Weak Monotone Comparative Statics

Yeon-Koo Che,[1] Jinwoo Kim,[2] Fuhito Kojima[3]

[1]Columbia University

[2]Seoul National University

[3]University of Tokyo

September 21, 2020

Motivation

- **Comparative Statics:** how predicted behavior changes as environment changes.
- **Monotone Comparative Statics:** Topkis (1979, 1998) and Milgrom and Shannon (1994) provide a method that captures essential properties driving comparative statics.
  - Since predictions are often nonunique, set order matters.
  - Existing theory uses strong set order

Strong Set vs Weak Set Order

- Consider a partial order $(X, \geq)$, which induces set orders.

Strong Set vs Weak Set Order

- Consider a partial order $(X, \geq)$, which induces set orders.
- **Strong Set Order:** $X'' \geq_{ss} X'$ (both subsets of X) if

$$\forall x' \in X', x'' \in X'', x' \vee x'' \in X'', x' \wedge x'' \in X'.$$

Strong Set vs Weak Set Order

- Consider a partial order $(X, \geq)$, which induces set orders.
- **Strong Set Order:** $X'' \geq_{ss} X'$ (both subsets of X) if

$$\forall x' \in X', x'' \in X'', x' \vee x'' \in X'', x' \wedge x'' \in X'.$$

- **Weak Set Order:** $X'' \geq_{ws} X'$ if
  - $X'' \geq_{uws} X' : \forall x' \in X'$, there exists $x'' \in X''$ with $x'' \geq x'$.
  - $X'' \geq_{lws} X' : \forall x'' \in X''$, there exists $x' \in X'$ with $x' \leq x''$.

  - Strong set order implies weak set order.

Strong Set vs Weak Set Order

- Consider a partial order $(X, \geq)$, which induces set orders.
- **Strong Set Order:** $X'' \geq_{ss} X'$ (both subsets of X) if

$$\forall x' \in X', x'' \in X'', x' \vee x'' \in X'', x' \wedge x'' \in X'.$$

- **Weak Set Order:** $X'' \geq_{ws} X'$ if
  - $X'' \geq_{uws} X' : \forall x' \in X'$, there exists $x'' \in X''$ with $x'' \geq x'$.
  - $X'' \geq_{lws} X' : \forall x'' \in X''$, there exists $x' \in X'$ with $x' \leq x''$.

  - Strong set order implies weak set order.

☐ The set $M(t) := \arg\max_{x \in X} u(x; t)$ increases in t in the strong set order if u satisfies **MS** conditions: *single crossing* in (x, t) and is *quasi-supermodular* in x.

☐ But beyond individual choices, MCS is difficult to achieve in the strong set order (e.g., social choice, games, and matching)

Illustration with Nash equilibria

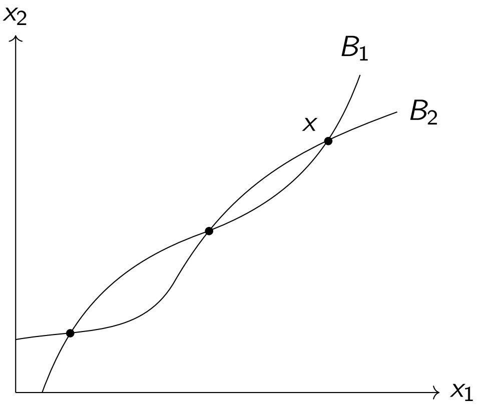


Figure: Failure of sMCS.

The MS conditions for payoffs guarantee monotonicity of best response.

Illustration with Nash equilibria

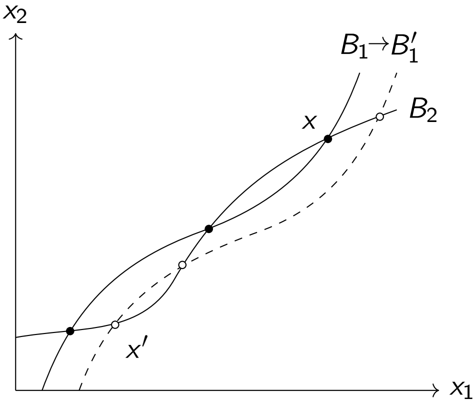


Figure: Failure of sMCS.

