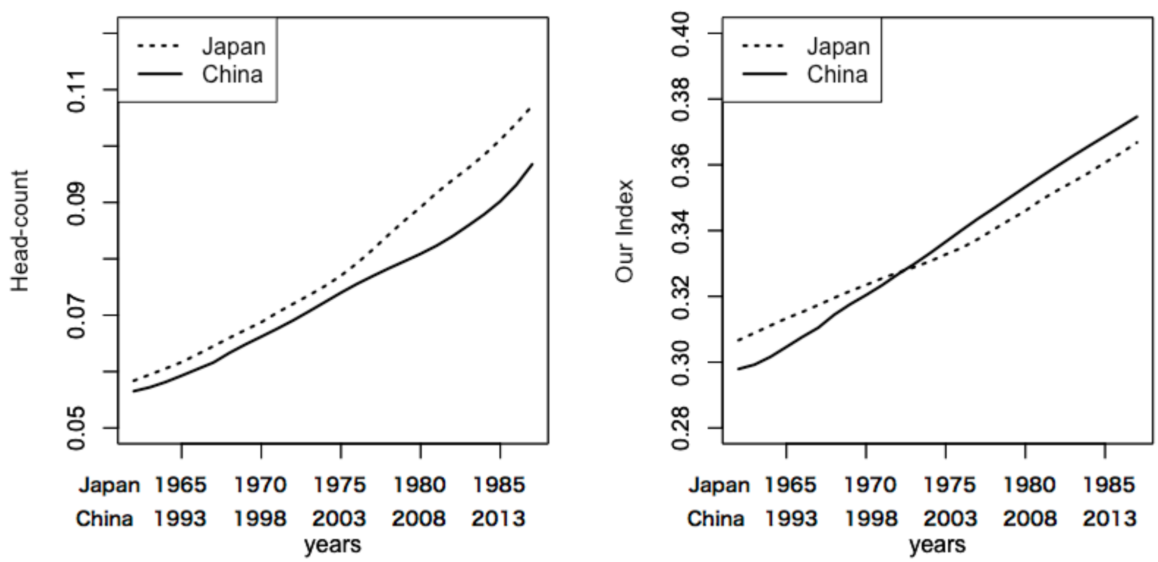

Figure 1: the head-count ratios and our measures

results of the head-count ratios and our measures using population data for China and Japan over the periods 1990–2015 and 1962–1987 data.[3] In the left plot, we

[2]Population data are sourced from United Nations (2017).

[3]We focus on the data for China over 1990–2015 for the following three reasons: (i) Population ageing had become a serious concern in this period. Certainly, in 2002, China was classified as an "ageing society" by the United Nations because its head-count ratio exceeded 7%. (ii) The year 1990 is about 10 years after the government of China officially enacted its unparalleled "one-child policy." (iii) At almost the same time, market-oriented economic reforms were instituted by the government, which induced several decades of rapid economic growth that also tended to decrease fertility rates in China (Zhang 2017). By focusing on this period, we can identify and explore policy impacts on population ageing in China. We use the data for Japan over 1962–1987 as a benchmark because population ageing in that country is particularly pronounced and this period also captures the period of economic growth in that country referred to as "Japanese economic miracle." Finally, the head-count ratio and our measure behave very

can see that the head-count ratio of China and Japan similarly increase over time, but the former is always lower than that the latter. However, in the right plot, by contrast, we observe that our measure for China increases at a faster rate compared to Japan, overtaking that country with respect to this measure.[4]

This empirical example suggests that if we use the head-count ratio, we might underestimate population ageing in China. On the contrary, our measure vividly captures China's rapid population ageing because it is more sensitive to the thickness of the potential working age population than the head-count ratio. Our measure for China sharply increases probably because of a decrease in the fertility rate associated with governmental policies in that country.

5 Concluding Remarks

Our measure satisfies two elementary properties, monotonicity, and the (strict) working age principle, whereas existing measures violate at least one of these properties. In this sense, our measure improves on what is currently available. However, our measure may not always supersede all of the extant alternative measures in all contexts. As already noted in Introduction, population ageing is a complex multidimensional phenomenon; it cannot be fully captured using only one measure. Our contribution is adding a new measure to the set of existing tools for measuring population ageing.

differently over these periods, which is fruitful for inter-method comparison.

