On the Uniqueness and Stability of the Equilibrium Price in Quasi-Linear Economies

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Abstract

In this paper, we show that if every consumer in a pure exchange economy has a quasi-linear utility function, then the normalized equilibrium price is unique, and it is locally stable with respect to the tâtonnement process. If the dimension of the consumption space is two, then this result can be expressed by the corresponding partial equilibrium model. Our study can be seen as that extends the results in partial equilibrium theory to economies with more than two dimensional consumption space.

JEL codes: C62, C61, D41, D51.

Keywords: Walrasian tâtonnement process, quasi-linear utility, excess demand function, uniqueness of equilibrium price, local stability.

1 Introduction

How many competitive equilibrium prices are there? Since Arrow and Debreu (1954) showed that there is at least one equilibrium price, this issue had become of great interest to economists. If the equilibrium price is unique, then it would greatly increase the predictive accuracy of the model. However, even in textbook-level examples, the economy can have multiple equilibria (e.g., see exercise 15.B.6 of Mas-Colell, Whinston, and Green (1995)). In this issue, economists had devided into two positions. The first position is one that tries to guarantee the uniqueness of the equilibrium price by making

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strong assumptions about the economy, and the second position is one that gives up the uniqueness of equilibrium prices. In particular, the position of Arrow and Debreu were clearly divided on this matter.

Gerard Debreu took the latter position. He said that all the known conditions for guaranteeing the uniqueness of equilibrium prices are "exceedingly strong" (Debreu, 1972). On the other hand, he proved the Sonnenschein-Mantel-Debreu theorem (Debreu, 1974). This theorem implies that any compact set in positive orthant could be included in the set of equilibrium prices when only the usual, widely accepted assumptions of economy were made. Thus, without any assumptions that he said to be "exceedingly strong", we know nothing about the number of equilibria. The number of equilibria may be one, a million, or infinity. To solve this problem, Debreu (1970) treated purturbation of the economy by initial endowment vector, and showed that in "almost all" economy, the number of equilibria is at least not infinite. This result had been refined and developed into the theory of regular economy.

On the other hand, Kenneth Arrow took the former position. He treated this problem in combination with another traditional problem, called the theory of tâtonnement process. In 19th century, Walras (1874) explained why equilibrium price is realized in the competitive equilibrium model as follows. First, when the price is higher than the equilibrium price, then supply will be high and demand will be low. This situation is called a state of excess supply. In this situation, a lot of goods remain unsold and inventory increases. Therefore, the price will go down. On the other hand, if the price is lower than the equilibrium price, the opposite will occur: that is, demand will be high and supply will be low. This situation is called a state of excess demand. In this instance, there will be many sellouts and the price will be high. As a result, the equilibrium price would attract the actual price, and the economy tends to trade by the equilibrium price. This idea was refined and expressed in differential equations, called **tâtonnement process**.

Arrow, Brock, and Hurwicz (1959) showed that if the excess demand function is gross substitute, then any equilibrium price is a globally stable steady state with respect to tâtonnement process. Because there can be only one globally stable steady state, they actually showed simultaneously that there is only one equilibrium price. Then, the next problem is to determine the conditions of the economy under which the excess demand function satisfies the gross substitution. However, this problem has not yet been resolved. As a result, Debreu's view that "there is a little that can be said about the number of equilibria in the general environment" is now common among economists.

Meanwhile, there are two theories of equilibrium, the general equilibrium theory and the partial equilibrium theory. The partial equilibrium theory has a foundation in general equilibrium theory, where there are only two types of goods, numeraire good and traded good, and consumer's utility function must be quasi-linear (see Hayashi (2017)). Instead, in partial equilibrium model, it is easy to show by drawing a diagram that there is only one equilibrium price and this price is globally stable (see our section 2). In other words, in a quasi-linear economy with two commodities, the equilibrium price is unique, and it is globally stable with respect to the tâtonnement process.

The problem in this paper is to extend this result to a quasi-linear economy with more than two commodities. That is, the purpose of this study is to see whether the above result holds when considering a general equilibrium model in which the utility remains quasi-linear and the dimension of the consumption space may be more than two. The results are as follows: first, the equilibrium price is unique in a pure exchange economy where all consumers have a quasi-linear utility function. Second, this equilibrium price is locally stable with respect to the tâtonnement process. Global stability could not be derived in this paper. This is related to the inherent difficulty of quasi-linear economies: see our discussion in subsection 3.5.

The structure of this paper is as follows. First, in order to help the readers better understand the purpose of this study, we provide an explanation of partial equilibrium theory and the general equilibrium model behind this theory in section 2. In section 3, we first define pure exchange economy, and then define the type of economy we call the "quasi-linear economy". There are two types of economies we call "quasi-linear economy", one permits negative consumption with respect to the numeraire good,[1] and the other assumes that the initial endowment of numeraire good is sufficiently large. On the basis of the above preparations, we prove the main result. The proofs are placed in section 4.

2 Motivation

Consider the classical partial equilibrium model. The market is described as the following figure. (Figure 1)

The decreasing line is called the **demand curve**, and the increasing line is called the **supply curve**. Two curves crosses at (p^*, x^*). p^* is called the **equilibrium price**, and x^* is called the **equilibrium output**. In this model, trade is done by equilibrium price, and the amount of traded commodity is determined by x^*.

[1] Note that, this is one of the traditional treatment of quasi-linear utility function. For example, see the definition of the quasi-linear preference in Mas-Colell, Whinston, and Green (1995).

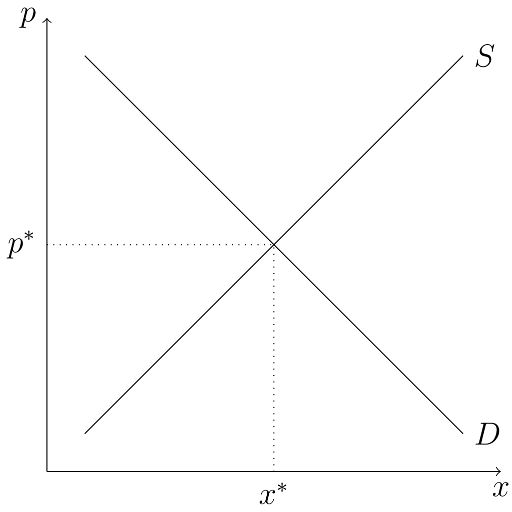

Figure 1: The Market

We explain the hidden structure that determines both curves. The hidden structure is written by general equilibrium model with two commodities. The first commodity is that is traded in this market, and the second commodity is the money. We assume that the price of the money is always one. Let p denote the price of the first commodity.

First, suppose that there are M suppliers of the first commodity, where the cost function of i-th supplier is denoted by $c_i(x_i)$. We assume that c_i is twice differentiable. Then, by first- and second-order necessary condition of profit maximization, we must have,

$$p = c_i'(x_i),\ c_i''(x_i) \ge 0. \tag{1}$$

The supply curve $S(x)$ is determined as follows:

$$S(x) = p \Leftrightarrow \exists x_1, ..., x_M \text{ s.t. } x = \sum_{i=1}^{M} x_i,\ c_i'(x_i) = p.$$

If $S'(x) < 0$, then $c_i''(x_i) < 0$ for some i, which contradicts (1). Therefore, we have that $S'(x) \ge 0$, and thus the supply curve is nondecreasing.

Second, suppose that there are N consumers, and every consumer has a quasi-linear utility function: that is, the utility function $U_i(x_i, y_i)$ is written as $u_i(x_i)+y_i$. We assume that U_i is strictly quasi-concave, which is equivalent to the strict concavity of u_i. Moreover, we assume that u_i is twice continuously differentiable. By Lagrange's first-order condition, if (x_i^*, y_i^*) maximizes the utility, then there exists $\lambda > 0$ such that

$$\begin{aligned} u_i'(x_i) - \lambda p &= 0, \\ 1 - \lambda &= 0, \\ m - px + y &= 0, \end{aligned}$$

which implies that $\lambda = 1$, and thus $p = u_i'(x_i)$. Now, because u_i is strictly concave, we have that u_i' is decreasing. Moreover, the demand curve $D(x)$ is determined by the following equation:

$$D(x) = p \Leftrightarrow x = \sum_{i=1}^{N} (u_i')^{-1}(p),$$

which implies that $D(x)$ is decreasing.