But equilibria do not shift in the strong set order. They do shift monotonically in the weak set order.

What We Do

- We consider **weak monotone comparative statics (wMCS)**
  - based on weak set order (cf. strong set order in MS and most others)
  - weaker requirements; easier to satisfy

What We Do

- We consider **weak monotone comparative statics (wMCS)**
  - based on weak set order (cf. strong set order in MS and most others)
  - weaker requirements; easier to satisfy
- Look for conditions for wMCS in the context of:
  - Individual choices
  - Pareto optimal choices
  - Games
  - Two-sided matching

What We Do

- We consider **weak monotone comparative statics (wMCS)**
  - based on weak set order (cf. strong set order in MS and most others)
  - weaker requirements; easier to satisfy
- Look for conditions for wMCS in the context of:
  - Individual choices
  - Pareto optimal choices
  - Games
  - Two-sided matching
- In the process, we make progress on
  - existence of fixed points and Nash equilibria in games
  - characterization and existence of stable matching in two-sided matching
- Expand applications of game theory and matching: to allow for indidviduals with incomplete preferences and multidivisional organizations.

Individual Choices

— characterizations along the lines of Milgrom and Shannon (1994) and Quah and Strulovici (2007)

— Omitted due to time constraint

Pareto Optimal Choices

Pareto Optimal Choices

- I: finite set of individuals
- X: set of possible (social) choices; a *poset* with $\geq$
- $u_i : X \to \mathbb{R}$ payoff function for $i \in I$;
  $\mathbf{u} = (u_i)$ profile of payoff functions
- $P(\mathbf{u})$: set of Pareto optimal choices (POC) under $\mathbf{u}$.

Pareto Optimal Choices

- I: finite set of individuals
- X: set of possible (social) choices; a *poset* with $\geq$
- $u_i : X \to \mathbb{R}$ payoff function for $i \in I$;

  $\mathbf{u} = (u_i)$ profile of payoff functions
- $P(\mathbf{u})$: set of Pareto optimal choices (POC) under $\mathbf{u}$.
  - POC can be interpreted as individual choice by a single agent with incomplete preferences (see Eliaz and Ok (2006))

Pareto Optimal Choices

- I: finite set of individuals
- X: set of possible (social) choices; a *poset* with $\geq$
- $u_i : X \to \mathbb{R}$ payoff function for $i \in I$;

  $\mathbf{u} = (u_i)$ profile of payoff functions
- $P(\mathbf{u})$: set of Pareto optimal choices (POC) under $\mathbf{u}$.
  - POC can be interpreted as individual choice by a single agent with incomplete preferences (see Eliaz and Ok (2006))
- We identify sufficient conditions on $\mathbf{u}$ and $\mathbf{v}$ such that

$$P(\mathbf{u}) \leq_{ws} P(\mathbf{v}).$$

Pareto Optimal Choices

- I: finite set of individuals
- X: set of possible (social) choices; a *poset* with $\geq$
- $u_i : X \to \mathbb{R}$ payoff function for $i \in I$;

  $\mathbf{u} = (u_i)$ profile of payoff functions
- $P(\mathbf{u})$: set of Pareto optimal choices (POC) under $\mathbf{u}$.
  - POC can be interpreted as individual choice by a single agent with incomplete preferences (see Eliaz and Ok (2006))
- We identify sufficient conditions on $\mathbf{u}$ and $\mathbf{v}$ such that

$$P(\mathbf{u}) \leq_{ws} P(\mathbf{v}).$$

- *Does MS condition for individuals imply wMCS of POCs?*

Pareto Optimal Choices

- I: finite set of individuals
- X: set of possible (social) choices; a *poset* with $\geq$
- $u_i : X \to \mathbb{R}$ payoff function for $i \in I$;