[4]In this computation, we fix $x = 20$, $z = 65$, $\bar{y} = 100$, and $\alpha = 1$ for simplicity. However, similar results are generated if we moderately change the values of these exogenous constants. For example, if we set $x_{China} = 15$ in accordance with conventions for starting age of working in China, while x_{Japan} remains at 20, our measure for China will decrease because the size of the labor force has increased, but qualitative trends remains the same.

Appendix

Proof of Theorem 1

Proof. One can easily check that (ii) implies (i). Let us show that (i) implies (ii). Consider any index function $I : \mathscr{D} \to \mathbb{R}$ that satisfies continuity, monotonicity, subgroup consistency, replication invariance, and the working age principle.

Step 1. Let us show that there exist $f : [0, \bar{y}] \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ such that for each $\boldsymbol{y} \in \mathscr{D}$,

$$I(\boldsymbol{y}) = F\left[\frac{1}{n(\boldsymbol{y})}\sum_{j=1}^{n(\boldsymbol{y})} f(y_j)\right], \tag{1}$$

where f is *continuous* and *non-decreasing* and F is *continuous* and *strictly increasing*. We first check that I is *symmetric*, that is, for each $\boldsymbol{y} \in \mathscr{D}$ and each $i, j \in \{1, 2, \ldots, n(\boldsymbol{y})\}$,

$$I(y_i, y_j, \boldsymbol{y}_{-i,j}) = I(y_j, y_i, \boldsymbol{y}_{-i,j}). \tag{2}$$

Take any $\boldsymbol{y} \in \mathscr{D}$. Then, obviously $y_i + y_j = y_j + y_i$ and $a(y_i) + a(y_j) = a(y_j) + a(y_i)$. Therefore, by working age principle, equation (2) holds.

Since I satisfies continuity, monotonicity, subgroup consistency, replication invariance, and symmetry, we can apply Proposition 1 by Foster and Shorrocks (1991); there exist $f : [0, \bar{y}] \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ that satisfy conditions listed in equation (1).

Step 2. Let us show that there exist $b_1, c_1 \in \mathbb{R}$ such that for each $y \in [0, x)$, $f(y) = b_1 y + c_1$. Note that for each $y, y' \in [0, x)$,

$$a(y) + a(y') = 2(z - x) = a\Big(\frac{y + y'}{2}\Big) + a\Big(\frac{y + y'}{2}\Big).$$

Then, by the working age principle, for each $y, y' \in [0, x)$,

$$F\left[\frac{1}{2}f(y) + \frac{1}{2}f(y')\right] = I(y, y') = I\left(\frac{y + y'}{2}, \frac{y + y'}{2}\right) = F\left[f\Big(\frac{y + y'}{2}\Big)\right],$$

that is,

$$\frac{1}{2}f(y) + \frac{1}{2}f(y') = f\Big(\frac{y+y'}{2}\Big).$$

Solving this Jensen equation (Aczel 1966; p. 46), it follows that there exist $b_1, c_1 \in \mathbb{R}$ such that $f(y) = b_1 y + c_1$ for all $y \in [0, x)$.

Step 3. Let us show that there exist $b_2, c_2 \in \mathbb{R}$ such that for each $y \in [x, z]$, $f(y) = b_2 y + c_2$. Note that for each $y, y' \in [x, z]$,

$$a(y) + a(y') = y + y' - 2x = a\Big(\frac{y+y'}{2}\Big) + a\Big(\frac{y+y'}{2}\Big).$$

Since $I : [0, \bar{y}]^n \to \mathbb{R}$ satisfies the working age principle, for each $y, y' \in [x, z]$,

$$F\left[\frac{1}{2}f(y) + \frac{1}{2}f(y')\right] = I(y, y') = I\left(\frac{y+y'}{2}, \frac{y+y'}{2}\right) = F\left[f\Big(\frac{y+y'}{2}\Big)\right],$$

that is, f yields Jensen equation

$$\frac{1}{2}f(y) + \frac{1}{2}f(y') = f\Big(\frac{y+y'}{2}\Big).$$

Therefore, there exist $b_2, c_2 \in \mathbb{R}$ such that $f(y) = b_2 y + c_2$ for all $y \in [x, z]$.

