On the Uniqueness and Stability of the Equilibrium Price in Quasi-Linear Economies

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Motivation: Partial Equilibrium Theory (1)

Consider the following diagram of partial equilibrium analysis:

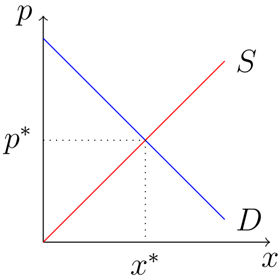

Figure: Partial Equilibrium Theory

where the blue decreasing line D denotes the demand curve and the red increasing line S denotes the supply curve.

Motivation: Partial Equilibrium Theory (2)

In this diagram, these curves cross at a unique point (p^*, x^*). This point is called an **equilibrium**, and p^* is called the equilibrium price and x^* is called the equilibrium output, respectively.
In the usual settings, the demand curve must be decreasing and the supply curve must be nondecreasing. (Explained later.) Therefore, the equilibrium must be unique.

Motivation: Partial Equilibrium Theory (3)

Next, choose a price $p' > p^*$.

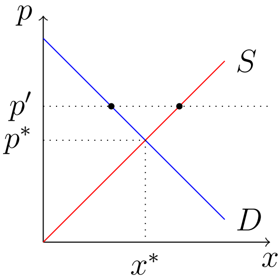

Figure: Excess supply

Then, at the height p', the supply curve is to the right of the demand curve. This phenomenon is called the **excess supply**.

Motivation: Partial Equilibrium Theory (4)

In the case of excess supply, a lot of products remain unsold in the market. In this situation, the price is expected to fall.

Motivation: Partial Equilibrium Theory (5)

Similarly, choose a price $p' < p^*$.

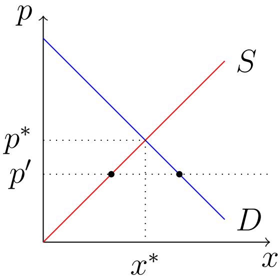

Figure: Excess demand

Then, at the height p', the supply curve is to the left of the demand curve. This phenomenon is called the **excess demand**.

Motivation: Partial Equilibrium Theory (6)

In this case, the market will be sold out and many consumers will not be able to purchase the product. Thus, the price is expected to rise. In conclusion, we found that the equilibrium price p^* has the power to attract other prices. We say that this price p^* is **stable**.

Motivation: Partial Equilibrium Theory (7)

If the demand curve is decreasing and the supply curve is nondecreasing, then the equilibrium price **must be unique and stable**. By the way, **why** is the demand curve decreasing and is the supply curve nondecreasing? The reason arises from the behind structure of partial equilibrium analysis.
In fact, there exists the foundation of the general equilibrium model behind this analysis.[1] In the behind model, the dimension of the commodity space must be two, and commodity 2 is called the **numeraire** good. The numeraire good is explained as a sort of money, and thus the price of this commodity is assumed to be 1. Therefore, we only consider the price of commodity 1, which is denoted by p.

[1] The explanation here is in abbereviated form. For more concrete knowledge, see Hayashi (2017).

Motivation: Partial Equilibrium Theory (8)

Consumers' utility function is assumed to be represented by the following form:

$$U_i(x_1, x_2) = u_i(x_1) + x_2,$$

where u_i is assumed to be strictly concave and twice continuously differentiable. This function is called the **quasi-linear utility**. Note that, because u_i is strictly concave, u_i' is decreasing. By Lagrange's multiplier rule, we can easily show that the first-order condition for optimality is described by

$$u_i'(x_1) = p.$$

Therefore, the demand curve $D(x)$ must satisfy the following:

$$D(x) = p \Leftrightarrow x = \sum_i (u_i')^{-1}(p),$$

which implies that $D(x)$ must be decreasing.

