Indirect Inference Estimation of Nonlinear Dynamic General Equilibrium Models: With an Application to Asset Pricing under Skewness Risk

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Abstract

This paper proposes an impulse-response matching procedure designed to estimate nonlinear dynamic models by indirect inference. The procedure uses a nonlinear auxiliary model based on Mittnik (1990), which like the economic model generates nonlinear impulse responses, and it is introduced here through its application to a nonlinear macro-finance model of asset pricing under skewness risk. Results show that 1) responses to productivity shocks are asymmetric in that negative shocks induce a larger response than positive shocks, 2) skewness risk accounts about for about one-quarter of the equity risk premium, and 3) the nonlinear model can endogenously generate conditional heteroskedasticity.

1. Introduction

Matching impulse responses is a widely used indirect inference procedure to estimate dynamic general equilibrium models. Theoretical foundations for this estimation strategy are provided in the seminal contributions by Gourieroux et al. (1993) and Smith (1993). Previous applications include Rotemberg and Woodford (1998), Christiano et al. (2005), Iacoviello (2005), Boivin and Giannoni (2006), Uribe and Yue (2006), DiCecio and Nelson (2007), Dupor et al. (2007), Jorda and Kozicki (2011), and Altig et al. (2011). Information criteria are developed by Hall et al. (2012). Bayesian versions are proposed by Christiano et al. (2011) and Guerron-Quintana et al. (2014). Although indirect inference was initially proposed as a method to estimate nonlinear models (see the examples in Gourieroux et al., 1993), most of the above applications concern linear or linearized models where impulse responses are independent of the sign, size, and timing of the shock. Since the response to a shock of size $+1$ (say, standard deviation) is one-half the response to a shock of size $+2$, is the mirror image of a response to a shock of size -1, and is independent of the state of the system when the shock takes place, it is sufficient to consider only one impulse response to describe the model dynamics.

With advances in nonlinear solution methods and the increase in computing power, nonlinear dynamic models in macroeconomics and finance are now often estimated, rather than calibrated. However, the use of impulse-response matching in this setup must address the fact that in nonlinear systems a single response does not completely characterize the dynamic effects of a shock. Instead, the effect depends on the sign, size, and timing of the shock (see Gallant et al., 1993, and Koop et al., 1996). Of course, under the conditions in Gourieroux et al. (1993) and Dridi et al. (2007), ignoring nonlinearity—that is, using a linear auxiliary model (e.g., a vector autoregression) and a single impulse response as a binding function—delivers consistent estimates of the structural parameters, but this approach may be inefficient when the data-generating process is nonlinear.

As an alternative, I consider here a nonlinear auxiliary model based on Mittnik (1990), which, like the economic model, generates nonlinear impulse responses. Specifically, the auxiliary model is a projection on a higher-order polynomial of observable state variables of economic model. Nonlinear impulse responses exploit information on the curvature of the model to provide more comprehensive information about the model dynamics compared with a single linear response. Because the loss in efficiency of the indirect inference estimator, compared to the efficiency of the maximum likelihood estimator, depends on the size of the correct score vector that is not spanned by the estimating equations of the auxiliary model, the use of a nonlinear, rather than a linear, auxiliary model may deliver gains in statistical efficiency. Monte-Carlo results reported here support this conjecture.

The proposed estimation method is introduced through its application to a nonlinear macro-finance model of asset pricing under skewness risk. This application is important in its own right. In contrast to literature based on either the capital asset pricing model or exogenous consumption processes, this paper considers an equilibrium asset-pricing model where consumption is endogenous and structural estimation tries to reconcile stock returns with key macroeconomic aggregates like productivity and investment. Traders inhabit a production economy where firms use labor and capital as inputs, and the financial assets are a riskless bond and shares or claims on the dividends of firms. The only source of risk is a productivity shock. However, rather than treating productivity as a latent variable, this project uses the series on U.S. productivity constructed by Fernald (2014) as one of the observable variables.

I provide evidence of departures from Gaussianity in the data and, in particular, show that productivity innovations, consumption, investment, and stock returns are all negatively skewed. The model is estimated under the assumption that productivity innovations are drawn from an asymmetric generalized extreme value (GEV) distribution (Jenkinson, 1955). This distribution is attractive because it nests three extreme value distributions—namely, the Gumbel, Fréchet, and Weibull distributions—as special cases. Since firms are subject to potentially large negative realizations from the long tail of the distribution, shareholders are subject to skewness risk. The model is solved using a nonlinear perturbation method that makes explicit the dependence of asset returns on the second- and third-order moments of productivity innovations.

Results show that the responses to productivity shocks are asymmetric in that negative shocks induce larger responses than positive shocks (in absolute value). Since the unconditional skewness of productivity innovations is much smaller than that of the other variables, it follows that the nonlinear propagation mechanism plays a substantial role in amplifying the different way in which negative and positive shocks are transmitted through the economy. Results also show that skewness risk accounts for approximately one-quarter of the equity risk premium and that the nonlinear model can endogenously generate conditional heteroskedasticity despite the fact that shocks are homoskedastic. Finally, the use of a nonlinear auxiliary model, rather than a linear model, for the estimation of the model turns out to make a meaningful difference for the economic conclusions that can be drawn from the analysis: Both auxiliary models suggest that investment is irreversible, but only estimates under the nonlinear model are precise enough to allow the rejection of the hypothesis that increasing and decreasing the capital stock are equally costly.

This paper is organized as follows. Section 2 presents a macro-finance model of asset pricing in a production economy subject to skewness risk. Section 3 proposes a simple nonlinear time-series model designed specifically to play the role of auxiliary model in the indirect inference estimation

of nonlinear dynamic equilibrium models, and discusses the use of nonlinear impulse responses as binding function. Section 4 reports results from Monte Carlo experiments that examine the performance of the proposed estimator. Section 5 describes the data used to estimate the model, reports evidence of departures from Gaussianity and parameter estimates, and examines the economic implications of asymmetric shocks and skewness risk. Section 6 concludes.

2. Asset Prices in a Production Economy with Skewness Risk

The representative trader has recursive preferences over consumption (Epstein and Zin, 1989),

$$U_t = \left((1 - \beta) \left(C_t \right)^{1 - 1/\psi} + \beta \left(E_t \left(U_{t+1}^{1-\gamma} \right) \right)^{(1-1/\psi)/(1-\gamma)} \right)^{1/(1-1/\psi))}, \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor, C_t is consumption, E_t is the expectation conditional on information available at time t, γ is the coefficient of risk aversion, and ψ is the intertemporal elasticity of substitution (IES). Time is discrete. In every period, the trader supplies a fixed time endowment, N, in a competitive labor market and participates in a financial market where shares and bonds are bought and sold. Shares are claims on the dividends of firms and bonds are private, riskless, one–period contracts that pay one unit of consumption at maturity.