Step 4. Let us show that there exist $b_3, c_3 \in \mathbb{R}$ such that for each $y \in (z, \bar{y}]$, $f(y) = b_3 y + c_3$. Note that for each $y, y' \in (z, \bar{y}]$,

$$a(y) + a(y') = 0 = a\Big(\frac{y+y'}{2}\Big) + a\Big(\frac{y+y'}{2}\Big).$$

Then, by the working age principle, for each $y, y' \in (z, \bar{y}]$,

$$\frac{1}{2}f(y) + \frac{1}{2}f(y') = f\Big(\frac{y+y'}{2}\Big).$$

Therefore, solving this Jensen equation, there exist $b_3, c_3 \in \mathbb{R}$ such that $f(y) = b_3 y + c_3$ for all $y \in (z, \bar{y}]$.

Step 5. Let us show that $0 < b_1 = b_3 \leq b_2$. Clearly, $b_1, b_2, b_3 > 0$ by monotonicity. We shall show that $b_1 = b_3$. Take any $y \in [0, x)$ and any $y' \in (z, \bar{y}]$. Let $\epsilon > 0$ be such that $\epsilon < \min\{x - y, y' - z\}$. Then, $y + \epsilon \in [0, x)$ and $y' - \epsilon \in (z, \bar{y}]$. Moreover,

$$a(y) + a(y') = z - x = a(y + \epsilon) + a(y' - \epsilon).$$

Therefore, by the working age principle,

$$F\left[\frac{1}{2}\big(f(y)+f(y')\big)\right]=I(y,y')=I\left(y+\epsilon,y'-\epsilon\right)=F\left[\frac{1}{2}\big(f(y+\epsilon)+f(y'-\epsilon)\big)\right],$$

that is,

$$f(y)+f(y')=f(y+\epsilon)+f(y'-\epsilon).$$

Hence by Steps 2 and 4,

$$b_1y+c_1+b_3y'+c_3=b_1(y+\epsilon)+c_1+b_3(y'-\epsilon)+c_3.$$

This equation implies that $b_1=b_3$.

We next show that $b_3\le b_2$. Take any $y\in[x,z)$ and any $y'\in(z,\bar{y}]$. Let $\epsilon>0$ be such that $\epsilon<\min\{z-y,y'-z\}$. Then, $y+\epsilon\in[x,z]$ and $y'-\epsilon\in(z,\bar{y}]$. Moreover,

$$a(y)+a(y')=z-y>z-(y+\epsilon)=a(y+\epsilon)+a(y'-\epsilon).$$

Therefore, by the working age principle,

$$F\left[\frac{1}{2}\big(f(y)+f(y')\big)\right]=I(y,y')\le I\left(y+\epsilon,y'-\epsilon\right)=F\left[\frac{1}{2}\big(f(y+\epsilon)+f(y'-\epsilon)\big)\right],$$

that is,

$$f(y)+f(y')\le f(y+\epsilon)+f(y'-\epsilon).$$

Hence by Steps 3 and 4,

$$b_2y+c_2+b_3y'+c_3\le b_2(y+\epsilon)+c_2+b_3(y'-\epsilon)+c_3.$$

This equation implies that $b_3\le b_2$.

Step 6. Let us show that $c_2=-\big(b_2-b_1\big)x+c_1$, and $c_3=\big(b_2-b_1\big)(z-x)+c_1$. By continuity of f at x, it follows that

$$b_1x+c_1=b_2x+c_2.$$

Therefore,

$$c_2 = -\big(b_2 - b_1\big)x + c_1.$$

Similarly, by continuity of f at z, it follows that

$$b_2 z + c_2 = b_3 z + c_3.$$

Therefore, by $b_1 = b_3$,

$$c_3 = \big(b_2 - b_3\big)z + c_2 = \big(b_2 - b_1\big)z - \big(b_2 - b_1\big)x + c_1 = \big(b_2 - b_1\big)(z - x) + c_1.$$

