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A BINOMIAL ASSET PRICING MODEL IN A CATEGORICAL SETTING

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Abstract. Adachi and Ryu introduced a category **Prob** of probability spaces whose objects are all probability spaces and whose arrows correspond to measurable functions satisfying an absolutely continuous requirement in [Adachi and Ryu, 2019]. In this paper, we develop a binomial asset pricing model based on **Prob**. We introduce generalized filtrations with which we can represent situations such as some agents forget information at some specific time. We investigate the valuations of financial claims along this type of non-standard filtrations.

1. Introduction

Adachi and Ryu introduced the category **Prob** as an *adequate* candidate of the category of probability spaces with *good* arrows. They show the existence of the conditional expectation functor from **Prob** to **Set**, which is a natural generalization of the classical notion of conditional expectation ([Adachi and Ryu, 2019]).,

In this paper, we develop a binomial asset pricing model based on the category **Prob**. Generalized filtrations defined in this setting change not only σ-algebras but also probability measures and even underlying sets throughout time. We introduce a few types of generalized filtrations. Each of them represents a subjective filtration of an agent. In other words, each agent has not only her subjective probability measure but also her own subjective filtration. For example, some filtration represents the situation in which she forgets the information generated at a specific time. This paper investigate the valuations of financial claims along these non-standard filtrations.

First, in Section 2, we review the concept of categorical probability theory and introduce generalized filtrations and adapted processes and martingales along them. In this setting, our probability spaces are

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changing as time goes on. For example, we may have a bigger underlying set in future than that in past. This case allows us to have unknown future elementary events. Section 3 is the heart of this paper in which we develop a concrete binomial asset pricing model and investigate a few generalized filtrations and possibility of valuations along them.

2. Generalized Filtrations

In this section, we introduce some basic concepts of categorical probability theory which was mainly introduced in [Adachi and Ryu, 2019] as a preparation for Section 3.

Let $\bar{X} = (X, \Sigma_X, \mathbb{P}_X)$, $\bar{Y} = (Y, \Sigma_Y, \mathbb{P}_Y)$ and $\bar{Z} = (Z, \Sigma_Z, \mathbb{P}_Z)$ be probability spaces throughout this paper.

Definition 2.1. [Null-preserving functions [Adachi and Ryu, 2019]] A measurable function $f : \bar{Y} \to \bar{X}$ is called ***null-preserving*** if $f^{-1}(A) \in \mathcal{N}_Y$ for every $A \in \mathcal{N}_X$, where $\mathcal{N}_X := \mathbb{P}_X^{-1}(0) \subset \Sigma_X$ and $\mathcal{N}_Y := \mathbb{P}_Y^{-1}(0) \subset \Sigma_Y$.

Definition 2.2. [Category **Prob** [Adachi and Ryu, 2019]] A category **Prob** is the category whose objects are all probability spaces and the set of arrows between them are defined by

$$\mathbf{Prob}(\bar{X},\bar{Y}) := \{f^- \mid f : \bar{Y} \to \bar{X} \text{ is a null-preserving function.}\},$$

where f^- is a symbol corresponding uniquely to a function f.

We write Id_X for an identity *measurable* function from $\bar{X}$ to $\bar{X}$, while writing id_X for an identity function from X to X. Therefore, the identity arrow of a **Prob**-object $\bar{X}$ is Id_X^-.

Definition 2.3. [Generalized Filtrations] Let $\mathcal{T}$ be a fixed small category which we sometimes call the ***time domain***. A ***$\mathcal{T}$-filtration*** is a functor $F : \mathcal{T} \to \mathbf{Prob}$.

$$\begin{tikzcd}
\mathcal{T} \arrow[d, "F"'] & t_0 \arrow[r, "i_0"'] \arrow[rr, bend left, "i_1 \circ i_0"] & t_1 \arrow[r, "i_1"'] & t_2 \arrow[r, "i_2"'] & \cdots \\
\mathbf{Prob} & Ft_0 \arrow[r, "Fi_0"] \arrow[rr, bend right, "F(i_1 \circ i_0) = Fi_1 \circ Fi_0"'] & Ft_1 \arrow[r, "Fi_1"] & Ft_2 \arrow[r, "Fi_2"] & \cdots
\end{tikzcd}$$

Figure 2.1. $\mathcal{T}$-filtration

When we say filtrations in the classical setting, we keep using a same underlying set Ω throughout time. This situation can be represented by the following diagram.

$$\begin{tikzcd}
\mathcal{T} & t_0 \arrow[r] & t_1 \arrow[r] & t_2 \arrow[r] & \dots \\
& \mathcal{F}_{t_0} \arrow[r, "Id_\Omega^{-1}"] & \mathcal{F}_{t_1} \arrow[r, "Id_\Omega^{-1}"] & \mathcal{F}_{t_2} \arrow[r, "Id_\Omega^{-1}"] & \dots \\
& \Omega & \Omega \arrow[l, "Id_\Omega"'] & \Omega \arrow[l, "Id_\Omega"'] & \dots \arrow[l, "Id_\Omega"']
\end{tikzcd}$$

However, in our new setting, the filtration can change not only σ-fields but also probability measures and underlying sets as the following diagram shows.

$$\begin{tikzcd}
\mathcal{T} \arrow[d, "F"'] & t_0 \arrow[r] & t_1 \arrow[r] & t_2 \arrow[r] & \dots \\
\mathbf{Prob} & \bar{X}_{t_0} \arrow[r, "f_0^-"] & \bar{X}_{t_1} \arrow[r, "f_1^-"] & \bar{X}_{t_2} \arrow[r, "f_1^-"] & \dots \\
& X_{t_0} & X_{t_1} \arrow[l, "f_0"'] & X_{t_2} \arrow[l, "f_1"'] & \dots \arrow[l, "f_2"']
\end{tikzcd}$$

One of the implications of this generalization is that we can think possibly distorted filtrations by using adequate null-preserving function f_t.

