# Signaling under Double-Crossing Preferences

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# Motivation

- There are a few assumptions in economics that have become "gold standard."
- Single-crossing property (aka Spence-Mirrlees condition) is one of them.
- In the classic example of education signaling, this amounts to saying that the marginal cost of education is lower for higher types.
- Many insights we learn from analyses of signaling are deeply rooted in this property.

# Motivation

- At a glance, SCP "looks" reasonable.
- SCP is handy in many ways.
  - Local incentive compatibility ensures global incentive compatibility.
  - Uniqueness under some refinement concepts (such as D1).
  - But with a byproduct that equilibrium is always monotone.
- Once we relax SCP in any way, we will naturally lose those properties as well.

# Motivation

- But,... economists do not always think of SCP as an accurate reflection of reality, especially in early days; it is rather a convenient assumption for tractability and clarity.
- Mailath (1987): "in many applications, it is difficult, if not impossible, to verify that the single crossing condition is satisfied for all [signaling and reputation levels]."
- Horner (2008): "Little is known about equilibria when single-crossing fails, as may occur in applications."

# Motivation

- To motivate our analysis, consider a model with news or "grades."
- Suppose that on top of the signal level, there is an exogenous source of information (test score, interview, etc).
- High-ability agents may have less incentive to signal, knowing that their type will be partially revealed anyway.
- SCP may fail to hold in this environment.
- More on this point later.

# Motivation

- SCP is not as robust as generally believed, nor is it innocuous.
- We provide more examples to make this point, and more broadly the underlying principle which tends to break down SCP.
- It is crucial to broaden the scope of analysis to circumstances that are not constrained by SCP.
- We take a step in this direction, by providing a general analysis of signaling under *double-crossing property*.

# Outline

1. Model setup
2. Examples
3. Characterization: Low types Separate High types Pairwise-Pool (LSHPP) equilibrium
4. Existence

# Main takeaways

- A general framework to capture double-crossing preferences.
- Examples of double-crossing preferences.
- A full characterization of equilibria under D1 criterion.
  - Any D1 equilibrium is LSHPP.
  - Equilibrium is possibly non-monotone (counter-signaling).
- An algorithm to construct an LSHPP equilibrium.
  - Equilibrium existence by construction

# Main takeaways

- LSHPP equilibrium is a generalization of Low types Separate High types Pool (LSHP) equilibrium introduced by Kartik (2009).
- Equilibrium under double-crossing preferences exhibits a particular form of pooling (pairwise pooling) at the higher end of types.
    - There is a threshold type below which types are fully separated and above which they are clustered in possibly non-monotonic ways.
- Our notion of pairwise pooling is closely related to a phenomenon known as *counter-signaling* in the literature.

# Main takeaways

- Counter-signaling (Feltovich et al., 2002; Araujo et al., 2007): low and high types pool by refraining from costly signaling while middle types separate from them.
  - "Too cool for school"
  - However, establishing a counter-signaling equilibrium in a general environment is difficult.
  - Our understanding of counter-signaling has been limited to specific contexts.
- Our equilibrium construction generalizes the notion of counter-signaling to that of pairwise-pooling and enables us to establish its existence under weak conditions.

# Setup

- Continuum of types: $\theta \in [\underline{\theta}, \overline{\theta}]$ which is agent's private information.
- Action: $a \in \mathbb{R}_+$ which is publicly observable.
- Reputation: $t = \mathbb{E}[\theta \mid a]$.
- Utility: $u(a, t, \theta)$.

**Assumption 1:** $u : \mathbb{R}_+ \times [\underline{\theta}, \overline{\theta}]^2$ is twice continuously differentiable, and is strictly increasing in $t$.

# Setup

- We make heavy use of the marginal rate of substitution (MRS) between action $a$ and reputation $t$:

$$m(a, t, \theta) := -\frac{u_a(a, t, \theta)}{u_t(a, t, \theta)}.$$

- If we let $t = \phi(a, u, \theta)$ represent the indifference curve of type $\theta$ at utility level $u$, MRS gives its slope, i.e., $\phi_a(a, u, \theta) = m(a, \phi(a, u, \theta), \theta)$.

# Setup

- SCP (in the usual way) suggests that if a low type $\theta''$ is indifferent between a higher action $a_1$ and lower one $a_2$, a higher type $\theta' > \theta''$ strictly prefers the higher action.
- This is equivalent to $m(a, t, \theta') < m(a, t, \theta'')$ for all $(a, t)$.
- Requiring this globally is indeed a strong restriction to impose on the structure of preferences.