Step 7. Let us show that there exist $\alpha \geq 0$ and a *continuous* and *strictly increasing* function $F' : \mathbb{R} \to \mathbb{R}$ such that for each $\boldsymbol{y} \in \mathscr{D}$,

$$I(\boldsymbol{y}) = F'\left[I_\alpha(\boldsymbol{y})\right].$$

By Steps 2–6, for each $y_i \in [0, \bar{y}]$,

$$f(y_i) = \begin{cases} b_1 y_i + c_1 & \text{if } y_i < x, \\ b_2 y_i - \big(b_2 - b_1\big)x + c_1 & \text{if } x \leq y_i \leq z, \\ b_1 y_i + \big(b_2 - b_1\big)(z - x) + c_1 & \text{if } z < y_i. \end{cases}$$

Let $\alpha \equiv \frac{b_2}{b_1} - 1$. Since $b_2 \geq b_1$, we have $\alpha \geq 0$. Then, for each $\boldsymbol{y} \in \mathscr{D}$, it follows that

$$\frac{1}{n(\boldsymbol{y})} \sum_{j=1}^{n(\boldsymbol{y})} f(y_j) = b_1 I_\alpha(\boldsymbol{y}) + c_1. \tag{3}$$

Let $F' : \mathbb{R} \to \mathbb{R}$ be such that $F'[u] = F[b_1 u + c_1]$ for all $u \in \mathbb{R}$. Then, F' is continuous and strictly increasing. In addition, by Step 1 and equation (3),

$$I(\boldsymbol{y}) = F\left[\frac{1}{n(\boldsymbol{y})} \sum_{j=1}^{n(\boldsymbol{y})} f(y_j)\right] = F'\left[I_\alpha(\boldsymbol{y})\right].$$

□

Proof of Corollary 1

Proof. One can easily check that (ii) implies (i). Note that if $I(\cdot) = F[I_0(\cdot)]$ for some *continuous* and *strictly increasing* function $F : \mathbb{R} \to \mathbb{R}$, then I violates the strict working age principle. Therefore, by Theorem 1, (i) implies (ii). $\square$

Proof of Corollary 2

Proof. One can easily check that (ii) implies (i). We show that (i) implies (ii). By Theorem 1, that there exist $\alpha \geq 0$ and a *continuous* and *strictly increasing* function $F : \mathbb{R} \to \mathbb{R}$ such that for each $\boldsymbol{y} \in \mathscr{D}$, $I(\boldsymbol{y}) = F[I_\alpha(\boldsymbol{y})]$. Remember that $I_\alpha(\boldsymbol{y}) = \frac{1}{n(\boldsymbol{y})} \sum_{j=1}^{n(\boldsymbol{y})} f_\alpha(y_j)$. Then, since I satisfies normalization, for each $y \in [0, \bar{y}]$, $F[f_\alpha(y)] = I(y) = \frac{y}{\bar{y}}$, and hence $f_\alpha(y) = F^{-1}\left[\frac{y}{\bar{y}}\right] = F^{-1}\left[\frac{f_\alpha^{-1}[f_\alpha(y)]}{\bar{y}}\right]$. It in turn implies that for each $u \in f_\alpha([0, \bar{y}])$, $F[u] = \frac{1}{\bar{y}} f_\alpha^{-1}(u)$. Note that by definition of f_α, $I_\alpha(\mathscr{D}) = f_\alpha([0, \bar{y}])$. Therefore, we have $I(\boldsymbol{y}) = \frac{1}{\bar{y}} \cdot f_\alpha^{-1}[I_\alpha(\boldsymbol{y})]$. $\square$

Tightness of the axioms

- Let $I_1 : \mathscr{D} \to \mathbb{R}$ be an index function such that

$$I_1(\boldsymbol{y}) = \frac{1}{2}\left(H(\boldsymbol{y}) + I_\alpha(\boldsymbol{y})\right) \quad \text{for all } \boldsymbol{y} \in \mathscr{D}.$$

- Let $I_2 : \mathscr{D} \to \mathbb{R}$ be an index function such that

$$I_2(\boldsymbol{y}) = \frac{1}{n(\boldsymbol{y})} \sum_{j=1}^{n(\boldsymbol{y})} \big(z - x - a(y_j)\big) \quad \text{for all } \boldsymbol{y} \in \mathscr{D}.$$

