A claims problem associated with international river management

Takayuki Oishi*

Department of Economics, Meisei University,

2-1-1, Hodokubo, Hino-city, Tokyo, 191-8506, Japan.

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Abstract

We analyze a claims problem applied to the sharing water model proposed by Ambec and Sprumont (2002, Journal of Economic Theory 107, 453-462). Unlike the Ambec and Sprumont model, the present water model describes a situation where a river flows through several states with water shortage that is derived from endowments and minimal amounts of water to save people in each state. In the water claims problem, each state has a claim to benefit derived from its usage of waters. First, we show a unique downstream incremental distribution, which is the solution in the Ambec and Sprumont model. Next, we axiomatize the family of convex combinations of the proportional and equal awards rules for water claims problems. Finally, under a situation where the family of convex combinations of these rules is employed, we give a necessary and sufficient condition under which the downstream incremental distribution is emerged as the outcome chosen by the majority voting.

Keywords: international river; claims problems; axiomatization; proportional rules; equal awards rules; majority voting

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* *E-mail address:* takayuki1q80@gmail.com, Tel: 81-42-591-5921.

1 Introduction

An international river is a transboundary watercourse that passes through at least two states. More than 260 river basins are international river basins. International rivers are managed by environmental law.[1] Under international environmental law, international river management is done by international commissions whose members are the watercourse states involved. As stated in LeMarquand (1977), an international river is a common property resource shared among the basin states, but the property rights over the waters through each basin state are not well defined. There have been ongoing conflicts over transboundary waters, e.g., the Jordan River (Israel vs. Lebanon), the Euphrates River (Turkey vs. Syria), and the Indus River (India vs. Pakistan). However, international tensions are currently decreasing, through management by international commission. For instance, competition for the waters of the Nile River between Egypt, Sudan, Ethiopia, and the Lake Victoria basin states has been replaced by cooperation through the Nile River basin commission.

As shown in the stylized fact mentioned, an international commission aims to adjust conflicts among watercourse states involved. There are two cases where each watercourse state claims its rights over the waters of an international river. In the first case, each watercourse state has a claim to quantity of water to consume. In the second case, each watercourse state has a claim to benefit derived from its usage of waters. One can consider each case by using the framework of "claims problems". Roughly speaking, claims problems deal with a situation where rights of each claimant has to be allocated among all the claimants, and there is not enough to honor the claims of all claimants (for instance, see O'Neill (1982), Aumann and Maschler (1985), Chun (1988), Thomson (2003), Moreno-Ternero (2006), and Ju, Miyagawa, and Sakai (2007)). The problem is to determine how rights should be assigned to each claimant. In the first case mentioned, Ansink and Weikard (2012) analyzed a claims problem of an international river, and characterized the class of sequential sharing rules, including the proportional rule, for claims problems for watercourse states. In Ansink and Weikard (2012), each state has an estate as the total available waters in its territory, and it has a claim to quantity of water to consume. On the other hand, in the literature the second case has not been investigated by using the framework of claims problems. This is the topic of the present study. In the literature of economic analysis of international river management, how

[1] The following environmental law play a very significant role in the management of international rivers: the Helsinki Rules on the uses of the waters of international rivers (for short, the Helsinki Rules), and the United Nations Convention on the law of the non-navigational uses of international watercourses (for short, the United Nations Convention). The Helsinki Rules are adopted by the International Law Association in 1966, and the United Nations Convention is formulated by the International Law Commission in 1997. For the details, see Birnie, Boyle, and Redgwell (2009).

to reallocate benefits among the watercourse states has been the main issue (for instance, see Ambec and Sprumont (2002), Ambec and Ehlers (2008), and van den Brink, van der Laan and Moes (2012)). The purpose of the present study is to investigate this issue by using an axiomatic approach in claims problems.

For this end, we analyze a claims problem applied to the sharing water model proposed by Ambec and Sprumont (2002). Ambec and Sprumont (2002) is the seminal work for the economic analysis of water problems. The Ambec and Sprumont model describes a situation where each state is located along an international river and has a source of water as the endowment. Each state's benefit is derived from its water consumption. Since the benefit function is strictly increasing, it has no satiation point. In this sense, all the watercourse states face with water shortage. The water model in the present study follows from the Ambec and Sprumont model. Unlike the Ambec and Sprumont model, however, the present model describes a situation where an international river has water shortage from a different view. In the present model, each state needs at minimal the amounts of water, referred to as the essential water consumption, to save people. States that are downstream of the most upstream state cannot attain their essential water consumptions if whenever they cannot utilize the endowment of the most upstream state. Such a kind of water shortage is known as a major reason for international conflicts over transboundary waters, and the 21st century is said to be "the age of water war" (see Postel (2006)).

A water claims problem is derived from the present water sharing model. Unlike Ansink and Weikard (2012), the water claims problem is formalized in the standard manner: The estate is the sum of benefits of all the states in a unique efficient water allocation, and each state has a claim against the estate. We assume that each state claims at least a "potential claim". The potential claim is related to a "downstream incremental distribution", that is introduced in the Ambec and Sprumont (2002). The downstream incremental distribution is defined as the marginal contribution vector corresponding to the ordering of the states along an international river. Ambec and Sprumont (2002) showed that only the benefit allocation satisfying lower and upper bounds on international law doctrine (i.e. absolute territorial sovereignty, and unlimited territorial integrity) is the downstream incremental distribution. The potential claim is the expected downstream incremental distribution in a situation where all the orderings of states along the river occur with the same probability. For the potential claim, we allow for anonymity on geographic position.

We consider an allocation rule that is a mapping that associates with each water claims problem an allocation. We carry out an axiomatic analysis of an allocation rule using the following properties. "Efficiency" requires that the estate should be distributed among the watercourse states. "Reallocation-proofness" requires that watercourse states should have no incentive to transfer their claims among themselves. "Anonymity" requires that the outcome chosen by a rule should depend only on the list of claims. These properties are well known in the literature of claims problems. We introduce a monotonicity property associated with the potential claim. "Monotonicity" requires that under a situation where all the states except for any two states claim the potential claim if the amount that one of the two states claims increases weakly, then the state's outcome chosen by a rule should increase weakly. Using all the properties mentioned, we axiomatize the family of convex combinations of the *proportional* and *equal awards rules*. This family is referred to as the "α-egalitarian proportional rules." Here, the share ratio $\alpha \in [0, 1]$ is the weight faced on the proportional rule, and the share ratio $1 - \alpha$ is the weight faced on the equal awards rule. The proportional and equal awards rules are the most popular rules for claims problems in practice. For other characterizations of this family of rules, for instance, see Moulin (1987) and Giménez-Gómez and Peris (2014). These axiomatizations are different from our result, and they are not associated with water sharing models. On the other hand, our result is related to an axiomatization of the family of certain allocation rules shown by Chun (1988). For the relation between the present study and Chun (1988), see Section 4.

Next, under a situation where the α-egalitarian proportional rules are employed, we investigate how the downstream incremental distribution is emerged. This investigation enables us to connect between our result and the result appearing in Ambec and Sprumont (2002). Our approach is as follows: Suppose that under a situation where the α-egalitarian proportional rules are employed all the states are making decision about where to put a share ratio α on the interval $[0, 1]$ by the majority voting. That is, each state i votes for putting a share ratio α_i on the interval $[0, 1]$; and then a share ratio $\alpha \in \{(\alpha_i)_{i \in N}\}$ is determined by the majority voting among all the states in N. On the class of water problems associated with international doctrine (i.e. absolute territorial sovereignty and unlimited territorial integrity), we give a necessary and sufficient condition under which the downstream incremental distribution is emerged as the outcome chosen by the majority voting. More precisely, on this class the majority voting always determines α such that the α-egalitarian proportional rule is the downstream incremental distribution if whenever each state claims constant scaled individual downstream incremental distribution.

It is worth comparing our study with other related papers. Since the seminal paper by Ambec and Sprumont (2002), the axiomatic literature on water problems has been growing. Under a model where each state's benefit function exhibits a satiation point, Ambec and Ehlers (2008) characterized a welfare distribution that coincides with the downstream incremental distribution. Using the assumptions of benefit functions appearing in Ambec and Ehlers, van den

Brink, van der Laan and Moes (2012) characterized the set of certain welfare distributions including the downstream incremental distribution in the case of *multiple* watercourses. Under the assumptions of concavity and continuity of benefit functions, van den Brink, Estévez-Fernández, van der Laan, and Moes (2014) characterized certain fair allocation rules by independent axioms imposed on water problems. These papers do not deal with axiomatizations of the α-egalitarian proportional rules for water problems.

The rest of this paper is organized as follows. In Section 2, we introduce a model of water problems. In Section 3, we show a unique downstream incremental distribution. In Section 4, we introduce a water claims problem derived from the sharing water model, and axiomatize the family of convex combinations of the proportional and equal awards rules. In Section 5, under a situation where the α-egalitarian proportional rules are employed we uncover how the downstream incremental distribution is emerged as the outcome chosen by majority voting. Finally, Section 6 contains some concluding remarks. In the Appendix, we show the unique existence of downstream incremental distribution, and logical independence of the axioms proposed.

2 A model of water problems

Let $\mathcal{U} \subseteq \mathbb{N}$ be a universe of agents with at least two agents.[2] We denote by $N \subseteq \mathcal{U}$ a finite non-empty subset of $\mathcal{U}$, and $n \equiv |N|$.

