Equity and time consistency for intertemporal social decisions*

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Abstract

In this paper, we study intertemporal social welfare evaluations when agents' discount factors may be different. We first show that there exists a conflict between efficiency, equity, and time consistency, even if all agents share the same discount factor. We argue that this impossibility result is due to a tension between equity and time consistency regarding how the past should be taken into account when evaluating intertemporal distributions. Then, we instead introduce weaker requirements of time consistency and equity. On the one hand, using the weaker version of time consistency, we characterize a maximin social welfare ordering which completely ignores the past information. On the other hand, using the weaker form of equity, we characterize another maximin social welfare ordering which completely respects the past information.

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1 Introduction

Many economic decisions are intertemporal. For instance, people save, borrow, invest, and choose consumption plans. Individuals would make different decisions because their time preferences are different. They could enjoy different levels of well-being in different periods. Various public policies such as social security reform and reduction of the emission of greenhouse gases also have dynamic features. Thus, it would be important to have criteria to evaluate intertemporal distributions.

In this paper, we study how distributions of lifetime utilities should be evaluated in terms of equity. In the problem, it would be important to consider how much the past should be taken into account. Consider the following example. There are two persons, Al and Bill, with the same age. Suppose that Al was born in a rich family and have lived a very happy life, while Bill was born in a poor family and have lived a severe life. However, Al consumed up the family's wealth while Bill works hard, and now Bill is more affluent. In this case, considering the past situations, it is not clear whether it is desirable to redistribute from Bill to Al.

The above example shows that evaluation of lifetime distributions could depend on the past or history. In dynamics, most studies consider time invariant decision rules, that is, decisions are independent of the past. This property is desirable for individual decision making, but not for social evaluation (Hayashi, 2016; Millner and Heal, 2018). It also appears too harsh to leave Al in a severe situation, for instance, if he is in huger and distress. Thus, it may be desirable to ignore the past, to some degree, and redistribute to the starving person. We analyze what conditions are related to social attitudes toward history.

In this paper, we study social welfare criteria satisfying equity and time consistency, as well as efficiency as the weak Pareto principle.[1] These three principles would be essential for intertemporal social decisions. In our model, individuals may have different discount

[1]The weak Pareto principle claims that social welfare should improve if every agent's lifetime utility increases.

factors, reflecting heterogeneity in time preferences. We discuss that a tension between equity and time consistency is related to how much the past should be respected. To our knowledge, with few exceptions, this problem has not been widely studied.

In the context of intertemporal choice, time consistency is a basic rationality postulate. This condition requires that decisions at different points in time should not be contradictory. Equity is another important property when assessing social situations. We introduce an equity axiom, *Limited Equity* requiring that, if distributions do not change over time from the present, then certain forms of redistribution from better-off to worse-off should be socially accepted.[2]

Our first result shows that even if all agents share the same discount factor, there exist no social orderings satisfying the weak Pareto principle, time consistency, and *Limited Equity*. The interpretation is as follows. Considering the example of Al and Bill above, the equity axiom requires to reduce the inequality independently of the past situations, while time consistency insists that the past should be respected for consistency of choices. Then, from these axioms, the redistribution from Al to Bill should be approved also when they were younger, despite Al was much better off than Bill at the point in time. Under the weak Pareto condition, this argument leads to a cycle of social preference.

Thus, in order to obtain possibility results, we must weaken either time consistency or *Limited Equity*. First, we examine the implication of a weaker time consistency, *Time Consistency for Equals*, which implies consistency of intertemporal choices only under equal situations. In other words, this axiom does not require to take unequal past distributions into account. Then, using the weak Pareto principle, *Limited Equity*, *Time Consistency for Equals*, and an auxiliary axiom, we characterize a maximin social ordering focusing on the worst average lifetime utilities in the society. This maximin ordering starts evaluates only distributions from the present period, and completely ignores the past.

Second, we consider a weaker axiom of equity, *Limited Equity for Equal Past*, which

[2]Formally, this is a weaker version of Hammond's (1976) equity axiom where the amount taken from the better-off is limited by an upper bound.

requires the redistribution only when both past distributions (from the first period) and discount factors are equal. Then, using the weak Pareto principle, *Limited Equity for Equal Past*, *Time Consistency*, and auxiliary axioms, we characterize another maximin social ordering focusing on the worst average lifetime utilities from the first period. The second maximin ordering evaluates average discounted sums of lifetime utilities from the first period, and thus takes the history into account.

These two results contrast the implications of equity and time consistency. On the one hand, if we require redistribution ignoring past distributions, with weakening time consistency, the social preference should also ignore the past. On the other hand, if we require full time consistency, with the weaker equity condition, past distributions should be strongly taken into consideration. Thus, how much the society should respect the history would depend largely on the balance between equity and time consistency.

The remainder of the paper is organized as follows. In Section 2, we present the model. Section 3 shows the conflicts between equity, efficiency, and time consistency. Section 4 gives the characterization of maximin. Section 5 discusses the related literature. Section 6 provides some concluding remarks.