Actually, the biggest aim of this paper is to investigate this kind of non-standard filtrations by using, as a first example, a simple binomial asset pricing model.

Before going into our concrete example, we will define adapted processes and martingales over this generalized filtrations.

Let F be a fixed $\mathcal{T}$-filtration throughout this section.

Definition 2.4. [F-Adapted Processes] ***An F-adapted process*** is a collection of natural transformations

$$\tau := \{\tau_s : \mathcal{T}(s, -) \dot{\rightarrow} L \circ F\}_{s \in Obj(\mathcal{T})} \tag{2.1}$$

For a **Prob**-arrow $\varphi : \bar{X} \to \bar{Y}$, there exists a measurable function $f : Y \to X$ such that $\varphi = f^-$ by its definition. We write φ^+ for this f. That is, $(\varphi^+)^- = \varphi$.

Now Let τ be an F-adapted process and $i : s \to t$ be a $\mathcal{T}$-arrow. Then, we have the following commutative diagram.

$$\begin{tikzcd}
 & Id_s \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \tau_{s,s}(Id_s) \arrow[ddd, mapsto] \\
s \arrow[d, "i"'] & & \mathcal{T}(s,s) \arrow[r, "\tau_{s,s}"] \arrow[d, "\mathcal{T}(s,i)"'] & L(Fs) \arrow[d, "L(Fi)"] & \\
t & & \mathcal{T}(s,t) \arrow[r, "\tau_{s,t}"'] & L(Ft) & \\
 & i \arrow[rr, mapsto] & & \tau_{s,t}(i) \; = & L(Fi)(\tau_{s,s}(Id_s))
\end{tikzcd}$$

For $s \in Obj(\mathcal{T})$ pick a random variable v_s satisfying $[v_s]_{\sim_{\mathbb{P}_{Fs}}} = \tau_{s,s}(Id_s)$. Then, we have

$$\tau_{s,t}(i) = [v_s \circ (Fi)^+]_{\sim_{\mathbb{P}_{Ft}}}. \tag{2.2}$$

That is, $\tau_{s,t}(i)$ is (Fi)-measurable.

Proposition 2.5. *Let $AP(F)$ be the set of all F-adapted processes. Then,*

$$AP(F) \cong \prod_{t \in Obj(\mathcal{T})} L(Ft). \tag{2.3}$$

Proof. By Yoneda Lemma, we have for $t \in Obj(\mathcal{T})$,

$$y_t : \mathrm{Nat}(\mathcal{T}(t, -), L \circ F) \cong (L \circ F)t. \tag{2.4}$$

Then, $\prod_{t \in Obj(\mathcal{T})} y_t$ is an isomorphism denoting (2.3).

□

For $x \in AP(F)$, we sometimes write

$$x = \{x_t\}_{t \in Obj(\mathcal{T})} \tag{2.5}$$

where

$$x_t := x(t) \in L(Ft). \tag{2.6}$$

Remark 2.6. For an arrow $i : s \to t$ in $\mathcal{T}$, in general, Fs and Ft are different probability spaces. So we cannot (for example) add two random variables $x_s \in L^1(Fs)$ and $x_t \in L^1(Ft)$ whose domains are $\tilde{F}s$

and $\tilde{F}t$.

$$\begin{tikzcd}
s \arrow[r, "i"] & t \\
Fs \arrow[r, "Fi"] & Ft \\
\tilde{F}s \arrow[d, "x_s"'] & \tilde{F}t \arrow[l, "(Fi)^+"'] \arrow[d, shift right, "x_t"'] \arrow[d, shift left, "x_s \circ (Fi)^+"] \\
\mathbb{R} & \mathbb{R}
\end{tikzcd}$$

In order to import x_s into $L^1(Ft)$, we take $x_s \circ (Fi)^+$ as its proxy. This fact allows us to treat $L^1(Ft)$ as a vector space containing all preceding random variables $x_s \in L^1(Fs)$ with $s \leq t$.

Next, we go into the definition of martingales. In order to make it possible, we need a concept of conditional expectations in the category **Prob** which was introduced in [Adachi and Ryu, 2019].