# Setup

- We deviate from this standard paradigm to allow for "double-crossing preferences."

**Definition 1** [Double-crossing property] For any $\theta' > \theta''$, there exists a continuous function $D(\cdot; \theta', \theta'') : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ such that:

(a) if $a < a_0 \leq D(t_0; \theta', \theta'')$, then

$$u(a, t, \theta'') \leq u(a_0, t_0, \theta'') \Rightarrow u(a, t, \theta') < u(a_0, t_0, \theta');$$

(b) if $a > a_0 \geq D(t_0; \theta', \theta'')$, then

$$u(a, t, \theta'') \leq u(a_0, t_0, \theta'') \Rightarrow u(a, t, \theta') < u(a_0, t_0, \theta').$$

# Setup

- The key to characterizing DCP is obviously the function $D(\cdot;\theta',\theta'')$, which we often call the *dividing line*.
- For an arbitrary pair of types $(\theta',\theta'')$, $D(\cdot)$ divides the $(a,t)$-space into two distinct regions.
- To the left of $D(\cdot)$, SCP holds in the usual way.
- To the right, SCP holds in the reverse way.
- The indifference curves are tangent at $(D(t;\theta',\theta''),t)$.

# Setup

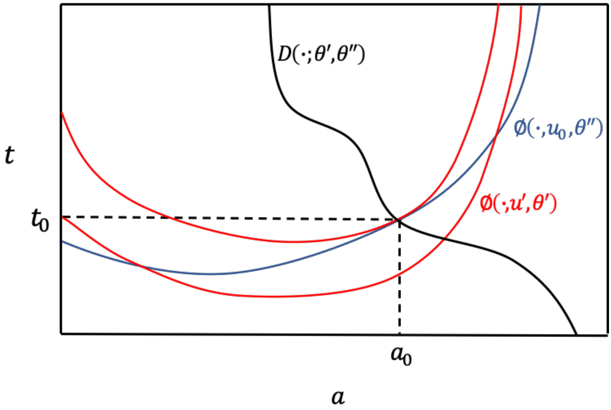


Figure: Double-crossing property. The indifference curve of a higher type $\theta'$ crosses that of a lower type $\theta''$ from above to the left of the dividing line $D(\cdot;\theta',\theta'')$, and crosses it again from below to the right of the dividing line. Along the dividing line, higher types have more convex indifference curves.

# SCP vs DCP

- SCP: Indifference curves of two types are single-crossing.
- DCP: MRSs (slopes of indifference curves) of two types are single-crossing.
    - Between any two types $\theta' > \theta''$, MRS of type $\theta'$ crosses that of type $\theta''$ from below to above.
    - For a given $t$, the two indifference curves cross at $a = D(t; \theta', \theta'')$.

# Setup

**Assumption 2:** $u(\cdot)$ satisfies DCP.

- Formally, the difference in MRSs between two types is single-crossing along an indifference curve: for $\theta' > \theta''$,

$$m(a, \phi(a, u_0, \theta''), \theta') - m(a, \phi(a, u_0, \theta''), \theta'') \begin{cases} \leq 0 & \text{if } a \leq a_0 \leq D(t_0; \theta', \theta''), \\ \geq 0 & \text{if } a \geq a_0 \geq D(t_0; \theta', \theta''), \end{cases}$$

  where type $\theta''$ attains utility $u_0$ at $(a_0, t_0)$.
- Assumption 2 holds if and only if there exists $D(\cdot)$ that satisfies this condition, so this can be adopted as an alternative assumption of DCP.

# Setup

- Assumptions 1 and 2 are sufficient if there are only two types.
- With a continuum of types, we need an additional restriction on how $D(\cdot;\theta',\theta'')$ shifts with respect to $\theta'$ and $\theta''$.

**Assumption 3:** For any $t$, $D(t;\theta',\theta'')$ strictly decreases in $\theta'$ and $\theta''$.

# Setup

- Define

$$D(t;\theta,\theta) := \lim_{\theta''\to\theta^-} D(t;\theta,\theta) = \lim_{\theta'\to\theta^+} D(t;\theta,\theta).$$

**Definition 2:** $(a,t)$ is in the SC-domain of type $\theta$ if it belongs to the set $SC(\theta) := \{(a,t) : a < D(t;\theta,\theta)\}$; and it is in the RSC-domain of type $\theta$ if it belongs to $RSC(\theta) := \{(a,t) : a > D(t;\theta,\theta)\}$.