The above arguments says that $S(x)$ is nondecreasing and $D(x)$ is decreasing, and thus the equation $D(x) = S(x)$ has at most one solution x^*. Define $p^* = D(x^*) = S(x^*)$. Then, (p^*, x^*) is the unique equilibrium in this market. Moreover, this is **stable in Walras' sense** by the following reason. Choose any $p > p^*$, and suppose that $D(\hat{x}) = S(\bar{x}) = p$. Then, by the above arguments, we must have that $\hat{x} < \bar{x}$. That is, this economy is in the state of excess supply. Hence, we expect that price goes down. Meanwhile, choose any $p < p^*$, and suppose that $D(\hat{x}) = S(\bar{x}) = p$. Then, we must have that $\hat{x} > \bar{x}$, and thus, this economy is in the state of excess demand. Hence, we expect that price goes up. This arguments implies that the equilibrium price has a power that attracts actual price, and thus this price is called **stable**, and we can consider that p^* is the realized price in long-run economy. In conclusion, in quasi-linear economy with two commodities, the equilibrium is unique and stable in Walras' sense.

Is this result general in equilibrium theory? The answer is NO, even in pure exchange economies. Debreu (1974) showed that for **every compact subset** in $\{p \in \mathbb{R}^n_{++} | \|p\| = 1\}$, there exists a pure exchange economy in which the set of equilibrium prices includes this set. Therefore, the uniqueness of the equilibrium is broken. Moreover, even if there uniquely exists an equilibrium price, this price may be **unstable**. Therefore, the above result is broken in a general situation.

Meanwhile, Arrow, Brock, and Hurwicz (1959) showed that if the excess demand function is **gross substitute**, then there uniquely exists an equilibrium price vector, and it is stable on the **tâtonnement process**. Therefore, if the excess demand function is gross substitute, then the results in partial equilibrium analysis revive. However, this result is criticized by Debreu

(1972). Debreu said that the requirement of gross substitution is "exceedingly strong". Actually, there is no known natural assumption on an economy under which the excess demand function becomes gross substitute.

Our next question is as follows: is the above result is general under **quasi-linear economy with more than two commodities**? Our purpose is to answer this question affirmatively. Consider a pure exchange economy, and suppose that the utility function is quasi-linear for every consumer. The purpose of this work is to show that there uniquely exists a normalized equilibrium price vector, and it is **locally stable** on Walrasian tâtonnement process. In other words, we will show that almost all results in partial equilibrium theory also holds for every quasi-linear pure exchange economy.

3 Model and Results

3.1 Preliminary: General Setups of Pure Exchange Economies

In this paper, for $x, y \in \mathbb{R}^M$, $x \geq y$ means $x_i \geq y_i$ for every $i \in \{1, ..., M\}$, and $x \gg y$ means $x_i > y_i$ for every $i \in \{1, ..., M\}$, respectively. Define the sets $\mathbb{R}^M_+ = \{x \in \mathbb{R}^M | x \geq 0\}$ and $\mathbb{R}^M_{++} = \{x \in \mathbb{R}^M | x \gg 0\}$ as usual. If $M = 1$, then we abbreviate this symbol, and simply write these set as $\mathbb{R}_+$ and $\mathbb{R}_{++}$. The notation e_j denote the j-th unit vector.

Let $L \geq 2$. We call a quadruplet $E = (N, (\Omega_i)_{i\in N}, (U_i)_{i\in N}, (\omega^i)_{i\in N})$ a **pure exchange economy** if,

(1) $N = \{1, ..., n\}$ is a finite set of consumers,

(2) $U_i : \Omega_i \to \mathbb{R}$ denotes the utility function of i-th consumer, where the set $\Omega_i \subset \mathbb{R}^L$ denotes the set of all possible consumption plans for i-th consumer, and

(3) $\omega^i \in \Omega_i$ denotes the initial endowment of i-th consumer.

Consider the following utility maximization problem:

$$\begin{aligned} \max \quad & U_i(x), \\ \text{subject to.} \quad & x \in \Omega_i, \\ & p \cdot x \leq m, \end{aligned} \tag{2}$$

where $p \gg 0$ and $m > 0$. Let $f^i(p, m)$ denote the set of all solutions of (2). This set-valued function $f^i(p, m)$ is called the **demand function** of consumer i. For given demand function f^i, define

$$X^i(p) = f^i(p, p \cdot \omega^i) - \omega^i.$$

This function is called the **excess demand function of consumer** i, and the following function

$$\zeta(p) = \sum_{i \in N} X^i(p)$$

is called the **excess demand function** in this economy. We call $p^* \in \mathbb{R}^L_{++}$ an **equilibrium price** if $0 \in \zeta(p^*)$.

If the demand function f^i is single-valued and differentiable at (p, m), then we can define

$$s^i_{jk}(p, m) = \frac{\partial f^i_j}{\partial p_k}(p, m) + \frac{\partial f^i_j}{\partial m}(p, m) f^i_k(p, m).$$

The $L \times L$ matrix-valued function $S_{f^i}(p, m) = (s^i_{jk}(p, m))^L_{j,k=1}$ is called the **Slutsky matrix**.

Consider the following differential inclusion:

$$\dot{p}_j(t) \in a_j \zeta_j(p(t)),\ p_j(0) = p_{0j},\ j \in \{1, ..., L\}, \tag{3}$$

where $a_1, ..., a_L > 0$ and $p_0 \in \mathbb{R}^L_{++}$. This inclusion is called the **tâtonnement process** in this economy E. We call a set $I \subset \mathbb{R}$ an **interval** if it is convex and contains at least two different points. A function $p : I \to \mathbb{R}^L_{++}$ is called a **solution** of (3) if I is an interval including 0, $p(t)$ is absolutely continuous on any compact set $C \subset I$, $p(0) = p_0$, and

$$\dot{p}_j(t) \in a_j \zeta_j(p(t)),$$

for all $j \in \{1, ..., L\}$ and almost all $t \in I$. It is well-known that if ζ is single-valued and continuous, then every solution of (3) is continuously differentiable.

An equilibrium price p^* is called **locally stable** if there exists an open neighborhood U of p^* such that 1) for every $p_0 \in U$, there exists a solution $p : I \to \mathbb{R}^L_{++}$ of (3) such that $\mathbb{R}_+ \subset I$, and 2) for every solution $p : I \to \mathbb{R}^L_{++}$ of (3) such that $\mathbb{R}_+ \subset I$, $\lim_{t \to \infty} p(t) = ap^*$ for some $a > 0$. If we can choose $U = \mathbb{R}^L_{++}$, then p^* is called **globally stable**.

Finally, throughout this paper, the symbol $Dh(x)$ refer to either the Fréchet derivative of h at x or its transpose, whichever is more convenient. We believe that there is no danger of causing confusion by this omission of the transposition sign.

3.2 Economies with Quasi-Linear Environments

We make assumptions for economies with quasi-linear environment. However, there are two styles on quasi-linear environment. The first style is in Mas-Colell, Whinston, and Green (1995), in which the consumption space Ω_i is assumed to be $\mathbb{R}^{L-1}_+ \times \mathbb{R}$. That is, in this style, consuming the negative amount of the numeraire good is admitted.[2] In the second style, we assume that as usual $\Omega_i = \mathbb{R}^n_+$, and the initial endowment of the numeraire good is sufficiently large for every consumer. We treat both styles, and thus we separates assumptions for these environments.

Throughout this paper, for every $x \in \mathbb{R}^L$, $\tilde{x}$ denotes $(x_1, ..., x_{L-1}) \in \mathbb{R}^{L-1}$. Here, we make assumptions on economies with quasi-linear environment.