Motivation: Partial Equilibrium Theory (9)

It is assumed that every supplier produce commodity 1, and the profit of the supplier j can be represented by the following function:

$$\pi_j(y_j) = py_j - c_j(y_j),$$

where $c_j(y_j)$ denotes the cost to produce y_j. This twice continuously differentiable function c_j is called the **cost function**. By the first-order condition and the second-order necessary condition for optimality, we must have that if j maximizes his/her profit, then $p = c_j'(y_j)$ and $c_j''(y_j) \ge 0$. Thus, roughly speaking, we have that c_j' must be nondecreasing around the mamimum point of the profit. Therefore, the demand curve $S(x)$ must satisfy the following:

$$S(x) = p \Leftrightarrow x = \sum_j (c_j')^{-1}(p),$$

which implies that the supply curve must be nondecreasing.

Partial Equilibrium vs. General Equilibrium (1)

Hence, in the partial equilibrium theory, the equilibrium price is unique and stable. Can we extend this result into more general economy? The answer is NO.

Partial Equilibrium vs. General Equilibrium (2)

Debreu (1974) showed the following result.

Sonnenschein-Mantel-Debreu Theorem

Choose any small $\varepsilon > 0$, and let $f : \Delta_\varepsilon \to \mathbb{R}^L$ be any continuous function that satisfies $p \cdot f(p) = 0$, where

$$\Delta_\varepsilon = \left\{ x \in \mathbb{R}^L \,\middle|\, \sum_i x_i = 1,\ x_i \geq \varepsilon \right\}.$$

Then, there exists a pure exchange economy with L consumers in which 1) every consumer has a continuous, strictly quasi-concave, and increasing utility function, and 2) $\zeta(p) = f(p)$ on Δ_ε, where ζ is the excess demand function ζ of this economy.

Partial Equilibrium vs. General Equilibrium (3)

By this theorem and Urysohn's lemma, we can easily show that the set of equilibrium prices can include any compact subset of the unit simplex, and thus the number of normalized equilibrium prices can be one, ten, a billion, or infinity. Moreover, Debreu also presented an example of the economy in which the normalized equilibrium price is unique but unstable. Therefore, both uniqueness and stability are broken in general equilibrium theory.

Meanwhile, Arrow, Block, and Hurwicz (1959) showed that if the excess demand function satisfies the gross substitution, then the normalized equilibrium price must be unique and stable. However, there is no known assumption of economy that assures the gross substitution of the excess demand function, and thus, the uniqueness and stability of the equilibrium price cannot be assured in general setup, even when the number of commodities is two.

Partial Equilibrium vs. General Equilibrium (4)

In conclusion, we found the following:

- If the number of commodities is two and every consumer has a quasi-linear utility, then the normalized equilibrium price is unique and stable.
- If the quasi-linear assumptions do not hold, then the normalized equilibrium price need not be unique, and even if it is unique, it need not be stable.

Today's Question: If every consumer has a quasi-linear utility but the number of commodities is more than two, then does the former result still remain true?

Today's Answer: YES, but partially. The normalized equilibrium price is unique and **locally stable** in this situation.

Today's Model: an Overview

- To simplify the arguments, we treat a pure exchange economy with L commodities, where $L \geq 2$.
- It is assumed that every consumer has a quasi-linear utility function
$$U_i(x) = u_i(x_1, ..., x_{L-1}) + x_L.$$
- It is assumed that either the negative amount $x_L < 0$ of the numeraire good L can be consumed, or the initial endowment ω_L^i of the numeraire good L for every consumer i is sufficiently large (for ensuring the inner solution at equilibrium price).
- We use several assumptions for consumers that assure the differentiability of the demand function.

Pure Exchange Economy (1)

We call a quadruplet $E = (N, (\Omega_i)_{i \in N}, (U_i)_{i \in N}, (\omega^i)_{i \in N})$ a **pure exchange economy** if,

(1) $N = \{1, ..., n\}$ is a finite set of consumers,

(2) $U_i : \Omega_i \to \mathbb{R}$ denotes the utility function of i-th consumer, where the set $\Omega_i \subset \mathbb{R}^L$ denotes the set of all possible consumption plans for i-th consumer, and

(3) $\omega^i \in \Omega_i$ denotes the initial endowment of i-th consumer.