# Setup

- So, what does Assumption 3 mean?
- The assumption is not easy to interpret in terms of preferences; the following is useful for relating it to MRS.

**Lemma 1:** Suppose preferences satisfy DCP. Then Assumption 3 holds if and only if $m(a, t, \theta)$ is strictly quasi-convex in $\theta$ for any $(a, t)$.

# Setup

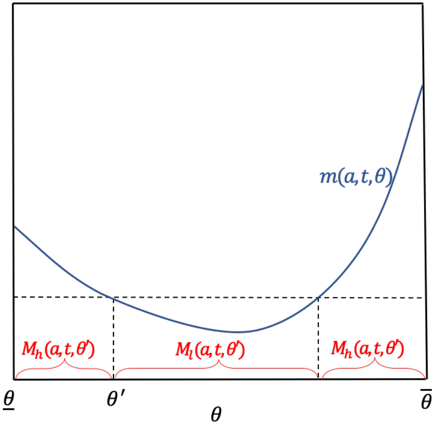


Figure: The marginal rate of substitution is quasi-convex in $\theta$ for any $(a, t)$.

# Setup

- Given this, an alternative way to state Definition 2 is that $(a, t)$ belongs to the SC-domain of type $\theta$ if $m(a, t, \cdot)$ is locally decreasing at $\theta$ and to the RSC-domain if $m(a, t, \cdot)$ is locally increasing.
- In the standard setup, MRS strictly decreases in type.
- In contrast, type $\theta$ has the lowest MRS at $(D(t; \theta, \theta), t)$.
- A symbolic feature of double-crossing preferences is that the marginal costs of signaling are (often) lowest for intermediate types.

# Setup

- The probability distribution over types is $F$ with full support.
- Signaling models typically exhibit a plethora of equilibria, and we adopt D1 to restrict off-path beliefs.
- Under D1, the standard setup predicts the least-cost separating equilibrium, which is distribution-free.
- This is not the case here, where D1 equilibria often entail some pooling.
- As a consequence, the distribution of types has a nontrivial impact on the form of equilibrium.

# Examples

- While our specification is a natural way to define double-crossing preferences, Assumption 2 do impose economically meaningful restrictions.
- It means that the indifference curves of higher types are more "convex."
  - Under SCP, the relevant issue is which type has a higher MRS.
  - Under DCP, the issue is of higher order: we need to determine how the slope of MRS is related to agent type, for which there appears to be no *a priori* obvious specification.

# Examples

- To better motivate our modeling choices, we provide several examples (here, two) of double-crossing preferences.
- This exercise serves two purposes.
  - It shows that despite its widespread use in economic analysis, SCP is not as robust as generally believed.
  - It also justifies our assumptions, i.e., that higher types tend to have more convex indifference curves in many economic environments.

# Signaling with News

- Several works have pointed out that SCP fails in signaling models with news or "grades" (Feltovich et al., 2002; Araujo et al., 2007; Daley and Green, 2014).
- Consider an environment with two sources of information: a signaling action and a test score.
- Suppose that the test score is binary, either pass or fail, and the agent passes the test with probability $\beta_0 + \beta\theta$.
- If the agent passes the test; he will be promoted and earn $V$; if he fails, his payoff is determined by his outside option $t$.

# Signaling with News

- The agent's payoff is given by

$$u(a, t, \theta) = (\beta_0 + \beta\theta)\lambda V + [1 - (\beta_0 + \beta\theta)]\lambda t - \left(\frac{\gamma a}{\theta} + \frac{a^2}{2}\right).$$

- This model, which captures the essence of the previous literature, satisfies both Assumptions 2 and 3.
- As Feltovich et al. (2002) illustrate, this type model often leads to counter-signaling.
- Our analysis shows this in a more general framework, showing that counter-signaling is a common feature of DCP.

# Risky Experimentation

- This example is adapted from our previous work (Chen et al., 2020).
- Suppose that an agent engages in risky experimentation with a hidden state of nature.
- If the state is good, success arrives stochastically with Poisson rate $\theta$; if not, success never arrives.
- The prior probability that the state is good is $\pi$.
- No one can observe the state, but the arrival rate $\theta$ is the agent's private information.

# Risky Experimentation

- The model is an optimal stopping problem with reputation concerns.
- For instance, consider an early-career economic theorist who is in obvious need of impressing coauthors, senior colleagues, and the entire academic community of his analytical prowess.
- Suppose that he works on a conjecture which may or may not be true.
- When should he abandon the project in case success has not arrived?