Imagine a line divided into n segments indexed by $i = 1, 2, \cdots, n$ with $n \geq 2$. Each segment i corresponds to state i. A watercourse flows from state 1 (i.e. the most upstream state) to state n (i.e. the most downstream state). We say that j is **downstream** of state i if $j > i$. On the other hand, we say that state j is **upstream** of state i if $j < i$. The set of states is denoted by N.

Each state $i \in N$ has a source of water as the **endowment**. We denote by e_i the quantity of water at state i's *endowment*. For each $i \in N$, let $e_i > 0$. The river picks up quantity of water along its course: The quantity of water is increased by e_i when the river flows through state i. Water is a private good. Each state $i \in N$ consumes x_i units of water. Each state i needs at minimal amounts $\bar{x}_i$ units of water to save people. The amount $\bar{x}_i$ is referred to as the **essential water consumption** of state i.

We put the following assumption on endowments. This assumption, referred to as the **endowment assumption**, says that states that are downstream of state 1 suffer from water shortage if whenever they cannot utilize

[2] We use $\subseteq$ for weak set inclusion, and $\subset$ for strict set inclusion.

waters of e_1: For each $i \in N$,

$$\sum_{k=1}^{i} \bar{x}_k < \sum_{k=1}^{i} e_k,$$

and for each $i', j' \in N \backslash \{1\}$ with $i' \leq j'$

$$\sum_{k=i'}^{j'} e_k < \sum_{k=i'}^{j'} \bar{x}_k.$$

The *endowment assumption* implies the following water structure: the source of water at state 1 is crucial for the essential water consumption of the states that are downstream of state 1. This water structure is not appeared in the Ambec and Sprumont model.

State i's benefit is derived from its water consumption. Let state i's **benefit function** be given by $\pi_i : \mathbb{R}_+ \to \mathbb{R}$. The benefit function is *strictly increasing, strictly concave,* and *differentiable at each* $x_i > 0$. Assume that its derivative $\pi_i'(x_i)$ goes to infinity as x_i tends to zero. **Extraction cost of water per unit**, denoted $\bar{\mathbf{c}}$, is *non-negative* and *constant.* Notice that in the Ambec and Sprumont model $\bar{\mathbf{c}} = 0$. For each state $i \in N$, the marginal benefit with respect to the *essential water consumption* is larger than the *marginal cost*: $\pi_i'(\bar{x}_i) > \bar{\mathbf{c}}$. This assumption is referred to as the **marginal cost assumption**.

For each pair $\{i, j\}$ such that $i, j \in N$ and $i > j$, consider that state i enjoys the essential water consumption $\bar{x}_i$ and its upstream state j does any water consumption x_j such that $x_j > \bar{x}_j$. Imagine that if the upstream state j transfers small amount of water ϵ such that $0 < \epsilon \leq x_j - \bar{x}_j$ to its downstream states, then state i can enjoy additional amount of water ϵ' such that $0 < \epsilon' \leq \epsilon$. Then we assume that $\pi_i'(\bar{x}_i + \epsilon') > \pi_j'(x_j - \epsilon)$. This assumption, referred to as the **marginal benefit assumption of water shortage**, is interpreted as follows: Each state suffering from a water shortage wants more water than its upstream states that do not suffer from a water shortage.

The marginal benefit assumption of water shortage and the marginal cost assumption are not appeared in the Ambec and Sprumont model. These assumptions play an important role for the results mentioned below (Proposition 1 and Theorem 1).

Money is available in unbounded quantity to perform side-payments. States value money and water. State i's **utility**, from consuming x_i units of water and receiving a net money transfer t_i, is given by $u_i : \mathbb{R}^2 \to \mathbb{R}$ such that $u_i(x_i, t_i) = \pi_i(x_i) - \bar{\mathbf{c}} \cdot x_i + t_i$.

We refer to $w \equiv (N, e, \bar{x}, \pi, \bar{\mathbf{c}})$, where $e = (e_1, e_2, \cdots, e_n)$, $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ and $\pi = (\pi_1, \pi_2, \cdots, \pi_n)$, as a **water problem** on $\mathcal{U}$. Let $\mathcal{W}$ be the set of all the water problems on $\mathcal{U}$.

An **allocation** is a vector $(x, t) = (x_1, \cdots, x_n, t_1, \cdots, t_n) \in \mathbb{R}^n_+ \times \mathbb{R}^n$ satisfying the *feasibility* constraints:

$$\sum_{i \in N} t_i \leq 0, \quad x_j \geq \bar{x}_j \text{ for each } j \in N,$$
$$\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} e_i \quad \text{for } j = 1, \cdots, n.$$

An allocation (x^*, t^*) is **efficient** if and only if it maximizes the sum of all states' benefits and wastes no money.

Proposition 1 *For each water problem $w \in \mathcal{W}$, there is a unique efficient water consumption.*

Proof. Consider the following problem (P):

$$(P) \quad : \quad \max_{x,t} \left(\sum_{i \in N} (\pi_i(x_i) - \bar{\mathbf{c}} \cdot x_i) + \sum_{i \in N} t_i \right)$$
$$s.t. \sum_{i \in N} t_i \leq 0, \, x_j \geq \bar{x}_j, \, \sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} e_i \quad \text{for } j = 1, \cdots, n.$$

Let $\mathcal{L}$ be the Lagrangian derived from Problem (P), namely

$$\mathcal{L} \equiv -\sum_{j \in N} \pi_j(x_j) + \bar{\mathbf{c}} \sum_{j \in N} x_j + \sum_{j \in N} \lambda_j (-x_j + \bar{x}_j) + \sum_{j \in N} \gamma_j \sum_{k=1}^{j} (x_k - e_k).$$

By the Kuhn-Tucker condition, a pair (x^*, t^*) is an optimal solution for problem (P) if and only if for each $j \in N$,

$$\lambda_j \geq 0, \, \gamma_j \geq 0, \, -x_j^* + \bar{x}_j \leq 0, \, \sum_{k=1}^{j} (x_k^* - e_k) \leq 0,$$
$$\lambda_j \left(-x_j^* + \bar{x}_j\right) = 0, \, \gamma_j \sum_{k=1}^{j} (x_k^* - e_k) = 0, \text{ and } -\pi_j'(x_j^*) + \bar{\mathbf{c}} - \lambda_j + \sum_{k=j}^{n} \gamma_k = 0.$$

We consider three steps.

Step 1 There exists $i \in N$ x^* such that $x_i^* > \bar{x}_i$.

Suppose not, that is, for each $i \in N$ $x_i^* = \bar{x}_i$. Notice that for each $j \in N$ $\sum_{k=1}^{j} (x_k^* - e_k) < 0$. This is because if $\sum_{k=1}^{j} (x_k^* - e_k) = 0$, then

$\sum_{k=j+1}^{j}(x_k^* - e_k) = 0$, which implies that $\sum_{k=j+1}^{j}\bar{x}_k = \sum_{k=j+1}^{j} e_k$, a contradiction to the endowment assumption. By this observation and the fact that for each $j \in N$ $\gamma_j \sum_{k=1}^{j}(x_k^* - e_k) = 0$, $\gamma_j = 0$. Since $-\pi_j'(x_j^*) + \bar{\mathbf{c}} - \lambda_j + \sum_{k=j}^{n}\gamma_k = -\pi_j'(x_j^*) + \bar{\mathbf{c}} - \lambda_j = 0$, $\pi_j'(x_j^*) = \bar{\mathbf{c}} - \lambda_j$. By the marginal cost assumption, $\lambda_j < 0$, a contradiction to $\lambda_j \geq 0$.

Step 2 For all $i \in N$, $x_i^* > \bar{x}_i$.

Let $\bar{\imath}$ be the smallest $i \in N$ such that $x_i^* > \bar{x}_i$. By step 1, such an $\bar{\imath}$ exists. Suppose that there exists a downstream state of $\bar{\imath}$, denoted $j > \bar{\imath}$, such that $x_j^* = \bar{x}_j$. By the marginal benefit assumption of water shortage, the value of the objective function for (P) can increase by transferring a small positive ϵ such that $0 < \epsilon \leq x_{\bar{\imath}}^* - \bar{x}_{\bar{\imath}}$ from state $\bar{\imath}$ to state j. Therefore, $x_j^* > \bar{x}_j$. By this observation, for each $j > \bar{\imath}$, $x_j^* > \bar{x}_j$. Next, we show that $\bar{\imath} = 1$. Suppose not. It suffices to consider $\bar{\imath} = 2$. Since $\gamma_2 \sum_{k=1}^{2}(x_k^* - e_k) = \gamma_2(\bar{x}_1 - e_1) + \gamma_2(x_2^* - e_2) = 0$, $e_1 > \bar{x}_1$, and $x_2^* > \bar{x}_2 > e_2$, $\gamma_2 = 0$. For each $i \geq 3$, $\gamma_i \sum_{k=1}^{i}(x_k^* - e_k) = \gamma_i(\bar{x}_1 - e_1) + \gamma_i \sum_{k=2}^{i}(x_k^* - e_k) = 0$. Suppose $\sum_{k=2}^{i}(x_k^* - e_k) = 0$. This implies that $x_1^* = e_1$. Then states that are downstream of state 1 suffer from water shortage even if they can utilize waters of e_1, a contradiction to the endowment assumption. By this observation together with the fact that $e_1 > \bar{x}_1$, for each $i \geq 3$, $\gamma_i = 0$. Then, $\pi_2'(x_2^*) = \bar{\mathbf{c}} - \lambda_2 + \sum_{k=2}^{n}\gamma_k = \bar{\mathbf{c}} - \lambda_2$. By the marginal cost assumption, $\lambda_2 < 0$, a contradiction to $\lambda_2 \geq 0$.

