INTERIM RATIONALIZABLE IMPLEMENTATION OF FUNCTIONS

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What is (Full) Implementation?

A social choice function (SCF) is **(fully) implementable** in a given solution concept if there exists a **mechanism (or game-form)** that satisfies the following two requirements:

Existence: The solution is always nonempty; and

Uniqueness: "every" outcome induced by the solution coincides with that specified by the SCF.

Motivating Question and Previous Results

- How does equilibrium implementation compare to rationalizable implementation?

- For complete information, Bergemann-Morris-Tercieux (2011) shows strict Maskin monotonicity to be necessary for rationalizable implementation of SCFs.

- Still for complete information, but considering correspondences, Kunimoto and Serrano (2019) shows that uniform monotonicity, much weaker than Maskin's and reducing to it in SCFs, is the only necessary condition for rationalizable implementation; see also Jain (2020).

Our Contribution

1. We make the theory of implementation more robust by applying rationalizability to incomplete information environments.

2. We establish the precise relationship between rationalizable implementation, Bayesian implementation, and **double implementation**, i.e., rationalizable implementation by a mechanism having a Bayesian equilibrium.

3. However, our analysis is confined to single-valued social choice **functions**. We plan to extend our findings to multi-valued social choice **sets** in a separate paper.

Double Implementation V.S. Bayesian Implementation

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c}\text{Double}\\ \text{Implementation}\end{array} \arrow[r, Rightarrow, "\text{By Def.}"] \arrow[d, Rightarrow, shift right=1ex] & \begin{array}{c}\text{Bayesian}\\ \text{Implementation}\end{array} \arrow[d, Rightarrow, shift right=1ex] \\
\begin{array}{c}\text{Interim Rationalizable}\\ \text{Monotonicity (IRM)}\end{array} \arrow[r, Rightarrow] \arrow[u, Rightarrow, shift right=1ex, "\text{NWR}"'] & \begin{array}{c}\text{Bayesian Incentive}\\ \text{Compatibility (BIC) \&}\\ \text{Bayesian Monotonicity (BM)}\end{array} \arrow[u, Rightarrow, shift right=1ex, "\text{NWR}"']
\end{tikzcd}
$$

Double Implementation	:=	Rationalizable Implementation by a Mechanism that has a Bayesian (Nash) equilibrium
NWR	:=	The No-Worst-Rule Condition

Double and Rationalizable Implementation

$$\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c}\text{Double}\\\text{Implementation}\end{array} \arrow[r, Rightarrow, shift left, "\text{By Def.}"] \arrow[d, Rightarrow, shift right] & \begin{array}{c}\text{Rationalizable}\\\text{Implementation}\end{array} \arrow[l, Rightarrow, shift left, "\text{Finite}"', "\text{Mechanisms}"] \arrow[d, Rightarrow, shift right] \\
\text{IRM} \arrow[r, Rightarrow, shift left, "\text{By Def.}"] \arrow[u, Rightarrow, shift right, "\text{NWR}"'] \arrow[d, Rightarrow] & \text{Weak IRM} \arrow[l, Rightarrow, shift left, "\text{Responsive SCFs}"] \arrow[u, Rightarrow, shift right, "\begin{array}{c}\text{Weak}\\\text{NWR}\end{array}"'] \arrow[d, Rightarrow] \\
\text{BM} & \begin{array}{c}\text{Strict-if-Responsive}\\\text{BIC (SIRBIC)}\end{array}
\end{tikzcd}$$

Preliminaries

- $I = \{1, \ldots, n\}$: finite set of agents.

- $T = \prod_i T_i$: finite set of states $t = (t_1, \ldots, t_n)$, where $t_i \in T_i$ is agent i's type.

- $T^* \subseteq T$:

$$\{t \in T : \exists i \in I \text{ s.t. } \pi_i(t_i)[t_{-i}] > 0\} \subseteq T^*,$$

where $\pi_i(t_i)$ denotes t_i's interim belief.

- A: countable set of pure outcomes.

- $\Delta(A)$: set of prob. distrib. over A.