2 The Model

We adopt a discrete time model. Let $\mathcal{T} = \mathbb{Z}_+$ denote the set of periods, where $\mathbb{Z}_+$ is the set of non-negative integers. $N = \{1, \cdots, n\}$ is the (finite) set of agents such that $|N| \geq 2$. We denote by $u_{it} \in \mathbb{R}$ agent i's instantaneous utility at period t, where $\mathbb{R}$ is the set of real numbers. It is assumed that the instantaneous utility numbers are fully measurable and interpersonally comparable. Let $\boldsymbol{u}_t = (u_{it})_{i \in N} \in \mathbb{R}^N$ be a distribution of utility at $t \in \mathcal{T}$. Especially, $\boldsymbol{u}_t 0 = \boldsymbol{x}_0$ is given and fixed. We also denote $U_i^T = (u_{it})_{t \geq T}$ given $T \geq 1$.

Given U_i^T, agent i's lifetime interests are evaluated by $W(U_i^T; \delta_i) \equiv \sum_{t \geq T} \delta_i^{t-T} u_{it}$, where $\delta_i \in (0, 1)$ is agent i's exponential discount factor. In this paper, we consider situations where agents' discount factors may be different because of heterogeneous time preferences.[3]

Given $T \geq 1$, $h^T = (\boldsymbol{u}_t)_{t<T} \in \mathbb{R}^{NT}$ is a history of distribution at evaluation period $T \geq 1$. Note that $h^1 = \boldsymbol{x}_0$. Let $H^T = \{(\boldsymbol{u}_t)_{t<T} | \boldsymbol{u}_t \in \mathbb{R}^N\}$ be the set of histories at T. Define the set of histories as $H = \bigcup_{T \geq 1} H_T$.

Given any history $h^T \in H$, the set of distributions of lifetime utilities from T is isomorphic to $\mathbb{R}^\infty$. Hence, slightly abusing the notation, we denote X the set of distributions of utility streams defined as follows.

$$X = \left\{ U^T \in (R_+^N)^\infty | T \geq 1, \text{ and } \sup_{t \in \mathcal{T}} |u_{it}| < +\infty \text{ for all } i \in N \right\}.$$

$\bar{U}_i^T$ is said to be *constant* if $\bar{U}_i^T = (u_{it})_{t \in \mathcal{T}}$ is such that $u_{it} = u_{it'}$ for all $t, t' \in \mathcal{T}$. Without any risk of confusion, the instantaneous utility in $\bar{U}_i^T$ is denoted by u_i. Similarly, $\bar{U}^T = (\bar{U}_i^T)_{i \in N}$ is called a *constant distribution.* Let $\bar{X}$ be the set of constant distributions.

We often use the following notations. Given $U^T \in X$ and $a \in \mathbb{R}$, let $aU^T = ((au_{it})_{t \geq T})_{i \in N}$. Moreover, given $U^T \in X$ and $\boldsymbol{w} \in \mathbb{R}^N$, $(\boldsymbol{w}, U^T)$ is a distribution where U^T follows after the distribution $\boldsymbol{w}$.

Let $\mathcal{D} = (0,1)^N$ be the set of profiles of discount factors. Moreover, $\mathcal{D}^E = \{\delta \in \mathcal{D} | \delta_i = \delta j$ for all $i, j \in N\}$ is the set of profiles of *uniform* discount factors.

Given any $h^T \in H$, the social evaluator's problem is to rank distributions of utility streams from T, based on agents' discount factors and the history. A social welfare function $\succsim$ is a mapping that determines a binary relation over distributions of utility streams for every discount factor profile and history: Given $\delta = (\delta_i)_{i \in N} \in \mathcal{D}$ and $h^T \in H$, $\succsim_{h^T}^{\delta}$ denotes a binary relation over X. We assume that $\succsim_{h^T}^{\delta}$ is complete, transitive, and continuous with respect to the sup norm. $\succ_{h^T}^{\delta}$ and $\sim_{h^T}^{\delta}$ are asymmetric and symmetric parts of $\succsim_{h^T}^{\delta}$, respectively.

Note that we do not assume that discounted sums $W(U_i^T; \delta_i)$ are interpersonally comparable. This is because discount factors (time preferences) are different among agents.

[3] Because of this, we do not assume interpersonal comparability of discounted sums of lifetime utility, as explained in the last paragraph of this section.

For instance, suppose that both Ann and Bob have the same constant utility, but Ann is more patient than Bob, that is, $\delta_A > \delta_B$. Then, $W(\bar{U}_A^T; \delta_A) > W(\bar{U}_B^T; \delta_B)$ (if compared), despite $\bar{U}_A^T = \bar{U}_B^T$. In this case, if we redistribute from Ann to Bob, the reason is that Bob is more impatient. This redistribution would not be compelling. This example shows that it is not necessarily desirable to assume interpersonal comparison of discounted sums of lifetime utility. In this paper, we rather derive criteria for interpersonal comparison from our axioms.

3 Efficiency, Equity, and Time Consistency

In this section, we first introduce axioms of efficiency, equity, and time consistency. We also show that these axioms are incompatible.

The first axiom is the standard Pareto condition.

Weak Pareto. For all $\delta \in \mathcal{D}$, all $h^T \in H$, and all $U^T, V^T \in X$, if $W(U_i; \delta_i) > W(V_i; \delta_i)$ for all $i \in N$, then $U \succ_{h^T}^{\delta} V$.

This axiom implies that an unanimous improvement should be socially preferred. Note that by continuity of social ranking, *Weak Pareto* implies the following *Pareto Indifference* principle.

Pareto Indifference. For all $\delta \in \mathcal{D}$, all $h^T \in H$, and all $U^T, V^T \in X$, if $W(U_i; \delta_i) = W(V_i; \delta_i)$ for all $i \in N$, then $U \sim_{h^T}^{\delta} V$.