Step 3 There is a unique efficient water consumption x^*.

Since $\pi_j'(0) \to \infty$, for each $j \in N$ $x_j^* > 0$. For each $j \in N$, since $x_j^* > \bar{x}_j$ by Step 2, $\lambda_j = 0$ and $\pi_j'(x_j^*) = \bar{\mathbf{c}} + \sum_{k=j}^{n}\gamma_k$. For each $j \in N$, let $\alpha_j \equiv \pi_j'(x_j^*)$. Since for each $j \in N$ $\gamma_j \geq 0$, for each pair $\{i, i'\}$ such that $i, i' \in N$ and $i < i'$ $\alpha_i \geq \alpha_{i'}$. Let $i_1 \equiv \min\{i \in N : \gamma_i > 0\}$, $i_2 \equiv \min\{i \in N : i > i_1, \gamma_i > 0\}$, $\cdots$, $i_K \equiv \min\{i \in N : i > i_{K-1}, \gamma_i > 0\}$, where $i_K = n$. We have the partition of N given by $N_1 \equiv \{1, \cdots, i_1\}$, $N_2 \equiv \{i_1 + 1, \cdots, i_2\}$, $\cdots N_K \equiv \{i_{K-1} + 1, \cdots, i_K\}$. For each $i \in N_k$ $(k = 1, \cdots, K)$, let $\gamma_i \equiv \gamma_{i_k} > 0$. Since for each i, j such that $i, j \in N \backslash \{1\}$ and $i \leq j$ $\sum_{k=i}^{j} e_k < \sum_{k=i}^{j} x_k^*$, we have that $N_1 = N$, $\sum_{i \in N}(x_i^* - e_i) = 0$, $\gamma_n > 0$, and for each $i \neq n$ $\gamma_i = 0$. Since for each $i \in N$ $\pi_i'(x_i^*) = \bar{\mathbf{c}} + \gamma_n < \pi_i'(\bar{x}_i)$ and $\pi_i'(\bar{x}_i) > \bar{\mathbf{c}}$ by the marginal cost assumption, there is a positive number γ_n such that $\gamma_n < \min_{i \in N}\pi_i'(\bar{x}_i) - \bar{\mathbf{c}}$. Therefore, there is a unique solution for problem (P). ■

3 Downstream incremental distribution

Ambec and Sprumont (2002) introduced a **downstream incremental distribution** by using the notions of the **core lower bounds** and the **aspiration**

upper bounds. The core lower bound is inspired from an international law doctrine called "**Absolute Territorial Sovereignty**" (**ATS** for short). This lower bound property requires that no coalition should get less than the welfare attainable by the water the coalition controls. The *aspiration upper bound*, on the other hand, is inspired from another international law doctrine called "**Unlimited Territorial Integrity**" (**UTI** for short). This upper bound property requires that no coalition should get a welfare higher than what it can achieve in the absence of the remaining states.

Let U_i be the set of upstream states of state i, namely $U_i \equiv \{j \in N : j < i\}$ with $U_1 = \emptyset$. Let $U_i^0 \equiv U_i \cup \{i\}$. A coalition $S \subseteq N$ is *consecutive* if $k \in S$ whenever $i, j \in S$ and $i < k < j$. Let $\mathcal{P}_S$ be the unique coarsest partition *of* S into consecutive components.

For each coalition $S \subseteq N$, let $z^*(S) \in \mathbb{R}_+^S$ be a consumption plan of waters under absolute territorial sovereignty that maximizes $\sum_{i \in S} (\pi_i(z_i) - \bar{\mathbf{c}} \cdot z_i)$ subject to the constraints: (a) for each $T \in \mathcal{P}_S$ and each $j \in T$, $\sum_{i \in U_j^0 \cap T} (z_i - e_i) \leq 0$; (b) for $T \in \mathcal{P}_S$ such that $1 \in T$ and for each $i \in T$, $z_i \geq \bar{x}_i$, and for $T' \in \mathcal{P}_S$ such that $1 \notin T'$ and for each $i \in T'$, $z_i \geq 0$. Condition (a) is the water consumption feasibility of coalition S under absolute territorial sovereignty. Condition (b) says that under absolute territorial sovereignty the members of the consecutive coalition $T \in \mathcal{P}_S$ including state 1 consume at least the essential waters since they enjoy the source of water at state 1. This condition also says that water consumptions of the members of coalition S are non-negative. We can verify easily that for each water problem $w \in \mathcal{W}$ and each $S \subseteq N$, there is a unique consumption plan under absolute territorial sovereignty $z^*(S) \in \mathbb{R}_+^S$.[3]

For each coalition $S \subseteq N$, let $z^{**}(S) \in \mathbb{R}_+^S$ be a consumption plan of waters under unlimited territorial integrity that maximizes $\sum_{i \in S}(\pi_i(z_i) - \bar{\mathbf{c}} \cdot z_i)$ subject to the constraints: for each $j \in S$, (c) $\sum_{i \in U_j^0 \cap S} z_i \leq \sum_{i \in U_j^0} e_i$, and (d) $z_j \geq \bar{x}_j$. Condition (c) is the water consumption feasibility of coalition S under unlimited territorial integrity. Condition (d) says that under unlimited territorial integrity the members of coalition S consume at least the essential waters since they always enjoy the source of water at state 1. We can verify easily that for each water problem $w \in \mathcal{W}$ and each $S \subseteq N$, there is a unique consumption plan under unlimited territorial integrity $z^{**}(S) \in \mathbb{R}_+^S$.[4]

An n-dimensional vector $b = (b_1, b_2, \cdots, b_n)$ satisfies the *core lower bounds* if for each $S \subseteq N$ $\sum_{i \in S} b_i \geq \sum_{i \in S}(\pi_i(z_i^*(S)) - \bar{\mathbf{c}} \cdot z_i^*(S))$. On the other hand, an n-dimensional vector $b = (b_1, b_2, \cdots, b_n)$ satisfies the *aspiration upper bounds* if for each $S \subseteq N$ $\sum_{i \in S} b_i \leq \sum_{i \in S}(\pi_i(z_i^{**}(S)) - \bar{\mathbf{c}} \cdot z_i^{**}(S))$.

[3] For $T \in \mathcal{P}_S$ such that $1 \in T$ there exists a unique $(z_i^*(S))_{i \in T}$ since the proof is the same as that of Proposition 1. For $T' \in \mathcal{P}_S$ such that $1 \notin T'$, there exists a unique $(z_i^*(S))_{i \in T'}$ since the proof is the same as that appearing in Ambec and Sprumont (2002, pp.456-457).

[4] The proof is the same as that of Proposition 1.

Definition 1 ***(Downstream incremental distribution, Ambec and Sprumont 2002)*** *For each water problem $w \in \mathcal{W}$, a downstream incremental distribution is an n-dimensional vector satisfying the core lower bounds and the aspiration upper bounds.*

The following theorem shows a unique downstream incremental distribution.

Theorem 1 *For each water problem $w \in \mathcal{W}$, there exists a unique downstream incremental distribution $b^* \in \mathbb{R}^n_{++}$: For each $w \in \mathcal{W}$ and each $i \in N$,*

$$b_i^* = \sum_{j \in U_i^0} \left(\pi_j(z_j^*(U_i^0)) - \bar{\mathbf{c}} \cdot z_j^*(U_i^0)\right) - \sum_{j \in U_i} \left(\pi_j(z_j^*(U_i)) - \bar{\mathbf{c}} \cdot z_j^*(U_i)\right) > 0$$

or, equivalently

$$b_i^* = \sum_{j \in U_i^0} \left(\pi_j(z_j^{**}(U_i^0)) - \bar{\mathbf{c}} \cdot z_j^{**}(U_i^0)\right) - \sum_{j \in U_i} \left(\pi_j(z_j^{**}(U_i)) - \bar{\mathbf{c}} \cdot z_j^{**}(U_i)\right) > 0$$

Proof. See Appendix A. ■

A unique downstream incremental distribution is shown in the Ambec and Sprumont model. It is useful to point out that the present model and the Ambec and Sprumont model are different in the sense of the water structure. This is because in the Ambec and Sprumont model the essential water consumption of each state $i \in N$ is assumed to be $\bar{x}_i = 0$, and this assumption and the endowment assumption in the present model are incompatible. This is because for each $i, j \in N \setminus \{1\}$ with $i \leq j$ $\sum_{k=i}^{j} e_k > 0 = \sum_{k=i}^{j} \bar{x}_k$, which is a contradiction to the endowment assumption that $\sum_{k=i}^{j} e_k < \sum_{k=i}^{j} \bar{x}_k$. Therefore, Theorem 1 and Proposition 1 in the present study are not derived directly from the results in Ambec and Sprumont (2002). Also, unlike the Ambec and Sprumont model, "superadditivity" of the coalitional function $v(S) \equiv \sum_{i \in S} (\pi_i(z_i^*(S)) - \bar{\mathbf{c}} \cdot z_i^*(S))$ is not trivial.[5] As shown in the proof (see Appendix A), the marginal benefit assumption of water shortage is employed to show *superadditivity*.

4 Claims problems among watercourse states

Next, we apply the sharing water model to a claims problem (O'Neill 1982; Aumann and Maschler 1985)[6]. This claims problem is referred to as a **water**

[5] For the definition of superadditivity, see Appendix A.