Theorem 2.7. [Conditional Expectation [Adachi and Ryu, 2019]] *Let $f^- : \bar{X} \to \bar{Y}$ be a* **Prob**-*arrow. For all $v \in \mathcal{L}^1(\bar{Y})$ and $A \in \Sigma_X$, there exists $u \in \mathcal{L}^1(\bar{X})$ satisfying the following equation.*

$$\int_A u \, d\mathbb{P}_X = \int_{f^{-1}(A)} v \, d\mathbb{P}_Y . \tag{2.7}$$

*We call u a **conditional expectation** along f^- and denote it by $E^{f^-}(v)$.*

Theorem 2.8. [Conditional Expectation Functor [Adachi and Ryu, 2019]] *There exists a functor $\mathcal{E} : \mathbf{Prob}^{op} \to \mathbf{Set}$ as following:*

$$\begin{tikzcd}
X & \bar{X} \arrow[d, "f^-"'] \arrow[r, mapsto, "\mathcal{E}"] & \mathcal{E}\bar{X} & := & L^1(\bar{X}) & \ni & {[E^{f^-}(v)]_{\sim_{\mathbb{P}_X}}} \\
Y \arrow[u, "f"] & \bar{Y} \arrow[r, mapsto, "\mathcal{E}"] & \mathcal{E}\bar{Y} \arrow[u, "\mathcal{E}f^-"'] & := & L^1(\bar{Y}) & \ni & {[v]_{\sim_{\mathbb{P}_Y}}} . \arrow[u, mapsto, "\mathcal{E}f^-"']
\end{tikzcd}$$

*We call $\mathcal{E}$ a **conditional expectation functor**.*

Definition 2.9. [F-Martingales] Let $F : \mathcal{T} \to$ **Prob** be a functor. An F***-martingale*** is an F-adapted process $x \in AP(F)$ such that for every $\mathcal{T}$-arrow $i : s \to t$,

$$(\mathcal{E} \circ F)i(x(t)) = x(s). \tag{2.8}$$

3. A Binomial Asset Pricing Model

In this section, we introduce a binomial asset pricing model based on the category **Prob**. First, we define a general scheme of our model by introducing a filtration $\mathcal{B}$.

$$\begin{tikzcd}
s \arrow[r, mapsto, "F"] \arrow[d, "i"'] & Fs \arrow[r, mapsto, "\mathcal{E}"] \arrow[d, "Fi"'] & \mathcal{E}(Fs) & := & L^1(Fs) & \ni & x_s = [E^{Fi}(v)]_{\sim_{\mathbb{P}_{Fs}}} \\
t \arrow[r, mapsto, "F"] & Ft \arrow[r, mapsto, "\mathcal{E}"] & \mathcal{E}(Ft) \arrow[u, "\mathcal{E}(Fi)"'] & := & L^1(Ft) & \ni & x_t = [v]_{\sim_{\mathbb{P}_{Ft}}} . \arrow[u, mapsto, "\mathcal{E}(Fi)"']
\end{tikzcd}$$

FIGURE 2.2. F-martingale

Definition 3.1. [Filtration $\mathcal{B}$] Let ω be the category whose objects are all integers starting with 0 and for each pair of integers m and n with $m \leq n$ there is a unique arrow $*_{m,n} : m \to n$. That is, ω is the category corresponding to the integer set $\mathbb{N}$ with the usual total order. Let $\mathbf{p} := \{p_i\}_{i=1,2,\ldots}$ be an infinite sequence of real numbers $p_i \in [0, 1]$. We define an ω-filtration $\mathcal{B} := \mathcal{B}^{\mathbf{p}} : \omega \to \mathbf{Prob}$ in the following way.

For an object n of ω, $\mathcal{B}n$ is a probability space $\bar{B}_n := (B_n, \Sigma_n, \mathbb{P}_n)$ whose components are defined as follows:

(1) $B_n := \{0, 1\}^n$, the set of all binary numbers of t digits,
(2) $\Sigma_n := 2^{B_n}$,
(3) for $a := d_1 d_2 \ldots d_n \in B_n$ where $d_i \in \{0, 1\}$ $(i = 1, 2, \ldots n)$. $\mathbb{P}_n : \Sigma_n \to [0, 1]$ is the probability measure defined by

$$\mathbb{P}_n(\{a\}) := \prod_{i=1}^{n} p_i^{d_i}(1 - p_i)^{1-d_i}. \tag{3.1}$$

For integers m and n with $m < n$, we define

$$\mathcal{B}(*_{m,n}) := f^-_{m,n} := (f_m \circ f_{m+1} \circ \cdots \circ f_{n-1})^- \tag{3.2}$$

where $f_n := (\mathcal{B}(*_{n,n+1}))^+$ is a predefined null-preserving function from B_{n+1} to B_n.

Note that any function from B_n is measurable since Σ_n is a powerset of B_n.

$$\begin{tikzcd}
\omega \arrow[d, "\mathcal{B}"'] & 0 \arrow[r, "i_0"] & 1 \arrow[r, "i_1"] & \ldots \arrow[r, "i_{n-1}"] & n \arrow[r, "i_n"] & n+1 \arrow[r, "i_{n+1}"] & \ldots \\
\mathbf{Prob} & \bar{B}_0 \arrow[r, "f_0^-"] & \bar{B}_1 \arrow[r, "f_1^-"] & \ldots \arrow[r, "f_{n-1}^-"] & \bar{B}_n \arrow[r, "f_n^-"] & \bar{B}_{n+1} \arrow[r, "f_{n+1}^-"] & \ldots
\end{tikzcd}$$

As we introduced, the functor $\mathcal{B}$ is a generalized filtration, representing a filtration over the classical binomial model, for example developed in [Shreve, 2005].