- $u_i : \Delta(A) \times T \to \mathbb{R}$: i's state dependent von Neumann-Morgenstern utility function.

- Interim EU of the SCF f for type t_i:

$$U_i(f|t_i) \equiv \sum_{t_{-i}} \pi_i(t_i)[t_{-i}]u_i(f(t_i, t_{-i}), (t_i, t_{-i}))$$

$$U_i(f; t_i'|t_i) \equiv \sum_{t_{-i}} \pi_i(t_i)[t_{-i}]u_i(f(t_i', t_{-i}), (t_i, t_{-i}))$$

SCFs, SCSs, and Mechanisms

- A (stochastic) **social choice function (SCF)**: $f : T \to \Delta(A)$.

- **Social choice set (SCS)** F: collection of SCFs.

- A **mechanism** $\Gamma = ((M_i)_{i \in I}, g)$: nonempty countable message space M_i for each $i \in I$, and a (stochastic) outcome function $g : M \to \Delta(A)$, where $M = \times_{i \in I} M_i$.

Interim Correlated Rationalizability (Dekel, Fudenberg, and Morris (2007))

- Message correspondence profile $S(t) = (S_1(t_1), \ldots, S_n(t_n))$, where each $S_i(t_i) \in 2^{M_i}$.

- The collection of message correspondence profiles is denoted by $\mathcal{S}$: complete lattice with the natural ordering of set inclusion: $S \leq S'$ if $S_i(t_i) \subseteq S'_i(t_i)$ for all $i \in I$ and $t_i \in T_i$.

- Largest element $\bar{S} = (M_1, \ldots, M_n)$.

- Smallest element $\underline{S} = (\emptyset, \ldots, \emptyset)$.

- Operator $b : \mathcal{S} \to \mathcal{S}$ to iteratively eliminate never best responses.

- $b(S) = (b_1(S), \ldots, b_n(S))$, with

$$b_i(S)[t_i] \equiv \left\{ m_i \,\middle|\, \begin{array}{l} \exists \lambda_i \in \Delta(T_{-i} \times M_{-i}) \text{ such that} \\ (1)\lambda_i(t_{-i}, m_{-i}) > 0 \Rightarrow m_j \in S_j(t_j)\ \forall j \neq i; \\ (2) m_i \in \arg\max_{m_i'} \sum_{t_{-i}, m_{-i}} \lambda_i(t_{-i}, m_{-i}) \\ \times u_i(g(m_i', m_{-i}(t_{-i})); (t_i, t_{-i})) \end{array} \right\}$$

- b is increasing: $S(t) \leq S^{'}(t) \Rightarrow b(S(t)) \leq b(S^{'}(t))$. By Tarski's fixed point theorem, there is a largest fixed point of b, $S^{\Gamma(t)}$, which gives us existence.

Interim Rationalizable Implementation of SCFs

SCFs f and f' are said to be **equivalent**, $f \approx f'$, whenever $f(t) = f'(t)$ for every $t \in T^*$.

Definition 3.1: An SCF f is implementable in interim rationalizable strategies whenever there exists $f' \approx f$ for which one can find a mechanism Γ with the following two conditions:

1. **Nonemptiness**: $S_i^{\Gamma(T)}(t_i) \neq \emptyset$ for all $t_i \in T_i$ and $i \in I$.

2. **Uniqueness**: for all $t \in T$, if $m \in S^{\Gamma(T)}(t)$, then $g(m) = f'(t)$.

Deceptions and Unacceptable Deceptions

- A **deception** is a profile of correspondences $\beta = (\beta_1, \ldots, \beta_n)$ such that $\beta_i : T_i \to 2^{T_i} \setminus \emptyset$ and $t_i \in \beta_i(t_i)$ for all $t_i \in T_i$ and $i \in I$.

- β is **unacceptable for an SCF** f if there exist $t \in T$ and $t' \in \beta(t)$ such that $f(t) \neq f(t')$; otherwise, β is **acceptable for** f.