The trader's budget constraint is

$$C_t + Q_t S_{t+1} + P_t B_{t+1} = X_t N + (Q_t + D_t) S_t + B_t, \tag{2}$$

where Q_t is the price of a share, S_t is the number of shares, P_t is the price of a bond, B_t is the number of bonds, X_t is the hourly wage, N is hours worked, and D_t is dividends. Since consumption is the numeraire, P_t, Q_t, and X_t are real prices in terms of units of consumption. The Euler equations that characterize the trader's optimal demand for shares and bonds are

$$Q_t = E_t \left(\beta \Lambda_{t,t+1} (Q_{t+1} + D_{t+1}) \right), \tag{3}$$

$$P_t = E_t \left(\beta \Lambda_{t,t+1} \right), \tag{4}$$

respectively, where $\Lambda_{t,t+1} = (V_{t+1}/W_t)^{1/\psi - \gamma} (C_{t+1}/C_t)^{-1/\psi}$, $V_t \equiv \max U_t$ is the value function, and $W_t \equiv \left(E_t V_{t+1}^{1-\gamma} \right)^{1/(1-\gamma)}$ is the certainty-equivalent future utility. The left-hand side of each of these equations is the price of the asset and the right-hand side is its expected payoff evaluated using the pricing kernel, $\beta \Lambda_{t,t+1}$.

Denote the gross return on a share and on a bond purchased at time t by $R_{t+1} = (Q_{t+1} + D_{t+1})/Q_t$ and $r_{t+1} = 1/P_t$, respectively. Rewrite equation (3) as

$$1 = E_t \left(\beta \Lambda_{t,t+1} \right) E_t (R_{t+1}) + \Gamma_t, \tag{5}$$

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where $\Gamma_t = cov_t\left(\beta\Lambda_{t,t+1}, R_{t+1}\right)$ is the risk premium. Then, use equations (4) and (5) to write the excess return of equity over the safe asset as

$$E_t(R_{t+1}) - r_{t+1} = -r_{t+1}\Gamma_t. \tag{6}$$

Equation (6) has the usual implication that the excess return is positive if the stock return, R_{t+1}, is negatively correlated with the pricing kernel, $\beta\Lambda_{t,t+1}$, which means that the return is high when the marginal utility of consumption is low.

The representative firm produces output using the technology

$$Y_t = (Z_t L_t)^{1-\alpha}\left(K_t\right)^{\alpha}, \tag{7}$$

where Y_t is output, Z_t is labor productivity, L_t is labor input, K_t is capital, and $\alpha \in (0,1)$ is a constant parameter. Productivity growth follows the process

$$\Delta\ln Z_{t+1} = (1-\rho)\zeta + \rho\Delta\ln Z_t + \epsilon_{t+1}, \tag{8}$$

where Δ is the difference operator, $\rho \in (-1,1)$ is a constant coefficient, ζ is the unconditional mean of productivity growth, and ϵ_t is an independent and identically distributed (i.i.d.) innovation with mean zero, constant conditional variance, and non-zero skewness. The assumption that productivity innovations have non-zero skewness implies that the firm may be subject to large realizations from the long tail of the shock distribution and, hence, shareholders bear skewness risk. In the empirical part of the paper, I assume that ϵ_t is drawn from a (reverse) generalized extreme value distribution (GEV) with mean zero, scale parameter η, and shape parameter ϑ. Depending on whether the shape parameter is zero, larger than zero, or smaller than zero, the distribution corresponds to either the Gumbel, the Weibull, or the Fréchet distribution (see Jenkinson, 1955).

The firm directly owns its capital stock. The law of motion for capital is

$$K_{t+1} = (1-\delta)K_t + I_t, \tag{9}$$

where $\delta \in (0,1)$ is the rate of depreciation and I_t is investment. The firm does not issue new shares and all investment is financed through retained earnings. Adjusting the capital stock involves a convex cost of the form

$$\Phi_t = \Phi(I_t/K_t) = \phi\left(\frac{\exp\left(-\kappa\left(I_t/K_t - \delta^*\right)\right) + \kappa\left(I_t/K_t - \delta^*\right) - 1}{\kappa^2}\right)K_t, \tag{10}$$

where $\phi \geq 0$ is a cost parameter and $\delta^* = \delta + (\zeta - 1)$. The adjustment cost is proportional to the capital stock and concerns investment beyond that required to replace depreciated capital and

to keep up with the growth of labor productivity. The functional form (10) is the linex function proposed by Varian (1974). Under this specification adjustment costs depend on both the sign and the magnitude of the adjustment. For instance in the case where $\kappa > 0$, a decrease in the capital stock involves a larger cost than an increase of the same magnitude. Thus, investment is partly irreversible in that the firm would find it very costly to recover the value of installed capital if it were to disinvest. In the case where $\kappa \to \infty$, the cost function takes the shape of an "L" and investment is completely irreversible. In the case where $\kappa \to 0$, the cost function converges to the quadratic function and, thus, capital decreases and increases of the same magnitude involve the same cost. This implies that 1) it is straightforward to test for asymmetry in capital adjustment costs by testing whether the parameter κ is statistically difference from zero or not, and 2) rejecting the hypothesis that κ is equal to zero against the alternative that it is larger than zero would be evidence of irreversibility in investment.

The firm chooses inputs to maximize

$$E_\iota \sum_{t=\iota}^{\infty} \beta^{t-\iota} \Lambda_{\iota,t} D_t, \tag{11}$$

with the maximization subject to the technology (7), the law of motion (9), and the cost function (10). Profits, which are transferred to shareholders in the form of dividends, are

$$D_t = Y_t - X_t L_t - I_t - \Phi_t. \tag{12}$$

Due to the assumption of constant returns to scale, dividends are just the return on capital net of new investment and adjustment costs.

In equilibrium, share holdings add up to 1 ($S_t = 1$), bonds are not held ($B_t = 0$), the labor market clears ($L_t = N$), and the goods market clears ($C_t + I_t + \Phi_t = Y_t$). Because the level of productivity is non-stationary and there is long-run growth in this economy, the model is rendered stationary by rescaling the variables by the lagged productivity shock, Z_{t-1}.