Pure Exchange Economy (2)

The following optimization problem is called the **utility maximization problem**:

$$\begin{aligned} &\max && U_i(x), \\ &\text{subject to.} && x \in \Omega_i, \\ & && p \cdot x \le m, \end{aligned} \tag{1}$$

where $p \gg 0$ and $m > 0$. The set of the solution of the above problem is denoted by $f^i(p, m)$, and the set-valued function $f^i : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \twoheadrightarrow \Omega_i$ is called the **demand function**.[2]

[2]Later, we assume several assumptions of U_i that assures the single-valuedness of f^i.

Pure Exchange Economy (3)

Define the **excess demand function** X^i of consumer i as

$$X^i(p) = f^i(p, p \cdot \omega^i) - \omega^i,$$

and the **excess demand function** ζ of this economy as

$$\zeta(p) = \sum_i X^i(p).$$

We call p^* an **equilibrium price** if $0 \in \zeta(p^*)$.
Note that, because ζ satisfies homogeneity of degree zero:

$$\zeta(ap) = \zeta(p) \text{ for all } a > 0,$$

we have that if p^* is an equilibrium price, then ap^* is also an equilibrium price for all $a > 0$. Therefore, if there is no normalization, the number of equilibrium prices is always infinity.

Pure Exchange Economy (4)

The following differential inclusion is called the **tâtonnement process**.

$$\dot{p}(t) \in \zeta(p(t)),\ p(0) = p_0.$$

An equilibrium price p^* is called **locally stable** if there exists a neighborhood U of p^* such that if $p_0 \in U$, then there exists a solution $p(t)$ of the above differential inclusion defined on $\mathbb{R}_+$, and for every such a solution, $\lim_{t\to\infty} p(t) = bp^*$ for some $b > 0$.
Note that, if each f^i is a usual single-valued continuously differentiable function, then ζ is also single-valued and continuously differentiable, and thus the above inclusion becomes a usual ordinary differential equation

$$\dot{p}(t) = \zeta(p(t)),\ p(0) = p_0. \tag{2}$$

Pure Exchange Economy (5)

Note. Actually, many researches treat the tâtonnement process as the following differential equation:

$$\dot{p}_j(t) = a_j\zeta_j(p(t)),\ p_j(0) = p_{0j} \text{ for } j = 1, ..., L,$$

where $a_1, ..., a_L > 0$. We can consider this tâtonnement process, and the reason why we omit the existence of a_j in this talk is just for simplification of our arguments.

Note also that, if ζ is single-valued and continuously differentiable, then there uniquely exists a nonextendable solution $p : I \to \mathbb{R}^L_{++}$ of (2), where I is some interval.[3] In this case, an equilibrium price p^* is locally stable if and only if there exists a neighborhood U of p^* such that if $p_0 \in U$, then the domain I of the nonextendable solution $p(t)$ of (2) includes $\mathbb{R}_+$, and $\lim_{t\to\infty} p(t) = bp^*$ for some $b > 0$.

[3]In this study, a set $I \subset \mathbb{R}$ is called **interval** if it is a convex set including at least two different points.

Quasi-Linear Economy (1)

Throughout this talk, we assume that for a vector $x \in \mathbb{R}^L$, $\tilde{x} = (x_1, ..., x_{L-1}) \in \mathbb{R}^{L-1}$.
The following assumption is one of the formal definition of quasi-linear preference.[4]

Assumption F

For every $i \in N$, $\Omega_i = \mathbb{R}^{L-1}_+ \times \mathbb{R}$ and

$$U_i(x) = u_i(\tilde{x}) + x_L. \tag{3}$$

Moreover, u_i is concave, nondecreasing, continuous on $\mathbb{R}^{L-1}_+$, twice continuously differentiable on $\mathbb{R}^{L-1}_{++}$, and $Du_i(\tilde{x}) \gg 0$ and $D^2u_i(\tilde{x})$ is negative semi-definite for all $\tilde{x} \in \mathbb{R}^{L-1}_{++}$.