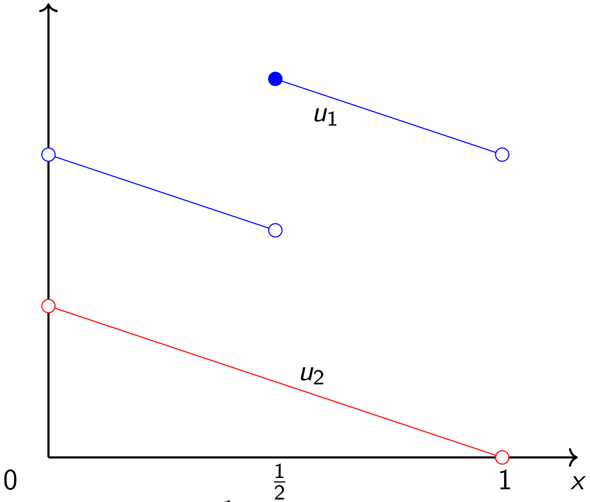
  $\mathbf{u} = (u_i)$ profile of payoff functions
- $P(\mathbf{u})$: set of Pareto optimal choices (POC) under $\mathbf{u}$.
  - POC can be interpreted as individual choice by a single agent with incomplete preferences (see Eliaz and Ok (2006))
- We identify sufficient conditions on $\mathbf{u}$ and $\mathbf{v}$ such that

$$P(\mathbf{u}) \leq_{ws} P(\mathbf{v}).$$

- *Does MS condition for individuals imply wMCS of POCs?* Not without additional condition.

Example

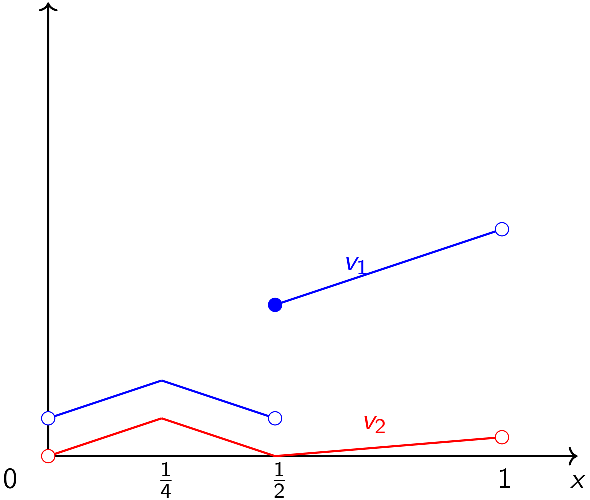
- Suppose $X = (0, 1)$.



- Unique Pareto optimum $= \frac{1}{2}$.

Example: after a single crossing dominating shift

- Suppose $X = (0, 1)$.



- Unique Pareto optimum $= \frac{1}{4}$ — Pareto optimum falls!!

wMCS of POC: one-dimensional X

If X is totally ordered, the condition is simple:

Theorem

Suppose

(i) X is compact and $\mathbf{u}$ and $\mathbf{v}$ are upper semicontinuous;

(ii) $\mathbf{v}$ single-crossing dominates $\mathbf{u}$.

Then, $P(\mathbf{u}) \leq_{ws} P(\mathbf{v})$.

wMCS of POC: one-dimensional X

If X is totally ordered, the condition is simple:

Theorem

Suppose

(i) X is compact and $\mathbf{u}$ and $\mathbf{v}$ are upper semicontinuous;

(ii) $\mathbf{v}$ single-crossing dominates $\mathbf{u}$.

Then, $P(\mathbf{u}) \leq_{ws} P(\mathbf{v})$.