Since the model does not have an exact analytical solution, I use a perturbation method to compute an approximate nonlinear solution. Perturbation methods start from the exact solution to a simplified form of the original problem and then use a power series (or perturbation) to characterize deviations from this exact solution.[1] In particular, I use a perturbation method based on Jin and Judd (2002) that involves taking a third-order expansion of the policy functions around the deterministic steady state and characterizing the local dynamics. I use this solution method for several reasons. First, I want to focus on the policy functions that solve the complete model, in

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[1]For an introduction to perturbation methods in economics see Judd (1998). Additional references include Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), Andreasen (2012), and Ruge-Murcia (2012).

contrast to the approach of previous research in finance, which typically focuses on Euler equations alone. I do this because, as we will see below, policy functions make explicit the dependence of prices and quantities on the state variables of the model and on the moments of the innovations. The second reason is that, as shown by Caldera et al. (2012), for models with recursive preferences, a third-order perturbation is as accurate as projection methods in the range of interest but is much faster to compute. Third, a third-order perturbation is necessary to fully identify the coefficient of risk aversion in models with Epstein-Zin preferences. As discussed in van Binsbergen et al. (2012), the only difference between Epstein-Zin preferences and preferences with constant relative risk aversion (CRRA) in a second-order perturbation is a constant, while in a third-order perturbation the difference is also reflected in the dynamics. Finally, a third-order perturbation is necessary to accurately capture the effect of the skewness on the solution.

Denote the policy function of a generic j variable in the model by $[p(s_t, \sigma)]^j$, where s_t is the vector of state variables and σ is a perturbation parameter that takes value zero in the deterministic steady state. The state variables here are the current capital stock and productivity growth. The deterministic steady state is computed analytically. Deviations from the steady state are computed by taking a third-order Taylor series expansion around it, with the derivatives taken with respect to s_t and σ. These derivatives are high-dimensional arrays that are cumbersome to represent using standard notation and for this reason researchers express the solution using tensor notation,

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\begin{aligned}
[p(s_t, \sigma)]^j = {} & [p(s, 0)]^j + [p_s(s, 0)]_a^j [(s_t - s)]^a \tag{13} \\
& + (1/2)[p_{ss}(s, 0)]_{ab}^j [(s_t - s)]^a [(s_t - s)]^b \\
& + (1/2)[p_{\sigma\sigma}(s, 0)]^j [\sigma][\sigma] + (1/2)[p_{s\sigma\sigma}(s, 0)]_a^j [(s_t - s)]^a [\sigma][\sigma] \\
& + (1/6)[p_{sss}(s, 0)]_{abc}^j [(s_t - s)]^a [(s_t - s)]^b [(s_t - s)]^c + (1/6)[p_{\sigma\sigma\sigma}(s, 0)]^j [\sigma][\sigma][\sigma],
\end{aligned}
$$

where a, b, and c are indices, $[p(s, 0)]^j$ is the value of the variable j in the deterministic steady state, and $[p_s(s, 0)]_a^j$, $[p_{ss}(s, 0)]_{ab}^j$, $p_{\sigma\sigma}(s, 0)]^j$, $[p_{sss}(s, 0)]_{abc}^j$, $[p_{s\sigma\sigma}(s, 0)]_a^j$, and $p_{\sigma\sigma\sigma}(s, 0)]^j$ are coefficients that depend on the structural parameters of the model. The approximate policy function includes linear, quadratic, and cubic terms in the state variables and its cross-products. It also depends on higher-order moments of the innovations in the form of a risk-adjustment factor that is proportional to their variance and skewness. The effect of the skewness is given by the term $(1/6)[p_{\sigma\sigma\sigma}(s, 0)]^j [\sigma][\sigma][\sigma]$, which is non-zero in the case where innovations follow an asymmetric distribution. In addition to this direct effect on the ergodic mean of the variables, the skewness of the innovations also effects the dynamics through the asymmetry it induces in the state (and, hence, in the control) variables.

3. Indirect Inference Estimation

Indirect inference requires the choice of an auxiliary model and a binding function, and these choices depend on the economic model to be estimated. To examine the possible benefits of using a nonlinear auxiliary model to estimate a nonlinear structural model, I consider here a nonlinear time series model based on Mittnik (1990). The most general formulation of the model I have in mind takes the form

$$y_t = \Omega_0 + \Omega_1 x_t + \Omega_2(x_t \otimes x_t) + \ldots + \Omega_m(x_t \otimes \ldots \otimes x_t) + \xi_t, \tag{14}$$

where y_t is a $k \times 1$ vector of observable variables, x_t is an $nk \times 1$ vector with n lags of each of the k variables in y_t, ξ_t is a $k \times 1$ vector of residuals, Ω_0 is a $k \times 1$ vector of constants, Ω_i for $i = 1, 2, \ldots, m$ are conformable matrices with fixed parameters, and $\otimes$ denotes the Kronecker tensor product. This multivariate nonlinear model is closely related to the univariate generalized autoregression (GAR) due to Mittnik (1990), in which the conditional mean of the variable is a function of its lagged values and a polynomial of lagged cross-products. Mittnik notes that since the relation between the variable and its lags is linear and the error term is separable, it is possible to use least squared methods to estimate the GAR coefficients. These conditions also hold for the model in (14). Thus, in the empirical application and Monte Carlo experiments below, I use ordinary least squares (OLS) equation by equation to estimate the coefficients of the auxiliary model. Note that (14) meets one of the key attributes that an auxiliary model should have for the purpose of indirect inference, namely that it should be relatively easy to estimate.

In practice I use a heavily restricted version of (14) for the estimation of the model. I do so for several reasons. First, the number of parameters increases rapidly with the number of variables, the number of lags, and polynomial order. Second, it is difficult to impose conditions to insure the stability and stationarity of the model. Finally, an unrestricted auxiliary model ignores features of the economic model that can provide a tighter link between structural and auxiliary models. In what follows, I illustrate the restrictions in the context of the asset-pricing model in section 2 and assume that the econometrician has access to data on the growth rates of productivity, consumption, and investment and on stock and bond returns to estimate the model. That is, $y_t = [\Delta \ln(Z_t), \Delta \ln(C_t), \Delta \ln(I_t), R_t, r_t]'$.[2]

Some restrictions on (14) are motivated by the process of the productivity shock in the economic model. Because $\Delta \ln(Z_t)$ follows an exogenous linear process that depends only on its own lags,

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[2] I use the growth rates of consumption and investment for the estimation of the model, rather than the rescaled variables C_t/Z_{t-1} and I_t/Z_{t-1}, because they have a clearer empirical interpretation. It is straightforward to construct the growth rates implied by the model from the simulated series of Z_t/Z_{t-1}, C_t/Z_{t-1} and I_t/Z_{t-1} in order to match the variables in the data.