The classical version requires the terminal time horizon T for determining the underlying set $\Omega := \{0, 1\}^T$ while our version does not require it since the time variant probability spaces can evolve without any limit. That is, our version allows unknown future elementary events, which, we believe, shows a big philosophical difference from the Kolmogorov world.

In order to see a variety of filtrations, we introduce two candidates of f_n.

Definition 3.2. [Candidates of f_n]

(1) f_n^{full}

$$\begin{array}{ccc} B_{n+1} & \xrightarrow{f_n^{full}} & B_n \\ \Cup & & \Cup \\ d_1 \dots d_n d_{n+1} & \xmapsto{f_n^{full}} & d_1 \dots d_n \end{array}$$

(2) f_n^{drop}

$$\begin{array}{ccc} B_{n+1} & \xrightarrow{f_n^{drop}} & B_n \\ \Cup & & \Cup \\ d_1 \dots d_{n-1}, d_n d_{n+1} & \xmapsto{f_n^{drop}} & d_1 \dots d_{n-1}\, 0 \end{array}$$

The function f_n^{drop} can be interpreted to forget what happens at time n.

Note that the function f_n^{full} is always null-preserving while f_n^{drop} is null-preserving if and only if $p_n = 0$.

Example 3.3. [Filtrations] As we mentioned in Definition 3.1, all we need to determine the filtration is to specify $f_n : B_{n+1} \to B_n$. We have three examples of filtration $\mathcal{B}$. For $j = 1, 2, \dots, n$,

(1) Classical filtration:
$$f_n := f_n^{full}.$$
(2) Drop-k:
$$f_n := \begin{cases} f_n^{drop} \text{ if } n = k, \\ f_n^{full} \text{ if } n \neq k. \end{cases}$$
(3) Elderly person: For fixed numbers $k_0, k_1 \in \mathbb{N}$,
$$f_n := \begin{cases} f_n^{drop} \text{ if } k_0 \leq n \leq T - k_1 \\ f_n^{full} \text{ if } 0 \leq n < k_0 \,\text{or}\, T - k_1 < n \leq T. \end{cases}$$

Proposition 3.4. *For a* **Prob***-arrow* $f_n^- : \bar{B}_n \to \bar{B}_{n+1}$*,* $v \in L^1(\bar{B}_{n+1})$ *and* $a \in B_n$*,*

$$E^{f_n^-}(v)(a)\mathbb{P}_n(\{a\}) = \sum_{b \in f_n^{-1}(a)} v(b)\mathbb{P}_{n+1}(\{b\}). \tag{3.3}$$

Especially, with the classical filtration, we have

$$f_n^{-1}(a) = (f_n^{full})^{-1}(a) = \{a0, a1\}. \tag{3.4}$$

Hence

$$\begin{aligned} E^{f_n^-}(v)(a) &= v(a0)\frac{\mathbb{P}_{n+1}(\{a0\})}{\mathbb{P}_n(\{a\})} + v(a1)\frac{\mathbb{P}_{n+1}(\{a1\})}{\mathbb{P}_n(\{a\})} \\ &= v(a0)(1-p_{n+1}) + v(a1)p_{n+1}. \end{aligned} \tag{3.5}$$

Definition 3.5. [$\mathcal{B}$-Adapted Process ξ_n] For $n = 1, 2, \ldots$ define a $\mathcal{B}$-adapted process ξ_n by

$$\begin{array}{ccc} B_n & \xrightarrow{\quad \xi_n \quad} & \mathbb{R} \\ \cup & & \cup \\ d_1 d_2 \ldots d_n & \xmapsto{\quad \xi_n \quad} & 2d_n - 1 \end{array}$$

Proposition 3.6. *For $a \in B_n$ with $\mathbb{P}_n(a) \neq 0$,*

$$\begin{aligned} E^{f_n^-}(\xi_{n+1})(a) &= \sum_{e \in I_n(1,a)} \frac{\mathbb{P}_{n+1}(e)}{\mathbb{P}_n(a)} - \sum_{e \in I_n(0,a)} \frac{\mathbb{P}_{n+1}(e)}{\mathbb{P}_n(a)} \\ &= \#(f_n^{-1}(a))p_{n+1} - \#I_n(0,a) \end{aligned}$$

where

$$I_n(j,a) := \{e \in f_n^{-1}(a) \mid (e)_{n+1} = j\}$$

for $j = 0, 1$, and $\#A$ denotes the cardinality of the set A.

Now we define two instruments tradable in our market.

Definition 3.7. [Stock and Bond Processes] Let μ, σ, r be three positive constants satisfying

$$|\mu - r| < \sigma. \tag{3.6}$$

(1) A ***stock process*** $S_n : B_n \to \mathbb{R}$ over $\mathcal{B}$ is defined by

$$S_0(\langle\rangle) := s_0, \quad S_{n+1} := (S_n \circ f_n)(1 + \mu + \sigma\xi_{n+1}) \tag{3.7}$$

where $\langle\rangle \in B_0$ is the empty sequence.