Next, we introduce an equity axiom.

Limited Equity. There exists $\alpha \in \mathbb{R}_{++}$ such that, for all $\delta \in \mathcal{D}$, all $h^T \in H$, and all $\bar{U}^T, \bar{V}^T \in \bar{X}$, if there exist $j, k \in N$ such that $v_k < u_k < u_j < v_j$, $v_j - u_j < \alpha$, and $u_i = v_i$ for all $i \neq j, k$, then $\bar{U}^T \succsim_{h^T}^{\delta} \bar{V}^T$.

This requirement implies that any rank preserving redistribution in constant distributions should be accepted as long as the amount taken from the better-off is limited by α.[4] This is a weaker version of Hammond equity in the sense that the "sacrifice" of the better-off is limited. Our results do not depend on α: It can be arbitrarily small.

The third axiom is a well-known condition of time consistency.

Time Consistency. For all $\delta \in \mathcal{D}$, all $h^T \in H$, all $\boldsymbol{w}_T \in \mathbb{R}^N$ and all $U^{T+1}, V^{T+1} \in X$,

$$(\boldsymbol{w}_T, U^{T+1}) \succsim^{\delta}_{h^T} (\boldsymbol{w}_T, V^{T+1}) \Longleftrightarrow U^{T+1} \succsim^{\delta}_{(h^T, \boldsymbol{w}_T)} V^{T+1}.$$

This axiom states that decisions at periods T and $T+1$ should be consistent. By repeated applications, this condition implies that social decisions at all $t \leq T$ and $T+1$ are consistent. This fact is used in the proof of Theorem 2 below.

The following proposition shows that those axioms are incompatible even if all agents have the same discount factor.

Proposition. On $\mathcal{D}^E$, there exists no $\succsim$ satisfying *Weak Pareto, Limited Equity,* and *Time Consistency*.[5]

Proof. We show the result by a two-person example. Let $N = \{1, 2\}$, δ be such that $\delta_i = \frac{1}{2}$ for all $i \in N$, and α be the limitation of sacrifice in the statement of *Limited Equity*. We consider $\bar{U}^T, \bar{V}^T \in \bar{X}$ such that $v_1 < u_1 < u_2 < v_2$, $v_2 - u_2 < \alpha$, and $u_1 - v_1 = (v_2 - u_2)/2$. Then, by *Limited Equity*, we have $\bar{U}^T \succsim^{\delta}_{\hat{h}^T} \bar{V}^T$ for all $\hat{h}^T \in H$, and thus $\bar{U}^T \succsim^{\delta}_{(h^{T-1}, \boldsymbol{w}_{T-1})} \bar{V}^T$ with $(h^{T-1}, \boldsymbol{w}_{T-1}) \in H$ and $\boldsymbol{w}_{T-1} = (w_1, 0)$ where $w_1 > v_2$. *Time Consistency* implies

$$\bar{U}^T \succsim^{\delta}_{(h^{T-1}, \boldsymbol{w}_{T-1})} \bar{V}^T \Longleftrightarrow (\boldsymbol{w}_{T-1}, \bar{U}^T) \succsim^{\delta}_{h^{T-1}} (\boldsymbol{w}_{T-1}, \bar{V}^T).$$

[4] In a different setting, Zuber (2018) introduces relevant equity conditions with the restriction of redistribution.

[5] We can obtain a similar result by replacing *Limited Equity* with the following condition. (The proof is available upon request.)

Weak Pigou-Dalton Transfer. For all $\delta \in \mathcal{D}$, all $h^T \in H$, and all $\bar{U}^T, \bar{V}^T \in \bar{X}$, if there exist $j, k \in N$ such that j and k are respectively the best-off and the worst-off in both $\bar{U}^T$ and $\bar{V}^T$, $u_i = v_i$ for all $i \neq j, k$, and for all $t \in \mathbb{R}_{++}$, $[u_j = v_j - t > v_k + t = u_k]$ implies $U^T \succ^{\delta}_{h^T} V^T$.

This requires that any Pigou-Dalton transfer from the best-off to the worst off in constant distributions should be socially preferred.

Note that

$$
\begin{aligned}
W\left((w_1, \bar{U}_1^T); \frac{1}{2}\right) &= w_1 + \sum_{t=2}^{\infty} \frac{1}{2^{t-1}} u_1 = w_1 + u_1, \\
W\left((w_1, \bar{V}_1^T); \frac{1}{2}\right) &= w_1 + \sum_{t=2}^{\infty} \frac{1}{2^{t-1}} v_1 = w_1 + v_1, \\
W\left((0, \bar{U}_2^T); \frac{1}{2}\right) &= 0 + \sum_{t=2}^{\infty} \frac{1}{2^{t-1}} u_2 = u_2, \\
W\left((0, \bar{V}_2^T); \frac{1}{2}\right) &= 0 + \sum_{t=2}^{\infty} \frac{1}{2^{t-1}} v_2 = v_2.
\end{aligned}
$$

Now consider $\bar{U'}^{T-1}, \bar{V'}^{T-1}, \bar{V''}^{T-1} \in \bar{X}$ such that

$$
\boldsymbol{u}' = \left(\frac{w_1 + u_1}{2}, \frac{u_2}{2}\right), \ \boldsymbol{v}' = \left(\frac{w_1 + v_1}{2}, \frac{v_2}{2}\right), \ \boldsymbol{v}'' = \left(\frac{w_1 + v_1 - \epsilon}{2}, \frac{v_2 - \epsilon}{2}\right),
$$

where $\epsilon > 0$ is small enough that

$$
\frac{w_1 + u_1}{2} - \frac{w_1 + v_1 - \epsilon}{2} = \frac{u_1 - (v_1 + \epsilon)}{2} < \alpha.
$$