# Risky Experimentation

- The model also satisfies both Assumptions 2 and 3.
- This is because of the conflicting effects of experimentation.
  - Higher types are more likely to achieve success *if the project is good*.
  - Higher types also learn faster that the project is not promising.
- The value of experimentation is higher for higher types early on, but becomes lower later.

# Why Double Crossing?

- DCP is more likely to emerge under two conditions.

1. The gains from signaling are not unbounded.
2. Higher types reach the point of diminishing returns faster than lower types.

- Under these two conditions, the marginal gain from signaling is not necessarily higher for higher types, as assumed under SCP.

# Characterization

- A characterization of perfect Bayesian equilibria under D1.
- $S : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$: sender's strategy
- $T : [\underline{\theta}, \overline{\theta}] \to [\underline{\theta}, \overline{\theta}]$: equilibrium reputation
- $Q(a) := \{\theta : S(\theta) = a\}$: the set of types that choose $a$
- We refer to $Q(a)$ as a pooling set if it is not a singleton.

# Full Separation

- Let $s^*(\cdot)$ be a fully separating strategy for some interval, where $T(\theta) = \theta$ in this interval.
- Incentive compatibility requires

$$u(s^*(\theta), \theta, \theta) \geq u(s^*(\theta + \epsilon), \theta + \epsilon, \theta).$$

- In the limit, we have

$$s^{*'}(\theta) = \frac{1}{m(s(\theta), \theta, \theta)}.$$

# Full Separation

- In general, the lowest cannot do worse than revealing his own type.
- Under full separation, the initial condition must satisfy

$$s^*(\underline{\theta}) = \underline{a}^* := \arg\max_a u(a, \underline{\theta}, \underline{\theta}).$$

- Under SCP, the solution to the differential equation with this initial condition constitutes a fully separating equilibrium (Mailath, 1987).
- The solution is known as the *least cost separating equilibrium* or the *Riley outcome*.

# Full Separation

- In our model, there is a dividing line $D(\cdot;\cdot,\cdot)$ that separates the $(a, t)$-space into two distinct domains.
- It is easy to show that no fully separating solution can extend over the dividing line.

**Proposition 1** There is no fully separating equilibrium if there exists $\theta' < \overline{\theta}$ such that $s^*(\theta') = D(\theta';\theta',\theta')$.

# Pooling under D1

- Under DCP, some form of pooling can survive D1.
- For any $(a, t)$ and any $Q(a)$, let

$$\theta_{\max}(a, t; Q(a)) := \arg\max_{\theta \in Q(a)} m(a, t, \theta),$$
$$\theta_{\min}(a, t; Q(a)) := \arg\min_{\theta \in Q(a)} m(a, t, \theta).$$

- Consider some pooling at $(a_p, t_p)$.
- To satisfy D1, we must have

$$t_p \geq \max\{\theta_{\max}(a, t; Q(a_p)), \theta_{\min}(a, t; Q(a_p))\}.$$

# Pooling under D1

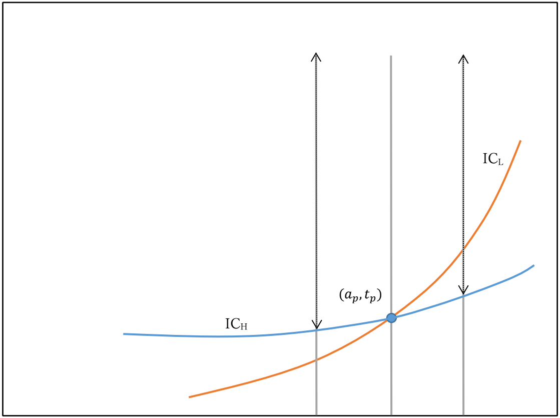


Figure: D1 assigns probability zero to a type if there is another type that benefits more from the deviation.

# LSHPP equilibrium

**Definition 3** A sender's strategy is LSHPP (Low types Separate High types Pairwise-Pool) if there is some $\theta_0 \in [\underline{\theta}, \overline{\theta}]$ such that:

(a) $S(\theta) = s^*(\theta)$ for $\theta \in [\underline{\theta}, \theta_0)$;

(b) $S(\theta)$ is discontinuous only at $\theta = \theta_0$, with an upward (resp. downward) jump if $s^*(\cdot)$ is increasing (resp. decreasing) on $[\underline{\theta}, \theta_0)$.

(c) $S(\theta)$ is weakly quasi-concave for $\theta \in [\theta_0, \overline{\theta}]$, with $S(\theta_0) = S(\overline{\theta})$.