I restrict to zero the coefficients of lagged higher-order terms in productivity and all terms in the other variables in the first equation of (14).[3] With these restrictions imposed, the first equation in the auxiliary model corresponds exactly to (8). Other restrictions are motivated by the model solution. The solution (13) shows that the control variables of the model are functions of a third-order polynomial on the state variables. Thus, $\Delta \ln (C_t)$, $\Delta \ln (I_t)$, R_t, and r_t in the auxiliary model should be functions of linear, quadratic and cubic terms in productivity growth. Because the process for productivity growth is stationary and all other variables are specified as functions of productivity growth only, the above restrictions imply that the nonlinear auxiliary model computed using data from a stationary model has a unique steady state and no explosive paths regardless of the shock size. To see this, recall that the first equation in the model is the process for productivity growth (8), which is stationary by assumption. The remaining equations of the model specify $\Delta \ln (C_t)$, $\Delta \ln (I_t)$, R_t, and r_t as functions of a constant and a third-order polynomial of lagged productivity growth. Provided the first six moments of productivity growth exist, these variables are stationary as well.

In related work, Aruoba et al. (2014) map the second-order perturbation solution for dynamic general equilibrium models into a quadratic autoregression (QAR) and characterize its impulse responses. My research complements their work by proposing a general nonlinear model that accommodates perturbation solutions of any order and can be used in a multivariate environment. Barnichon and Matthes (2014) construct nonlinear impulse responses by using Gaussian basis functions to parameterize the coefficients of an atheoretical moving average representation of a system. Depending on the function parameters, impulse responses may be asymmetric, be hump-shaped, and/or display overshooting and oscillations. Similarly, the impulse responses of the restricted version of (14) can display these features depending on the lag length and the order of the polynomial.

The binding function maps the parameters of the economic model into those of the auxiliary model. This role is played here by the nonlinear impulse responses. Recall that in linear models, impulse responses are exactly proportional to the sign and size of the shock and independent of its timing. Thus, one impulse response (of any size or sign) is sufficient to describe the model dynamics. However, in this paper the auxiliary model is nonlinear and its impulse responses depend on the sign, size, and timing of the shock in a non-trivial manner. Methods to compute impulse responses in nonlinear systems have been proposed by Gallant et al. (1993) and Koop et al. (1996). One

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[3]In addition to this model-based argument there is a statistical argument. During the estimation procedure, artificial data are generated from the structural model to compute a synthetic nonlinear auxiliary model and impulse responses. Since (8) holds in the model, the coefficients of the additional terms in the auxiliary model should not be statistically different from zero. Imposing these restrictions on (14) delivers sharper estimates of the coefficients and impulse responses, and avoids the extra "noise" that arises because the coefficients of the superfluous variables will not be identically equal to zero in a finite sample.

could also consider computing impulse responses using local-projections or using shocks located in different percentiles of the innovation distribution. In the latter case, shocks are of different size and possibly a different sign, by construction. Any of these methods would provide a suitable characterization of the model responses for the purpose of indirect inference estimation.

With the above elements in place, we are ready to formulate the indirect inference estimator in this setup. Consider a nonlinear model with unknown parameters $\theta \in \Theta$, where θ is a $q \times 1$ vector and $\Theta \subset \Re^q$ is a compact set. The econometrician has at her disposal a sample of $T+n$ observations of k data series to estimate the model. Denote this sample by $\{y_t\}_{t=-(n-1)}^{T}$, where y_t an $k \times 1$ vector. Assume that y_t is stationary and ergodic, possibly as a result of a prior transformation of the raw data by means of a detrending procedure. Denote by $w_t(\theta)$ the $k \times 1$ vector with artificial data simulated from the economic model using parameter values θ and assume that $w_t(\theta)$ is stationary and ergodic for all $\theta \in \Theta$. The size of the simulated sample is $\tau T + n - 1$ with $\tau \geq 1$ because, in general, the simulated sample may be larger than the actual sample. Under the null hypothesis, there exists a unique $\theta_0 \in \Theta$, where θ_0 is an interior point of Θ, such that the random sequences $\{y_t\}_{t=-(n-1)}^{T}$ and $\{w_t(\theta)\}_{t=-(n-1)}^{\tau T}$ have identical distributions.

Using the simulated sample $\{w_t(\theta)\}_{t=-(n-1)}^{\tau T}$, estimate the nonlinear auxiliary model and compute impulse responses. Denote the impulse responses by $\gamma(\theta, \tau T, h)$, where h is the horizon of the responses, and rearrange them as a $\mu h \times 1$ vector where μ is the number of responses computed. Similarly for the actual data, estimate the same nonlinear auxiliary model and compute the impulse responses $\gamma(T, h)$. Then, the indirect inference estimator is

$$\hat{\theta} = \arg \min_{\theta \in \Theta} \left(\gamma(T, h) - \gamma(\theta, \tau T, h) \right)' W \left(\gamma(T, h) - \gamma(\theta, \tau T, h) \right), \tag{15}$$

where W is a $\mu h \times \mu h$ weighting matrix. Intuitively, the indirect inference estimator minimizes the weighted distance between the impulse responses computed using actual data and using artificial data simulated from the economic model. Under the conditions in Gourieroux et al. (1993),

$$\sqrt{T}(\hat{\theta} - \theta_0) \rightarrow N(0, (1 + 1/\tau)(J'WJ)^{-1}J'WSWJ(J'WJ)^{-1}), \tag{16}$$

where $J = E(\partial \gamma(\theta, \tau T, h)/\partial \theta)$ is a finite matrix of dimension $\mu h \times q$ and full column rank and S is the asymptotic variance of $\sqrt{T} \left(\gamma(T, h) - \gamma(h) \right)$. The multiplicative term $(1 + 1/\tau) > 1$ captures the effect of simulation uncertainty on the estimates.

4. Monte Carlo Experiments

This section reports the results of Monte-Carlo experiments used to evaluate the performance of the indirect inference estimator. The experiments examine the properties of the estimator as a

function of the sample size and compare the efficiency of estimators using linear and nonlinear auxiliary models. Results show that 1) estimates are quantitatively close to the true values even in small samples and regardless of whether one uses a linear or a nonlinear auxiliary model, but 2) standard errors are smaller when one uses a nonlinear, rather than a linear, model.