The Strictly Lower Contour Set of f for Type t_i

- Given an SCF f, for each $i \in I$ and $t_i \in T_i$, define

$$Y_i[t_i, f] \equiv \left\{ y : T_{-i} \to \Delta(A) \;\middle|\; \begin{array}{cl} \text{either} & y(t_{-i}) = f(t_i, t_{-i}), \forall t_{-i} \\ \text{or} & U_i(f|t_i) > U_i(y|t_i) \end{array} \right\}.$$

- We say $t_i \sim_i^f t_i'$ if $f(t_i, t_{-i}) = f(t_i', t_{-i})$ for any $t_{-i} \in T_{-i}$. Otherwise, we say $t_i \not\sim_i^f t_i'$.

Weak Refutability of Deceptions

Definition 4.3: A deception β that is unacceptable for an SCF f is **weakly refutable** if there exist $i \in I$, $t_i \in T_i$, and $t_i^{'} \in \beta_i(t_i)$ satisfying $t_i^{'} \not\approx_i^f t_i$ such that for all $\psi_i \in \Delta(T_{-i} \times T)$ satisfying $\psi_i(t_{-i}, \tilde{t}) > 0 \Rightarrow \tilde{t}_{-i} \in \beta_{-i}(t_{-i})$ and $\pi_i(t_i)[t_{-i}] = \sum_{\tilde{t} \in T} \psi_i(t_{-i}, \tilde{t})$ for all $t_{-i} \in T_{-i}$, there exists an SCF $f^{'}$ such that $f^{'}(\tilde{t}_i, \cdot) \in Y_i[\tilde{t}_i, f]$ for all $\tilde{t}_i \in T_i$ and

$$\sum_{t_{-i}, \tilde{t}} \psi_i(t_{-i}, \tilde{t}) u_i\Big(f^{'}(\tilde{t}), (t_i, t_{-i})\Big) > \sum_{t_{-i}, \tilde{t}} \psi_i(t_{-i}, \tilde{t}) u_i\Big(f(t_i^{'}, \tilde{t}_{-i}), (t_i, t_{-i})\Big).$$

Strong refutability: same, but in stead of $f^{'}$, there exists $y : T_{-i} \to \Delta(A)$ such that $y \in \bigcap_{\tilde{t}_i \in T_i} Y_i[\tilde{t}_i, f]$.

Weak Interim Rationalizable Monotonicity

Definition 4.4: A SCF f satisfies **weak IRM** if every deception β that is unacceptable for f is weakly refutable.

- An SCF f satisfies **IRM** if every deception β that is unacceptable for f is **strongly** refutable.

Necessity

Theorem 4.5: If an SCF f is implementable in interim rationalizable strategies, then there exists an SCF $\widehat{f} \approx f$ that satisfies weak IRM.

Bayesian Incentive Compatibility (BIC)

Definition 5.1: An SCF f satisfies **Bayesian incentive compatibility (BIC)** if, for all $i \in I$ and $t_i \in T_i$,

$$U_i(f|t_i) \geq U_i(f; t_i^{'}|t_i), \ \forall t_i^{'} \in T_i.$$

If these inequalities are strict whenever $t_i \not\approx_i^f t_i^{'}$, then we say that f satisfies **strict-if-responsive Bayesian incentive compatibility (SIRBIC)**.

Lemma 5.2: If an SCF f satisfies weak IRM, then it satisfies SIRBIC.

The Weakly Lower Contour Set of f for Type t_i

- For each $i \in I$ and $t_i \in T_i$, define

$$Y_i^w[t_i, f] \equiv \{y : T_{-i} \to \Delta(A) : U_i(f|t_i) \geq U_i(y|t_i)\}.$$

- Notice that $Y_i[t_i, f]$ is a subset of $Y_i^w[t_i, f]$.