[4] For example, see ch.3 of Mas-Colell, Whinston, and Green (1995).

Quasi-Linear Economy (2)

In Assumption F, we admit the negative consumption of the numeraire good L. However, many researches treat Ω_i simply as $\mathbb{R}^L_+$, and thus this assumption may seem to be somewhat odd. Therefore, we make an alternative assumption.

Assumption S1

For every $i \in N$, $\Omega_i = \mathbb{R}^L_+$ and U_i satisfies the same assumptions as in Assumption F.

Quasi-Linear Economy (3)

Assumption S1 admits the appearance of the corner solution in an equilibrium, which makes the analysis difficult. Thus, we put an additional assumption.

Assumption S2

Define

$$\alpha_i = \sup\{u_i(\tilde{x}^i) - u_i(\tilde{\omega}^i) | \sum_j (x^j - \omega^j) = 0,$$
$$U_j(x^j) \geq U_j(\omega^j) \text{ for every } j \neq i\}.$$

Then, $\omega_L^i > \alpha_i$ for all $i \in N$.

This assumption ensures that if x^* is equilibrium allocation, then $x_L^{i*} > 0$ for all $i \in N$.

Quasi-Linear Economy (4)

We need two more assumptions.

Assumption Q

For every $\tilde{p} \in \mathbb{R}^{L-1}_{++}$ and $m > 0$, the following problem

$$\begin{aligned} \max \quad & u_i(\tilde{x}) \\ \text{subject to.} \quad & \tilde{x} \in \mathbb{R}^{L-1}_{+}, \\ & \tilde{p} \cdot \tilde{x} \le m \end{aligned} \tag{4}$$

has an inner solution $\tilde{x}^* \gg 0$. Moreover, the following equation $Du_i(\tilde{x}) = \tilde{p}$ also has an inner solution $\tilde{x}^+ \gg 0$. If $\tilde{x} \in \mathbb{R}^{L-1}_{+}$ and $u_i(\tilde{x}) > u_i(0)$, then u_i is strongly increasing on $\tilde{x} + \mathbb{R}^{L-1}_{+}$.

Quasi-Linear Economy (5)

Assumption U

For all $i \in N$, $\omega^i \geq 0$ and $\omega^i \neq 0$, and moreover, $\sum_i \omega^i \gg 0$.

Note. To ensure the existence of the inner solution of the problem (4), microeconomic theorists usually assume the **boundary condition**: that is, they assume that for every $c \in \mathbb{R}$, $u_i^{-1}(c) \cap \mathbb{R}^{L-1}_{++}$ is closed in the topology of $\mathbb{R}^{L-1}$. This assumption, however, excludes the following utility function $u_i(x) = \sqrt{x_1} + \sqrt{x_2}$. In contrast, Assumption Q admits both $u_i(x) = (x_1 x_2)^{1/3}$ and $u_i(x) = \sqrt{x_1} + \sqrt{x_2}$. I think that this assumption is sufficiently weak, and many utility functions satisfy this assumption.

Quasi-Linear Economy (6)

Definition

- A pure exchange economy E is called a **first-type quasi-linear economy** if it satisfies Assumptions F, Q, and U.
- A pure exchange economy E is called a **second-type quasi-linear economy** if it satisfies Assumptions S1, S2, Q, and U.
- A pure exchange economy E is called a **quasi-linear economy** if it is either a first-type quasi-linear economy or a second-type quasi-linear economy.

Main Result

Theorem 1

Suppose that E is a quasi-linear economy. Then, there exists an equilibrium price in this economy. Moreover, if p^* is an equilibrium price, then the following results hold.

(1) every equilibrium price is proportional to p^*, and

(2) p^* is locally stable.

Therefore, we recover the results of partial equilibrium analysis in quasi-linear economies with L commodities.

Preliminary Results (1)

Here, I present a sketch of the proof of Theorem 1. First, I prepare several preliminary results.