- In the example: If $X = [0, 1]$, then

$$P(\mathbf{u}) = \{0, \tfrac{1}{2}\} \leq_{ws} \{\tfrac{1}{4}, 1\} = P(\mathbf{v}).$$

Proof Sketch

- Any $x < \inf P(\mathbf{u})$ is Pareto dominated under $\mathbf{u}$
- In particular, it is Pareto dominated by some $x' \in P(\mathbf{u})$ (due to compactness), so $x' > x$;
- $\Leftrightarrow x$ Pareto dominated by x' under $\mathbf{u}$,.
- By SCP, x Pareto dominated (by x') under $\mathbf{v}$
- $\inf P(\mathbf{u}) \leq \inf P(\mathbf{v})$.

Similar argument shows $\sup P(\mathbf{u}) \leq \sup P(\mathbf{v})$. With a little more care, the result follows. $\square$

wMCS of POC: General X

Theorem

Suppose

(i) X is a convex, compact lattice

(ii) $\mathbf{u}$ and $\mathbf{v}$ are upper semicontinuous, concave, supermodular; and $\mathbf{v}$ increasing-difference dominates $\mathbf{u}$.

Then, $P(\mathbf{u}) \leq_{ws} P(\mathbf{v})$.

wMCS of POC: General X

Theorem

Suppose

(i) X is a convex, compact lattice

(ii) $\mathbf{u}$ and $\mathbf{v}$ are upper semicontinuous, concave, supermodular; and $\mathbf{v}$ increasing-difference dominates $\mathbf{u}$.

Then, $P(\mathbf{u}) \leq_{ws} P(\mathbf{v})$.

- Supermodularity: cardinal strengthening of quasi-supermodularity
- Increasing differences: cardinal strengthening of single crossing

wMCS of POC: General X

Theorem

Suppose

(i) X is a convex, compact lattice

(ii) $\mathbf{u}$ and $\mathbf{v}$ are upper semicontinuous, concave, supermodular; and $\mathbf{v}$ increasing-difference dominates $\mathbf{u}$.

Then, $P(\mathbf{u}) \leq_{ws} P(\mathbf{v})$.

- Supermodularity: cardinal strengthening of quasi-supermodularity
- Increasing differences: cardinal strengthening of single crossing

- Upshot: Conditions guaranteeing sMCS for individual choices give wMCS for POCs, in a "well-behaved" environment.

Proof Skech

We utilize our new characterization of POC.

Theorem (Che, Kim, Kojima and Ryan, 2020)

Given our conditions, $x \in P(\mathbf{u})$ if and only if there exists a sequence $\{\phi^k\}_{k=1}^K$ of nonnegative welfare weights, ϕ^K strictly positive, such that $x \in X^k(\mathbf{u})$ for all $k = 1, ..., K$, where

$$X^0(\mathbf{u}) := X \text{ and } X^k(\mathbf{u}) := \arg \max_{x' \in X^{k-1}(\mathbf{u})} \sum_i \phi_i^k u_i(x'). \quad \Rightarrow$$

Proof Skech

We utilize our new characterization of POC.

Theorem (Che, Kim, Kojima and Ryan, 2020)

Given our conditions, $x \in P(\mathbf{u})$ if and only if there exists a sequence $\{\phi^k\}_{k=1}^K$ of nonnegative welfare weights, ϕ^K strictly positive, such that $x \in X^k(\mathbf{u})$ for all $k = 1, ..., K$, where

$$X^0(\mathbf{u}) := X \text{ and } X^k(\mathbf{u}) := \arg \max_{x' \in X^{k-1}(\mathbf{u})} \sum_i \phi_i^k u_i(x'). \quad \Rightarrow$$

- Fix any sequence $\{\phi^k\}$. Apply MS result inductively to get

$$P_{\{\phi^k\}}(\mathbf{u}) := X^K(\mathbf{u}) \leq_{ss} X^K(\mathbf{v}) =: P_{\{\phi^k\}}(\mathbf{v}).$$

Proof Skech

We utilize our new characterization of POC.