Tables 1A and 1B report results of experiments using samples of 100, 200, 400, $1,000$, and $5,000$ observations with innovations drawn from GEV and normal distributions, respectively. The data generating process (DGP) is the asset-pricing model in section 2 solved using a third-order perturbation and the auxiliary model is nonlinear, as described in section 3. The binding functions are impulse responses to productivity shocks in the 5th and 95th percentiles of the innovation distribution. The shocks take place when the variables are at the mean of the ergodic distribution and their effect is traced out by simulating the auxiliary model for $h = 10$ periods. The estimated parameters are the intertemporal elasticity of substitution (ψ), the coefficient of risk aversion (γ), the capital adjustment cost parameters (ϕ and κ), and the parameters of the productivity shock process. The true parameter values (see the top row of the tables) are similar to those obtained from the estimation of the model reported in section 5. The discount rate is fixed to $\beta = 0.998$, depreciation rate is fixed to $\delta = 0.0225$, the labor share ($1 - \alpha$) is fixed to 0.65, and the mean gross rate of productivity growth is fixed to $\zeta = 1.0026$. Results are based on 200 replications with artificial samples five times larger than the actual sample (that is, $\tau = 5$). Table 1C report results of experiments where the auxiliary model is linear and the binding function is the impulse response to a productivity shock in the 95th percentile of the innovation distribution. Since the DGP is exactly the same as in table 1A, comparing tables 1A and 1C is informative about the performance of the estimator across auxiliary models.

The tables show that regardless of whether one uses a normal or a GEV distribution, and a linear or a nonlinear auxiliary model, indirect inference delivers point estimates that are quantitatively close to the true parameter values, even when the sample size is small. This result is due to the fact that the indirect inference estimator is consistent for the structural parameters (see Gourieroux et al., 1993, and Dridi et al., 2007). Notice, however, by comparing tables 1A and 1C that standard errors are smaller in the former table, where the auxiliary model is nonlinear, than in the latter one, where the auxiliary model is linear. This observation supports the conjecture that estimates based on the nonlinear auxiliary model are generally more efficient than those based on the linear model. Finally note that, as one would expect, standard errors generally decrease as the sample size increases in all tables.

5. Empirical Application

This section illustrates the application of indirect inference to the estimation of the nonlinear asset pricing model in section 2. In particular, this section provides empirical evidence for relaxing the assumption of normally distributed productivity shocks, discusses the implementation of impulse-response matching using the auxiliary nonlinear model, reports parameter estimates, and examines the economic implications of the model.

5.1 Data

The asset pricing model is estimated using quarterly observations of stock returns, bond returns, productivity growth, consumption growth, and investment growth from 1960:Q1 to 2017:Q4. Productivity growth is measured using the series on the growth rate of total factor productivity (TFP) constructed by John Fernald (see Fernald, 2014), which is available from the Federal Reserve Bank of San Francisco website (*www.frbsf.org*).[4] Consumption is measured by personal consumption expenditures on non-durable goods and services. Investment is measured by the sum of personal consumption expenditures on durable goods and private non-residential fixed investment. Both series were divided by the consumer price index (CPI) and the civilian non-institutional population to transform them into real per-capita variables. Bond returns are measured by the average return of the three-month Treasury Bill in each quarter. The raw data used to construct the series of consumption, investment, and bond returns were taken from the Federal Reserve Bank of St. Louis website (*www.stlouisfed.org*). Stock returns were constructed using the quarterly value-weighted index (including distributions) of the New York Stock Exchange, available from the Center for Research in Security Prices (*www.crsp.com*). Nominal stock and bond returns were converted into real returns using CPI inflation.

Figure 1 plots the data (expressed in quarterly rates) in the upper panel and their histograms in the lower panel. Table 2 reports descriptive statistics. In this figure and table, productivity innovations are the residuals of an ordinary least squares (OLS) regression of productivity growth on a constant and its lagged value (see equation (8)). During the sample period, the average growth rate of U.S. productivity is about 0.26% per quarter (that is, about 1% per year), while that of consumption and investment are 0.37% and 0.34% per quarter, respectively. The average return on stocks and bonds are 1.74% per quarter (or 7% per year) and 0.2% per quarter (or 0.8% per year), respectively. As it is well know, consumption growth and bond returns have lower standard

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[4]Fernald's measure of productivity is TFP, while productivity in the model is labor productivity. However, under the assumption that the elasticity of labor in the production function (that is, $1 - \alpha$) is constant, the rate of growth of both productivity measures is exactly proportional.

deviations than investment growth and stock returns, and there is a large quantitative difference between the return of stocks and bonds, with the equity premium being 1.54% per quarter (or about 6.2% per year). The histograms and statistics show that all series are negatively skewed (except bond returns) and leptokurtic meaning that their tails are fatter than those of a normal distribution. To see the latter point, recall that the kurtosis of the normal distribution is three and note that all kurtoses in table 2 are larger than three.

The hypothesis that the data follow a normal distribution is tested using Jarque-Bera and Lilliefors tests. The Jarque-Bera test is based on sample estimates of the skewness and excess kurtosis, both of which should be zero if the data are normally distributed. The Lilliefors test is a version of the Kolmogorov-Smirnov test used to evaluate the null hypothesis that the data are drawn from a normal distribution with unknown mean and standard deviation. The p-values of these tests are reported in table 2. These results provide statistical evidence of departures from Gaussianity in the U.S. data in that the hypothesis of normality is rejected at the 5% significance level in all cases. The only exceptions are productivity growth and bond returns for which the hypothesis is rejected at the 10% level by the Jarque-Bera and Lilliefors tests, respectively.

Although table 2 shows that the departures from Gaussianity in U.S. productivity growth are statistically significant, the table also shows that the unconditional skewness of productivity innovations and productivity growth are quantitatively smaller than that of the other series. This observation suggests that a nonlinear propagation mechanism plays a substantial role in amplifying the asymmetric effects of productivity shocks on asset returns and on the rates of growth of consumption and investment.

5.2   Estimation

The asset-pricing model is estimated using the indirect inference strategy proposed in section 3— that is, by finding the parameters that minimize the distance between impulse responses generated by a projection of the U.S. data on a third-order polynomial of productivity growth and the responses generated by the same projection based on data simulated from the model. The binding function are impulse responses to productivity shocks in the 5th and 95th percentiles of the innovation distribution for a horizon of 10 periods. For the computation of the impulse responses, shocks are assumed to take place when the variables are at the mean of their ergodic distribution. The simulated sample is 20 times larger than the U.S. sample. I estimate two versions of the model with different distributions for productivity innovations, namely the GEV and the normal distributions. Productivity growth is modeled as an AR(1), with this lag length selected using the Bayes information criterion (BIC).