Proposition 1

Suppose that E is a quasi-linear economy. Then, for every $i \in N$, f^i is a single-valued continuous function. Moreover, if $x = f^i(p, m)$, then $\tilde{x} \in \mathbb{R}^{L-1}_{++}$. Further, Walras' law

$$p \cdot f^i(p, m) = m \tag{5}$$

and homogeneity of degree zero

$$f^i(ap, am) = f^i(p, m) \text{ for all } a > 0 \tag{6}$$

hold.

Preliminary Results (2)

Proposition 2

Suppose that E is a quasi-linear economy. If either E is first-type or $f_L^i(p,m) > 0$, then f^i is continuously differentiable at (p,m), and

$$\frac{\partial f_j^i}{\partial m}(p,m) = \begin{cases} 0 & \text{if } 1 \le j \le L-1, \\ \frac{1}{p_L} & \text{if } j = L. \end{cases} \tag{7}$$

Preliminary Results (3)

Proposition 3

Suppose that E is a quasi-linear economy, and either E is first-type or $f^i_L(p,m) > 0$. Then, the Slutsky matrix $S_{f^i}(p,m)$ satisfies the following three properties.

(R) The rank of $S_{f^i}(p,m)$ is $L-1$. Moreover, $p^T S_{f^i}(p,m) = 0^T$ and $S_{f^i}(p,m)p = 0$.
(ND) For every $v \in \mathbb{R}^L$ such that $v \neq 0$ and $p \cdot v = 0$, $v^T S_{f^i}(p,m)v < 0$.
(S) The matrix $S_{f^i}(p,m)$ is symmetric.

Preliminary Results (4)

Proposition 4

Suppose that E is a quasi-linear economy, and ζ is the excess demand function of this economy. Then, this function ζ satisfies the following **Walras' law**

$$p \cdot \zeta(p) = 0, \tag{8}$$

and the **homogeneity of degree zero**

$$\zeta(ap) = \zeta(p) \text{ for all } a > 0. \tag{9}$$

Moreover, if $\zeta(p^*) = 0$, then ζ is continuously differentiable around p^*, and

$$D\zeta(p^*) = \sum_{i=1}^{n} S_{f^i}(p^*, p^* \cdot \omega^i). \tag{10}$$

Preliminary Results (5)

Note. Actually, equation (10) is crucial for Theorem 1. This equation arises from equation (7), and (7) arises from the quasi-linear assumption of U_i. Therefore, Theorem 1 holds for only quasi-linear economies.

Preliminary Results (6)

Proposition 5

Suppose that E is a second-type quasi-linear economy and let p^* be an equilibrium price of this economy. Then, $f^i_L(p^*, p^* \cdot \omega^i) > 0$ for every $i \in N$.

Now, suppose that Theorem 1 holds for any first-type quasi-linear economy. Let E be a second-type quasi-linear economy, and define $\hat{\Omega}_i = \mathbb{R}^{L-1}_+ \times \mathbb{R}$ and $\hat{E} = (N, (\hat{\Omega}_i)_{i\in N}, (U_i)_{i\in N}, (\omega^i)_{i\in N})$. Then, $\hat{E}$ is a first-type quasi-linear economy. Using Proposition 5, we can easily show that any equilibrium price in E is also that in $\hat{E}$, and the excess demand function of $\hat{E}$ is the same as that of E on some neighborhood of this equilibrium price, which implies that Theorem 1 also holds for any second-type quasi-linear economy. Therefore, it suffices to show that Theorem 1 holds for a first-type quasi-linear economy.

Modified Economy (1)

Let $E = (N, (\Omega_i)_{i \in N}, (U_i)_{i \in N}, (\omega^i)_{i \in N})$ be a first-type quasi-linear economy. Choose any $s \geq 0$, and define $\Omega_i^s = \{x \in \Omega_i | x_j \geq -s \text{ for all } j\}$. Then, we can define the new economy $E_s = (N, (\Omega_i^s)_{i \in N}, (U_i)_{i \in N}, (\omega^i)_{i \in N})$. Let ζ_s be the excess demand function of the economy E_s.