Assumption F. For every $i \in N$, $\Omega_i = \mathbb{R}^{L-1}_+ \times \mathbb{R}$, and the function U_i can be written as follows:

$$U_i(x) = u_i(\tilde{x}) + x_L, \tag{4}$$

where u_i is continuous, nondecreasing, and concave on $\mathbb{R}^{L-1}_+$. Moreover, u_i is twice continuously differentiable on $\mathbb{R}^{L-1}_{++}$, $Du_i(\tilde{x}) \gg 0$, and the Hessian matrix $D^2u_i(\tilde{x})$ is negative definite for every $\tilde{x} \in \mathbb{R}^{L-1}_{++}$.[3]

Assumption S1. For every $i \in N$, $\Omega_i = \mathbb{R}^L_+$ and the function U_i can be written as (4), where u_i is continuous, nondecreasing, and concave on $\mathbb{R}^{L-1}_+$. Moreover, u_i is twice continuously differentiable on $\mathbb{R}^{L-1}_{++}$, $Du_i(\tilde{x}) \gg 0$ and the Hessian matrix $D^2u_i(\tilde{x})$ is negative definite for every $\tilde{x} \in \mathbb{R}^{L-1}_{++}$.

Assumption S2. Define

$$\alpha_i = \sup\left\{ u_i(\tilde{x}^i) - u_i(\tilde{\omega}^i) \left| \sum_{j \in N} (x^j - \omega^j) = 0,\ U_j(x^j) \ge U_j(\omega^j) \text{ for every } j \neq i \right. \right\}.$$

Then, $\omega^i_L > \alpha_i$ for every $i \in N$.[4]

Assumption Q. For every $i \in N$, $\tilde{p} \in \mathbb{R}^{L-1}_{++}$, and $m > 0$, the following

[2] If we consider that x_L denotes the amount of money, the negative x_L indicates a debt.

[3] This assumption is equivalent to the non-zero Gaussian curvature requirement of Debreu (1972) for every indifference hypersurface of U_i passing through the interior of Ω_i. Debreu (1972) showed that this requirement is equivalent to the differentiability of the demand function at any price and money such that every coordinate of the demand is positive.

[4] For example, this assumption is satisfied if Assumption U holds and $\omega^i_L > u_i(\tilde{\omega}) - u_i(\tilde{\omega}^i)$ for every $i \in N$, where $\omega = \sum_{j \in N} \omega^j$.

problem

$$\begin{array}{rl} \max & u_i(\tilde{x}) \\ \text{subject to.} & \tilde{x} \in \mathbb{R}^{L-1}_+, \\ & \tilde{p} \cdot \tilde{x} \le m \end{array} \tag{5}$$

has an inner solution $\tilde{x}^* \gg 0$. Moreover, the equation

$$Du_i(\tilde{x}) = \tilde{p} \tag{6}$$

also has an inner solution $\tilde{x}^+ \gg 0$. In addition, if $u_i(\tilde{x}) > u_i(0)$, then u_i is strictly increasing on $\tilde{x} + \mathbb{R}^{L-1}_+$.[5]

Assumption U. $\omega^i \in \mathbb{R}^L_+ \setminus \{0\}$ and $\sum_{i \in N} \omega^i \gg 0$.

We call a pure exchange economy E that satisfies Assumptions F, Q, and U a **first-type quasi-linear economy**, and that satisfies Assumptions S1, S2, Q, and U a **second-type quasi-linear economy**, respectively. We call E a **quasi-linear economy** if it is either a first-type quasi-linear economy or a second-type quasi-linear economy.

3.3 Propositions

We first present several basic results on quasi-linear economies.

Proposition 1. Suppose that E is a quasi-linear economy. Then, for every $i \in N$, f^i is a single-valued continuous function. Moreover, if $x = f^i(p, m)$, then $\tilde{x} \in \mathbb{R}^{L-1}_{++}$. Further, Walras' law

$$p \cdot f^i(p, m) = m \tag{7}$$

and homogeneity of degree zero

$$f^i(ap, am) = f^i(p, m) \text{ for all } a > 0 \tag{8}$$

hold.

Proposition 2. Suppose that E is a quasi-linear economy. If either E is first-type or $f^i_L(p, m) > 0$, then f^i is continuously differentiable at (p, m), and

$$\frac{\partial f^i_j}{\partial m}(p, m) = \begin{cases} 0 & \text{if } 1 \le j \le L-1, \\ \frac{1}{p_L} & \text{if } j = L. \end{cases} \tag{9}$$

[5]This requirement admits both $u_i(x) = (x_1 x_2)^{1/3}$ and $u_i(x) = \sqrt{x_1} + \sqrt{x_2}$.

Proposition 3. Suppose that E is a quasi-linear economy, and fix $(p, m) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$. Suppose also that either E is first-type or $f^i_L(p, m) > 0$. Then, the Slutsky matrix $S_{f^i}(p, m)$ satisfies the following three properties.

(R) The rank of $S_{f^i}(p, m)$ is $L - 1$. Moreover, $p^T S_{f^i}(p, m) = 0^T$ and $S_{f^i}(p, m)p = 0$.

(ND) For every $v \in \mathbb{R}^L$ such that $v \neq 0$ and $p \cdot v = 0$, $v^T S_{f^i}(p, m)v < 0$.

(S) The matrix $S_{f^i}(p, m)$ is symmetric.

Recall that the definition of the excess demand function ζ:

$$\zeta(p) = \sum_{i \in N} X^i(p) = \sum_{i \in N}[f^i(p, p \cdot \omega^i) - \omega^i].$$

Because of Proposition 1, we have that ζ is a single-valued continuous function defined on $\mathbb{R}^L_{++}$.

Proposition 4. Suppose that E is a quasi-linear economy, and ζ is the excess demand function of this economy. Then, this function ζ satisfies the following **Walras' law**

$$p \cdot \zeta(p) = 0, \tag{10}$$

and the **homogeneity of degree zero**

$$\zeta(ap) = \zeta(p) \text{ for all } a > 0. \tag{11}$$

Moreover, there is at least one equilibrium price p^* in this economy.

Proposition 5. Suppose that E is a second-type quasi-linear economy and let p^* be an equilibrium price of this economy. Then, $f^i_L(p^*, p^* \cdot \omega^i) > 0$ for every $i \in N$.

3.4 Main Results

Suppose that E is a quasi-linear economy, and recall the tâtonnement process

$$\dot{p}_j(t) = a_j \zeta(p(t)),\ p_j(0) = p_{0j}, \tag{12}$$

where $a_1, ..., a_L > 0$ and $p_0 \in \mathbb{R}^L_{++}$. Note that, because the excess demand function ζ is single-valued in quasi-linear economies, the tâtonnement process is not a differential inclusion but an ordinary differential equation in such economies, and thus we can use usual techniques on ordinary differential equations. In particular, any solution $p : I \to \mathbb{R}^L_{++}$ of (12) is continuously differentiable.

We now complete the preparation of our main result.

Theorem 1. Suppose that E is a quasi-linear economy, and ζ is the excess demand function of this economy. Choose any equilibrium price p^* in this economy. Then, every equilibrium price is proportional to p^*. Moreover, this equilibrium price is locally stable.

3.5 Remarks on Global Stability of Equilibria

Actually, we want to show the global stability of an equilibrium price p^*. That is, we want to show that **for every** $p_0 \in \mathbb{R}^L_{++}$, there exists a solution $p(t)$ of (12) defined on $\mathbb{R}_+$, and for such a solution, $\lim_{t\to\infty} p(t)$ exists and is proportional to p^*. However, there are two hard tasks, and both could not easily be solved.