The number of parameters in the auxiliary nonlinear model is 18. The parameters are five intercepts, the autoregressive coefficient of lagged productivity growth in the equation for productivity growth, and the coefficients for the linear, quadratic, and cubic terms of lagged productivity growth in the equations for consumption growth, investment growth, stock returns, and bond returns. Note that the parameters in the auxiliary equation for productivity growth correspond exactly to the structural parameters in (8). The number of estimated structural parameters in the model with normal innovations is six and in the model with GEV innovations is seven.

I also estimate the model with GEV innovations using a linear auxiliary model whereby productivity growth, consumption growth, investment growth, stock returns, and bond returns are projected on a constant and lagged productivity growth. In this case the binding function are impulse responses to a productivity shock in the 95th percentile of the innovation distribution. The number of parameters in the auxiliary model is 10: an intercept and a coefficient of lagged productivity growth for each of the five equations in the auxiliary model.

The weighting matrix, W, is the identity matrix. The Jacobian matrix, J, is computed by taking numerical derivatives of $\gamma(\theta, \tau T, h)$ with respect to θ at the optimum. For the estimation of the variance matrix, S, I follow Hall et al. (2012). Their bootstrap procedure involves estimating synthetic regressions constructed using draws with replacement from the residuals of the regression estimated using the actual data, deriving the implied impulse responses, and computing the variance-covariance matrix of these responses. I use 500 synthetic regressions, but results appear robust to using similar values. For the minimization of the statistical objective function (equation (15)), I use the derivative-free Nelder–Mead and switch to a gradient-based algorithm after 400 function evaluations to sped up convergence. For each model, I ran the estimation procedure several times with starting values randomly chosen in a grid and report in section 5.3 the estimates that correspond to the minimum value of the objective function across all runs.

Figure 2 plots the responses of U.S. consumption growth, investment growth, stock returns, and bond returns to productivity shocks in the 5th and 95th percentiles of the innovation distribution. Since the mean of the distribution is zero, the productivity shock in the 5th percentile is a negative shock, while the shock in the 95th percentile is a positive shock. The responses are generated from the estimated nonlinear auxiliary model and the plots are deviations from the mean of their respective ergodic distributions. The horizontal axis in the figure is quarters and the vertical axis are quarterly rates (in percent). The most striking feature of this figure is the asymmetry in the effects of positive and negative productivity shocks. In particular, the quantitative effects of the negative shock are larger than those of the positive shock in all cases. The positive shock to productivity growth induces a temporary increase in all variables, except in stock returns which

decline on impact but increase thereafter approaching its long-run mean from above. The negative shock induces the converse effects but their magnitude is larger. Consumption growth decreases −0.39 after the negative shock, but increases 0.17 after the positive shock; investment growth decreases −1.11 after the negative shock, but increases 0.80 after the positive shock; and bond returns decrease −0.11 after the negative shock, but increase 0.04 after the positive shock. The asymmetry is the largest in the case of stock returns: the negative shock induces a decrease of −2.29, while the positive shock has a non-monotonic effect and induces an initial decrease of −0.39 followed by an increase of 0.12.

5.3   Parameter Estimates

Table 3 reports estimates of the intertemporal elasticity of substitution (ψ), the coefficient of risk aversion (γ), the capital-adjustment cost parameters (ϕ and κ), the autoregressive coefficient of productivity growth (ρ), and the parameters of the distribution of productivity innovations. During the estimation procedure the discount rate (β) was fixed to 0.998, the depreciation rate (δ) was fixed to 0.0225 meaning that the annual depreciation rate is approximately 9%, the mean of the gross rate of quarterly productivity growth (ζ) was fixed to 1.0026 which is its mean during the sample period, and the capital elasticity in the production function (α) was fixed to 0.35. The latter figure is consistent with data from the National Income and Product Accounts (NIPA) that show that the share of capital in total income is approximately 35%.

Estimates of the intertemporal elasticity of substitution (IES) are about 0.012 for all auxiliary models and distributions and they are statistically different from zero and one. These estimates are consistent with values reported by Hall (1988), Epstein and Zin (1991), Vissing-Jørgensen (2002), and Yogo (2006): Hall reports estimates between 0.07 and 0.35, Epstein and Zin between 0.18 and 0.87 depending on the measure of consumption and instruments used, Vissing-Jørgensen between 0.30 and 1 depending on the households' asset holdings, and Yogo between 0.023 and 0.024 depending on the moments used to estimate the model. The meta-analysis of 169 studies reported in Havranek (2015) suggests that the IES for asset holders is around 0.35. The coefficient of relative risk aversion varies from 83.1 (when innovations are GEV and the auxiliary model is linear) to 155.1 (when innovations are normal and the auxiliary model is nonlinear). These estimates are comparable to the estimate of 79 reported by van Binsbergen et al. (2012), but higher than the value of 10 used as the upper limit by calibration studies in the finance literature. Overall, these results—low IES and large risk aversion—are typical of the macro-finance literature with a representative agent.

The estimate of the parameter that determines the asymmetry of the capital adjustment-cost

function (κ) is positive and statistically different from zero for the model where innovations are GEV and the auxiliary model is nonlinear, but it is not statistically different from zero for the other two models. This means that for the latter models, it is not possible to reject the hypothesis that decreases and increases in the capital stock of the same magnitude are equally costly (that is, that $\kappa = 0$). In contrast, for the former model the hypothesis can be rejected at standard significance levels in favor of the alternative $\kappa > 0$. The alternative implies that a decrease in the capital stock is more costly than an increase of the same magnitude and, in this sense, investment is irreversible. I examine this result in more detail below in section 5.4.3. Overall, however, this result illustrates the importance of using a nonlinear auxiliary model for the estimation of nonlinear structural models, because its more efficient estimates can make a meaningful difference for inference, in this case about the irreversibility of investment.

Estimates of the autoregressive coefficient of productivity growth and the standard deviation of the innovations are similar across the three versions of the model. For the version where innovations are GEV and the auxiliary model is nonlinear, the shape parameter is positive but quantitatively small and not statistically different from zero. Thus, among extreme value distributions, the one that best describes productivity innovations is the (reverse) Gumbel distribution. For the version where innovations are GEV and the auxiliary model is linear, the shape parameter is positive and statistically different from zero, which implies that productivity innovations follow a (reverse) Fréchet distribution. However, the point estimate is quantitatively very close to zero and, thus, the distribution resembles the (reverse) Gumbel distribution. In both cases the implied skewness is relatively small, as it is the case in the U.S. data.