[6] Claims problems deal with the situation where the liquidation value of a bankrupt firm has to be allocated among its creditors, but there is not enough to honor the claims of all

claims problem. In water claims problems, each state's claim is its benefit, not its water consumption.

Let E be the sum of benefits of all the states in a unique efficient allocation x^* for each water problem $w \in \mathcal{W}$ (as shown in Proposition 1), that is,

$$E \equiv \sum_{i \in N} \left(\pi_i(x_i^*) - \bar{\mathbf{c}} \cdot x_i^*\right).$$

Fix an arbitrary water problem $w \in \mathcal{W}$. Let E be the **estate** derived from the water problem w. Let c_i be state i's **claim** (or right) against the estate E, that is, each state $i \in N$ claims the amount c_i. For $S \subseteq N$, let $c_S \equiv \sum_{i \in S} c_i$. As shown in Theorem 1, there exists a unique downstream incremental distribution b^* for the water problem w. Let $m \equiv \frac{1}{n}\sum_N b_j^*$. We call m a **potential claim** for the water problem w. We assume that each state claims at least the potential claim m, namely for each $i \in N$ $c_i \geq m$. There is not enough to honor the claims of all states, namely $c_N \geq E$.

A justification of the lower bond is that we allow for **anonymity on geographic position á la veil of ignorance**.[7] For an illustration, imagine three states 1, 2, and 3 are the watercourse states of an international river. Let $N = \{1, 2, 3\}$. Consider that **U** is the upstream position, **M** is the midstream position, and **D** is the downstream position. Assume that at each geographic position, the endowment, the essential water consumption, and the benefit function are given. Assume that each state is located at each geographic position with an equal probability. Since all the possibilities of the tuple of states at **U**, **M**, and **D**, respectively, are (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1), it is natural to consider that each state $i \in N$ potentially claims

$$\frac{2}{6}b_U^* + \frac{2}{6}b_M^* + \frac{2}{6}b_D^* = \frac{1}{3}\sum_N b_j^*.$$

Notice that b_U^*, b_M^*, and b_D^* are the downstream incremental distribution at each geographic position. By the same argument, each state $i \in N = \{1, 2, \cdots, n\}$ potentially claims $\frac{1}{n}\sum_N b_j^*$, namely m.

For each $w \in \mathcal{W}$, a **water claims problem** is a pair $(c, E) \in \mathbb{R}^{n+1}_{++}$. Let $\mathcal{P}$ be the set of water claims problems on $\mathcal{W}$. For each water problem $w \in \mathcal{W}$, let $X(w)$ be the set of allocations: $X(w) \equiv \{x \in \mathbb{R}^N_+ : \sum_{i \in N} x_i \leq E\}$. For each water problem $w \in \mathcal{W}$, an **allocation rule** (simply, a **rule**) is a mapping, denoted φ, that associates with each water claims problem $(c, E) \in \mathcal{P}$ an allocation $x \in X(w)$.

creditors. The problem is to determine how the creditors should share the liquidation value.

[7] The notion of "veil of ignorance" was introduced in Rawls (1971).

For management of an international river, the environmental law recommends management by a commission that consists of the watercourse states of an international river. Imagine that the commission designs an allocation rule φ that satisfies several desirable properties. The list of properties are as follows.

Efficiency requires that for each water claims problem the whole value of the *estate* should be distributed among the states.

Efficiency (Eff): For each $w \in \mathcal{W}$, and each $(c, E) \in \mathcal{P}$, $\sum_{i \in N} \varphi_i(c, E) = E$.

The following property requires that the outcome chosen by a rule should depend only on the list of claims, not on who holds them.

Anonymity (AN): For each $w \in \mathcal{W}$, each $(c, E) \in \mathcal{P}$, each permutation $\sigma : N \to N$ and each $i \in N$, $\varphi_i(c, E) = \varphi_{\sigma(i)}(c_\sigma, E)$, where $c_\sigma \equiv (c_{\sigma(i)})_{i \in N}$.

There is the stylized fact that the environmental law recommends equitable management of an international river.[8] From this reason, it may be natural for the commission to design an allocation rule satisfying that its members never benefit from transferring their claims among themselves. In the literature on claims problems, this property is well known as **reallocation-proofness** (see Thomson 2003; Ju, Miyagawa, and Sakai 2007).

Reallocation-proofness (RAP): For each $w \in \mathcal{W}$, each $(c, E) \in \mathcal{P}$, and $T \subset N$ with $T \neq \emptyset$,

$$\sum_{i \in T} \varphi_i(c, E) = \sum_{i \in T} \varphi_i((c'_i)_{i \in T}, (c_i)_{i \in N \setminus T}, E),$$

where $((c'_i)_{i \in T}, (c_i)_{i \in N \setminus T}, E) \in \mathcal{P}$ such that $c_T = c'_T$.

We introduce a monotonicity property associated with the potential claim. **Monotonicity** requires that under a situation where all the states except for any two states claim the potential claim if the amount that one of the two states claims increases weakly, then the state's outcome chosen by a rule should increase weakly.

Monotonicity (Mon): For any fixed pair $\{i, j\}$ such that $i, j \in N$, each $w \in \mathcal{W}$, and each $(c, E), (c', E) \in \mathcal{P}$ such that

$$\begin{aligned} c &= (m, \cdots, m, c_i, m, \cdots, m, c_j, m \cdots, m) \text{ and} \\ c' &= (m, \cdots, m, c'_i, m, \cdots, m, c'_j, m \cdots, m), \end{aligned}$$

[8] See the Helsinki Rules, Article IV and the United Nations Convention, Article 5.

where $c'_N = c_N$ and $m \equiv \frac{1}{n}\sum_N b_j^*$, if $c'_i \geq c_i$, then

$$\varphi_i(c', E) \geq \varphi_i(c, E).$$

The **proportional rule** is the commonly used rule for claims problems in practice. For each $w \in \mathcal{W}$, each $(c, E) \in \mathcal{P}$, and each $i \in N$, it is defined by

$$PR_i(c, E) \equiv \frac{c_i}{c_N}E.$$

The **equal awards rule** is one of the most important rules for claims problems in the literature. For each $w \in \mathcal{W}$, each $(c, E) \in \mathcal{P}$, and each $i \in N$, it is defined by

$$EA_i(c, E) \equiv \frac{E}{n}.$$

We consider the family of convex combinations of the *proportional* and *equal awards rules*. We refer it to as the **α-egalitarian proportional rule**. Let $\alpha \in [0, 1]$. For each $w \in \mathcal{W}$, and each $(c, E) \in \mathcal{P}$, the α-egalitarian proportional rule, denoted φ^α, is defined by

$$\varphi^\alpha(c, E) \equiv \alpha PR(c, E) + (1 - \alpha)EA(c, E).$$

We characterize the α-egalitarian proportional rules for water claims problems as follows:

Theorem 2 *For each $w \in \mathcal{W}$ such that $n \geq 3$, and each $(c, E) \in \mathcal{P}$, a rule satisfies efficiency, anonymity, monotonicity, and reallocation-proofness if and only if there is $\alpha \in [0, 1]$ such that the rule is the α-egalitarian proportional rule.*

Proof. If there is $\alpha \in [0, 1]$ such that a rule is the α-egalitarian proportional rule φ^α, then it is clear that φ^α satisfies the four properties. We show that if a rule satisfies efficiency, anonymity, continuity, and reallocation-proofness then there is $\alpha \in [0, 1]$ such that the rule is the α-egalitarian proportional rule. Let $N = \{1, 2, \cdots, n\}$ with $n \geq 3$ and $c \equiv (c_1, c_2, \cdots, c_n)$ be given. Let $m \equiv \frac{1}{n}\sum_N b_j^*$.

Claim 1 For each $i \in N$, $\varphi_i(c, E) = \frac{c_i}{c_N}E - \frac{1}{c_N}(nc_i - c_N)\, g(c_N, E)$, where $g(c_N, E) \equiv \varphi_1(m, c_N - (n-1)m, m, \cdots, m, E)$.

Let φ be a rule satisfying the four axioms. Now let $c' \equiv (c_1 + c_2 - m, m, c_3, c_4, \cdots, c_n)$. Note that $c_1 + c_2 - m \geq m$. We have

$$\varphi_1(c, E) + \varphi_2(c, E) \overset{\textbf{RAP}}{=} \varphi_1(c', E) + \varphi_2(c', E). \tag{1}$$

Let $c'' \equiv (c_1, c_{N\backslash\{1\}} - (n-2)m, m, \cdots, m)$, where for each $N' \subseteq N$ $c_{N\backslash N'} \equiv \sum_{j \in N\backslash N'} c_j$, and $c_N \equiv \sum_{j\in N} c_j$. Note that $c_{N\backslash\{1\}} - (n-2)m \geq (n-1)m - (n-2)m = m$. Let $N' \equiv N\backslash\{1\}$. We have

$$\sum_{i\in N'} \varphi_i(c, E) \overset{\textbf{RAP}}{=} \sum_{i\in N'} \varphi_i(c'', E) \tag{2}$$

By this observation,

$$\varphi_1(c, E) \overset{\textbf{Eff}}{=} \varphi_1(c'', E). \tag{3}$$

Similarly, for each $i \in N$

$$\begin{aligned} \varphi_i(c, E) &\overset{(3),\textbf{AN}}{=} \varphi_1(c_i, c_{N\backslash\{i\}} - (n-2)m, m, \cdots, m, E), \\ \varphi_1(c', E) &\overset{(3)}{=} \varphi_1(c_1 + c_2 - m, c_{N\backslash\{1,2\}} - (n-3)m, m, \cdots, m, E), \text{ and} \\ \varphi_2(c', E) &\overset{(3),\textbf{AN}}{=} \varphi_1(m, c_N - (n-1)m, m, \cdots, m, E). \end{aligned} \tag{4}$$

Notice that $c_{N\backslash\{1,2\}} - (n-3)m \geq m$ and $c_N - (n-1)m \geq m$.