First, suppose that $p(t)$ is a solution of (12) for some $p_0 \in \mathbb{R}^L_{++}$ defined on $\mathbb{R}_+$. We want to show that $\lim_{t\to\infty} p(t)$ exists. However, this problem is difficult if p_0 is too far from the half-line $\{ap^*|a > 0\}$, and we could not prove this result. For example, we could not exclude the possibility that the trajectory of $p(t)$ consists of a cycle. If $L = 2$, then we can show that such a case vanishes, because, by the intermediate value theorem, the sign of $\zeta_1(p_1, p_2)$ coincides with that of $\frac{p_2}{p_1} - \frac{p_2^*}{p_1^*}$. However, this is a too simple case, and if $L > 2$, such a rough argument cannot be done.

The second problem is more serious. That is, we could not show the existence of a solution $p(t)$ of (12) defined on $\mathbb{R}_+$. The essential problem is the following: because $\mathbb{R}^L_{++}$ is open, $C \subset \mathbb{R}^L_{++}$ may be not compact even if it is bounded and closed in relative topology of $\mathbb{R}^L_{++}$. The existence of the a solution of (12) whose trajectory is included in $\mathbb{R}^L_{++}$ is called the 'viability problem'. To solve this viability problem, we can use the **strong boundary condition** of the excess demand function. Consider a pure exchange economy E such that the excess demand function ζ is a single-valued function. We say that ζ satisfies the strong boundary condition if and only if for every sequence (p^k) on $\mathbb{R}^L_{++}$ such that $p^k \to p \in \mathbb{R}^L_+ \setminus (\mathbb{R}^L_{++} \cup \{0\})$ as $k \to \infty$, if $J = \{j|p_j = 0\}$, then

$$\sum_{j\in J} \zeta_j(p^k) \to +\infty.$$

If the set Ω_i is bounded from below, then ζ usually satisfies the strong boundary condition. Theorem 7 of Hosoya and Yu (2013) states that if ζ is single-valued, continuous, and homogeneous of degree zero, and it satisfies Walras' law and the strong boundary condition, then there exists a solution of (12) defined on $\mathbb{R}_+$. Therefore, if the economy E is second-type quasi-linear, then we can solve the viability problem positively.

However, in first-type quasi-linear economies, there is no known method for solving the viability problem. Therefore, it is hard to prove the existence of a solution $p(t)$ of (12) defined on $\mathbb{R}_+$, and thus the global stability is also difficult to verify.

4 Proofs

4.1 Lemmas

In this subsection, we show several lemmas.

First, suppose that E is a first-type quasi-linear economy and choose any $s \ge 0$. Let

$$\Omega_i^s = \mathbb{R}_+^{L-1} \times [-s, +\infty[.$$

Consider the modified economy $E_s = (N, (\Omega_i^s)_{i\in N}, (U_i)_{i\in N}, (\omega^i)_{i\in N})$. Note that a second-type quasi-linear economy can be treated as E_0 for some first-type quasi-linear economy E. Let f_s^i be the demand function of consumer i in the economy E_s, and ζ_s be the excess demand function of the economy E_s.

Lemma 1. Suppose that E is a quasi-linear economy. Then, there exists a continuously differentiable function $\tilde{x}^i : \mathbb{R}_{++}^{L-1} \to \mathbb{R}_{++}^{L-1}$ such that

$$\tilde{y} = \tilde{x}^i(\tilde{p}) \Leftrightarrow Du_i(\tilde{y}) = \tilde{p}.$$

Proof. Let $\tilde{p} \in \mathbb{R}_{++}^{L-1}$. By Assumption Q, there exists a solution $\tilde{x}^* \in \mathbb{R}_{++}^{L-1}$ of the following equation:

$$Du_i(\tilde{x}) = \tilde{p}. \tag{13}$$

Consider the following optimization problem:

$$\begin{aligned} \max \quad & u_i(\tilde{x}) - \tilde{p}\cdot\tilde{x} \\ \text{subject to.} \quad & \tilde{x} \in \mathbb{R}_{++}^{L-1}. \end{aligned}$$

Because of either Assumption F or Assumption S1, we have that u_i is strictly concave on $\mathbb{R}_{++}^{L-1}$, and thus 1) any solution of (13) is also a solution of this problem, and 2) the solution of the above problem is unique. Therefore, $\tilde{x}^*$ is the unique solution of the equation (13), and thus we can define $\tilde{x}^i(\tilde{p}) = \tilde{x}^*$. Because $D^2u_i(\tilde{x}^*)$ is negative definite, it is regular, and thus by the inverse function theorem, we have that $\tilde{x}^i(\tilde{p})$ is continuously differentiable. This completes the proof. ■

For any $\tilde{p} \in \mathbb{R}^{L-1}_{++}$ and $m \in \mathbb{R}_{++}$, let

$$x^i_L(\tilde{p}, m) = m - \tilde{p} \cdot \tilde{x}^i(\tilde{p}).$$

Lemma 2. Suppose that E is a first-type quasi-linear economy and $s \geq 0$, and choose any $(p, m) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$. Define $(q, w) = p_L^{-1}(p, m)$. Then,

$$f^i(p, m) = (\tilde{x}^i(\tilde{q}), x^i_L(\tilde{q}, w)),$$

and if $x^i_L(\tilde{q}, w) \geq -s$, then

$$f^i_s(p, m) = (\tilde{x}^i(\tilde{q}), x^i_L(\tilde{q}, w)).$$

Proof. By Assumption F, we have that $u_i(\tilde{x})$ is strictly concave on $\mathbb{R}^{L-1}_{++}$. Because u_i is continuous, it is concave on $\mathbb{R}^{L-1}_{+}$, and thus U_i is concave on Ω_i.

Define

$$x^* = (\tilde{x}^i(\tilde{q}), x^i_L(\tilde{q}, w)).$$

First, we show the latter claim of this lemma. Suppose that $x^*_L \geq -s$. Then, we have that $x^* \in \Omega^s_i$. Moreover,

$$p \cdot x^* = p_L(\tilde{q} \cdot \tilde{x}^i(\tilde{q}) + x^i_L(\tilde{q}, w)) = p_L w = m.$$

By Lagrange's multiplier rule, we have that $x^* \in f^i_s(p, m)$. Suppose that $y^* \in f^i_s(p, m)$ and $x^* \neq y^*$. Then, $U_i(x^*) = U_i(y^*)$. If $\tilde{x}^* = \tilde{y}^*$, then $x^*_L = y^*_L$ by equation $U_i(x^*) = U_i(y^*)$, which contradicts that $x^* \neq y^*$. Thus, we have that $\tilde{x}^* \neq \tilde{y}^*$. Define $z(t) = (1 - t)x^* + ty^*$. Then, for every $t \in [0, 1]$, we have that $p \cdot z(t) \leq m$, and thus $U_i(z(t)) \leq U_i(x^*)$. Because U_i is concave, we have that $U_i(z(t)) = U_i(x^*)$ for all $t \in [0, 1]$. Set $t_1 = \frac{1}{2}$ and $t_2 = \frac{1}{4}$, and let $z^* = z(t_1), z^+ = z(t_2)$. Then, $\tilde{z}^* \in \mathbb{R}^{L-1}_{++}$, $p \cdot z^* \leq m$, and $U_i(z^*) = U_i(x^*)$. However, because u_i is strictly concave on $\mathbb{R}^{L-1}_{++}$, we have that

$$u_i(\tilde{z}^+) > \frac{1}{2} u_i(\tilde{x}^*) + \frac{1}{2} u_i(\tilde{z}^*),$$

which implies that $U_i(z^+) > U_i(x^*)$. This contradicts that $U_i(z^+) = U_i(x^*)$. Therefore, $f^i_s(p, m) = \{x^*\}$, as desired.