(2) A ***bond process*** $b_n : B_n \to \mathbb{R}$ over $\mathcal{B}$ is defined by

$$b_0(\langle\rangle) := 1, \quad b_{n+1} := (b_n \circ f_n)(1 + r). \tag{3.8}$$

Proposition 3.8. *For any $a \in B_n$,*

(1) $E^{f_n^-}(S_{n+1}) = S_n\big((1+\mu)E^{f_n^-}(1_{B_{n+1}}) + \sigma E^{f_n^-}(\xi_{n+1})\big)$.

(2) $E^{f_n^-}(1_{B_{n+1}})(a) = \frac{\mathbb{P}_{n+1}(f_n^{-1}(a))}{\mathbb{P}_n(a)}$.

(3) $b_n(a) = (1+r)^n$.

Let us consider about the discounted stock process

$$S'_n := b_n^{-1} S_n. \tag{3.9}$$

We want to find an ω-filtration with which S'_n becomes a martingale. Here is the shape of the filtration whose detail we will determine.

Definition 3.9. [Filtration $\mathcal{C}$] Let n be an object of the category ω.

(1) $\mathbb{Q}_n : \Sigma_n \to [0,1]$ is a probability measure of (B_n, Σ_n),
(2) $\bar{C}_n := (B_n, \Sigma_n, \mathbb{Q}_n)$,
(3) $g_n := f_n$.

We define an ω-filtration $\mathcal{C}$ by for $n \in Obj(\omega)$,

$$\mathcal{C}(n) := \bar{C}_n, \quad \mathcal{C}(*_{n,n+1}) := g_n^-. \tag{3.10}$$

$$\begin{array}{ccc}
\omega & & 0 \xrightarrow{i_0} 1 \xrightarrow{i_1} \dots \xrightarrow{i_{n-1}} n \xrightarrow{i_n} n+1 \xrightarrow{i_{n+1}} \dots \\
\mathcal{C}\big\downarrow & & \\
\mathbf{Prob} & & \bar{C}_0 \xrightarrow{g_0^-} \bar{C}_1 \xrightarrow{g_1^-} \dots \xrightarrow{g_{n-1}^-} \bar{C}_n \xrightarrow{g_n^-} \bar{C}_{n+1} \xrightarrow{g_{n+1}^-} \dots
\end{array}$$

FIGURE 3.1. Filtration $\mathcal{C}$

Theorem 3.10. *A process S'_n is a $\mathcal{C}$-martingale , that is, for $n \in \mathbb{N}$, $E^{g_n^-}(S'_{n+1}) = S'_n$ if and only if for all $n \in \mathbb{N}$ and $a \in B_n$,*

$$\mathbb{Q}_n(\{a\}) = c_1\, \mathbb{Q}_{n+1}(I_n(1,a)) + c_0\, \mathbb{Q}_{n+1}(I_n(0,a)) \tag{3.11}$$

where for $j = 0, 1$

$$I_n(j,a) := \{e \in f_n^{-1}(a) \mid (e)_{n+1} = j\} \tag{3.12}$$

and

$$c_1 := \frac{1+\mu+\sigma}{1+r}, \quad c_0 := \frac{1+\mu-\sigma}{1+r}. \tag{3.13}$$

Proof. For $a \in B_n$

$$\begin{aligned}
& S'_n(a)\mathbb{Q}_n(\{a\}) = E^{g_n^-}(S'_{n+1})(a)\mathbb{Q}_n(\{a\}) \\
= & \sum_{e \in f_n^{-1}(a)} S'_{n+1}(e)\mathbb{Q}_{n+1}(\{e\}) \\
= & \sum_{e \in f_n^{-1}(a)} b_{n+1}^{-1}(e)(S_n \circ f_n)(e)(1+\mu+\sigma\xi_{n+1}(e))\mathbb{Q}_{n+1}(\{e\}) \\
= & \sum_{e \in f_n^{-1}(a)} (1+r)^{-(n+1)} S_n(a)(1+\mu+\sigma\xi_{n+1}(e))\mathbb{Q}_{n+1}(\{e\}) \\
= & S'_n(a) \sum_{e \in f_n^{-1}(a)} \frac{1+\mu+\sigma\xi_{n+1}(e)}{1+r}\mathbb{Q}_{n+1}(\{e\}).
\end{aligned}$$

if and only if

$$\begin{aligned}
\mathbb{Q}_n(\{a\}) = & \sum_{e \in I_n(1,a)} \frac{1+\mu+\sigma}{1+r}\mathbb{Q}_{n+1}(\{e\}) + \sum_{e \in I_n(0,a)} \frac{1+\mu-\sigma}{1+r}\mathbb{Q}_{n+1}(\{e\}) \\
= & c_1\, \mathbb{Q}_{n+1}(I_n(1,a)) + c_0\, \mathbb{Q}_{n+1}(I_n(0,a)).
\end{aligned}$$

□

In order to determine more detail of $\mathcal{C}$, we need the following condition for $\mathbb{Q}_n$.