This is possible because $v_2 - u_2 < \alpha$ and $u_1 - v_1 = (v_2 - u_2)/2$. Then, *Limited Equity* implies $\bar{V''}^{T-1} \succsim^{\delta}_{h^{T-1}} \bar{U'}^{T-1}$. However, since

$$
\begin{aligned}
W\left(\bar{U'}_1^{T-1}; \frac{1}{2}\right) &= \sum_{t=1}^{\infty} \frac{1}{2^{t-1}} \frac{w_1 + u_1}{2} = w_1 + u_1 = W\left((w_1, \bar{U}_1^T); \frac{1}{2}\right), \\
W\left(\bar{V'}_1^{T-1}; \frac{1}{2}\right) &= \sum_{t=1}^{\infty} \frac{1}{2^{t-1}} \frac{w_1 + v_1}{2} = w_1 + v_1 = W\left((w_1, \bar{V}_1^T); \frac{1}{2}\right), \\
W\left(\bar{U'}_2^{T-1}; \frac{1}{2}\right) &= \sum_{t=1}^{\infty} \frac{1}{2^{t-1}} \frac{u_2}{2} = u_2 = W\left((0, \bar{U}_2^T); \frac{1}{2}\right), \\
W\left(\bar{V'}_2^{T-1}; \frac{1}{2}\right) &= \sum_{t=1}^{\infty} \frac{1}{2^{t-1}} \frac{v_2}{2} = v_2 = W\left((0, \bar{V}_2^T); \frac{1}{2}\right),
\end{aligned}
$$

we have $\bar{U'}^{T-1} \sim^{\delta}_{h^{T-1}} (\boldsymbol{w}_{T-1}, \bar{U}^T)$ and $\bar{V'}^{T-1} \sim^{\delta}_{h^{T-1}} (\boldsymbol{w}_{T-1}, \bar{V}^T)$ by *Pareto Indifference* (implied by *Weak Pareto* and continuity), and thus $\bar{U'}^{T-1} \succsim^{\delta}_{h^{T-1}} \bar{V'}^{T-1}$ by $(\boldsymbol{w}_{T-1}, \bar{U}^T) \succsim^{\delta}_{h^{T-1}} (\boldsymbol{w}_{T-1}, \bar{V}^T)$ and transitivity. *Weak Pareto* implies $\bar{V'}^{T-1} \succ^{\delta}_{h^{T-1}} \bar{V''}^{T-1}$, and we obtain $\bar{U'}^{T-1} \succ^{\delta}_{h^{T-1}} \bar{V''}^{T-1}$ by transitivity. Therefore, we have a contradiction. $\square$

This proposition establishes that under *Weak Pareto*, *Time Consistency* conflicts with equity. The intuition behind the proof is as follows. *Time Consistency* requires to respect the past information even if the distribution in the previous period is very unequal. Moreover, *Limited Equity* means redistribution independently of the past. Thus, if Al is and will be worse-off than Bill from the present period, redistribution from Bill to Al is desirable even if Al was much better-off in the previous period. Thus, taking account of this information, from *Time Consistency* and *Pareto Indifference*, we would have a situation that Al was better-off than Bill when evaluated at the previous period. Then, we have a cycle of social preferences. We must weaken either *Time Consistency* or *Limited Equity* to avoid the difficulty, as in the next section.

4 Characterizations

In this section, we examine implications of weaker versions of equity and time consistency, and characterize social welfare criteria. To state the axioms in this section, we use the following notation.

$$H^E = \{h^T = (\boldsymbol{x}_t)_{t<T} \in H | x_{it} = x_{jt} \text{ for all } i, j \in N \text{ and all } t \geq 1\}.$$

H^E is the set of histories where all agents' instantaneous utilities are equal in every $t \geq 1$.

4.1 Weakening Time Consistency

We first introduce a weaker time consistency condition. As discussed there, *Time Consistency* requires to respect the information of unequal past distributions, which would cause the inconsistencies among the axioms. Therefore, we propose another time consistency requirement that respects only equal past distributions.

Time Consistency for Equals. For all $\delta \in \mathcal{D}^E$, all $h^T \in H$, all $\boldsymbol{w} \in \mathbb{R}^N$ such that $w_i = w_j$ for all $i, j \in N$, and all $U^{T+1}, V^{T+1} \in X$,

$$(\boldsymbol{w}_T, U^{T+1}) \succsim^{\delta}_{h^T} (\boldsymbol{w}_T, V^{T+1}) \Longleftrightarrow U^{T+1} \succsim^{\delta}_{(h^T, \boldsymbol{w}_T)} V^{T+1}.$$

We also introduce an auxiliary invariance condition to obtain a social criterion.

Invariance of Discount Factors in Constant Distributions (IDFC). For all $\delta, \delta' \in \mathcal{D}$, all $h^T \in H$, and all $\bar{U}^T, \bar{V}^T \in \bar{X}$,

$$\bar{U}^T \succsim_{h^T}^{\delta} \bar{V}^T \Longleftrightarrow \bar{U}^T \succsim_{h^T}^{\delta'} \bar{V}^T.$$

When comparing constant distributions from period T, we do not have to consider discounting because agents have the same utility in every period from T. Thus, this axiom requires social preference over the constant distributions to be invariant of discount factors.