We have that

$$\begin{aligned} &\varphi_1(c_1, c_{N\backslash\{1\}} - (n-2)m, m, \cdots, m, E) \\ &+\varphi_1(c_2, c_{N\backslash\{2\}} - (n-2)m, m, \cdots, m, E) \\ &\overset{(1),(4)}{=} \varphi_1(c_1 + c_2 - m, c_{N\backslash\{1,2\}} - (n-3)m, m, \cdots, m, E) \\ &+\varphi_1(m, c_N - (n-1)m, m, \cdots, m, E). \end{aligned} \tag{5}$$

Let $N' \subset N$, and $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\begin{aligned} f(c_{N'}, c_N, E) \;\equiv\; &\varphi_1(c_{N'} - (|N'| - 1)m, c_{N\backslash N'} - (|N\backslash N'| - 1)m, m, \cdots, m, E) \\ &-\varphi_1(m, c_N - (n-1)m, m, \cdots, m, E) \end{aligned} \tag{6}$$

and

$$g(c_N, E) \equiv \varphi_1(m, c_N - (n-1)m, m, \cdots, m, E). \tag{7}$$

We have that for all c_1, c_2, and E,

$$f(c_1, c_N, E) + f(c_2, c_N, E) \overset{(5),(6)}{=} f(c_1 + c_2, c_N, E).$$

Since $n \geq 3$, f is additive with respect to its first argument for each c_N and E.

By **Mon** together with the fact that f is additive, there exists a continuous function[9] $h : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(c_i, c_N, E) = c_i h(c_N, E). \tag{8}$$

By substituting (8) to (6),

$$\begin{aligned} c_i h(c_N, E) &= \varphi_1(c_i, c_{N\setminus\{i\}} - (n-2)m, m, \cdots, m, E) - g(c_N, E) \\ &\overset{(4)}{=} \varphi_i(c, E) - g(c_N, E), \end{aligned}$$

which implies

$$\varphi_i(c, E) = c_i h(c_N, E) + g(c_N, E). \tag{9}$$

Since $c_N h(c_N, E) + n g(c_N, E) \overset{\mathbf{Eff}}{=} E$,

$$h(c_N, E) = \frac{E - n g(c_N, E)}{c_N}. \tag{10}$$

For each $i \in N$,

$$\varphi_i(c, E) \overset{(9),(10)}{=} \frac{c_i}{c_N} E - \frac{1}{c_N} (n c_i - c_N)\, g(c_N, E). \tag{11}$$

Claim 2 There is $\alpha \in [0, 1]$ such that $g(c_N, E) = (1 - \alpha)\frac{E}{n}$.

By **Mon**, we claim that for each $i \neq 2$,

$$\varphi_2(m, c_N - (n-1)m, m, \cdots, m, E) \geq \varphi_i(m, c_N - (n-1)m, m, \cdots, m, E). \tag{12}$$

Notice that $c_N - (n-1)m \geq nm - (n-1)m = m$.

By **AN**, for each pair $\{i', j'\}$ such that $i', j' \in N$ with $i', j' \neq 2$,

$$\begin{aligned} \varphi_{i'}(m, c_N - (n-1)m, m, \cdots, m, E) &= \varphi_{j'}(m, c_N - (n-1)m, m, \cdots, m, E) \\ &\equiv r. \end{aligned} \tag{13}$$

By **Eff** together with (12) and (13),

$$\varphi_2(m, c_N - (n-1)m, m, \cdots, m, E) = E - (n-1)r \geq r,$$

or equivalently, $E \geq nr$. Since $\varphi(\cdot) \geq 0$, $r \in [0, \frac{E}{n}]$, which implies that there is $\alpha \in [0, 1]$ such that

$$\varphi_1(m, c_N - (n-1)m, m, \cdots, m, E) = (1 - \alpha)\frac{E}{n} \overset{(7)}{=} g(c_N, E).$$

[9] For the details, see Aczél (1966, p 85).

Claim 3 There is $\alpha \in [0,1]$ such that $\varphi(c,E) = \alpha PR(c,E) + (1-\alpha)EA(c,E)$.

By Claims 1 and 2, for each $i \in N$, there is $\alpha \in [0,1]$ such that

$$\begin{aligned}
\varphi_i(c,E) &= \frac{c_i}{c_N}E - \frac{1}{c_N}\left(nc_i - c_N\right)(1-\alpha)\frac{E}{n} \\
&= \frac{c_i}{c_N}E - \frac{c_i}{c_N}(1-\alpha)E + (1-\alpha)\frac{E}{n} \\
&= \alpha\frac{c_i}{c_N}E + (1-\alpha)\frac{E}{n},
\end{aligned}$$

which completes the proof. ■

For logical independence of the four axioms, see Appendix B. We remark on the number of basin states, and the relation between Theorem 2 in the present study and Theorem 1 in Chun (1988).

First, in the real world, many international rivers flow through at least three states. For instance, Human Development Report (2006) by United Nations Development Programme says that 14 states share the Danube, 11 the Nile and the Niger, and 9 the Amazon. Therefore, the assumption that $n \geq 3$ appearing in Theorem 1 is appropriate for this stylized fact.

Next, Theorem 1 in Chun (1988) shows that on the domain of all claims problems where the number of claimants is at least three all solutions satisfying *efficiency*, *anonymity*, *continuity*, and *reallocation-proofness* is characterized by $\varphi_i(c,E) = \frac{c_i}{c_N}E - \frac{1}{c_N}\left(nc_i - c_N\right)g(c_N,E)$ for each $i \in N$, where $g(c_N,E) \equiv \varphi_1(0,c_N,0,\cdots,0,E)$.[10] This family is the same as in Claim 1 in the proof of Theorem 2 in the present model when $m=0$. However, the present model does not allow for $m=0$ since the potential claim m is positive. More importantly, we replace *continuity* by *monotonicity* associated with the potential claim. Furthermore, we show that the family of rules appearing in Claim 1 is the family of α-egalitarian proportional rules. This point is not shown in the proof of Theorem 1 in Chun (1988).

5 Majority voting and downstream incremental distribution

In this section, under a situation where the α-egalitarian proportional rules are employed, we give a necessary and sufficient condition under which the

[10] "Continuity" requires that a small change in the claims should not lead to a large change in the outcome chosen by a rule.

downstream incremental distribution is emerged as the outcome chosen by the majority voting.

Let the number of states be given by $n \equiv 2n'+1$, where $n' \in \mathbb{N}$ with $n' \geq 1$. For each $w \in \mathcal{W}$, each $(c, E) \in \mathcal{P}$, and each $i \in N$, the outcome chosen by the *α-egalitarian proportional rules* is given by

$$\varphi_i^\alpha(c, E) = \alpha \frac{c_i}{c_N} E + (1-\alpha)m,$$

where $m = \frac{E}{n}$.

Suppose that under a situation where the α-egalitarian proportional rules are employed all the states are making decision about where to put a share ratio α on the interval $[0, 1]$ by the majority voting. The process is as follows.

Step 1: Each state $i \in N$ votes for putting a share ratio $\alpha_i \in [0, 1]$.

Step 2: A share ratio $\alpha \in \{(\alpha_i)_{i \in N}\}$ is determined by the majority voting among all the states.

Let w^* be a water problem in which for each state $i \in N$ either (i) or (ii) holds:

(i) For a unique optimal solution $z_i^*(\{i\})$ under **ATS**, $\pi_i(z_i^*\{\{i\}) - \bar{\mathbf{c}} \cdot z_i^*(\{i\}) > m$,
(ii) For a unique optimal solution $z_i^{**}(\{i\})$ under **UTI**, $\pi_i(z_i^{**}\{\{i\}) - \bar{\mathbf{c}} \cdot z_i^{**}(\{i\}) < m$.

Let $\mathcal{W}^*$ be a non-empty class of the water problems w^* in which each state i who satisfies that for a unique optimal solution $z_i^*(\{i\})$ under **ATS** $\pi_i(z_i^*(\{i\}) - \bar{\mathbf{c}} \cdot z_i^*(\{i\}) > m$ consists of a majority. Let $\mathcal{P}^*$ be the set of claims problems derived from $\mathcal{W}^*$.

Proposition 2 *On the class of $\mathcal{W}^*$, for each $(c, E) \in \mathcal{P}^*$, suppose that by using the majority voting all the states are making decision about where to put a share ratio $\alpha \in \{(\alpha_i)_{i \in N}\}$ involved in the α-egalitarian proportional rules. The majority voting always determines α such that $\varphi^\alpha(c, E) = b^*$ if and only if for each $i \in N$ $c_i = kb_i^*$, where $k \geq \frac{m}{b_i^*}$.*

Proof. We consider three steps.

Step 1: For each state $i \in N$ either $b_i^* > m$ or $b_i^* < m$.