Theorem (Che, Kim, Kojima and Ryan, 2020)

Given our conditions, $x \in P(\mathbf{u})$ if and only if there exists a sequence $\{\phi^k\}_{k=1}^K$ of nonnegative welfare weights, ϕ^K strictly positive, such that $x \in X^k(\mathbf{u})$ for all $k = 1, ..., K$, where

$$X^0(\mathbf{u}) := X \text{ and } X^k(\mathbf{u}) := \arg\max_{x' \in X^{k-1}(\mathbf{u})} \sum_i \phi_i^k u_i(x'). \quad \Rightarrow$$

- Fix any sequence $\{\phi^k\}$. Apply MS result inductively to get
$$P_{\{\phi^k\}}(\mathbf{u}) := X^K(\mathbf{u}) \leq_{ss} X^K(\mathbf{v}) =: P_{\{\phi^k\}}(\mathbf{v}).$$
- The result then follows since
$$P(\mathbf{u}) = \bigcup_{\{\phi^k\}} P_{\{\phi^k\}}(\mathbf{u}) \leq_{ws} \bigcup_{\{\phi^k\}} P_{\{\phi^k\}}(\mathbf{v}) = P(\mathbf{v}).$$

(Strong set order is NOT closed under $\cup$, but weak set order is.) $\square$

Example

Let $X = [0,6]^2$, $I = \{1,2\}$ and

$$u_1(x,y) = -(x-1)^2 - (y-1)^2, \quad u_2(x,y) = -(x-4)^2 - (y-1)^2$$
$$v_1(x,y) = -(x-1)^2 - (y-4)^2, \quad v_2(x,y) = -(x-4)^2 - (y-2)^2.$$

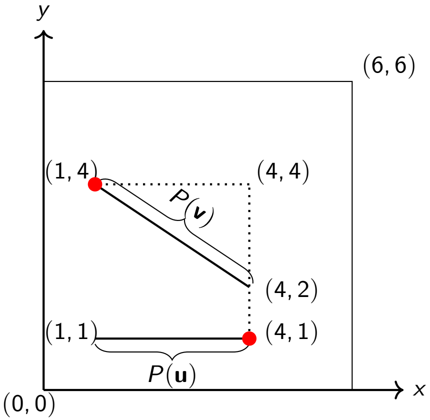


Figure: Failure of strong set monotonicity

Fixed Point Theorem and Applications

Tarski-Zhou Fixed Point Theorem

Theorem (Tarski-Zhou)

Suppose

- X: a complete lattice
- $F : X \rightrightarrows X$: non-empty, complete sublattice-valued, strong set monotonic

Then, the fixed point set is nonempty and a complete lattice.

New Fixed Point Theorem

Theorem (Tarski-Zhou)

Suppose

- X: a complete lattice
- $F : X \rightrightarrows X$: non-empty, complete sublattice-valued, strong set monotonic

Then, the fixed point set is nonempty and a complete lattice.

Theorem (Li-CKK)

Suppose

- X: partially ordered, and compact
- $F : X \rightrightarrows X$: non-empty, compact-valued, (upper) weak set monotonic
- regularity: $X_+(F)$ is non-empty.

Then, the fixed point set is nonempty and contains a maximal point.

- Note: analogous for "lower weak set monotonicity"

Comparison

Theorem (Tarski-Zhou)

Suppose

- X: a complete lattice
- $F : X \rightrightarrows X$: non-empty, complete sublattice-valued, strong set monotonic

Then, the fixed point set is nonempty and a complete lattice.

Theorem (Li-CKK)

Suppose

- X: partially ordered, and compact
- $F : X \rightrightarrows X$: non-empty, compact-valued, (upper) weak set monotonic
- regularity: $X_+(F)$ is non-empty.

Then, the fixed point set is nonempty and contains a maximal point.

- Note: analogous for "lower weak set monotonicity"

Comparison

Theorem (Tarski-Zhou)

Suppose

- X: a complete lattice
- $F : X \rightrightarrows X$: non-empty, complete sublattice-valued, strong set monotonic

Then, the fixed point set is nonempty and a complete lattice.