Next, we show the former claim of this lemma. Clearly $x^* \in \Omega_i$. Again by Lagrange's multiplier rule, we have that $x^* \in f^i(p, m)$. If $y^* \in f^i(p, m)$ and

$x^* \neq y^*$, then choose $s > 0$ so large that $x^*, y^* \in \Omega_i^s$. Then, $y^* \in f_s^i(p,m)$, which is a contradiction. This completes the proof. ■

Lemma 3. Suppose that E is a first-type quasi-linear economy, and $s \geq 0$. Choose any $(p,m) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, and define

$$x^* = (\tilde{x}^i(p_L^{-1}\tilde{p}), x_L^i(p_L^{-1}\tilde{p}, p_L^{-1}m)).$$

Suppose that $x_L^* < -s$. Let $\bar{m} = m + p_L s$, and $\tilde{x}^+ \in \mathbb{R}^{L-1}_{++}$ be a solution of the problem

$$\begin{aligned} \max \quad & u_i(\tilde{x}) \\ \text{subject to.} \quad & \tilde{x} \in \mathbb{R}^{L-1}_+, \\ & \tilde{p} \cdot \tilde{x} \leq \bar{m}, \end{aligned}$$

and define $x_L^+ = -s$. Then, $f_s^i(p,m) = (\tilde{x}^+, x_L^+)$.[6]

Proof. First, because of Assumption F, we have that u_i is continuous and strictly concave on $\mathbb{R}^{L-1}_{++}$. Therefore, we have that if $\tilde{y} \in \mathbb{R}^{L-1}_+$ and $\tilde{p} \cdot \tilde{y} \leq \bar{m}$, then either $\tilde{y} = \tilde{x}^+$ or $u_i(\tilde{y}) < u_i(\tilde{x}^+)$. Moreover, because u_i is increasing on $\mathbb{R}^{L-1}_{++}$, we have that $\tilde{p} \cdot \tilde{x}^+ = \bar{m}$. Because $f^i(p,m) = x^*$, we have that $U_i(x^*) > U_i(x^+)$.

Suppose that there exists $y^+ \in \Omega_i^s$ such that $x^+ \neq y^+$, $p \cdot y^+ \leq m$ and $U_i(y^+) \geq U_i(x^+)$. Then, $\tilde{p} \cdot \tilde{y}^+ \leq \bar{m}$. If $y_L^+ = -s$, then we have that $\tilde{y}^+ \neq \tilde{x}^+$, and thus $u_i(\tilde{y}^+) < u_i(\tilde{x}^+)$, which is a contradiction. Therefore, we have that $y_L^+ > -s$, and there exists $t \in]0,1[$ such that for $z^+ = (1-t)x^* + ty^+$, $z_L^+ = -s$. Because u_i is concave on $\mathbb{R}^{L-1}_+$, we have that $u_i(\tilde{z}^+) \geq (1-t)u_i(\tilde{x}^*) + tu_i(\tilde{y}^+)$, which implies that $U_i(z^+) > U_i(x^+)$, and thus $u_i(\tilde{z}^+) > u_i(\tilde{x}^+)$. However, $\tilde{p} \cdot \tilde{z}^+ \leq m + p_L s = \bar{m}$, which contradicts the definition of $\tilde{x}^+$. Thus, we have that $f_s^i(p,m) = \{x^+\}$, as desired. This completes the proof. ■

Lemma 4. Suppose that a function $\xi : \mathbb{R}^L_{++} \to \mathbb{R}^L$ satisfies the following properties.

1) ξ is continuous and satisfies (10) and (11).

2) There exists $s > 0$ such that $\xi_j(p) > -s$ for every $p \in \mathbb{R}^L_{++}$ and $j \in \{1, ..., L\}$.

3) If (p^k) is a sequence of $\mathbb{R}^L_{++}$ such that $p^k \to p \neq 0$ as $k \to 0$ and the set $J = \{j | p_j = 0\}$ is nonempty, then $\|\xi(p^k)\| \to \infty$ as $k \to \infty$.

[6] Note that, the existence of such $\tilde{x}^+$ is assumed in Assumption Q.

Then, there exists p^* such that $\xi(p^*) = 0$.

Proof. Omitted. See Proposition 17.C.1 of Mas-Colell, Whinston, and Green (1995). ■

Lemma 5. Suppose that E is a first-type quasi-linear economy and either $s > 0$, or $s = 0$ and $\omega_L^i > 0$ for all $i \in N$. Then, ζ_s is a single-valued function that satisfies 1)-3) of Lemma 4.

Proof. By Lemmas 2-3, we have that f_s^i is a single-valued function. Because of Berge's maximum theorem, we have that f_s^i is continuous, and thus ζ_s is also single-valued and continuous.

It is easy to prove that ζ_s satisfies (10) and (11), and thus we omit the proof of this fact. Thus, ζ_s satisfies 1).

Because $\zeta_s(p) \gg -\sum_{i\in N}\omega^i - (s+1, s+1, ..., s+1)$ for all $p \in \mathbb{R}^L_{++}$, we have that ζ_s satisfies 2).

Therefore, it suffices to show that 3) holds for ζ_s. Suppose that (p^k) is a sequence of $\mathbb{R}^L_{++}$ such that $p^k \to p \neq 0$ as $k \to 0$ and the set $J = \{j|p_j = 0\}$ is nonempty, but $\|\zeta_s(p^k)\| \not\to \infty$. By taking a subsequence, we can assume that $\zeta_s(p^k) \to x$ for some $x \in \mathbb{R}^L$. Let $x^k = \zeta_s(p^k)$ and $x^{ik} = f_s^i(p^k, p^k \cdot \omega^i)$. Because x^{ik} is also bounded, we can assume that $x^{ik} \to x^i$ as $k \to \infty$.

Suppose that $p_L = 0$. Because $p \cdot x^i = p \cdot \omega^i$ and $\sum_{i\in N}\omega^i \gg 0$, we have that there exists i such that $x_j^i > 0$ for some j with $p_j > 0$. Define $y^i = x^i - \varepsilon e_j + e_L$, where $\varepsilon > 0$ is sufficiently small that $U_i(y^i) > U_i(x^i)$. Then, $U_i(y^i) > U_i(x^{ik})$ and $p^k \cdot y^i < p^k \cdot x^{ik}$ for some k, which is a contradiction.

Therefore, we have that $p_L > 0$. Next, suppose that for some i, $u_i(\tilde{x}^i) = u_i(0)$. If $s > 0$, then because $U_i(x^i) \ge U_i(\omega^i)$, we have that $x_L^i \ge 0 > -s$. If $s = 0$ and $\omega_L^i > 0$, then by the same inequality, we have that $x_L^i > 0 = -s$. Therefore, in both cases, $x_L^i > -s$. Fix any $M > 2\frac{\|\tilde{p}\|}{p_L}$. By Assumption Q, there exists $\tilde{y} \in \mathbb{R}^{L-1}_{++}$ such that $\frac{\partial u}{\partial x_j}(\tilde{y}) > M$ for all j. Define $v_j = \frac{y_j}{\sum_{\ell=1}^{L-1} y_\ell}$ for $j = \{1, ..., L-1\}$ and $v_L = -\frac{2}{p_L}(\tilde{p} \cdot \tilde{v})$. Because u_i is strictly concave and increasing on $\mathbb{R}^{L-1}_{++}$, we have that $g(t) = u_i(t\tilde{v})$ is increasing and $\lim_{t\downarrow 0} g'(t) > M$. Thus, there exists $t > 0$ such that for $y^i = (0, x_L^i) + tv$, $U_i(y^i) > U_i(x^i)$, $y_L^i > -s$, and $p \cdot y^i < p \cdot x^i$. This implies that $U_i(y^i) > U_i(x^{ik})$ and $p^k \cdot y^i < p^k \cdot x^{ik}$ for some k, which is a contradiction. Therefore, for every i, $u_i(\tilde{x}^i) > u_i(0)$. Choose any j such that $p_j = 0$. Because ζ_s satisfies (10), there exists i and ℓ such that $p_\ell > 0$ and $x_\ell^i > 0$. Then, for $y^i = x^i + e_j - \varepsilon e_\ell$, we have that $y_\ell^i > 0$ and $U_i(y^i) > U_i(x^i)$ if $\varepsilon > 0$ is sufficiently small. Because $p \cdot y^i < p \cdot x^i$, we have that there exists k such that $U_i(y^i) > U_i(x^{ik})$ and $p^k \cdot y^i < p^k \cdot x^{ik}$, which is a contradiction. This completes the proof. ■

Lemma 6. Suppose that E is a first-type quasi-linear economy. Then, there exists $s > 0$ such that the following properties holds.