First, we consider one case where for a unique $z_i^*(\{i\})$, $\pi_i(z_i^*\{\{i\})-\bar{\mathbf{c}}\cdot z_i^*(\{i\})>m$. Let $v(S)\equiv\sum_S[\pi_i(z_i^*(S))-\bar{\mathbf{c}}\cdot z_i^*(S)]$. By Claim 2 in the proof of Theorem 1, $v(S)$ is superadditive. Then,

$$\begin{aligned} b_i^* &= v(U_i^0)-v(U_i)\geq v(\{i\}) \\ &= \pi_i(z_i^*(\{i\}))-\bar{\mathbf{c}}\cdot z_i^*(\{i\})>m. \end{aligned}$$

Next, we consider the other case where for a unique $z_i^{**}(\{i\})$ with $i\geq 2$, $\pi_i(z_i^{**}\{\{i\})-\bar{\mathbf{c}}\cdot z_i^{**}(\{i\})<m$. Let $w(S)\equiv\sum_S[\pi_i(z_i^{**}(S))-\bar{\mathbf{c}}\cdot z_i^{**}(S)]$. By Claim 4 in the proof of Theorem 1, for $S\subset T\subset N$ and $i>\max T$, $w(S\cup\{i\})-w(S)\geq w(T\cup\{i\})-w(T)$. Let $S=\emptyset$, and $T=\{1,2,\cdots,i-1\}$. Then,

$$\begin{aligned} m &> w(\{i\})-0\geq w(\{1,2,\cdots,i\})-w(\{1,2,\cdots,i-1\}) \\ &= w(U_i^0)-w(U_i)=b_i^*. \end{aligned}$$

Notice that on the class of $\mathcal{W}^*$ $w(\{1\})=\pi_i(z_i^{**}(\{1\}))-\bar{c}\cdot z_i^{**}(\{1\})=v(\{1\})>m$.

By the two cases, on the class of $\mathcal{W}^*$ for each state $i\in N$ either $b_i^*>m$ or $b_i^*<m$ holds.

Step 2: If for each $i\in N$ $c_i=kb_i^*$, where $k\geq\frac{m}{b_i^*}$, then the majority voting always determines α such that $\varphi^\alpha(c,E)=b^*$.

If for each $i\in N$ $c_i=kb_i^*$, where $k\geq\frac{m}{b_i^*}$, then $\frac{c_i}{c_N}E=\frac{b_i^*}{b_N^*}E=b_i^*$. Since each state i who satisfies that for a unique optimal solution $z_i^*(\{i\})$ under **ATS** $\pi_i(z_i^*(\{i\})-\bar{\mathbf{c}}\cdot z_i^*(\{i\})>m$ consists of a majority, each state i whose bliss point is $\alpha_i^*=1$ consists of the majority. Since each state j who satisfies that for a unique optimal solution $z_j^{**}(\{j\})$ under **UTI** $\pi_j(z_j^{**}(\{j\})-\bar{\mathbf{c}}\cdot z_j^*(\{j\})<m$ consists of the minority, each state j whose bliss point is $\alpha_j^*=0$ consists of a minority. By these observations together with Step 1, on the class $\mathcal{W}^*$ preferences of the states who claim $c=kb^*$ are single-peaked. Thanks to the Median Voter Theorem[11], a share ratio $\alpha\in\{(\alpha_i^*)_{i\in N}\}$ is determined as the median, namely, $\alpha=1$, which implies $\varphi^\alpha(c,E)=b^*$.

Step 3: If the majority voting always determines α such that $\varphi^\alpha(c,E)=b^*$, then for each $i\in N$ $c_i=kb_i^*$, where $k\geq\frac{m}{b_i^*}$.

Since on the class $\mathcal{W}^*$ the majority voting always determines α such that $\varphi^\alpha(c,E)=b^*$, it must hold that for each $i\in N$ $\frac{c_i}{c_N}E=b_i^*$, which implies that

[11] For the details of the Median Voter Theorem, for instance, see Austen-Smith and Banks (2000).

$\frac{c_i}{c_N} = \frac{b_i^*}{b_N^*}$. Therefore, for any constant number $l \in \mathbb{R}_{++}$ such that $l \geq \frac{b_N^*}{b_i^*} m$, $c_i = l\frac{b_i^*}{b_N^*} = kb_i^*$, where $k = \frac{l}{b_N^*}$, a desired claim. ■

6 Concluding remarks

We showed in the Ambec and Sprumont sharing water model with water shortage that is derived from endowments and essential water consumption there exists a unique downstream incremental distribution. We also axiomatized the α-egalitarian proportional rules for the water claims problem by *efficiency*, *anonymity*, *reallocation-proofness*, and *monotonicity* associated with the potential claim. Under a situation where the α-egalitarian proportional rules are employed we give a necessary and sufficient condition under which the downstream incremental distribution is emerged as the outcome chosen by the majority voting on where to put a share ratio α.

Finally, we remark on open questions. In Hougaard, Moreno-Ternero, and Østerdal (2013), claims problems in which claimants have baselines are introduced. An individual baseline can be interpreted as an objective entitlement, or a right that receives the highest priority. In the present water sharing model, the potential claim is regarded as a baseline of watercourse states. Whether or not we can generalize the results of the present study to water problems in the presence of various baselines may deserve investigation, which we leave to the future research. On the other hand, in van den Brink, He, and Huang (2017) a cost sharing problem of a polluted river in which there are multiple watercourses is investigated. Since the present study deals with only the case of a single watercourse, whether or not we can generalize the results of the present study to water problems with multiple watercourses may deserve investigation, which we also leave to the future research. For this future research, as used in van den Brink, He, and Huang (2017), an extension of our model by means of a permission structure, which is a game theoretic notion, may be useful.

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Appendix A: Proof of Theorem 1

For the proof, we have six claims.

Claim 1 If $(b_1, \cdots, b_n) \in \mathbb{R}^n$ satisfies the core lower bounds and the aspiration upper bounds, then for each $i \in N$ $b_i = b_i^*$.

Let $v(S) \equiv \sum_{i\in S} (\pi_i(z_i^*(S)) - \bar{\mathbf{c}} \cdot z_i^*(S))$, and $w(S) \equiv \sum_{i\in S} (\pi_i(z_i^{**}(S)) - \bar{\mathbf{c}} \cdot z_i^{**}(S))$. First, $v(\{1\}) = w(\{1\}) = b_1^*$. Therefore, $b_1 = b_1^*$. Next, fix j such that $j < n$. Suppose that for each $i \le j$ $b_i = b_i^*$. Since $v(U^0_{(j+1)}) = w(U^0_{(j+1)}) = \sum_{i\in U^0_{(j+1)}} b_i$, $b_{j+1} = v(U^0_{(j+1)}) - \sum_{i\in U^0_j} b_i$. By the supposition, $\sum_{i\in U^0_j} b_i = \sum_{i\in U^0_j} b_i^* = v(U_{(j+1)})$. Therefore, $b_{j+1} = v(U^0_{(j+1)}) - v(U_{(j+1)}) = b^*_{j+1}$.

Claim 2 v is "superadditive", that is, for each $S, T \subseteq N$ with $S \cap T = \emptyset$,

$$\begin{aligned} &\sum_{i\in S\cup T} (\pi_i(z_i^*(S\cup T)) - \bar{\mathbf{c}} \cdot z_i^*(S\cup T)) \\ \ge\ &\sum_{i\in S} (\pi_i(z_i^*(S)) - \bar{\mathbf{c}} \cdot z_i^*(S)) + \sum_{i\in T} (\pi_i(z_i^*(T)) - \bar{\mathbf{c}} \cdot z_i^*(T)) \,. \end{aligned}$$

Since $\sum_{i\in S\cup T} z_i^*(S\cup T) = \sum_{i\in S} z_i^*(S) + \sum_{i\in T} z_i^*(T) = \sum_{i\in S\cup T} e_i$, we show that

$$\sum_{i\in S\cup T} \pi_i(z_i^*(S\cup T)) \ge \sum_{i\in S} \pi_i(z_i^*(S)) + \sum_{i\in T} \pi_i(z_i^*(T)).$$

If $S\cup T$ is not consecutive, $\sum_{i\in S\cup T} \pi_i(z_i^*(S\cup T)) = \sum_{i\in S} \pi_i(z_i^*(S)) + \sum_{i\in T} \pi_i(z_i^*(T))$. If $S\cup T$ is consecutive and $1 \notin S\cup T$, by the the definition of z^*, $\sum_{i\in S\cup T} \pi_i(z_i^*(S\cup T)) \ge \sum_{i\in S} \pi_i(z_i^*(S)) + \sum_{i\in T} \pi_i(z_i^*(T))$. Without loss of generality, let $1 \in S$. It suffices to consider the case where S, T, and $S \cup T$ are consecutive. There is a pair of the lists of positive numbers $\{(\epsilon_i)_{i\in S}, (\epsilon'_i)_{i\in T}\}$ such that for each $i \in S$ $z_i^*(S) - \epsilon'_i > \bar{x}_i$ and $\sum_{i\in S} \epsilon'_i = \sum_{i\in T}(\bar{x}_i + \epsilon_i - z_i^*(T))$. This fact follows from the followings: Since for each $i \in T$ $\bar{x}_i > z_i^*(T)$, it suffices to show that $\sum_{i\in S}(z_i^*(S) - \bar{x}_i) > \sum_{i\in T}(\bar{x}_i - z_i^*(T))$. Suppose not, that is, for some S, T such that (i) S, T, and $S \cup T$ are consecutive, and (ii) $1 \in S$, $\sum_{i\in S}(z_i^*(S) - \bar{x}_i) \le \sum_{i\in T}(\bar{x}_i - z_i^*(T))$, which implies that $\sum_{i\in S}(e_i - \bar{x}_i) \le \sum_{i\in T}(\bar{x}_i - e_i)$. By this fact together with the assumption that $\sum_{i\in S\cup T} \bar{x}_i \le \sum_{i\in S\cup T} e_i$, $\sum_{i\in S\cup T}(e_i - \bar{x}_i) = 0$. If so, we have that $\sum_{i\in S\cup T} z_i^*(S\cup T) = \sum_{i\in S\cup T} \bar{x}_i$, a contradiction to that $\sum_{i\in S\cup T} z_i^*(S\cup T) > \sum_{i\in S\cup T} \bar{x}_i$ (by the same argument as in the proof of Proposition 1). Thus there is a pair of the lists of positive numbers $\{(\epsilon_i)_{i\in S}, (\epsilon'_i)_{i\in T}\}$ such that $\sum_{i\in S}(z_i^*(S) - \bar{x}_i) > \sum_{i\in T}(\bar{x}_i - z_i^*(T)) + \sum_{i\in T} \epsilon_i = \sum_{i\in S} \epsilon'_i$. For such a pair $\{(\epsilon_i)_{i\in S}, (\epsilon'_i)_{i\in T}\}$,