Modified Economy (2)

Lemma 1

Choose any $s \geq 0$, and suppose that either $s > 0$ or $\omega_L^i > 0$ for all i. Then, ζ_s satisfies the following properties:

1) ζ_s is a continuous function that satisfies (8) and (9).
2) ζ_s is bounded from below.
3) If (p^k) is a sequence in $\mathbb{R}^L_{++}$, $p^k \to p \neq 0$ as $k \to \infty$, and the set $J = \{j | p_j = 0\}$ is nonempty, then $\|\zeta_s(p^k)\| \to \infty$ as $k \to \infty$.

Modified Economy (3)

Lemma 2

Let ζ (resp. ζ_s) be the excess demand function of a first-type quasi-linear economy E (resp. E_s). If $s > 0$ is sufficiently large, then the set of equilibrium prices in E coincides with that in E_s, and for every equilibrium price p^* in E, there exists a neighborhood U of p^* such that $\zeta(p) = \zeta_s(p)$ for every $p \in U$ and ζ is continuously differentiable on U.

Note that, Proposition 17.C.1 of Mas-Colell, Whinston, and Green (1995) states that if a function $f : \mathbb{R}^L_{++} \to \mathbb{R}^L$ satisfies properties 1)-3) in Lemma 1, then there exists p^* such that $f(p^*) = 0$. Therefore, we conclude that if E is a quasi-linear economy, then there exists at least one equilibrium price.

Modified Economy (4)

Define $\bar{S} = \{p \in \mathbb{R}^L_{++} | \|p\| = 1\}$. By using Propositions 1-4, we can show the following proposition.

Lemma 3

If $s > 0$ satisfies the requirements in Lemma 2, then the following properties hold.

4) If $\zeta_s(p^*) = 0$ for some $p^* \in \bar{S}$, then ζ_s is continuously differentiable around p^*.
5) If $\zeta_s(p^*) = 0$, then

$$\begin{vmatrix} D\zeta_s(p^*) & p^* \\ (p^*)^T & 0 \end{vmatrix} \neq 0.$$

Modified Economy (5)

The following lemma is well-known in the theory of regular economies.[5]

Lemma 4

Suppose that $f : \mathbb{R}^L_{++} \to \mathbb{R}^L$ satisfies 1)-5) of Lemmas 1 and 3. For any $p^* \in \bar{S} \cap f^{-1}(0)$, define

$$g(p^*) = \begin{vmatrix} Df(p^*) & p^* \\ (p^*)^T & 0 \end{vmatrix},$$

and

$$\text{index}(p^*) = \begin{cases} +1 & \text{if } (-1)^L g(p) > 0, \\ -1 & \text{if } (-1)^L g(p) < 0. \end{cases}$$

Then, the set $\bar{S} \cap f^{-1}(0)$ is finite, and

$$\sum_{p^* \in \bar{S} \cap f^{-1}(0)} \text{index}(p^*) = +1.$$

[5] See Propositions 5.3.3, 5.3.4, and 5.6.1 of Mas-Colell (1985).

Modified Economy (6)

By using (10), we can show the following result.

Lemma 5

Suppose that $s > 0$ satisfies the requirements in Lemma 2. Then, for every equilibrium price $p^* \in \bar{S}$ in E_s,

$$\text{index}(p^*) = +1.$$

Therefore, by Lemmas 4 and 5, we have that the set $\bar{S} \cap (\zeta_s)^{-1}(0)$ is a singleton $\{p^*\}$. Because the set of equilibrium prices in E_s coincides with that in E (Lemma 2), we have that the set of equilibrium prices in E is $\{ap^* | a > 0\}$, and thus the first statement (uniqueness) of Theorem 1 is verified.