(i) The set of equilibrium prices in E coincides with that in E_s.

(ii) For every equilibrium price p^*, $\min_i f^i_L(p^*, p^* \cdot \omega^i) > -s$.

Proof. Let

$$\omega = \sum_{i \in N} \omega^i,$$

and choose $s > 0$ such that

$$s > \max_{i \in N}[u_i(\tilde{\omega}) - u_i(0)].$$

Suppose that p^* is an equilibrium price of E. Let $x^i = f^i(p^*, p^* \cdot \omega^i)$. Because

$$\sum_{i \in N} x^i = \omega,$$

we have that $\tilde{x}^i \le \tilde{\omega}$, and thus, if $x^i_L \le -s$, then

$$U_i(x^i) = u_i(\tilde{x}^i) + x^i_L \le u_i(\tilde{\omega}) - s < u_i(0) \le U_i(\omega^i),$$

which is a contradiction. Therefore, we have that $x^i_L > -s$. By Lemma 2, we have that $f^i_s(p^*, p^* \cdot \omega^i) = x^i$, and thus p^* is an equilibrium price of E_s.

Conversely, suppose that p^* is an equilibrium price of E_s. Let $x^i = f^i_s(p^*, p^* \cdot \omega^i)$. Then, by the same arguments as above, we can show that $x^i_L > -s$. By Lemma 2, we have that $f^i(p^*, p^* \cdot \omega^i) = x^i$, and thus p^* is an equilibrium price of E. This completes the proof. ■

Lemma 7. Suppose that $\xi : \mathbb{R}^L_{++} \to \mathbb{R}^L$ is a continuous function, and define

$$S^* = \{p \in \mathbb{R}^L_{++} | \|p\| = 1\}.$$

Suppose also that ξ satisfies the following five properties:

1) The function ξ satisfies (10) and (11).

2) There exists $s > 0$ such that $\xi_j(p) > -s$ for every $p \in \mathbb{R}^L_{++}$ and $j \in \{1, ..., L\}$.

3) If (p^k) is a sequence of $\mathbb{R}^L_{++}$ such that $p^k \to p \neq 0$ as $k \to 0$ and the set $J = \{j | p_j = 0\}$ is nonempty, then $\|\xi(p^k)\| \to \infty$ as $k \to \infty$.

4) If $\xi(p) = 0$, then ξ is continuously differentiable around p.

5) If $\xi(p) = 0$, then

$$\chi(p) = \begin{vmatrix} D\xi(p) & p \\ p^T & 0 \end{vmatrix} \neq 0.$$

Define $E = \xi^{-1}(0) \cap S^*$, and for $p \in E$,

$$\mathrm{index}(p) = \begin{cases} +1 & \text{if } (-1)^L \chi(p) > 0, \\ -1 & \text{if } (-1)^L \chi(p) < 0. \end{cases}$$

Then, the set E is finite, and

$$\sum_{p \in E} \mathrm{index}(p) = +1.$$

Proof. Omitted. See Propositions 5.3.3, 5.3.4, and 5.6.1 of Mas-Colell (1985).

4.2 Proof of Proposition 1

If E is first-type, then by Lemma 2, we have that f^i is single-valued and continuously differentiable, and $\tilde{f}^i(p, m) \in \mathbb{R}^{L-1}_{++}$. If E is second-type, then by Lemmas 2 and 3, f^i is single-valued and $\tilde{f}^i(p, m) \in \mathbb{R}^{L-1}_{++}$, and by Berge's maximum theorem, f^i is continuous.

It is clear that f^i is homogeneous of degree zero. Because U_i is locally non-satiated, we have that f^i satisfies Walras' law. This completes the proof. ■

4.3 Proof of Proposition 2

Let $\tilde{\Omega}_i$ denote the interior of Ω_i. Suppose that $x = f^i(p, m)$ and either the economy is first-type or $x_L > 0$. Then, $x \in \tilde{\Omega}_i$, and because of Lemma 2, we have that there exists an open neighborhood V of (p, m) such that if $(q, w) \in V$, then

$$f^i(q, w) = (\tilde{x}^i(q_L^{-1}\tilde{q}), x_L^i(q_L^{-1}\tilde{q}, q_L^{-1}w)).$$

Therefore, f^i is continuously differentiable on V and (9) holds. This completes the proof. ■

4.4 Proof of Proposition 3

First, choose an open neighborhood U of (p, m) such that f^i is single-valued and continuously differentiable on U. Because of Proposition 1, we have that

$$q \cdot f^i(q, w) = w$$

for every $(q, w) \in U$ and

$$f^i(ap, am) = f^i(p, m)$$

for every $a > 0$. Hence, by differentiation,

$$f^i_j(p, m) + \sum_{k=1}^{L} p_k \frac{\partial f^i_k}{\partial p_j}(p, m) = 0,$$

$$p \cdot \frac{\partial f^i}{\partial m}(p, m) = 1,$$

$$\sum_{j=1}^{L} p_j \frac{\partial f^i_k}{\partial p_j}(p, m) + m \frac{\partial f^i_k}{\partial m}(p, m) = 0,$$

and thus we have

$$p^T S_{f^i}(p, m) = 0^T, \ S_{f^i}(p, m)p = 0.$$

Second, for $x = f^i(p, m)$, define

$$E^x_i(q) = \inf\{q \cdot y | U_i(y) \ge U_i(x)\}.$$

It is well known that E^x_i is concave and continuously differentiable around p, and the following Shephard's lemma holds:[7]

$$DE^x_i(q) = f^i(q, E^x_i(q)), \ E^x_i(p) = m.$$

Because f^i is continuously differentiable around (p, m), to differentiate both sides of the above equality, we have that

$$D^2 E^x_i(p) = S_{f^i}(p, m).$$

Therefore, (S) holds because of Young's theorem, and $S_{f^i}(p, m)$ is negative semi-definite.

[7] See Lemma 1 of Hosoya (2020). Although this lemma assumes that $\Omega_i = \mathbb{R}^L_+$, the proof of this result is still valid for the case in which $\Omega_i = \mathbb{R}^{L-1}_+ \times \mathbb{R}$.

Third, let $\tilde{\Omega}_i$ denote the interior of Ω_i, and for $y \in \tilde{\Omega}_i$, define

$$g^i(y) = DU_i(y),$$

and for $j, k \in \{1, ..., L-1\}$, define

$$a^i_{jk}(y) = \frac{\partial g^i_j}{\partial x_k}(y) - \frac{\partial g^i_j}{\partial x_n}(y) g^i_k(y).$$

The $(L-1) \times (L-1)$ matrix-valued function $A_{g^i}(y) = (a^i_{jk}(y))_{j,k=1}^{L-1}$ is called **Antonelli matrix**. Samuelson (1950) showed that if $y = f^i(q, w) \in \tilde{\Omega}_i$, then $A_{g^i}(y)$ is regular, and the inverse matrix of $A_{g^i}(y)$ coincides with $(s^i_{jk}(q, w))_{j,k=1}^{L-1}$. By assumption of this proposition, we have that if $x = f^i(p, m)$, then $x \in \tilde{\Omega}_i$, and thus, (R) holds. Moreover, because $S_{f^i}(p, m)$ is symmetric, there exists an orthogonal matrix P such that

$$S_{f^i}(p, m) = P^T \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_L \end{pmatrix} P,$$

where each $\lambda_j \le 0$ is an eigenvalue of $S_{f^i}(p, m)$. Because of (R), we have that there exists exactly one j such that $\lambda_j = 0$, and $\lambda_k < 0$ whenever $k \neq j$. This implies that (ND) holds, which completes the proof. ■