An equilibrium is an *LSHPP equilibrium* if the sender's strategy is LSHPP.

# LSHPP equilibrium

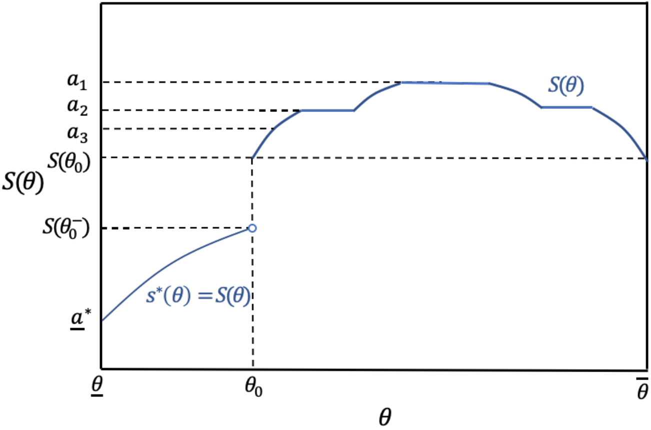


Figure: LSHPP strategy. Below the gap, $S(\cdot)$ coincide with the least cost separating strategy $s^*(\cdot)$. Above the gap, $S(\cdot)$ is quasi-concave.

# LSHPP equilibrium

- There is a (unique) threshold type $\theta_0$ below which types are fully separated and below which they are clustered in possibly non-monotonic ways.
- Above the "gap," two distinct types may pair up to choose the same action, or two distinct intervals of types pair up–therefore the term *pairwise-pooling*.
- Pairs further apart choose lower actions than pairs that are closer to one another.
- Our notion of LSHPP equilibrium includes as special cases:
  - *Full separation* ($\theta_0 = \underline{\theta}$)
  - *Full pooling* ($\theta_0 = \overline{\theta}$ with $S(\cdot)$ constant
  - *LSHP*: $\theta \in (\underline{\theta}, \overline{\theta})$ with $S(\cdot)$ constant above the gap

# LSHPP equilibrium

**Theorem 1** Any D1 equilibrium is LSHPP if Assumptions 1 to 3 are satisfied.

# Mass pooling vs atomless pooling

- Below the gap, there is fully separation.
- Our strategy is to identify restrictions on possible forms of pooling above the gap.

**Lemma 2** Suppose there is an interval $(\theta'', \theta')$ such that $S(\theta)$ is continuous and strictly monotone, and $Q(S(\theta))$ is a pooling set for some $\theta$ in this interval. Then, there exists $p(\cdot)$ such that, for all $\theta \in (\theta'', \theta')$, (a) $Q(S(\theta)) = \{\theta\} \cup \{p(\theta)\}$; and (b) $m(S(\theta), T(\theta), \theta) = m(S(\theta), T(\theta), p(\theta))$.

# Mass pooling vs atomless pooling

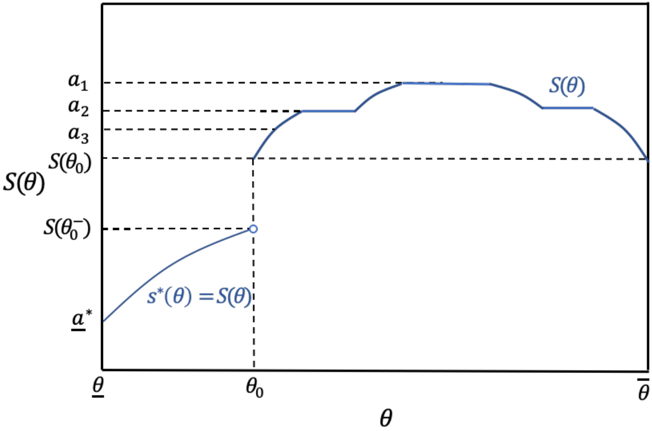


Figure: LSHPP strategy. Below the gap, $S(\cdot)$ coincide with the least cost separating strategy $s^*(\cdot)$. Above the gap, $S(\cdot)$ is quasi-concave.

# Mass pooling vs atomless pooling

- Two forms of pooling are feasible under DCP.
- *Atomless pooling*: Exactly two types pair up to choose the same action.
- *Mass pooling*: Intervals of types pair up to choose the same action.

# Mass pooling vs atomless pooling

- Atomless pooling is easy to characterize because any two types that are paired must have the same MRS.
- It is also easy to characterize mass pooling if it consists of one interval ($Q(a)$ is connected).
- The case of disconnected sets is much more complicated as it can take infinitely many forms.