We now obtain the following characterization.

Theorem 1. If $\succsim$ satisfies *Weak Pareto, Limited Equity, Time Consistency for Equals* and *IDFC*, then, for all $\delta \in \mathcal{D}$, all $h^T \in H$, and all $U^T, V^T \in X$,

$$U \succsim_{h^T}^{\delta} V \Longleftrightarrow \min_{i \in N}(1-\delta_i) \sum_{t \geq T} \delta_i^{t-1} u_{it} \geq \min_{i \in N}(1-\delta_i) \sum_{t \geq T} \delta_i^{t-1} v_{it}.$$

Proof. First, we prove the following lemma.

Lemma 1. Suppose $\succsim$ satisfies the axioms of Theorem 1. Then, for all $\delta \in \mathcal{D}$, all $h^T \in H$, and all $\bar{U}^T, \bar{V}^T \in \bar{X}$, if there exist $i, j \in N$ such that $v_i < u_i < u_j < v_j$ and $u_k = v_k$ for all $k \neq i, j$, then $\bar{U}^T \succsim_{h^T}^{\delta} \bar{V}^T$.

Proof. Since $\bar{U}^T, \bar{V}^T \in \bar{X}$, we can apply *IDFC* and arbitrarily modify discount factors. Let $\delta \in \mathcal{D}$ be such that $\delta_i = \delta_j = \delta_0$ for all $i, j \in N$.

Given $\mathbf{0} \in \mathbb{R}^N$, by *Weak Pareto* and continuity, we can invoke *Pareto Indifference* and obtain $\bar{U}^T \sim_{h^T}^{\delta} (\mathbf{0}, \delta_0^{-1}\bar{U}^T)$ and $\bar{V}^T \sim_{h^T}^{\delta} (\mathbf{0}, \delta_0^{-1}\bar{V}^T)$. Let $\delta' \in \mathcal{D}$ be such that $\delta'_i = \delta'_j = \delta'_0$ for all $i, j \in N$, and

$$\delta'_0 \delta_0^{-1} \max_{i \in N} |u_i - v_i| < \alpha,$$

where $\alpha > 0$ is the upper limit in *Limited Transfer*. Then,

$$\begin{aligned}
\bar{U}^T \succsim^{\delta}_{h^T} \bar{V}^T &\iff (\mathbf{0}, \delta_0^{-1}\bar{U}^T) \succsim^{\delta}_{h^T} (\mathbf{0}, \delta_0^{-1}\bar{V}^T) \\
&\iff \delta_0^{-1}\bar{U}^T \succsim^{\delta}_{(h^T,\mathbf{0})} \delta_0^{-1}\bar{V}^T \text{ by } \textit{Time Consistency for Equals} \\
&\iff \delta_0^{-1}\bar{U}^T \succsim^{\delta'}_{(h^T,\mathbf{0})} \delta_0^{-1}\bar{V}^T \ \text{ by } \textit{IDFC} \\
&\iff (\mathbf{0}, \delta_0^{-1}\bar{U}^T) \succsim^{\delta'}_{h^T} (\mathbf{0}, \delta_0^{-1}\bar{V}^T) \ \text{ by } \textit{Time Consistency for Equals.}
\end{aligned}$$

By *Pareto Indifference*, $(\mathbf{0}, \delta_0^{-1}\bar{U}^T) \sim^{\delta'}_{h^T} \delta'\delta_0^{-1}\bar{U}^T$ and $(\mathbf{0}, \delta_0^{-1}\bar{V}^T) \sim^{\delta'}_{h^T} \delta'\delta_0^{-1}\bar{V}^T$. Thus, from above, we obtain

$$\bar{U}^T \succsim^{\delta}_{h^T} \bar{V}^T \iff \delta'\delta_0^{-1}\bar{U}^T \succsim^{\delta'}_{h^T} \delta'\delta_0^{-1}\bar{V}^T.$$

By the assumption on δ', $\delta'\delta_0^{-1}(v_j - u_j) < \alpha$, and hence we can apply *Limited Transfer* to obtain $\delta'\delta_0^{-1}\bar{U}^T \succsim^{\delta'}_{h^T} \delta'\delta_0^{-1}\bar{V}^T$, which implies $\bar{U}^T \succsim^{\delta}_{h^T} \bar{V}^T$ as sought. □

Now we give the proof of Theorem 1. First, we show that, for $U^T, V^T \in X$,

$$\min_{i\in N}(1-\delta_i)\sum_{t\geq T}\delta_i^{t-1}u_{it} > \min_{i\in N}(1-\delta_i)\sum_{t\geq T}\delta_i^{t-1}v_{it} \Longrightarrow U^T \succ^{\delta}_{h^T} V^T.$$

Without loss of generality, we assume that

$$(1-\delta_1)W(V_1^T;\delta_1) = \min_{i\in N}(1-\delta_i)\sum_{t\geq T}\delta_i^{t-1}v_{it}.$$

Let $\bar{V'}^T \in \bar{X}$ be such that

$$\begin{aligned}
v'_1 &= (1-\delta_1)W(V_1^T,\delta_1) + \epsilon, \\
v'_i &= \max_{j\in N}(1-\delta_j)W(V_j^T,\delta_j) + \epsilon \ \text{ for all } i \neq 1,
\end{aligned}$$

where $\epsilon > 0$ and

$$(1-\delta_1)W(V_1^T,\delta_1) + (n+2)\epsilon < \min_{i\in N}(1-\delta_i)W(U_i^T,\delta_i).$$

Then, by *Weak Pareto*, we have $\bar{V'}^T \succ^{\delta}_{h^T} V^T$.