Figure 3 plots the estimated probability density function (PDF) of productivity innovations for the three versions of the model. In this figure, the units in the horizontal axis are standard deviations from the mean. The panels for the GEV distribution also plot the PDF of a symmetric normal distribution with the same variance (thin line). The negative skewness of the GEV distribution is clear from this figure: the distribution has less mass in the right tail and more mass in the left tail than does the normal distribution. This means that large negative productivity innovations are more likely than large positive innovations of the same magnitude. Since large negative draws from the long tail of the distribution reduce output and dividends, shareholders are subject to skewness risk.

5.4 Economic Implications

This section examines the implications of the model for the unconditional moments of the variables and the irreversibility of investment, it quantifies the contribution of skewness risk to the

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equity premium, and it shows that the nonlinear model can endogenously generate conditional heteroskedasticity in asset returns.

5.4.1 Moments

Figure 4 compares the moments predicted by the model (vertical axis) with those computed using the U.S. data (horizontal axis). The moments are the mean, standard deviation, skewness, and cross-correlations of productivity growth, consumption growth, investment growth, stock returns, and bond returns. The continuous line is the 45 degree line. If the model were to perfectly match the moments of the U.S. data, all dots would lie on this line. The figure shows that in general all models perform relatively well, and suggests that the version with GEV innovations and a nonlinear auxiliary model does marginally better than the two other models in matching the moments of the data. This impression is statistically confirmed by the root mean squared errors (RMSE) reported in the figure, which the lowest for this model. However, the overall difference in RMSE across models is quantitatively very small.

A key moment of interest in the literature is the mean return of stocks compared with the mean return of bonds. Their respective annual returns in the U.S. data are 6.98% and 0.79%. (These are the values reported in table 2 multiplied by four to convert them into an annual rate.) Thus, the excess return of stocks over the safe asset—that is, the equity premium—is approximately 6.2% per year. When innovations are GEV and the auxiliary model is nonlinear, the model predicts mean stock and bond returns of 7.92% and 1.02%, respectively, and, thus, an equity premium of 6.9%. When innovations are GEV and the auxiliary model is linear, the predicted mean returns are 6.64% and 0.90%, respectively, and when innovations are normal and the auxiliary model is nonlinear, they are 7.94% and 0.85%, respectively. Hence, all versions of the model generate mean stock and bond returns and equity premia in line with the historical data.

This result is primarily due to the large estimates of risk aversion coupled with low elasticity of intertemporal substitution and adjustment costs to capital, which generate a volatile stochastic discount factor (Jerman, 1998, and Campanale et al., 2010). Moreover, note that asymmetric adjustment costs *and* negatively skewed productivity shocks render capital "riskier" in that potentially large realization from the long tail of the shock distribution may require costly disinvestment and a large, and potentially persistent, drop in dividends. This is why the models with negatively skewed GEV innovations deliver the same equity premium as the model with normal innovations, but with a much lower coefficient of risk aversion.

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5.4.2 Composition of the Equity Premium

The third-order perturbation used to solve the model allows one to express the risk premia as a function of the variance and skewness of the shock innovations (see Andreasen, 2012). In terms of the solution written as in (13), the equity risk premium is

$$[p(s_t, \sigma)]^j = (1/2)[p_{\sigma\sigma}(s,0)]^j[\sigma][\sigma] + (1/2)[p_{s\sigma\sigma}(s,0)]^j_a[(s_t - s)]^a[\sigma][\sigma] \tag{17}$$
$$+ (1/6)[p_{\sigma\sigma\sigma}(s,0)]^j[\sigma][\sigma][\sigma].$$

The premium consists of a constant term in the variance of the productivity innovations, a time-varying term in the variance and the current value of the state variables, and a constant term in the skewness of the innovations. In the special case where the innovation distribution is symmetric, and skewness is zero, the equity premium depends only on the variance terms (that is, the first two terms in the right-hand side of (17)). Then, the key difference is the term $(1/6)[f_{\sigma\sigma\sigma}(s,0)]^j[\sigma][\sigma][\sigma]$, which is zero when the innovation distribution is symmetric and non-zero in the more general case where the distribution is skewed.

Writing the solution for the equity premium as (17) is helpful because it permits its decomposition into the parts attributable to the variance and to the skewness of productivity innovations in the three versions of the model, and hence, to quantify the importance of skewness risk. For the version where innovations are normal and skewness is zero, the model predicts (by construction) that all equity premium is due to variance risk. In contrast, for the version where innovations are GEV and the auxiliary model is nonlinear, estimates imply that skewness risk constitutes 24.4% of the equity premium. For the version where innovations are GEV and the auxiliary model is linear, estimates imply a similar share of 22%. These results suggests that skewness risk is a quantitatively important part of the equity premium in the U.S. data.

5.4.3 Investment Irreversibility

Figure 5 plots the capital adjustment-cost functions implied by the estimated parameters reported in table 3. The horizontal axis is the investment rate beyond that required to replace depreciated capital and to keep up with the growth of labor productivity (that is, $I_t/K_t - \delta^*$). The vertical axis is the cost as proportion of steady state output. As reported in table 3, for the model where innovations are GEV and the auxiliary model is nonlinear the estimate of the asymmetry parameter in the cost function (κ) is positive, quantitatively large, and statistically significant. The fact that κ is statistically different from zero implies that the hypothesis that the adjustment cost function is quadratic—and, thus, that increases and decreases of the capital stock of the same magnitude involve the same cost—can be rejected. Moreover, the large positive value of κ implies an L-shaped

cost function whereby increasing the capital stock involves relatively low adjustment costs, but decreasing the capital stock can be prohibitively costly for the firm. In this sense, investment is irreversible.

For the model where innovations are GEV and the auxiliary model is linear the estimate of the asymmetry parameter is also positive but not statistically different from zero. Thus, although the point estimate is suggestive of investment irreversibility, it is not possible to reject the hypothesis that the adjustment cost function is quadratic. Thus, the use of a nonlinear versus a linear auxiliary model for the indirect inference estimation of the model affects the economic conclusions that can be drawn about the irreversibility of investment and its implications for asset pricing.

Finally, for the model where innovations are normal and the auxiliary model is nonlinear the estimate of the asymmetry parameter is negative but not statistically different from zero.