Theorem (Li-CKK)

Suppose

- X: partially ordered, and compact
- $F : X \rightrightarrows X$: non-empty, compact-valued, (upper) weak set monotonic
- regularity: $X_+(F)$ is non-empty.

Then, the fixed point set is nonempty and contains a maximal point.

- Note: analogous for "lower weak set monotonicity"

wMCS of Fixed Point Set

Let $\mathcal{E}(F)$ be the fixed point set of F.

Theorem (CKK)

Suppose X is compact, both F and G satisfy CKK conditions. If $G(x) \geq_{uws} F(x)$ for all x, then $\mathcal{E}(G) \geq_{uws} \mathcal{E}(F)$.

- analogous for "lower weak set monotonic."

Theorem

With order continuity (satisfied if X is finite), a fixed point can be found iterating F from a regular point (i.e., X_+ or X_-).

- But, can't guarantee obtaining a maximal or minimal fixed point this way. $\Rightarrow$

Application: Games with Weak Strategic Complementarities

- $\Gamma = (I, X, (B_i)_{i \in I})$ a game where
  - I: finite set of players
  - X: set of strategy profiles
  - B_i: best response correspondence

- Γ is a game with weak strategic complementarity if
  - for each i, B_i is. nonempty, compact valued and upper weak set monotonic
  - $B = (B_i)$ satisfies regularity.

wMCS of Nash equilibria

Theorem

1. A game Γ with weak strategic complementarities has a nonempty set of Nash equilibria.
2. Suppose that Γ' and Γ are both games with weak strategic complementarities, and $B_i'(s_{-i}) \geq_{uws} B_i(s_{-i})$ for every $i \in I$ and $s_{-i} \in S_{-i}$. Then, $\mathcal{NE}(\Gamma') \geq_{uws} \mathcal{NE}(\Gamma)$.

wMCS of Nash equilibria

Theorem

1. A game Γ with weak strategic complementarities has a nonempty set of Nash equilibria.
2. Suppose that Γ' and Γ are both games with weak strategic complementarities, and $B_i'(s_{-i}) \geq_{uws} B_i(s_{-i})$ for every $i \in I$ and $s_{-i} \in S_{-i}$. Then, $\mathcal{NE}(\Gamma') \geq_{uws} \mathcal{NE}(\Gamma)$.

- Requirement weaker than standard "(quasi)supermodular" games (Milgrom and Shannon (1994))
- Preferences don't need to be complete: B_i can simply be Pareto optimal choices (recall results before)

Application: General Model of Two Sided Matching with Contracts

- W: finite set of workers
- F: finite set of firms
- X: finite set of contracts; a contract $x \in X$ specifies a worker w and a firm f and a contract term (salary).
- **choice correspondence**: $C_a(X')$ are optimal choices by agent $a \in F \cup W$ from X':
- **stable allocation** suitably defined—*Individually Rational* and *No Blocking*.

Conditions on C_a

1. **Weak Substitutability**: the rejection correspondence $R_a(X') = \{Z : Z = X'_a \setminus Y \text{ for some } Y \in C_a(X')\}$ is weak set monotonic with "$\supset$" as order.

Conditions on C_a

1. **Weak Substitutability**: the rejection correspondence $R_a(X') = \{Z : Z = X'_a \setminus Y \text{ for some } Y \in C_a(X')\}$ is weak set monotonic with "$\supset$" as order.

2. **Sen's α:** $Y \in C_a(X'')$ and $Y \subset X' \subset X'' \Rightarrow Y \in C_a(X')$.

Conditions on C_a

1. **Weak Substitutability**: the rejection correspondence $R_a(X') = \{Z : Z = X'_a \setminus Y \text{ for some } Y \in C_a(X')\}$ is weak set monotonic with "$\supset$" as order.