# Disconnected pooling sets

**Lemma 3** Suppose there is pooling at $(a_p, t_p)$ such that the pooling set $Q(a_p)$ is disconnected.

(a) $Q(a_p) = Q_L(a_p) \cup Q_R(a_p)$, where $Q_L(a_p)$ and $Q_R(a_p)$ are two disjoint intervals, with $(a_p, t_p) \in SC(\theta)$ for $\theta \in Q_L(a_p)$ and $(a_t, t_p) \in RSC(\theta)$ for $\theta \in Q_R(a_p)$.

(b) $S(\theta) \geq a_p$ for all $\theta \in [\min Q(a_p), \max Q(a_p)]$.

(c) $S(\theta)$ is continuous for all $\theta \in [\min Q(a_p), \max Q(a_p)]$.

# Disconnected pooling sets: from outside

- Suppose $S(\cdot)$ is continuous on $[\theta_0, \overline{\theta}]$, and there is a disconnected pooling set $Q(a_p)$ in this interval.
- There must be a path $S(\cdot)$ converging to $a_p$ as $\theta$ approaches $\underline{\theta}_p := \min Q(a_p)$.
- Lemma 2 suggests that there must be another path $(S(p(\cdot))$ converging to $a_p$ as $\theta$ approaches $\overline{\theta}_p := \min Q(a_p)$.
- At the end points, we must have $m(a_p, t_p, \underline{\theta}_p) = m(a_p, t_p, \underline{\theta}_p)$.

# Disconnected pooling sets: from outside

- Recall that by quasi-convexity of MRS, types outside of $[\underline{\theta}_p, \overline{\theta}_p]$ have higher MRS.
- They have more incentive to choose lower actions.
- When $S(\cdot)$ approaches $a_p$ from "outside," it must be increasing on the left and decreasing on the right.

# Disconnected pooling sets: from outside

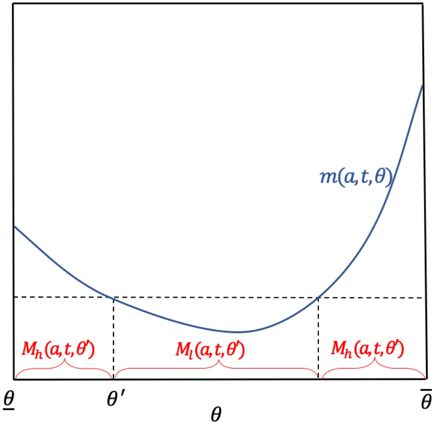


Figure: The marginal rate of substitution is quasi-convex in $\theta$ for any $(a, t)$.

# Disconnected pooling sets: from inside

- If $Q(a_p)$ is disconnected, we can define

$$J(a_p) := \{\theta : \theta \notin Q(a_p), \theta \in (\underline{\theta}_p, \overline{\theta}_p)\}.$$

- Let $\underline{\theta}_j := \inf J(a_p)$ and $\overline{\theta}_j := \sup J(a_p)$.
- If $S(\cdot)$ is continuous, there must be at least two more paths, $S(\cdot)$ and $S(p(\cdot))$, converging to $a_p$.

# Disconnected pooling sets: from inside

- Again, $m(a_p, t_p, \underline{\theta}_j) = m(a_p, t_p, \overline{\theta}_j)$.
- By quasi-concavity, types in $(\underline{\theta}_j, \overline{\theta}_j)$ have lower MRS.
- They more incentive to choose higher actions.

# Disconnected pooling sets: summary

- $S(\cdot)$ must be quasi-concave above the gap.
- Quasi-convexity of MRS strongly suggests quasi-concavity of $S(\cdot)$.
- Any disconnected pooling set must consist of two intervals (or $J(a_p)$ must be connected).
- This is also from (strict) quasi-convexity of MRS: at any $(a, t)$, there are at most two types with the same MRS.

# Existence

**Assumption 4** For any $\theta$, $u(\cdot, \theta, \theta)$ is quasi-concave, with a unique optimal action $a^*(\theta)$ such that $(a^*(\theta), \theta) \in SC(\theta)$.

**Assumption 5** $F : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ is continuously differentiable and strictly increasing.

# Existence

- For $\theta \leq \theta_0$, $S(\theta) = s^*(\theta)$, $T(\theta) = \theta$.
- Let $\theta_* \in \arg\max_{\theta \in [\theta_0, \bar{\theta}]} S(\theta)$ denote the "boundary type."
- Above $\theta_0$, there are three objects to be determined:
    - $\sigma$: signaling action taken by $\theta \in [\theta_0, \theta_*]$
    - $\tau$: equilibrium reputation
    - $p$: type that is "paired with" type $\theta$
- Construct a perfect Bayesian equilibrium under D1.