Local Stability (1)

Recall the definition of tâtonnement process:

$$\dot{p}(t) = \zeta(p(t)),\ p(0) = p_0.$$

If $p(t)$ is a solution of the above equation, then

$$\frac{d}{dt}\|p(t)\|^2 = 2p(t) \cdot \zeta(p(t)) = 0,$$

which implies that $\|p(t)\|$ is invariant. Let p^* be an equilibrium price. Choose any small $\varepsilon > 0$ such that $\|v\| = \varepsilon$ implies that $p^* + v \in \mathbb{R}^L_{++}$. Define

$$S = \{v \in \mathbb{R}^L | \|v\| = \varepsilon,\ p^* \cdot v = 0\},$$

and

$$p(t, v) = \frac{\|p^*\|}{\|p^* + tv\|}(p^* + tv).$$

Local Stability (2)

Lemma 6

Define $m_i^* = p^* \cdot \omega^i$ and consider

$$g^i(t,v) = \begin{cases} \frac{1}{t^2}(p(t,v)-p^*)\cdot(f^i(p(t,v),p(t,v)\cdot\omega^i)-f^i(p^*,m_i^*)) & \text{if } t\neq 0,\\ v^T DX^i(p^*)v & \text{if } t=0.\end{cases}$$

Then, g^i is continuous on $[-1,1]\times S$.

This lemma is actually proved in Kihlstrom, Mas-Colell, and Sonnenschein (1976). Because the domain of g^i is compact, g^i is uniformly continuous.

Local Stability (3)

As a corollary of Lemma 6, we can show the following result.

Lemma 7

There exists $\delta > 0$ such that if $0 < |t| \leq \delta$ and $v \in S$, then

$$(p(t,v) - p^*) \cdot (\zeta(p(t,v)) - \zeta(p^*)) < 0. \tag{11}$$

Local Stability (4)

Choose any sufficiently small $\varepsilon' > 0$. Then, for every $p \in \mathbb{R}^L_{++}$ such that $\|p - p^*\| < \varepsilon'$, there exists $t \in [-\delta, \delta]$ and $v \in S$ such that $p = b(p)p(t, v)$ for $b(p) = \frac{\|p\|}{\|p^*\|}$. Then, the following function

$$V(p) = \|p - b(p)p^*\|^2$$

satisfies the definition of Lyapunov function on the set $S^+ = \{q \in \mathbb{R}^L_{++} | \|q\| = \|p\|\}$: that is, for every solution $p(t)$ of (2), by (11),

$$\frac{d}{dt}V(p(t)) = b(p)(p(t, v) - p^*) \cdot (\zeta(p(t, v)) - \zeta(p^*)) < 0.$$

Therefore, we have that if $\|p_0 - p^*\| < \varepsilon'$, then the domain I of the nonextendable solution $p(t)$ of (2) includes $\mathbb{R}_+$, and $\lim_{t \to \infty} p(t) = b(p)p^*$. This completes the proof of Theorem 1.

Several Future Tasks (1)

There are several future tasks. First, I treat a pure exchange economy in this talk. In the context of partial equilibrium theory, this assumption implies that the supply curve becomes a vertical line.

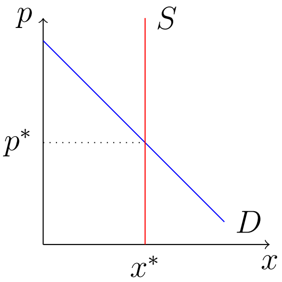

Figure: Pure Exchange Partial Equilibrium Diagram

Several Future Tasks (2)

If one wants to treat an upward supply curve, then he/she must introduce the production. However, an upward supply curve means that the technology set is decreasing return to scale, and thus a positive profit is realized. Then, the definition of the excess demand function becomes much more difficult.

Several Future Tasks (3)

If one chooses the horizontal supply curve, then the technology can be assumed to be constant return to scale, and thus the profit becomes zero. However, in this case the supply function should be set-valued, and moreover, the value of this function may be the empty set. This makes treating the tâtonnement process difficult. In conclusion, both upward and horizontal supply curves are difficult to treat in present. (I think that upward case is easier than horizontal case. But this is just a conjecture.)

Several Future Tasks (4)

Finally, I do not mention the surplus analysis in this talk. In partial equilibrium theory, the total surplus is related to Kaldor's improvement order and thus is very important for welfare analysis. I need to examine whether the same analysis can be done in our model.

Thank you for your attention.