Proposition 3.11. *The following conditions for $\mathbb{Q}_n$ are equivalent.*

(1) for all $n \in \mathbb{N}$, $a \in B_n$,

$$\mathbb{Q}_{n+1}(\{a0, a1\}) = \mathbb{Q}_n(\{a\}) \tag{3.14}$$

(2) for all $n \in \mathbb{N}$, f_n^{full} is measure-preserving w.r.t. $\mathbb{Q}_n$, that is,

$$\mathbb{Q}_n = \mathbb{Q}_{n+1} \circ (f_n^{full})^{-1}. \tag{3.15}$$

(3) there exists a sequence of functions $\{q_k : B_k \to [0,1]\}_{k=1,2,\dots}$ such that for all $n = 1, 2, \dots$ and $d_j = 0, 1$,

$$\mathbb{Q}_n(\{d_1 d_2 \dots d_n\}) = \prod_{k=1}^{n} q_k(d_1 d_2 \dots d_k) \tag{3.16}$$

such that for every $a \in B_{n-1}$, $q_n(a0) + q_n(a1) = 1$.

In the following discussion, we assume the following assumption which is the condition (2) of Proposition 3.11.

Assumption 3.12. For all $n \in \mathbb{N}$, f_n^{full} is measure-preserving w.r.t. $\mathbb{Q}_n$.

By Assumption 3.12 and (3) of Proposition 3.11, we have

$$\mathbb{Q}_{n+1}(\{d_1 d_2 \dots d_n d_{n+1}\}) = \mathbb{Q}_n(\{d_1 d_2 \dots d_n\}) q_{n+1}(d_1 d_2 \dots d_{n+1}).$$

In the rest of this subsection, we will investigate the shape of $\mathbb{Q}_n$ under the assumption that S'_n is $\mathcal{C}$-martingale.

3.1. **Classical Filtration.** First, we prepare a lemma for for the proof of the following propositions.

Lemma 3.13. *If $1 = c_1 x + c_0(1-x)$, then*

$$x = \frac{1}{2} + \frac{r-\mu}{2\sigma} \quad \textit{and} \quad 1 - x = \frac{1}{2} - \frac{r-\mu}{2\sigma}. \tag{3.17}$$

Proposition 3.14. *For a fixed $n \in \mathbb{N}$, assume that $f_n = f_n^{full}$. Then for $a \in B_n$ with $\mathbb{Q}_n(\{a\}) \neq 0$, we have*

$$q_{n+1}(a1) = \frac{1}{2} + \frac{r-\mu}{2\sigma},$$
$$q_{n+1}(a0) = \frac{1}{2} - \frac{r-\mu}{2\sigma}.$$

Note that the resulting probability depends neither on a nor on n.

Proof. By observing the following diagram

$$
\begin{tikzcd}
 & a1 \\
a \arrow[ur, dash] \arrow[dr, dash] & \\
 & a0
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_n & B_{n+1} \arrow[l, "f_n^{full}"'] \\
a \arrow[u, phantom, sloped, "\in"] & ad_{n+1} \arrow[u, phantom, sloped, "\in"] \arrow[l, mapsto, "f_n^{full}"']
\end{tikzcd}
$$

we have

$$
\begin{aligned}
(f_n^{full})^{-1}(a) &= \{a0, a1\} \\
I_n(1, a) &= \{a1\}, \\
I_n(0, a) &= \{a0\}
\end{aligned}
$$

By (3.11)

$$
\begin{aligned}
\mathbb{Q}_n(\{a\}) &= c_1 \mathbb{Q}_{n+1}(I_n(1, a)) + c_0 \mathbb{Q}_{n+1}(I_n(0, a)) \\
&= c_1 \mathbb{Q}_{n+1}(\{a1\}) + c_0 \mathbb{Q}_{n+1}(\{a0\})
\end{aligned}
$$

Now since

$$
\mathbb{Q}_{n+1}(\{ad_{n+1}\}) = \mathbb{Q}_n(\{a\}) q_{n+1}(ad_{n+1})
$$

and $\mathbb{Q}_n(\{a\}) \neq 0$, we have

$$
1 = c_1 q_{n+1}(a1) + c_0 q_{n+1}(a0).
$$

Hence by Lemma 3.13, we have

$$
q_{n+1}(a1) = \frac{1}{2} + \frac{r - \mu}{2\sigma}, \quad q_{n+1}(a0) = \frac{1}{2} - \frac{r - \mu}{2\sigma}.
$$

□

Corollary 3.15. *If $\mathcal{B}$ is the classical filtration, then for any $n \in \mathbb{N}$ and $a \in B_n$ we have*

$$
\mathbb{Q}_n(a) = \Big(\frac{1}{2} + \frac{r - \mu}{2\sigma}\Big)^{n(1,a)} \Big(\frac{1}{2} - \frac{r - \mu}{2\sigma}\Big)^{n(0,a)} \tag{3.18}
$$

where

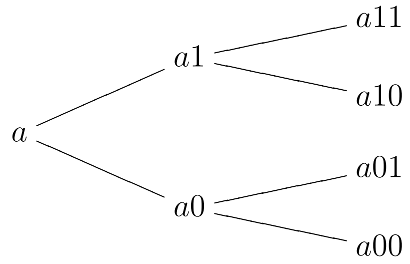
$$
n(j, a) := \#\{k \mid (a)_k = j\}. \tag{3.19}
$$