The Weak No-Worst-Rule Condition

Definition 6.1: The SCF f satisfies the **weak no-worst-rule** condition (**weak NWR**) if, for all $i \in I$, $t_i \in T_i$, and $\phi_i \in \Delta(T_{-i} \times T_{-i})$, there exist $y, y' \in Y_i^w[t_i, f]$ such that

$$\sum_{t_{-i}, t'_{-i}} \phi_i(t_{-i}, t'_{-i}) u_i\Big(y(t'_{-i}), (t_i, t_{-i})\Big) \neq \sum_{t_{-i}, t'_{-i}} \phi_i(t_{-i}, t'_{-i}) u_i\Big(y'(t'_{-i}), (t_i, t_{-i})\Big).$$

- **NWR**: same, but $y, y' \in \bigcap_{t_i \in T_i} Y_i^w[t_i, f]$.

Sufficiency

Theorem 6.3: For any SCF f, if there exists an SCF $\hat{f} \approx f$ such that $\hat{f}$ satisfies weak IRM and weak NWR, then the SCF f is implementable in interim rationalizable strategies.

Bayesian Monotonicity

A **single-valued** deception β^s is a profile of functions $(\beta^s_1, \ldots, \beta^s_n)$ such that $\beta^s_i : T_i \to T_i$ for all $i \in I$.

β^s is **unacceptable** for an SCF f if $f(\beta^s(t)) \neq f(t)$ for some $t \in T$; otherwise, β^s is **acceptable** for f.

Definition 5.7: An SCF f satisfies **Bayesian monotonicity (BM)** if, for every single-valued deception β^s that is unacceptable for f, there exist $i \in I$, $t_i \in T_i$, and $y : T_{-i} \to \Delta(A)$ such that $y \in \bigcap_{\tilde{t}_i \in T_i} Y_i^{w}[\tilde{t}_i, f]$ and

$$U_i(y \circ \beta^s_{-i} | t_i) > U_i(f \circ \beta^s | t_i).$$

Example

- $I = 1, 2$;
- $T_1 = \{t_1, t_1', t_1''\}$, $T_2 = \{t_2, t_2'\}$; interim beliefs such that $T^* = T$;
- A consists of six pure alternatives;
- f is the SCF that maximizes the sum of payoffs in each state.

The Example Continued

- We show that f satisfies weak IRM and weak NWR, and hence, by Theorem 6.3, f is implementable in interim rationalizable strategies.

- f violates IRM, hence contradicting an assertion in Oury and Tercieux (2012) as to the necessity of IRM for interim rationalizable implementation.

- f also violates Bayesian monotonicity, and hence it is “not” Bayesian implementable.

Other Results

Theorem 8.1: If an SCF f satisfies IRM and NWR, it is **doubly implementable**, i.e., implementable in interim rationalizable strategies by a mechanism that has a Bayesian equilibrium.

Lemma 5.8: If an SCF f satisfies IRM, it satisfies Bayesian monotonicity.

An SCF f is **responsive** if, for any $i \in I$ and $t_i, t_i^{'} \in T_i$,

$$t_i \neq t_i^{'} \Rightarrow t_i \not\sim_i^f t_i^{'}, \text{ i.e., } \exists t_{-i} \in T_{-i} \text{ s.t. } f(t_i, t_{-i}) \neq f(t_i^{'}, t_{-i}).$$

Theorem 8.3: Let f be an SCF that is responsive. Then, weak IRM and IRM are equivalent.

Summary

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{Double Implementation} \arrow[r, Rightarrow, shift left=2, "\text{By Def.}"] \arrow[d, Rightarrow, shift right=1] & \text{Rationalizable Implementation} \arrow[l, Rightarrow, shift left=2, "\text{Finite}"', "\text{Mechanisms}"] \arrow[d, Rightarrow, shift right=1] \\
\text{IRM} \arrow[u, Rightarrow, shift right=1, "\text{NWR}"'] \arrow[r, Rightarrow, shift left=2, "\text{By Def.}"] \arrow[d, Rightarrow] & \text{Weak IRM} \arrow[u, Rightarrow, shift right=1, "\text{Weak NWR}"'] \arrow[l, Rightarrow, shift left=2, "\text{Responsive SCFs}"] \arrow[d, Rightarrow] \\
\text{BM} & \text{Strict-if-Responsive BIC (SIRBIC)}
\end{tikzcd}
$$