Next, let $\bar{V''}^T \in \bar{X}$ such that $v_1'' = v_1' + (n+1)\epsilon$ and $v_i'' = v_1' + (n+2)\epsilon$ for all $i \neq 1$. By repeated applications of Lemma 2, we obtain $\bar{V''}^T \succ_{h^T}^{\delta} \bar{V'}^T$. *Weak Pareto* implies $U^T \succ_{h^T}^{\delta} \bar{V''}^T$, and transitivity implies $U^T \succ_{h^T}^{\delta} V^T$ as sought.

By the usual argument, we can apply continuity and the result above to obtain

$$\min_{i \in N}(1-\delta_i)\sum_{t \geq T} \delta_i^{t-1} u_{it} = \min_{i \in N}(1-\delta_i)\sum_{t \geq T} \delta_i^{t-1} v_{it} \Longrightarrow U^T \sim_{h^T}^{\delta} V^T.$$

The proof is straightforward, and thus can be safely omitted. $\square$

4.2 Weakening Equity

In this section, we investigate social orderings satisfying *Time Consistency* and the following equity condition weaker than *Limited Equity*.

Limited Equity with Equal Past. There exists $\alpha \in \mathbb{R}_{++}$ such that, for all $\delta \in \mathcal{D}^E$, all $h_T \in H^E$, and all $\bar{U}^T, \bar{V}^T \in \bar{X}$, if there exist $j, k \in N$ such that $v_k < u_k < u_j < v_j$, $v_j - u_j < \alpha$, and $u_i = v_i$ for all $i \neq j, k$, then $\bar{U}^T \succsim_{h^T}^{\delta} \bar{V}^T$.

This axiom argues that the limited redistribution as in *Limited Equity* should be accepted only when both the discount factors and the past distributions from the first period are equal. Thus, this axiom is compatible with *Time Consistency* under *Weak Pareto*.

We introduce two auxiliary invariance conditions to pin down a social criterion. To take histories into consideration for social evaluation, *IDFC* would be too demanding because it requires social preferences to be invariant of discount factors ignoring histories. Thus, we introduce weaker conditions.

Invariance after the Initial Period. For all $\delta, \delta' \in \mathcal{D}$, for all $\bar{U}^1, \bar{V}^1 \in \bar{X}$,

$$\bar{U}^1 \succsim_{h^1}^{\delta} \bar{V}^1 \Longleftrightarrow \bar{U}^1 \succsim_{h^1}^{\delta'} \bar{V}^1.$$

This axiom insists that the the social evaluation should be independent of discount factors only when comparing constant lifetime distributions at the first period. In this case, we do not have to worry that the society ignores some distributions from the first period. Note that the initial period is fixed at $\boldsymbol{x}_0$, the society has no choice but to take it as given.

The next axiom requires that social preferences over constant distributions should be invariant of *common* discount factors if past distributions are equal. If discount factors or past utilities are different among agents, we have to consider how the evaluations of the utilities change by different discount factors. In the case of the following invariance, we do not need to be concerned about this problem.

Invariance with Equal Past. For all $\delta, \delta' \in \mathcal{D}^E$, for all $h^T \in H^E$, for all $\bar{U}^T, \bar{V}^T \in \bar{X}$,

$$\bar{U}^T \succsim^{\delta}_{h^T} \bar{V}^T \Longleftrightarrow \bar{U}^T \succsim^{\delta'}_{h^T} \bar{V}^T.$$

We obtain the following characterization.

Theorem 2. If $\succsim$ satisfies *Weak Pareto, Limited Equity with Equal Past, Time Consistency, Invariance after the Initial Period*, and *Invariance with Equal Past*, then, for all $\delta \in \mathcal{D}$, all $h^T \in H$, and all $U^T, V^T \in X$,

$$U^T \succsim^{\delta}_{h^T} V^T \Longleftrightarrow \min_{i\in N}(1-\delta_i)\left[\sum_{t=1}^{T-1} \delta_i^{t-1} x_{it} + \sum_{t\geq T} \delta_i^{t-1} u_{it}\right] \geq \min_{i\in N}(1-\delta_i)\left[\sum_{t=1}^{T-1} \delta_i^{t-1} x_{it} + \sum_{t\geq T} \delta_i^{t-1} v_{it}\right],$$

where $h^T = (\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{T-1})$.

Proof. First, we prove the following lemma.

Lemma 2. Suppose $\succsim$ satisfies the axioms of Theorem 2. Then, for all $\delta \in \mathcal{D}$, all $h^1 \in H$, and all $\bar{U}^1, \bar{V}^1 \in \bar{X}$, if there exist $i, j \in N$ such that $v_i < u_i < u_j < v_j$ and $u_k = v_k$ for all $k \neq i, j$, then $\bar{U}^1 \succsim^{\delta}_{h^1} \bar{V}^1$.

Proof. Since $\bar{U}^1, \bar{V}^1 \in \bar{X}$, we can apply *Invariance after the Initial Period* and arbitrarily modify discount factors. Let $\delta \in \mathcal{D}$ be such that $\delta_i = \delta_j = \delta_0$ for all $i, j \in N$.