3.2. Drop-k Filtration.

Proposition 3.16. *For a fixed $n(= 1, 2, \dots)$, assume that $f_n = f_n^{drop}$. Then for $a \in B_{n-1}$ with $\mathbb{Q}_{n-1}(\{a\}) \neq 0$, we have*

$$
\begin{aligned}
q_n(a1) &= 0, \\
q_n(a0) &= 1, \\
q_{n+1}(a01) &= \frac{1}{2} + \frac{r - \mu}{2\sigma}, \\
q_{n+1}(a00) &= \frac{1}{2} - \frac{r - \mu}{2\sigma}.
\end{aligned}
$$

Proof. By observing the following diagram



$$\begin{tikzcd}
B_{n-1} & B_n \arrow[l, "f_{n-1}^{full}"'] & B_{n+1} \arrow[l, "f_n^{drop}"'] \\
a \arrow[u, phantom, "\Cup"] & a0 \arrow[u, phantom, "\Cup"] \arrow[l, mapsto, "f_{n-1}^{full}"'] & ad_nd_{n+1} \arrow[u, phantom, "\Cup"] \arrow[l, mapsto, "f_n^{drop}"']
\end{tikzcd}$$

we have

$$\begin{aligned}
(f_n^{drop})^{-1}(a1) &= \emptyset \\
I_n(1, a1) &= I_n(0, a1) = \emptyset \\
(f_n^{drop})^{-1}(a0) &= \{a00, a01, a10, a11\} \\
I_n(1, a0) &= \{a01, a11\} \\
I_n(0, a0) &= \{a00, a10\}
\end{aligned}$$

By (3.11)

$$\mathbb{Q}_n(\{a1\}) = c_1\mathbb{Q}_{n+1}(I_n(1, a1)) + c_0\mathbb{Q}_{n+1}(I_n(0, a1)) = 0.$$

Now since $\mathbb{Q}_n(\{ad_n\}) = \mathbb{Q}_{n-1}(\{a\})q_n(ad_n)$ and $\mathbb{Q}_{n-1}(\{a\}) \neq 0$, we have

$$q_n(a1) = 0, \quad q_n(a0) = 1 - q_n(a1) = 1.$$

Next, again by (3.11)

$$\begin{aligned}
\mathbb{Q}_n(\{a0\}) &= c_1\mathbb{Q}_{n+1}(I_n(1, a0)) + c_0\mathbb{Q}_{n+1}(I_n(0, a0)) \\
&= c_1\big(\mathbb{Q}_{n+1}(\{a01\}) + \mathbb{Q}_{n+1}(\{a11\})\big) \\
&\quad + c_0\big(\mathbb{Q}_{n+1}(\{a00\}) + \mathbb{Q}_{n+1}(\{a10\})\big)
\end{aligned}$$

By dividing both hands by $\mathbb{Q}_{n-1}(\{a\}) \neq 0$,

$$\begin{aligned}
q_n(a0) &= c_1\big(q_n(a0)q_{n+1}(a01) + q_n(a1)q_{n+1}(a11)\big) \\
&\quad + c_0\big(q_n(a0)q_{n+1}(a00) + q_n(a1)q_{n+1}(a10)\big)
\end{aligned}$$

Then, since $q_n(a1) = 0$ and $q_n(a0) = 1$,

$$1 = c_1q_{n+1}(a01) + c_0q_{n+1}(a00).$$

Hence, by Lemma 3.13, we have

$$q_{n+1}(a01) = \frac{1}{2} + \frac{r-\mu}{2\sigma}, \quad q_{n+1}(a00) = \frac{1}{2} - \frac{r-\mu}{2\sigma}.$$

$\square$

We have to check that both f_n^{full} and f_n^{drop} are null-preserving w.r.t. $\mathbb{Q}_n$.

$$\begin{tikzcd}
 & B_n \arrow[dd, dotted, "drop_n"] & \ni & d_1 \dots d_{n-1} d_n \arrow[dd, dotted, mapsto] \\
B_{n+1} \arrow[ur, "f_n^{full}"] \arrow[dr, "f_n^{drop}"'] & & & \\
 & B_n & \ni & d_1 \dots d_{n-1} 0
\end{tikzcd}$$

If $\mathbb{Q}_n(d_1 \dots d_{n-1}1) \neq 0$, then $drop_n$ is null-preserving, and so is f_n^{drop} since f_n^{full} is measure-preserving.

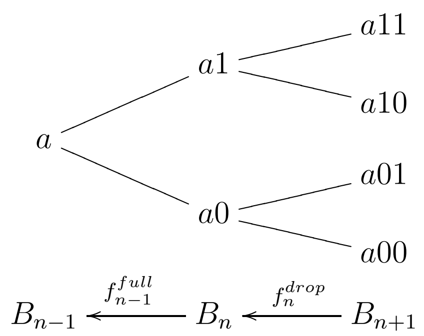


FIGURE 3.2. f_n^{drop}

Remark 3.17. We have the following remarks for Figure 3.2.