2. **Sen's α:** $Y \in C_a(X'')$ and $Y \subset X' \subset X'' \Rightarrow Y \in C_a(X')$.
   - Weaker than **WARP** = Sen's α + Sen's β.
   - Sen's β: $Y, Y' \in C_a(X'), Y \in C_a(X''), X' \subset X'' \Rightarrow Y' \in C_a(X'')$
   - Relaxing Sen's β accommodates incomplete preferences $\Rightarrow$

- cf. State of the art assumes a stronger version of 1 and WARP.

Fixed Point Characterization of Stability

Build a tâtonnement-like operator: $T(X', X'') = (T_1(X''), T_2(X'))$, for each $(X', X'') \in 2^X \times 2^X$, where

$$T_1(X'') = \{\tilde{X} \in 2^X : \tilde{X} = X \setminus \tilde{Y} \text{ for some } \tilde{Y} \in R_W(X'')\},$$
$$T_2(X') = \{\tilde{X} \in 2^X : \tilde{X} = X \setminus \tilde{Y} \text{ for some } \tilde{Y} \in R_F(X')\},$$

where R_W and R_F defined similarly to before.

Fixed Point Characterization of Stability

Build a tâtonnement-like operator: $T(X', X'') = (T_1(X''), T_2(X'))$, for each $(X', X'') \in 2^X \times 2^X$, where

$$T_1(X'') = \{\tilde{X} \in 2^X : \tilde{X} = X \setminus \tilde{Y} \text{ for some } \tilde{Y} \in R_W(X'')\},$$
$$T_2(X') = \{\tilde{X} \in 2^X : \tilde{X} = X \setminus \tilde{Y} \text{ for some } \tilde{Y} \in R_F(X')\},$$

where R_W and R_F defined similarly to before.

Theorem

Suppose C_a satisfies Sen's α for all a. Then, Z is stable if and only if there exists a fixed point (X', X'') of T such that $Z \in C_F(X') \cap C_W(X'')$.

- cf. The state of art assumes WARP.

Existence of Stability

Theorem

Suppose choice correspondences satisfy Sen's α and weak substitutability. Then, a stable allocation exists.

Existence of Stability

Theorem

Suppose choice correspondences satisfy Sen's α and weak substitutability. Then, a stable allocation exists.

Proof Sketch.

- Define a partial order set $(2^X \times 2^X, \geq)$ with $\geq = (\supset, \subset)$.
- Weak Substitutability: T is weak set monotonic.
- Fixed Point Theorem: T has a fixed point

By our characterization, a stable allocation exists. □

Remark: Gale-Shapley is an iterative version of Tarski that works for a simple environment. We are generalizing it.

weak MCS

Theorem

Suppose that a firm's choice correspondence becomes more permissive (in set inclusion). Then, workers become better off and firms become worse off in the weak set order sense (under original preferences).

weak MCS

Theorem

Suppose that a firm's choice correspondence becomes more permissive (in set inclusion). Then, workers become better off and firms become worse off in the weak set order sense (under original preferences).

Proof Sketch.

- Stable allocation = Fixed point of T
- Change in choice $\Rightarrow$ Change in T
- Use Comparative statics of fixed points

□

Applications:

1. Multidivisional organizations
2. Matching with Regional Constraints

Applications:

1. Multidivisional organizations

2. Matching with Regional Constraints

$\Rightarrow$ Modelled by choice correspondences selecting Pareto optimal choices among divisions or among hospitals within region:

Applications:

1. Multidivisional organizations

2. Matching with Regional Constraints

⇒ Modelled by choice correspondences selecting Pareto optimal choices among divisions or among hospitals within region:

Choice correspondence satisfies weak substitutability and Sen's α while violating WARP

Applications:

1. Multidivisional organizations

2. Matching with Regional Constraints

⇒ Modelled by choice correspondences selecting Pareto optimal choices among divisions or among hospitals within region:

Choice correspondence satisfies weak substitutability and Sen's α while violating WARP

- Corollaries: Existence of stable allocations, comparative statics: when the hiring constraint becomes more restrictive; all other firms benefit, workers are hurt.