Given $\mathbf{0} \in \mathbb{R}^N$, by *Pareto Indifference* (from *Weak Pareto* and continuity), we obtain $\bar{U}^1 \sim^{\delta}_{h^1} (\mathbf{0}, \delta_0^{-1}\bar{U}^1)$ and $\bar{V}^1 \sim^{\delta}_{h^1} (\mathbf{0}, \delta_0^{-1}\bar{V}^1)$. Let $\delta' \in \mathcal{D}$ be such that $\delta'_i = \delta'_j = \delta'_0$ for all $i, j \in N$, and

$$\delta'_0 \delta_0^{-1} \max_{i\in N} |u_i - v_i| < \alpha,$$

where $\alpha > 0$ is the upper limit in *Limited Transfer with Equal Past*. Then,

$$\begin{aligned}
\bar{U}^1 \succsim^{\delta}_{h^1} \bar{V}^1 &\iff (\mathbf{0}, \delta_0^{-1}\bar{U}^1) \succsim^{\delta}_{h^1} (\mathbf{0}, \delta_0^{-1}\bar{V}^1) \\
&\iff \delta_0^{-1}\bar{U}^T \succsim^{\delta}_{(h^1,\mathbf{0})} \delta_0^{-1}\bar{V}^T \text{ (by } \textit{Time Consistency}) \\
&\iff \delta_0^{-1}\bar{U}^1 \succsim^{\delta'}_{(h^1,\mathbf{0})} \delta_0^{-1}\bar{V}^1 \text{ (by } \textit{Invariance with Equal Past}) \\
&\iff (\mathbf{0}, \delta_0^{-1}\bar{U}^1) \succsim^{\delta'}_{h^1} (\mathbf{0}, \delta_0^{-1}\bar{V}^1) \text{ (by } \textit{Time Consistency}).
\end{aligned}$$

By *Pareto Indifference*, $(\mathbf{0}, \delta_0^{-1}\bar{U}^1) \sim^{\delta'}_{h^1} \delta'\delta_0^{-1}\bar{U}^1$ and $(\mathbf{0}, \delta_0^{-1}\bar{V}^1) \sim^{\delta'}_{h^1} \delta'\delta_0^{-1}\bar{V}^1$. Thus, from above, we obtain

$$\bar{U}^1 \succsim^{\delta}_{h^1} \bar{V}^1 \iff \delta'\delta_0^{-1}\bar{U}^1 \succsim^{\delta'}_{h^1} \delta'\delta_0^{-1}\bar{V}^1.$$

By the assumption on δ', $\delta'\delta_0^{-1}(v_j - u_j) < \alpha$, and hence we can apply *Limited Transfer with Equal Past* to obtain $\delta'\delta_0^{-1}\bar{U}^1 \succsim^{\delta'}_{(h^1,\mathbf{0})} \delta'\delta_0^{-1}\bar{V}^1$, which implies $\bar{U}^1 \succsim^{\delta}_{h^1} \bar{V}^1$ as sought. $\square$

Now we give the proof of Theorem 2. First, we show that, for $U^T, V^T \in X$,

$$\min_{i\in N}(1-\delta_i)\sum_{t\geq 1}\delta_i^{t-1}u_{it} > \min_{i\in N}(1-\delta_i)\sum_{t\geq 1}\delta_i^{t-1}v_{it} \Longrightarrow U^T \succ^{\delta}_{h^T} V^T.$$

Without loss of generality, we assume that

$$(1-\delta_1)\left[\sum_{t=1}^{T-1}\delta_i^{t-1}x_{1t} + \sum_{t\geq T}\delta_i^{t-1}v_{1t}\right] = \min_{i\in N}(1-\delta_i)\left[\sum_{t=1}^{T-1}\delta_i^{t-1}x_{it} + \sum_{t\geq T}\delta_i^{t-1}v_{it}\right].$$

Let $(y_i)_{i\in N}, (z_i)_{i\in N} \in \mathbb{R}^N_+$ be such that

$$\begin{aligned}
&y_i = (1-\delta_i)\left[\sum_{t=1}^{T-1}\delta_i^{t-1}x_{it} + \sum_{t\geq T}\delta_i^{t-1}v_{it}\right] \text{ for all } i \in N, \\
&z_1 = y_1 + \epsilon,\ z_i = \max_{j\in N} y_j + \epsilon \text{ for all } i \neq 1,
\end{aligned}$$

where $\epsilon > 0$ and

$$y_1 + (n+2)\epsilon < \min_{i\in N}(1-\delta_i)\left[\sum_{t=1}^{T-1}\delta_i^{t-1}x_{it} + \sum_{t\geq T}\delta_i^{t-1}u_{it}\right].$$

Let $\bar{Y}^1, \bar{Z}^1 \in \bar{X}$ be such that, in every $t \geq 1$, each agent i has y_i and z_i, respectively.

By *Time Consistency*, we obtain

$$U^T \succsim_{h^T}^{\delta} V^T \Longleftrightarrow (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{T-1}, U^T) \succsim_{h^1}^{\delta} (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{T-1}, V^T). \tag{1}$$

Note that *Pareto Indifference* implies $(x^1, \cdots, x^{T-1}, V^T) \sim_{h^1}^{\delta} Y^1$.