(1) Since the agent evaluates stock and bond along the function f_n^{drop}, she can recognise only the nodes $a0$, $a01$ and $a00$ and can not recognise the nodes $a1$, $a11$ and $a10$. We interpret these nodes $a1$, $a11$ and $a10$ as invisible.
(2) The values $q_{n+1}(a11) \in [0, 1]$ can be arbitrarily selected, and $q_{n+1}(a10)$ is computed by $1 - q_{n+1}(a10)$. That is, the probability measure $\mathbb{Q}_{n+1}$ is not determined uniquely, so is not the risk-neutral filtration $\mathcal{C}$.
(3) The probability measure $\mathbb{Q}_n$ is not equivalent to the original measure $\mathbb{P}_n$. Therefore, it is not an EMM.

Remark 3.18. Let $\mathcal{C} : \omega \to \mathbf{Prob}$ be a risk-neutral filtration, and $Y : B_T \to \mathbb{R}$ be a payoff at time T.

Then, for the agent who has a drop-k filtration as her subjective filtration, the price of Y at time n with a unique arrow $i : n \to T$ is given by

$$Y_n := E^{\mathcal{C}i}(b_T^{-1} Y).$$

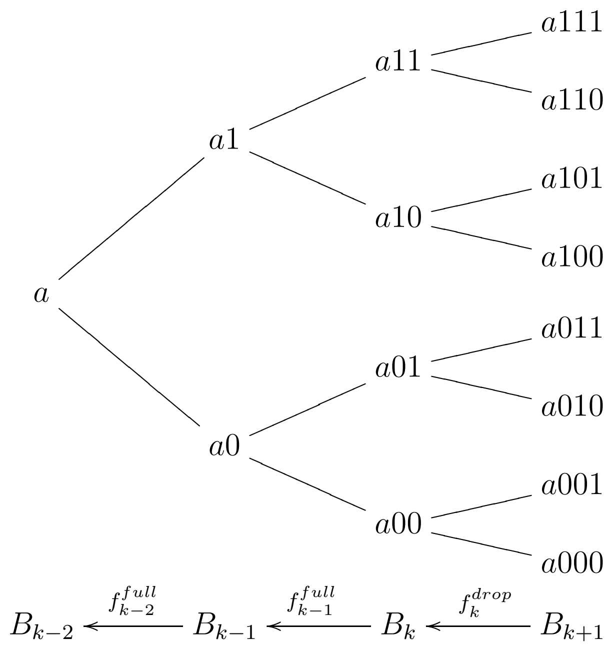


FIGURE 3.3. drop-k filtration

$$
\begin{tikzcd}
n \arrow[r, mapsto, "\mathcal{C}"] \arrow[d, "i"'] & \bar{C}_n \arrow[r, mapsto, "\mathcal{E}"] \arrow[d, "\mathcal{C}i"'] & \mathcal{E}(\bar{C}_n) & := & L^1(\bar{C}_n) & \ni & Y_n = E^{\mathcal{C}i}(b_T^{-1}Y) \\
T \arrow[r, mapsto, "\mathcal{C}"] & \bar{C}_T \arrow[r, mapsto, "\mathcal{E}"] & \mathcal{E}(\bar{C}_T) \arrow[u, "\mathcal{E}(\mathcal{C}i)"'] & := & L^1(\bar{C}_T) & \ni & b_T^{-1}Y \arrow[u, mapsto, "\mathcal{E}(\mathcal{C}i)"']
\end{tikzcd}
$$

You can see in Figure 3.4 that at time $n-1$ the value of $Y_n(a1)$ is discarded and use only the value of $Y_n(a0)$ for computing $Y_{n-1}(a)$.

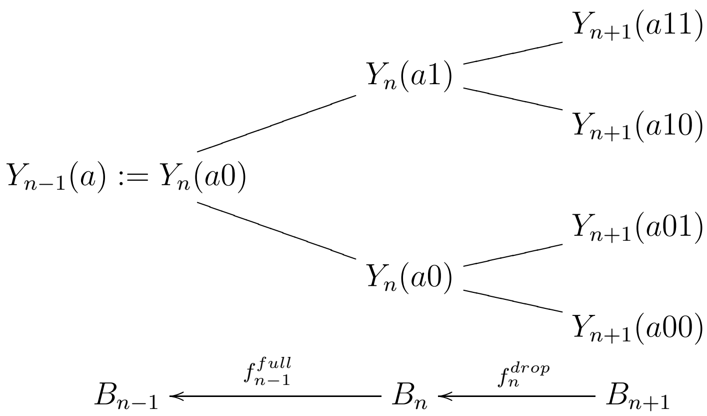


FIGURE 3.4. Valuation along f_n^{drop}

4. Concluding Remarks

We formulated an infinitely growing sequence of binomial probability spaces in the category **Prob**. We gave some concrete (possibly distorted) filtrations. We determined the shape of the risk-neutral filtrations to the above examples. We showed the valuations of claims given at time T through the distorted filtrations.

References

[Adachi and Ryu, 2019] Adachi, T. and Ryu, Y. (2019). A category of probability spaces. *J. Math. Sci. Univ. Tokyo*, to appear. arXiv:1611.03630 [math.PR].

[MacLane, 1997] MacLane, S. (1997). *Categories for the Working Mathematician.* Number 5 in Graduate Texts in Mathematics. Springer-Verlag, New York, 2nd edition.

[Shreve, 2005] Shreve, S. E. (2005). *Stochastic Calculus for Finance I; The Binomial Asset Pricing Model.* Springer-Verlag, New York.

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