- Let $I_3 : \mathscr{D} \to \mathbb{R}$ be an index function such that

$$I_3(\boldsymbol{y}) = \frac{1}{n(\boldsymbol{y})} \sum_{j=1}^{n(\boldsymbol{y})} y_j + \left(\frac{1}{n(\boldsymbol{y})} \sum_{j=1}^{n(\boldsymbol{y})} \big(z - x - a(y_j)\big)\right)^2 \quad \text{for all } \boldsymbol{y} \in \mathscr{D}.$$

- Let $I_4 : \mathscr{D} \to \mathbb{R}$ be an index function such that

$$I_4(\boldsymbol{y}) = n(\boldsymbol{y}) \cdot I_\alpha(\boldsymbol{y}) \quad \text{for all } \boldsymbol{y} \in \mathscr{D}.$$

- Let $I_5 : \mathscr{D} \to \mathbb{R}$ be an index function such that

$$I_5(\boldsymbol{y}) = \frac{1}{n(\boldsymbol{y})} \sum_{j=1}^{n(\boldsymbol{y})} y_j^2 \quad \text{for all } \boldsymbol{y} \in \mathscr{D}.$$

Our index function I_α cannot be monotonically transformed by these index functions. The satisfaction and the violation of axioms by these functions are summarized by Table 1. It shows the independence of the axioms in our Theorem 1 and Corollary 1.

	CON	MON	SUB	REP	WAP	SWAP
I_1	$-$	$+$	$+$	$+$	$+$	$+$
I_2	$+$	$-$	$+$	$+$	$+$	$+$
I_3	$+$	$+$	$-$	$+$	$+$	$+$
I_4	$+$	$+$	$+$	$-$	$+$	$+$
I_5	$+$	$+$	$+$	$+$	$-$	$-$

Table 1: Tightness of axioms

Let us only check that I_5 violates the working age principle. Note that $x+z = \frac{x+z}{2} + \frac{x+z}{2}$ and $a(x) + a(z) = a\left(\frac{x+z}{2}\right) + a\left(\frac{x+z}{2}\right)$. However,

$$\begin{aligned} I_5(x,z) - I_5\left(\frac{x+z}{2}, \frac{x+z}{2}\right) &= \tfrac{1}{2}\left(x^2 + z^2 - 2\left(\tfrac{x+z}{2}\right)^2\right) \\ &= \tfrac{1}{4}(z-x)^2 > 0. \end{aligned}$$

Therefore, I_5 violate the working age principle.

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References

Aczel, J. (1966) *Lectures on Functional Equations and Their Applications*, Academic Press, New York.

Chu, C.Y.C. (1997) "Age-Distribution Dynamics and Aging Indexes," *Demography*, Vol. 34(4), pp. 551–563.

Dalton, H. (1920) "The Measurement of the Inequality of Incomes," *Economic Journal*, Vol. 30, pp.348–361.

Foster, J.E. and Shorrocks, A.F. (1991) "Subgroup Consistent Poverty Indices," *Economterica*, Vol. 59(3), pp. 687–709.

Kitao, S. (2015) "Fiscal Cost of Demographic Transition in Japan," *Journal of Economic Dynamics & Control*,Vol. 54, pp.37–58.

Sanderson, W.C. and Scherbov, S. (2005) "Average remaining lifetimes can increase as human populations age," *Nature*, Vol. 435, pp. 811–813.

Sanderson, W.C. and Scherbov, S. (2010) "Remeasuring Aging," *Science*, Vol. 329, Issue 5997, pp. 1287–1288.

Sen, A. (1976) "Poverty: An Ordinal Approach to Measurement," *Econometrica*, Vol. 44(2), pp. 219–231.

United Nations (2002) *World Population Ageing: 1950-2050.* United Nations, Population Division.

United Nations (2017) *World Population Prospects 2017.* United Nations, Population Division.

Zhang, J. (2017) "The Evolution of China's One-Child Policy and Its Effects on Family Outcomes," *Journal of Economic Perspectives*, Vol. 31(1), pp. 141–160.