First, by *Weak Pareto*, we have $\bar{Z}^1 \succ_{h^1}^{\delta} \bar{Y}^1$. Next, let $\bar{W}^1 \in \bar{X}$ be such that $w_1 = y_1 + (n+1)\epsilon$ and $w_i = y_1 + (n+2)\epsilon$ for all $i \neq 1$. By repeated applications of Lemma 2, we have $\bar{W}^1 \succsim_{h^1}^{\delta} \bar{Z}^1$, and hence transitivity implies $\bar{W}^1 \succsim_{h^1}^{\delta} \bar{Y}^1$. *Weak Pareto* implies $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{T-1}, U^T) \succ_{h^1}^{\delta} \bar{W}^1$, and by transitivity,

$$(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{T-1}, U^T) \succ_{h^T}^{\delta} \bar{Y}^1 \sim_{h^T}^{\delta} (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{T-1}, V^T).$$

By transitivity again, and by (1), we have obtained the desired result.

By the usual argument, we can apply continuity and the result above to obtain

$$\min_{i \in N}(1-\delta_i) \sum_{t \geq 1} \delta_i^{t-1} u_{it} = \min_{i \in N}(1-\delta_i) \sum_{t \geq 1} \delta_i^{t-1} v_{it} \Longrightarrow U \sim_{h^T}^{\delta} V.$$

The proof is straightforward, and thus can be safely omitted. $\square$

5 Related Literature

In this section, we discuss the related literature. First, we discuss consistency conditions. In the literature of intertemporal decision making, aside from *Time Consistency*, two conditions below are often required.

Stationarity (Koopmans, 1960). For all $\delta \in \mathcal{D}$, all $h^T \in H$, all $\boldsymbol{w} \in \mathbb{R}^N$ and all $U, V \in X$,

$$U \succsim_{h^T}^{\delta} V \Longleftrightarrow (w, U) \succsim_{h^T}^{\delta} (w, V).$$

Time Invariance. For all $\delta \in \mathcal{D}$, all $h^T, \hat{h}^T \in H$, and all $U^T, V^T \in X$,

$$U \succsim_{h^T}^{\delta} V \Longleftrightarrow U \succsim_{\hat{h}^T}^{\delta} V.$$

Stationarity says that social judgments are independent of the same first periodic distributions. *Time Invariance* requires that decision making should be independent of what happened in the past. Note that impossibility results similar to our Propositions 1 and 2 can be obtained by replacing *Time Consistency* with *Stationarity.* Halevy (2015) shows that, among *Time Consistency, Stationarity,* and *Time Invariance* for social preference, any two of these imply the remaining condition.

Zuber (2011) shows that when agents have common discount factors, social orderings satisfying the strong Pareto principle, *Time Consistency*, and *Time Invariance* should be additive. If discount factors are heterogeneous, there exists no such social ordering. In the setting where all agents have the same consumptions in every period, Jackson and Yariv (2015) show impossibility results that a "time consistent" social preference satisfying *Strong Pareto* should be dictatorial. Millner and Heal (2018) argue that the time consistency condition of Jackson and Yariv (2015) is the conjunction of *Time Invariance* and *Stationarity.*[6]

Hayashi (2016) insists that *Time Invariance* is a good property for individual decision making, but not for social decisions, especially when evaluating intertemporal distributions. Millner and Heal (2018) also argue that for dynamic social decision making in a group of living individuals, time invariance is a problematic feature for social preference, both normatively and positively, and thus time consistency is more suitable. They also claim that the three consistency conditions are conflated in the literature. In this paper, therefore, we require only time consistency conditions following these arguments.

Next, we discuss on equity. To our knowledge, equity conditions have received little attention in the context of intertemporal social decision. An exception is Bommier and Zuber (2012, Theorem 2), showing that any two time consistent and strongly Paretian social evaluation functions cannot have different degrees of inequality aversion based on

[6]In the same model, Adams et al. (2014) develop a revealed preference approach to analyze time inconsistencies of household consumption adopting a collective choice setting. Their time consistency is also a mix of *Time Invariance* and *Stationarity*, according to Millner and Heal (2018).

Hammond equity (Hammond, 1976). Another exception is Hayashi (2016), who characterizes a large class of social orderings by the strong Pareto principle, *Time Consistency*, equity as convexity of social welfare function, separability (of irrelevant agents), and and some invariance conditions. Hayashi's (2016) social ordering consistently updates weights on utilities depending on the past distributions.

6 Concluding Remarks

In this paper, we have considered social welfare ordering over distributions of lifetime utilities. We have shown inconsistencies between *Weak Pareto, Time Consistency*, and *Limited Equity* in the situation of common discount factors. Note that *Limited Equity* is weaker than Hammond's (1976) equity principle because the sacrifice of better-off is limited. To avoid the difficulty, we have introduced weaker conditions, *Time Consistency for Equals* and *Limited Equity with Equal Past.* Then, we have characterized the two maximin social ordering based on average lifetime utility. By weakening *Time Consistency*, we have obtained the maximin ordering which ignores the past distributions because of *Limited Equity.* In contrast, by weakening *Limited Equity*, we have derived another maximin which respects the past information because of *Time Consistency.*

The two social orderings are extreme in the sense that those orderings either completely ignore or respect the past information. There could be various degrees to which we can take the past into account. It remains for future research to have social orderings that have intermediate properties between the two directions. Another future research topic is to weaken Pareto principle under the condition that agents are partially not responsible for their discount factors. A resent study by Hayashi and Lombardi (2019) considers this problem, and derives a social discount factor. Since discount factor could be influenced by environments and social backgrounds, this problem would be important for intertemporal social evaluation.

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