4.5 Proof of Proposition 4

To show (10) and (11) is easy, and thus we omit its proof. If E is second-type, then by Lemmas 4 and 5, there exists an equilibrium price p^*. Thus, we assume that E is first-type. Take $s > 0$ that satisfies requirements in Lemma 6. Then, by Lemmas 4 and 5, there exists p^* such that $\zeta_s(p^*) = 0$, which implies that p^* is an equilibrium price in E. This completes the proof. ■

4.6 Proof of Proposition 5

First, note that if $x^i = f^i(p^*, p^* \cdot \omega^i)$, then

$$\sum_{i \in N} (x^i - \omega^i) = 0.$$

Because $p^* \cdot \omega^i \le p^* \cdot \omega^i$, we must have $U_i(x^i) \ge U_i(\omega^i)$. This implies that $u_i(\tilde{x}^i) \le u_i(\tilde{\omega}^i) + \alpha_i < U_i(\omega^i)$, and thus $x^i_L > 0$. This completes the proof. ■

4.7 Proof of Theorem 1

First, suppose that $E = (N, (\Omega_i)_{i\in N}, (U_i)_{i\in N}, (\omega^i)_{i\in N})$ is a second-type quasi-linear economy. Let $\hat{\Omega}_i = \mathbb{R}^{L-1}_+ \times \mathbb{R}$, and consider the alternative economy $\hat{E} = (N, (\hat{\Omega}_i)_{i\in N}, (U_i)_{i\in N}, (\omega^i)_{i\in N})$. Then, $\hat{E}$ is a first-type quasi-linear economy, and $E = \hat{E}_0$. Let f^i (resp. $\hat{f}^i$) be the demand function of i in economy E (resp. $\hat{E}$). Because of Lemma 2 and Proposition 5, we have that if p^* is an equilibrium price of E, then there exists a neighborhood U of p^* such that $f^i(p, p\cdot\omega^i) = \hat{f}^i(p, p\cdot\omega^i)$ for every $p \in U$. In particular, p^* is also an equilibrium price of $\hat{E}$. Therefore, if the claim of this theorem holds for first-type quasi-linear economies, then this theorem also holds for second-type quasi-linear economies. Hence, we assume without loss of generality that E is a first-type quasi-linear economy. By Lemma 2, we have that f^i is continuously differentiable.

We separate the proof into several steps. First, recall that

$$X^i(p) = f^i(p, p\cdot\omega^i) - \omega^i.$$

Because f^i is continuously differentiable, we have that X^i is also continuously differentiable.

Step 1. For every $p \in \mathbb{R}^L_{++}$,

$$\frac{\partial X^i_j}{\partial p_k}(p) = \begin{cases} s^i_{jk}(p, p\cdot\omega^i) & \text{if } j \neq L, \\ s^i_{jk}(p, p\cdot\omega^i) - \frac{X^i_k(p)}{p_L} & \text{if } j = L. \end{cases} \tag{14}$$

Proof of Step 1. Let $m = p\cdot\omega^i$. If $j \neq L$, then by Proposition 2,

$$\frac{\partial X^i_j}{\partial p_k}(p) = \frac{\partial f^i_j}{\partial p_k}(p, m) = s^i_{jk}(p, m),$$

as desired. If $j = L$, then by Walras' law (7),

$$X^i_j(p) = f^i_j(p, p\cdot\omega^i) = \frac{p\cdot\omega^i}{p_L} - \frac{1}{p_L}(\tilde{p}\cdot\tilde{f}^i(p, m)),$$

and because $\frac{\partial\tilde{f}^i}{\partial m}(p, m) = 0$, we have that for $k \neq L$,

$$\frac{\partial X^i_j}{\partial p_k}(p) = \frac{\omega^i_k}{p_L} - \frac{f^i_k(p, m)}{p_L} - \frac{1}{p_L}\left(\tilde{p}\cdot\frac{\partial\tilde{f}^i}{\partial p_k}(p, p\cdot\omega^i)\right).$$

Because of (R), we have that $\sum_{\ell=1}^{L} p_\ell s_{\ell k}(p,m) = 0$, and thus,

$$s_{jk}^i(p,m) = -\frac{1}{p_L}\left(\tilde{p} \cdot \frac{\partial \tilde{f}^i}{\partial p_k}(p,m)\right).$$

This implies that

$$\frac{\partial X_j^i}{\partial p_k}(p) = s_{jk}^i(p,m) - \frac{X_k^i(p)}{p_L},$$

as desired. Finally, suppose that $j = k = L$. Because of (R), we have that $\sum_{\ell=1}^{L} s_{j\ell}^i(p,m)p_\ell = 0$, and because of (8), we have that $X^i(p) = X^i(ap)$ for $a > 0$. Therefore,

$$\sum_{\ell=1}^{L} \frac{\partial X_j^i}{\partial p_\ell}(p)p_\ell = 0.$$

Moreover,

$$\sum_{\ell=1}^{L} X_\ell^i(p)p_\ell = p \cdot \omega^i - p \cdot \omega^i = 0.$$

Hence,

$$s_{jk}^i(p,m) = -\frac{1}{p_L}\sum_{\ell=1}^{L-1} s_{j\ell}^i(p,m)p_\ell,$$

and

$$\sum_{\ell=1}^{L-1} \frac{X_\ell^i(p)}{p_L}p_\ell = -X_k^i(p),$$

which implies that

$$\begin{aligned}
\frac{\partial X_j^i}{\partial p_k}(p) = & -\frac{1}{p_L}\sum_{\ell=1}^{L-1} \frac{\partial X_j^i}{\partial p_\ell}(p)p_\ell \\
= & -\frac{1}{p_L}\sum_{\ell=1}^{L-1}\left(s_{j\ell}^i(p,m) - \frac{X_\ell^i(p)}{p_L}\right)p_\ell \\
= & \; s_{jk}^i(p,m) - \frac{X_k^i(p)}{p_L},
\end{aligned}$$

as desired. This completes the proof of Step 1. ■

Step 2. Define $S^* = \{p \in \mathbb{R}_{++}^L | \|p\| = 1\}$. Then, there uniquely exists $p^* \in S^*$ that is an equilibrium price of this economy.

Proof of Step 2. Choose $s > 0$ that satisfies all requirements in Lemma 6. Then, $\zeta(p^*) = 0$ if and only if $\zeta_s(p^*) = 0$. Because (11) holds, by Proposition 4, we have that there exists an equilibrium price $p^* \in S^*$ in economy E_s. Fix such a p^*. By Lemma 2, there exists an open neighborhood U of p^* such that if $p \in U$, then $\zeta_s(p) = \zeta(p)$. In particular, ζ_s is continuously differentiable around p^*, and

$$D\zeta_s(p^*) = D\zeta(p^*).$$

By Step 1, we have that for every $v \neq 0$ with $p^* \cdot v = 0$,

$$v^T D\zeta(p^*)v = \sum_{i \in N} v^T DX^i(p^*)v = \sum_{i \in N} v^T S_{f^i}(p^*, p^* \cdot \omega^i)v < 0,$$

by (ND). Therefore, we have that[8]

$$(-1)^L \begin{vmatrix} D\zeta_s(p^*) & p^* \\ (p^*)^T & 0 \end{vmatrix} > 0.$$

This implies that ζ_s satisfies assumptions 1)-5) of ξ in Lemma 7, and thus the set $E = \zeta_s^{-1}(0) \cap S^*$ is finite, and

$$\text{index}(p^*) = +1$$

for all $p^* \in S^*$, where index(p^*) is defined in Lemma 7. This implies that E is a singleton, as desired. This completes the proof of Step 2. ■.

Choose any equilibrium price p^* in this economy. By Step 2, the set of all equilibrium prices coincides with $\{ap^* | a > 0\}$.