$$
\begin{aligned}
& \sum_{i \in S} \pi_i(z_i^*(S)) + \sum_{i \in T} \pi_i(z_i^*(T)) \\
\leq\ & \sum_{i \in S} \pi_i(z_i^*(S) - \epsilon_i') + \sum_{i \in T} \pi_i(\bar{x}_i + \epsilon_i) \\
\leq\ & \sum_{i \in S \cup T} \pi_i(z_i^*(S \cup T)) \text{ (by the definition of } z^*\text{)},
\end{aligned}
$$

which is a desired claim. Notice that the first inequality is derived from the marginal benefit assumption of water shortage.

Claim 3 b^* satisfies the core lower bounds.

Since v is *superadditive* by Claim 2, it suffices to show that the core lower bounds hold for *consecutive* coalitions. Let $\min S$ and $\max S$ be the smallest member of S and the largest member of S, respectively. For any consecutive S such that $1 \notin S$, $\{S, U_{\min S}\}$ is a partition of $U^0_{\max S}$. By the definitions of b^* and v, $\sum_{i \in S} b_i^* = v(U^0_{\max S}) - v(U_{\min S})$. Since v is superadditive, $v(U^0_{\max S}) - v(U_{\min S}) \geq v(S)$, which implies that b^* satisfies the core lower bounds.

Claim 4 For $S \subset T \subset N$ and $i > \max T$, $w(S \cup \{i\}) - w(S) \geq w(T \cup \{i\}) - w(T)$.

The proof of Claim 4 consists of two steps:

Step 1 If $\emptyset \neq S \subset T \subset N$, then $z^{**}(S) \geq (z_k^{**}(T))_{k \in S}$.

It suffices to show that $z^{**}(S) \geq (z_k^{**}(T))_{k \in S}$ whenever $\emptyset \neq S \neq N$ and $t \in N \backslash S$. Write $z^{**}(S) = x$ and $(z_k^{**}(S \cup \{t\}))_{k \in S} = y$. We claim $\sum_{k \in S}(y_k - x_k) \leq 0$. Suppose $\sum_{k \in S}(y_k - x_k) > 0$. By the definition of w, $\sum_{k \in S} x_k = \sum_{k \in U^0_{\max S}} e_k$, which implies $\sum_{k \in S} y_k > \sum_{k \in U^0_{\max S}} e_k$, a contradiction to the constraint $\sum_{k \in S} y_k \leq \sum_{k \in U^0_{\max S}} e_k$. Let $k_1 \leq \cdots \leq k_L$ be those $k \in S$ such that $x_k \neq y_k$ (if none exists, there is nothing to prove). We claim $y_{k_L} - x_{k_L} < 0$. Suppose, by contradiction, $y_{k_L} - x_{k_L} \geq 0$ and $x_{k_L} \neq y_{k_L}$. Let j^* be the largest member in U_{k_L} such that $y_{j^*} - x_{j^*} < 0$. (Notice that if $j^* = k_L$, $y_{k_L} - x_{k_L} < 0$, there is nothing to prove by using contradiction. If $j^* \neq k_L$, j^* necessarily exists since $\sum_{k \in S}(y_k - x_k) \leq 0$ and $y_{k_L} - x_{k_L} > 0$.) Let $y^\epsilon \in \mathbb{R}^S_+$ such that $y^\epsilon_{k_L} \equiv y_{k_L} - \epsilon$, $y^\epsilon_{j^*} \equiv y_{j^*} + \epsilon$, and $y^\epsilon_k \equiv y_k$ for $k \neq k_L, j^*$. Since $y_{j^*} < x_{j^*}$, $x_{k_L} < y_{k_L}$ and $\pi'_{j^*}(x_{j^*}) = \pi'_{k_L}(x_{k_L})$ (by the argument in the proof of Proposition 1), $\pi'_{j^*}(y_{j^*}) > \pi'_{j^*}(x_{j^*}) = \pi'_{k_L}(x_{k_L}) > \pi'_{k_L}(y_{k_L})$. Using this observation and the strict concavity of benefit functions, we can choose $\epsilon > 0$ small enough so that

$$\begin{aligned}
&\sum_{k\in S}[(\pi_k(y_k^\epsilon) - \bar{\mathbf{c}}\cdot y_k^\epsilon) - (\pi_k(y_k) - \bar{\mathbf{c}}\cdot y_k)] \\
&= [\pi_{j^*}(y_{j^*}^\epsilon) - \bar{\mathbf{c}}\cdot y_{j^*}^\epsilon - (\pi_{j^*}(y_{j^*}) - \bar{\mathbf{c}}\cdot y_{j^*})] + [\pi_{k_L}(y_{k_L}^\epsilon) - \bar{\mathbf{c}}\cdot y_{k_L}^\epsilon - (\pi_{k_L}(y_{k_L}) - \bar{\mathbf{c}}\cdot y_{k_L})] \\
&= [\pi_{j^*}(y_{j^*}^\epsilon) - \pi_{j^*}(y_{j^*})] + [\pi_{k_L}(y_{k_L}^\epsilon) - \pi_{k_L}(y_{k_L})] - \bar{\mathbf{c}}\cdot(y_{j^*}^\epsilon - y_{j^*}) - \bar{\mathbf{c}}\cdot(y_{k_L}^\epsilon - y_{k_L}) \\
&= [\pi_{j^*}(y_{j^*}^\epsilon) - \pi_{j^*}(y_{j^*})] + [\pi_{k_L}(y_{k_L}^\epsilon) - \pi_{k_L}(y_{k_L})] \\
&> 0,
\end{aligned}$$

while y^ϵ meets the same constraints as y. Notice that the inequality is derived from $y_{j^*}^\epsilon > y_{j^*}$, $y_{k_L}^\epsilon < y_{k_L}$, $\pi'_{j^*}(y_{j^*}) > \pi'_{k_L}(y_{k_L})$, and strict concavity of benefit functions. Thus, we have a contradiction to the optimal solution y. Because $y_{k_L} - x_{k_L} < 0$, it follows that $y_{k_l} - x_{k_l} < 0$ successively for $l = L-1, \cdots, 1$.

Step 2 For $S \subset T \subset N$ and $i > \max T$, $w(S\cup\{i\}) - w(S) \geq w(T\cup\{i\}) - w(T)$.

Let $S \subset T \subset N$, and $i > \max T$. Let $z' \in \mathbb{R}_+^{S\cup\{i\}}$ such as $z'_i = z_i^{**}(T\cup\{i\})$, and for each $j \in S$ $z'_j = z_j^{**}(T\cup\{i\}) + z_j^{**}(S) - z_j^{**}(T)$. By Step 1, for each $j \in S$ $z_j^{**}(T\cup\{i\}) \leq z_j^{**}(T) \leq z_j^{**}(S)$. Therefore, for each $j \in S$ $0 \leq z_j^{**}(T\cup\{i\}) \leq z'_j \leq z_j^{**}(S)$. Since for each $j \in S$ $z'_j \leq z_j^{**}(S)$, state j's consumption plan z'_j for $S\cup\{i\}$ satisfies the same constraints as $z_j^{**}(S)$. By the definition of z^{**}, for each $j \in S$ $z_j^{**}(S)$ satisfies the same constraints as $z_j^{**}(S\cup\{i\})$. Again by the definition of z^{**}, $z_i^{**}(S)\cup\{i\}$ satisfies the same constraints as $z_i^{**}(T\cup\{i\})$. Therefore, the consumption plan z' for $S\cup\{i\}$ satisfies the same constraints as $z^{**}(S\cup\{i\})$, namely, for each $l \in S\cup\{i\}$ $\sum_{k\in U_l^0\cap(S\cup\{i\})} z'_k \leq \sum_{k\in U_l^0} e_k$. By this observation together with the definition of w, $w(S\cup\{i\}) \geq \sum_{j\in S\cup\{i\}}\left(\pi_j(z'_j) - \bar{\mathbf{c}}\cdot z'_j\right)$, which implies that