# Bayes' rule

- Belief consistency requires that for any interval $[\theta_E, \theta_B]$,

$$\int_{\theta_E}^{\theta_B} \tau(\theta)\,\mathrm{d}F(\theta) + \int_{p(\theta_B)}^{p(\theta_E)} \tau(\theta)\,\mathrm{d}F(\theta) = \int_{\theta_E}^{\theta_B} \theta\,\mathrm{d}F(\theta) + \int_{p(\theta_B)}^{p(\theta_E)} \theta\,\mathrm{d}F(\theta).$$

- Taking the limit, we obtain a "pointwise" belief:

$$\tau(\theta_E) = \frac{f(\theta_E)}{f(\theta_E) + f(p(\theta_E))\ |p'(\theta_E)|}\theta_E + \frac{f(p(\theta_E))\ |p'(\theta_E)|}{f(\theta_E) + f(p(\theta_E))\ |p'(\theta_E)|}p(\theta_E).$$

- Solving this for $p'$ yields

$$p'(\theta) = \frac{f(\theta)}{f(p(\theta))}\frac{\theta - \tau(\theta)}{p(\theta) - \tau(\theta)}.$$

# Incentive compatibility

- In equilibrium, no type has an incentive to mimic adjacent types:

$$u(\sigma(\theta), \tau(\theta), \theta) \geq u(\sigma(\theta+\epsilon), \tau(\theta+\epsilon), \theta).$$

- In the limit, we obtain

$$\sigma'(\theta) = \frac{\tau'(\theta)}{m(\sigma(\theta), \tau(\theta), \theta)}.$$

- This is local incentive compatibility; we still need to check global incentive compatibility.

# Pairwise matching

- When there is mass pooling, incentive compatibility must be satisfied for both $\theta$ and $p(\theta)$.
- This boils down to the restriction that the two paired types must have the same MRS:

$$m(\sigma(\theta), \tau(\theta), p(\theta)) - m(\sigma(\theta), \tau(\theta), \theta) = 0.$$

- Taking derivative with respect to $\theta$ gives

$$[\hat{m}_a(\cdot) - m_a(\cdot)]\,\sigma'(\cdot) + [\hat{m}_t(\cdot) - m_t(\cdot)]\,\tau'(\cdot) = m_\theta(\cdot) - \hat{m}_\theta(\cdot)p'(\cdot),$$

where $\hat{m}(\cdot)$ is MRS evaluated at $(\sigma(\theta), \tau(\theta), p(\theta))$.

# Indifference at the gap

- The three differential equations allow us to obtain a candidate equilibrium strategy above some threshold $\theta_0$.
- To pin down an equilibrium, type $\theta_0$ must be indifferent between choosing $s^*(\theta)$ and jumping to $\sigma(\theta_0)$.
- Define

$$\Delta_u(\theta_*) := u(s^*(\theta_0), \theta_0, \theta_0) - u(\sigma(\theta_0), \tau(\theta_0), \theta_0),$$

  where $\theta_0$ is taken as an implicit function of $\theta_*$.
- Equilibrium requires $\Delta_u(\theta_*) \leq 0$ (and $\Delta_u(\theta_*) = 0$ if there is an interior solution $\theta_0 \in (\underline{\theta}, \overline{\theta})$).

# Atomless pooling

- Where there is atomless pooling, we have

$$[\hat{m}_a(\cdot) - m_a(\cdot)]\,\sigma'(\cdot) + [\hat{m}_t(\cdot) - m_t(\cdot)]\,\tau'(\cdot) = m_\theta(\cdot) - \hat{m}_\theta(\cdot)p'(\cdot).$$

- Under DCP, the left-hand side is always strictly positive.
- Atomless pooling requires $m_\theta(\cdot) - \hat{m}_\theta(\cdot)p'(\cdot) > 0$.
- Once the right-hand side turns from positive to zero, the solution cannot be extended further back.

