Minimax Predictive Dinsities of Dynamic Linear Model under Kullback-Leibler Loss

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Consider a time series $(y_t, \boldsymbol{x}_t)$, where $y_t \in \mathbb{R}$ and $\boldsymbol{x}_t$ is a countably infinite vector. Assume that the data generating process of y_t can be represented as a dynamic linear models:

$$y_t = \mu_t + \sum_{i=1}^{\infty} \beta_i x_{i,t}$$

$$x_{i,t}; \text{Markov processes,}$$

where μ_t is the drift. Observed processes are $\{y_t\}$ and $\{x_{i,t}\}_{i \in I}$ where I is an arbitrary set of finite i. Set $\mathcal{F}_t = \sigma(y_s, x_{i,s}; s \leq t, i \in I)$, the smallest σ-algebra that makes all of the observable processes measurable.

Based on the information $\mathcal{F}_t$, we consider the problem of obtaining a predictive density $\hat{p}(y_{t+1} | \mathcal{F}_t)$ for y_{t+1} that is close to $p(y_{t+1})$. We measure this closeness by Kullback–Leibler (KL) loss,

$$L(p, \hat{p}(\cdot | \mathcal{F}_t)) = \int \log \frac{p(y_{t+1})}{\hat{p}(y_{t+1} | \mathcal{F}_t)} p(y_{t+1})\, dy_{t+1}$$

and evaluate $\hat{p}$ by its expected loss or risk function

$$R_{KL}(p, \hat{p}) = \int L(p, \hat{p}(\cdot | \mathcal{F}_t))\, d\mu_{Y,X}$$

where $\mu_{Y,X}$ is cyrindrical mesure for $\{y_s, x_s\}_{0 \leqq s \leqq t}$.

We construct a predictive density by dynamic linear model(DLM), that is,

$$y_t = F_t^\top \boldsymbol{\theta}_t + v_t, \quad v_t \sim \mathcal{N}(0, V_t)$$

$$\boldsymbol{\theta}_t = G_t^\top \boldsymbol{\theta}_{t-1} + \boldsymbol{w}_t, \boldsymbol{w}_t \sim \mathcal{N}(0, W_t)$$

where $F_t = [1, x_{1,t}, \cdots, x_{I,t}]^\top$, $\boldsymbol{\theta}_t = [\theta_{0,t}, \cdots, \theta_{I,t}]^\top$ is $(1+I)$ vector, G_t is known matrix, V_t is known scalar, and W_t is known matrix. Our purpose is to show that when $G_t = I$, random walk DLM, the predictive density has minimaxity with respect to KL divergence loss.

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