Recall the tâtonnement process (12):

$$\dot{p}_j(t) = a_j \zeta_j(p(t)),\ p_j(0) = p_{0j}.$$

We had assumed that $a_1, ..., a_L > 0$. Because p^* is an equilibrium price in this economy, we have that p^* is a steady state of (12). Define

$$h(p) = \sqrt{a_1^{-1} p_1^2 + ... + a_L^{-1} p_L^2}.$$

We note that $h(p)$ satisfies all requirements of the norm. In particular, we have that $h(ap) = ah(p)$ for every p and $a > 0$, and thus $Dh(p)p = h(p)$ if $p \neq 0$. Define

$$S(b) = \{p \in \mathbb{R}^L_{++} | h(p) = h(bp^*)\}.$$

[8]See Debreu (1952).

Choose a sufficiently small $\varepsilon > 0$ such that if $\|v\| \le \varepsilon$ and $t \in [-1, 1]$, then $p^* + tv \in \mathbb{R}^L_{++}$. Define

$$S = \{v \in \mathbb{R}^L | \|v\| = \varepsilon,\ Dh(p^*)v = 0\},$$

and

$$p(t, v) = \frac{h(p^*)}{h(p^* + tv)}(p^* + tv).$$

Step 3. Define $m_i^* = p^* \cdot \omega^i$ and the following function

$$g^i(t, v) = \begin{cases} \frac{1}{t^2}(p(t, v) - p^*) \cdot (f^i(p(t, v), p(t, v) \cdot \omega^i) - f^i(p^*, m_i^*)) & \text{if } t \neq 0, \\ v^T DX^i(p^*)v & \text{if } t = 0. \end{cases}$$

Then, g^i is continuous on $[-1, 1] \times S$.

Proof of Step 3. Clearly, g^i is continuous at (t, v) if $t \neq 0$. Therefore, it suffices to show that g^i is continuous at $(0, v)$ for all $v \in S$.

Choose any $\varepsilon' > 0$. Note that, $p(0, v) = p^*$ and h is continuously differentiable without 0. We can easily check that

$$\frac{\partial p}{\partial t}(0, v) = v, \tag{15}$$

$$\frac{\partial p}{\partial v_j}(0, v) = 0 \text{ for all } j \in \{1, ..., L\}. \tag{16}$$

Define

$$q(t, v) = \|p(t, v) - (p^* + tv)\|.$$

By (15) and (16), we have that $q(0, v) = 0$ and $Dq(0, v) = 0^T$ for all $v \in S$. Hence, by the formula of finite increments, for all $v \in S$, there exists an open and convex neighborhood U_v of $(0, v)$ such that if $(t', v'), (t'', v'') \in U_v$, then

$$|q(t', v') - q(t'', v'')| \le \varepsilon' \|(t', v') - (t'', v'')\|/2. \tag{17}$$

We can assume without loss of generality that

$$U_v = \{(t', v') \in [-1, 1] \times S | |t'| < 2\delta_v,\ \|v' - v\| < \delta_v\}.$$

Define

$$W_v = \{v' \in S | \|v' - v\| < \delta_v\}.$$

Then, (W_v) is an open covering of S, and thus, there exists a finite subcovering $(W_{v_1}, ..., W_{v_M})$. Let $\delta^* = \min\{\delta_{v_1}, ..., \delta_{v_M}\}$. Then, we have that

$$\sup_{t \in]0, \delta^*]} \frac{q(t, v)}{t} < \varepsilon'$$

for all $v \in S$.

Fix any $v \in S$. Since $f^i(ap, am) = f^i(p, m)$, we have that

$$f^i(p(t,v), p(t,v) \cdot \omega^i) = f^i(p^* + tv, (p^* + tv) \cdot \omega^i).$$

Define

$$\hat{g}^i(t,v) = \frac{1}{t} v \cdot (f^i(p^* + tv, (p^* + tv) \cdot \omega^i) - f^i(p^*, m_i^*)).$$

Then, by chain rule, we can easily show that there exists $\delta > 0$ such that if $0 < |t| < \delta$ and $v \in S$, then

$$|\hat{g}^i(t,v) - v^T DX^i(p^*)v| < \varepsilon'.$$

Therefore, if $0 < |t| < \min\{\delta, \delta^*\}$, we have that

$$\begin{aligned}
|g^i(t,v) - v^T DX^i(p^*)v| &\le |g^i(t,v) - \hat{g}^i(t,v)| + |\hat{g}^i(t,v) - v^T DX^i(p^*)v| \\
&< \frac{1}{t}\varepsilon' \|f^i(p(t,v), p(t,v) \cdot \omega^i) - f^i(p^*, m_i^*)\| + \varepsilon' \\
&\le (M+1)\varepsilon',
\end{aligned}$$

where $M > 0$ is some constant independent of (t, v).[9] Let $M^* > 0$ be the operator norm of $DX^i(p^*)$. If $\|v - v'\| < \sqrt{\varepsilon'}$ and $0 \le |t| < \min\{\delta, \delta^*\}$, then

$$\begin{aligned}
|g^i(t,v') - g^i(0,v)| &\le |g^i(t,v') - g^i(0,v')| + |g^i(0,v') - g^i(0,v)| \\
&< (M + M^* + 1)\varepsilon'.
\end{aligned}$$

Therefore, g^i is continuous at $(0, v)$. This completes the proof of Step 3. ■

Note that, because $S_{f^i}(p^*, m^*)$ satisfies (ND), by Step 1, we have that $\max_{v \in S} \sum_{i \in N} g^i(0,v) < 0$. By Step 3, we have that if $\varepsilon > 0$ is sufficiently small, then $\sum_{i \in N} g^i(t,v) < 0$ for every $t \in [-1, 1]$ and $v \in S$. Thus, we assume without loss of generality that $\varepsilon > 0$ is so small that $\sum_{i \in N} g^i(t,v) < 0$ for every $t \in [-1, 1]$ and $v \in S$.

Step 4. There exists an open neighborhood U of p^* such that if $p_0 \in U$, then there uniquely exists a solution $p(t)$ of (12) defined on $\mathbb{R}_+$, and $\lim_{t \to \infty} p(t) = bp^*$ for $b = \frac{h(p_0)}{h(p^*)}$.[10]

Proof of Step 4. First, suppose that $p(t)$ is a solution defined on (12). Then,

$$\frac{d}{dt}(h(p(t)))^2 = \sum_{j=1}^{L} a_j^{-1} p_j(t) a_j \zeta(p_j(t)) = 0,$$

[9] Use the formula of finite increments.

[10] In this case, ζ is continuously differentiable, and thus the solution (12) is automatically unique by Picard-Lindelöf's theorem.

by Proposition 4. Therefore, we have that $h(p(t)) = h(p_0)$ for all t.

Next, let $U = \{p \in \mathbb{R}^L_{++} | h(p) = h(p^*),\ h(p - p^*) < \varepsilon'\}$, where $\varepsilon' > 0$ is so small that for all $p \in U$, there exists $v \in S$ and $t \in [0, 1]$ such that p is proportional to $p^* + tv$. Let $W = \{p \in \mathbb{R}^L_{++} | (h(p^*)/h(p))p \in U\}$. Define

$$V(p) = (h(p - (h(p)/h(p^*))p^*))^2.$$

Then, by Step 3 and our succeeding arguments, we have that

$$\frac{d}{dt} V(p(t)) < 0$$

for every solution $p(t)$ of (12) such that $p_0 \in U$. Because $V(p) > 0$ if p is not proportional to p^* and $V(bp^*) = 0$ for $b > 0$, we have that V is a Lyapunov function of (12) on $W \cap S(b)$. Therefore, if $p_0 \in W$, then there uniquely exists a solution $p(t)$ of (12) defined on $\mathbb{R}_+$, and $\lim_{t \to \infty} p(t) = bp^*$ for $b = \frac{h(p_0)}{h(p^*)}$. This completes the proof of Step 4. ■.

Steps 2 and 4 states our claim of Theorem 1 is correct. This completes the proof. ■

Acknowledgement

We are grateful for Shinichi Suda and Toru Maruyama for their helpful lectures on tâtonnement process.

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