# Mass pooling

- When atomless pooling is not feasible, we have mass pooling where an interval of types take the same action, i.e., $\sigma'(\cdot) = 0$ and $\tau'(\cdot) = 0$.
- Suppose all types in $[\theta_E, \theta_B] \cup [\hat{\theta}_B, \hat{\theta}_E]$ choose $(a_B, t_B)$.
- Given $(\theta_B, \hat{\theta}_B)$, we need to find $(\theta_E, \hat{\theta}_E)$ such that

$$m(a_B, t_B, \hat{\theta}_E) = m(a_B, t_B, \theta_E),$$
$$\mathbb{E}[\theta \mid \theta \in [\theta_E, \theta_B] \cup [\hat{\theta}_B, \hat{\theta}_E]] = t_B.$$

# Algorithm

- Letting $x = (p, \sigma, \tau)$, we have a (mostly) well defined system of differential equations of the form $x' = H(\theta, x)$.
- An initial condition is

$$\sigma(\theta_*) = D(\theta_*; \theta_*, \theta_*),\ \ \tau(\theta_*) = p(\theta_*) = \theta_*.$$

- This is the only valid initial condition if there is atomless pooling in a neighborhood of $\theta_*$.
- The solution to the system of differential equations with initial condition gives a candidate equilibrium.

# Algorithm

- Consider the following algorithm.

1. If atomless pooling is feasible, then extend the solution as far as possible (until $m_\theta(\cdot) - \hat{m}_\theta(\cdot)p'(\cdot)$ turns 0).
2. Once in mass pooling, find the largest $\theta_E$ that satisfies the two conditions and, if any, switch to atomless pooling.
3. Continues this process until we find a $\theta_E$ such that $p(\theta_E) = \overline{\theta}$.

# Algorithm

- This algorithm consistently gives us a well defined system of differential equations $x' = H(\theta, x)$ where $x = (p, \sigma, \tau)$.
- Following this algorithm, we can consistently find a "gap point" $\theta_0$ such that $p(\theta_0) = \overline{\theta}$ for any given $\theta_*$.
- Let $\zeta$ denote this mapping from $\theta_*$ to $\theta_0$.
- Continuity of $\Delta_u(\cdot)$ is ensured if this mapping is continuous with respect to $\theta_*$.

# Existence

**Theorem 2** An LSHPP equilibrium exists if Assumptions 1 to 5 are satisfied.

# Numerical examples

- Consider the "signaling with news" example.
- We choose parameters so that

$$u(a,t,\theta) = \lambda(\theta + (1-\theta)t) - \left(\frac{a}{\theta} + \frac{a^2}{2}\right).$$

- Let the distribution of types be given by $f(\theta) = 2.5 + \kappa(\theta - 0.3)$ for $\theta \in [0.1, 0.5]$.

# Numerical examples

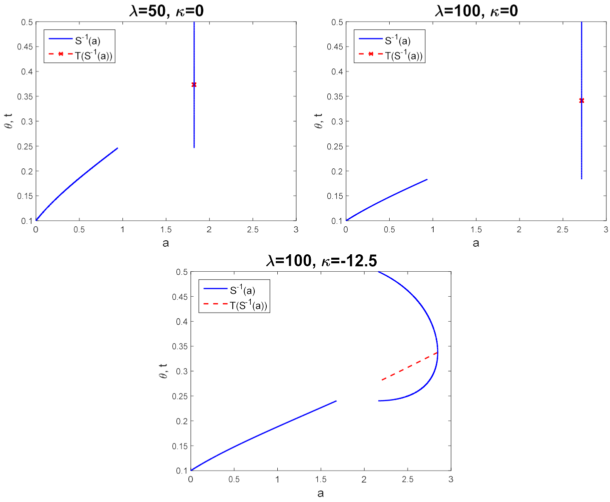


Figure: Equilibrium actions for different returns to signaling (parameter $\lambda$) and different type distributions (parameter $\kappa$).

# Discussion: multiplicity

- Equilibrium under DCP is not typically unique.
- There is a continuum of pooling equilibria.
- Even from a given $\theta_*$, there can be other algorithms which choose other equilibria.
- Any further characterization?

# Discussion: other variations of DCP

- The structure of preferences in our model is determined by Assumptions 2 and 3.
- Assumptions 2 says SC to the left of $D$ and RSC to the right; assumption 3 says mrs is quasi-convex in type.
- These assumptions capture independent aspects of DCP and can be relaxed one by one, thereby yielding four possible specifications.
- We argue that our current specification is most natural and economically.
- Incidentally, it also turns out to be most tractable and well behaved.

# Conclusion

- Despite its widespread use in economic analysis, SCP is not as robust as it is generally believed.
- Some examples are provided to make case for DCP.
- Equilibrium under DCP is fully characterized and shown to exhibit a particular form of pooling called LSHPP.
- Existence is established under mild conditions.