Conclusion

- We propose weak monotone comparative statics (wMCS)
- Requirement is weaker, so wider applicability
- Analyzed: individual choices, Pareto optimal choices, games with weak strategic complementarity, matching theory
- Future Research:
  - Weaker sufficient conditions for wMCS of Pareto optimal choices
  - More applications

Thank You!

Some References

Che, Yeon-Koo, Jinwoo Kim, Fuhito Kojima, and Chris Ryan. 2020. "Characterizing Pareto Optima: Sequential Utilitarian Welfare Maximization." mimeo.

Eliaz, Kfir, and Efe Ok. 2006. "Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences." *Games and Economic Behavior*, 56: 61–86.

Fleiner, Tamás. 2003. "A Fixed-Point Approach to Stable Matchings and Some Applications." *Mathematics of Operations Research*, 28: 103–126.

Li, Jinlu. 2014. "Fixed point theorems on partially ordered topological vector spaces and their applications to equilibrium problems with incomplete preferences." *Fixed Point Theory and Applications*, 192: 1–17.

Milgrom, Paul, and Chris Shannon. 1994. "Monotone Comparative Statics." *Econometrica*, 62: 157–180.

Quah, John K.-H., and Bruno Strulovici. 2007. "Comparative Statics with Interval Dominance Order: Some Extensions." mimeo.

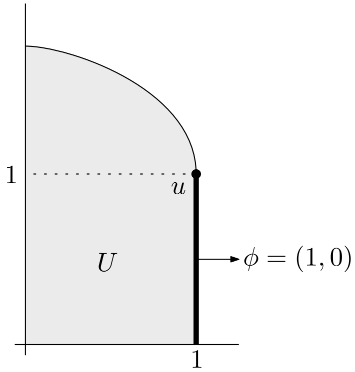
Tarski, Alfred. 1955. "A Lattice-Theoretical Fixpoint Theorem and Its Applications." *Pacific Journal of Mathematics*, 5: 285–309.

Topkis, Donald M. 1979. "Equilibrium Points in Nonzero-Sum n-Person Submodular Games." *SIAM Journal on Control and Optimization*, 17: 773787.

Topkis, Donald M. 1998. *Supermodularity and Complementarity.* Princeton, New Jersey:Princeton University.

Zhou, Lin. 1994. "The Set of Nash Equilibria of a Supermodular Game is a Complete Lattice." *Games and Economic Behavior*, 7: 295–300.

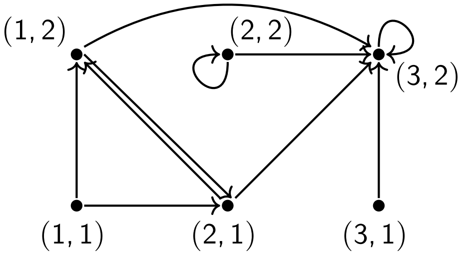
Illustration of Non-Exposed Pareto Optimum



- u is Pareto optimal but not exposed.

Return

Faiulure of any iteration to reach a minimal fixed point



- The minimal fixed point $(2, 2)$ cannot be reached from any iterative application of F starting from $(1, 1)$.

Return

Violation of Sen's β due to Preference Imcompleteness

- A firm f with two divisions, δ and δ', and three workers w, w', and w''.
- Workers are all acceptable to δ and δ' while $w'' \succ_{\delta'} w'$.
- Constrained to hire at most one worker across the divisions.
- No strict preferences over which division should hire a worker when both divisions have applicants.
- $C_f(\{(w,\delta),(w',\delta')\}) = \{\{(w,\delta)\},\{(w',\delta')\}\}$.
- But $C_f(\{(w,\delta),(w',\delta'),(w'',\delta')\}) = \{\{(w,\delta)\},\{(w'',\delta')\}\}$.

Return

Li (2014), Fleiner (2003), Che et al. (2020), Tarski (1955), Zhou (1994)