$$\begin{aligned}
w(S\cup\{i\}) - w(S) &\geq \sum_{l\in S\cup\{i\}}(\pi_l(z'_l) - \bar{\mathbf{c}}\cdot z'_l) - \sum_{j\in S}\left(\pi_j(z_j^{**}(S)) - \bar{\mathbf{c}}\cdot z_j^{**}(S)\right) \\
&= \pi_i(z'_i) - \bar{\mathbf{c}}\cdot z'_i + \sum_{j\in S}\left[\left(\pi_j(z'_j) - \pi_i(z_j^{**}(S))\right) - \bar{\mathbf{c}}\cdot\left(z'_j - z_j^{**}(S)\right)\right].
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& w(T\cup\{i\}) - w(T) \\
&= \sum_{l\in S\cup\{i\}} (\pi_l(z_l^{**}(T\cup\{i\})) - \bar{\mathbf{c}}\cdot z_l^{**}(T\cup\{i\})) + \sum_{l\in T\setminus S} (\pi_l(z_l^{**}(T\cup\{i\})) - \bar{\mathbf{c}}\cdot z_l^{**}(T\cup\{i\})) \\
&- \sum_{j\in S} \left(\pi_j(z_j^{**}(T)) - \bar{\mathbf{c}}\cdot z_j^{**}(T)\right) - \sum_{l\in T\setminus S} (\pi_l(z_l^{**}(T)) - \bar{\mathbf{c}}\cdot z_l^{**}(T)) \\
&= \pi_i(z_i') - \bar{\mathbf{c}}\cdot z_i' + \sum_{j\in S} \left[\left(\pi_j(z_j^{**}(T\cup\{i\})) - \pi_j(z_j^{**}(T))\right) - \bar{\mathbf{c}}\cdot\left(z_j^{**}(T\cup\{i\}) - z_j^{**}(T)\right)\right] \\
&+ \sum_{l\in T\setminus S} [(\pi_l(z_l^{**}(T\cup\{i\})) - \pi_l(z_l^{**}(T))) - \bar{\mathbf{c}}\cdot(z_l^{**}(T\cup\{i\}) - z_l^{**}(T))] \\
&\le \pi_i(z_i') - \bar{\mathbf{c}}\cdot z_i' + \sum_{j\in S} \left[\left(\pi_j(z_j^{**}(T\cup\{i\})) - \pi_j(z_j^{**}(T))\right) - \bar{\mathbf{c}}\cdot\left(z_j^{**}(T\cup\{i\}) - z_j^{**}(T)\right)\right],
\end{aligned}
$$

where the inequality follows from the fact that for each $l \in T\setminus S$,

$$(\pi_l(z_l^{**}(T)) - \pi_l(z_l^{**}(T\cup\{i\}))) - \bar{\mathbf{c}}\cdot(z_l^{**}(T) - z_l^{**}(T\cup\{i\})) \geq 0,$$

since $z_l^{**}(T\cup\{i\}) \le z_l^{**}(T)$ (by Step 1) and $\pi_l'(z_l^{**}(T)) \geq \bar{\mathbf{c}}$ (by the same argument as in the proof of Proposition 1), and benefit function π_l is strictly concave. Since for each $j \in S$ benefit function π_j is strictly concave, $z_j^{**}(T\cup\{i\}) \le z_j' \le z_j^{**}(S)$, $z_j' - z_j^{**}(S) = z_j^{**}(T\cup\{i\}) - z_j^{**}(T)$ (by the definition of z'), and $\pi_j'(z_j^{**}(T\cup\{i\})) \geq \pi_j'(z_j') \geq \bar{\mathbf{c}}$ (by the fact that $\pi_j'(z_j^{**}(S)) \geq \bar{\mathbf{c}}$, and continuity and strict concavity of π_j),

$$
\begin{aligned}
& \left(\pi_j(z_j') - \pi_i(z_j^{**}(S))\right) - \bar{\mathbf{c}}\cdot\left(z_j' - z_j^{**}(S)\right) \\
& \geq \left(\pi_j(z_j^{**}(T\cup\{i\})) - \pi_i(z_j^{**}(T))\right) - \bar{\mathbf{c}}\cdot\left(z_j^{**}(T\cup\{i\}) - z_j^{**}(T)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& w(S\cup\{i\}) - w(S) \\
\geq\ & \pi_i(z_i') - \bar{\mathbf{c}}\cdot z_i' + \sum_{j\in S} \left[\left(\pi_j(z_j') - \pi_i(z_j^{**}(S))\right) - \bar{\mathbf{c}}\cdot\left(z_j' - z_j^{**}(S)\right)\right] \\
\geq\ & \pi_i(z_i') - \bar{\mathbf{c}}\cdot z_i' + \sum_{j\in S} \left[\left(\pi_j(z_j^{**}(T\cup\{i\})) - \pi_i(z_j^{**}(T))\right) - \bar{\mathbf{c}}\cdot\left(z_j^{**}(T\cup\{i\}) - z_j^{**}(T)\right)\right] \\
\geq\ & w(T\cup\{i\}) - w(T),
\end{aligned}
$$

which completes the proof of the claim.

Claim 5 b^* satisfies the aspiration upper bounds.

By the definition of b^* and Claim 4, for each $S \subseteq N$,

$$\sum_{i \in S} b_i^* = \sum_{i \in S} [w(U_i^0) - w(U_i)] \leq \sum_{i \in S} [w(U_i^0 \cap S) - w(U_i \cap S)] = w(S),$$

where the inequality is derived from the fact that for each $i \in S$ $(U_i \cap S) \subset S$ together with Claim 4, and the last equality is derived from the fact that for each $i \in S$ $U_i \cap S = U^0_{(\max(U_i \cap S))} \cap S$, so that all terms except for $w(U^0_{(\max S)} \cap S)$ and $w(U_{(\min S)} \cap S)$ cancel out, and $w(U^0_{(\max S)} \cap S) = w(S)$ and $w(U_{(\min S)} \cap S) = w(\emptyset) = 0$. Therefore, z^* satisfies the aspiration upper bounds.

Claim 6 For each $i \in N$, $b_i^* > 0$.

By the definition of b_1^*, $b_1^* > 0$. Again, by the definition of b_i^* and the superadditivity of v, for each $i \in N \backslash \{1\}$,

$$b_i^* = v(U_i^0) - v(U_i) \geq v(\{i\}) > 0,$$

where the last inequality is derived from the fact that $z^*(\{i\}) > 0$. ■

Appendix B: Logical independence

For logical independence of the four axioms, we consider the following four rules.

- For each $w \in \mathcal{W}$ such that $n \geq 3$, each $(c, E) \in \mathcal{P}$, and each $i \in N$, let $\varphi_i^1(c, E) = \frac{E}{n+1}$. The mapping φ^1 satisfies all the axioms except for *efficiency*.

- For each $w \in \mathcal{W}$ such that $n \geq 3$, and each $(c, E) \in \mathcal{P}$, let $\varphi^2(c, E) = b^*$, where b^* is a (unique) downstream incremental distribution. The mapping φ^2 satisfies all the axioms except for *anonymity*.

- For each $w \in \mathcal{W}$ such that $n \geq 3$, each $(c, E) \in \mathcal{P}$, and any pair $\{i, j\}$ such that $i, j \in N$, let $\varphi^3(c, E) = EA(c, E)$ if $c_i = 2m + \epsilon$, $c_j = 2m - \epsilon$, where $0 < \epsilon \leq m$, and $c_k = m$ for all $k \neq i, j$; otherwise $\varphi^3(c, E) = PR(c, E)$. The mapping φ^3 satisfies all the axioms except for *monotonicity*. In fact, for $(c, E) \in \mathcal{P}$ such that $c_i = c_j = 2m$, and $c_k = m$ for all $k \neq i, j$, $\varphi_i^3(c, E) = PR_i(c, E) = \frac{2E}{n+2}$. On the other hand, for $(c', E) \in \mathcal{P}$ such that $c'_i = 2m + \epsilon$, $c'_j = 2m - \epsilon$, where $0 < \epsilon \leq m$, and $c'_k = m$ for all $k \neq i, j$ $\varphi_i^3(c', E) = EA_i(c', E) = \frac{E}{n}$. Notice that $c_N = c'_N$. Since $n \geq 3$, $\frac{2E}{n+2} - \frac{E}{n} = \frac{(n-2)E}{n(n+2)} > 0$, which implies that if $c'_i > c_i$, then $\varphi_i^3(c', E) < \varphi_i^3(c, E)$.

- For each $w \in \mathcal{W}$ such that $n \geq 3$, and each $(c, E) \in \mathcal{P}$, let $\varphi^4(c, E)$ be given by the **Talmud rule** (Aumann and Maschler 1985), denoted T, that is, for each $i \in N$ (1) if $\sum(c_i/2) \geq E$, then $T_i(c, E) \equiv \min\{c_i/2, \lambda\}$, where λ is chosen so that $\sum_N \min\{c_i/2, \lambda\} = E$; (2) if $\sum(c_i/2) \leq E$, then $T_i(c, E) \equiv c_i - \min\{c_i/2, \lambda\}$, where λ is chosen so that $\sum_N [c_i - \min\{c_i/2, \lambda\}] = E$. The mapping φ^4 satisfies all the axioms except for *reallocation-proofness*. In fact, for $(c, E) \in \mathcal{P}$ such that $c = (100, 200, 300)$ and $E = 200$, $T(c, E) = (50, 75, 75)$. On the other hand, for $(c, E) \in \mathcal{P}$ such that $c' = (150, 150, 300)$ and $E = 200$, $T(c', E) = (200/3, 200/3, 200/3)$. Therefore, $\sum_{i \in \{1,2\}} T(c, E) \neq \sum_{i \in \{1,2\}} T(c', E)$.

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