5.4.4   Conditional Heteroskedasticity

Most financial and macroeconomic series feature time-varying volatility and a large literature has developed in econometrics and finance to study this phenomenon. For the U.S. data examined here, table 5 reports p-values of the Lagrange Multiplier (LM) test of hypothesis of no conditional heteroskedasticity (Engle, 1982). The test is carried out on the residuals of the first-order autoregression of each series and the statistic is calculated as the product of the number of observations and the uncentered R^2 of the OLS regression of squared residuals on a constant and two of its lags. Under the null hypothesis, the statistic is distributed chi-square with 2 degrees of freedom. As in previous literature, the hypothesis can be rejected for stock returns, bond returns, and consumption growth, but it cannot be rejected for investment growth and productivity growth.

Table 5 also reports results of tests carried out on artificial data generated from the models. The length of the artificial data is twenty time larger than the actual U.S. data. For all versions of the model, the hypothesis of no conditional heteroskedasticity cannot be rejected for productivity growth, but it can be rejected for stock returns, bond returns, and investment growth. The key observation here is that the conditional heteroskedasticity in asset returns arises despite the fact that productivity shocks are conditionally homoskedastic (both in the models and the data) and it is instead the endogenous result of the nonlinear propagation of the model. The result that a nonlinear model can generate ARCH effects, even when shocks are i.i.d. and parameters are time-invariant, was first made by Granger and Machina (2006) and the result reported here is a real-world illustration of their conjecture.

In order to understand this result further, it is helpful to refer back to the generic formulation of the policy function in (13) and to notice that it includes a time-varying term in the variance,

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$(1/2)[f_{x\sigma\sigma}(x,0)]_a^j[(x_t - x)]^a[\sigma][\sigma]$. This term makes the function resemble the ARCH-M model use by Engle et al. (1987) to study the term structure, where the conditional variance directly affects the mean. However, the key difference is that while in the ARCH-M model the conditional variance is time-varying and its coefficient is constant, in this model the conditional variance is constant (by assumption) and its coefficient is time-varying because it is a linear function of the state variables.

6. Conclusions

This paper proposes an impulse-response matching procedure for the estimation of nonlinear dynamic models by indirect inference. The procedure uses as auxiliary model a simple nonlinear model based on Mittnik (1990) where variables are projected on a higher-order polynomial of observable state variables of economic model. Monte Carlo experiments indicate that estimates based on the nonlinear auxiliary model are more efficient than those based on a linear auxiliary model. The proposed estimation method is introduced through its application to a nonlinear macro-finance model of asset pricing under skewness risk. The motivating evidence for studying skewness risk is the empirical observation that productivity innovations, consumption growth, investment growth, and stock returns are negatively skewed, and that test results reject the hypothesis that these data are drawn from a symmetric, normal distribution.

Results show that the responses to productivity shocks are asymmetric in that negative shocks induce larger responses than positive shocks. Since the skewness of productivity innovations is relatively small, I conclude that a nonlinear propagation mechanism amplifies the different way in which positive and negative productivity shocks are transmitted through the economy. In addition, I find that skewness risk accounts about for about approximately one-quarter of the equity risk premium and that the nonlinear model can endogenously generate conditional heteroskedasticity. Finally, the use of a nonlinear auxiliary model, rather than a linear model, for the estimation of the model turns out to make here a meaningful difference for the economic conclusions that can be drawn from the analysis because only in the former case statistical inference supports the notion that investment is irreversible.

In future research I examine the small sample properties of the proposed indirect inference estimator and seek to relax the assumption that variables are projected on a polynomial of observable state variables. This assumption is extremely convenient because it makes it easier to insure that the nonlinear auxiliary model is stable and has no explosive paths regardless of the shock size, but it may be restrictive in nonlinear models where all state variables are latent. Another important issue to be addresses in future research is developing information criteria for the horizon of impulse responses in the case where the auxiliary model is nonlinear.

Table 1A. Monte Carlo Results:

GEV Innovations and Nonlinear Auxiliary Model

Parameter	ρ	η	ψ	γ	ϕ	κ	ϑ
True value	0.4000	0.0050	0.0200	80.000	200.000	1,000.00	0.0950
				$T = 100$			
Mean	0.3976	0.0052	0.0193	77.367	192.133	1,061.13	0.1006
Median	0.4036	0.0051	0.0193	78.176	185.651	1,038.14	0.0969
S.E.	0.0026	0.0001	0.0003	1.291	2.795	42.03	0.0071
				$T = 200$			
Mean	0.3898	0.0050	0.0191	81.002	182.573	1,005.61	0.0980
Median	0.3917	0.0050	0.0191	81.153	177.785	1,022.24	0.0949
S.E.	0.0018	0.0001	0.0002	1.029	1.565	43.25	0.0061
				$T = 400$			
Mean	0.4062	0.0049	0.0196	79.127	191.116	907.37	0.1074
Median	0.4063	0.0050	0.0197	79.170	186.324	872.52	0.1019
S.E.	0.0014	0.0001	0.0001	0.850	1.428	31.35	0.0057
				$T = 1000$			
Mean	0.4037	0.0050	0.0199	78.106	187.447	1,037.26	0.1049
Median	0.4048	0.0050	0.0199	78.063	184.824	1,049.27	0.1004
S.E.	0.0009	0.0001	0.0001	0.5689	0.804	26.09	0.0036
				$T = 4000$			
Mean	0.4010	0.0050	0.0202	80.124	198.835	1,033.57	0.0917
Median	0.4009	0.0050	0.0204	80.228	198.048	1,013.45	0.0929
S.E.	0.0003	0.0001	0.0001	0.179	0.292	6.63	0.0012

Notes: T is the sample size, Mean and Median are respectively the mean and median of the estimated coefficients, and S.E. is standard error. In all experiments the following parameters were fixed: $\beta = 0.998$, $\delta = 0.0225$, $\alpha = 0.35$, and $\zeta = 1.0026$. Artificial samples are five times larger than the actual sample size ($\tau = 5$). The number of replications is 200.

Table 1B. Monte Carlo Results:
Normal Innovations and Nonlinear Auxiliary Model

Parameter	ρ	η	ψ	γ	ϕ	κ
True value	0.4000	0.0050	0.0200	80.000	200.000	$1,000.00$
			$T = 100$			
Mean	0.3795	0.0058	0.0209	74.669	234.067	935.20
Median	0.3622	0.0056	0.0207	71.552	212.403	974.80
S.E.	0.0022	0.0001	0.0004	2.315	2.953	77.42
			$T = 200$			
Mean	0.3987	0.0052	0.0208	78.599	214.080	$1,017.30$
Median	0.3940	0.0050	0.0202	77.920	201.354	979.46
S.E.	0.0016	0.0001	0.0003	1.829	1.995	57.44
			$T = 400$			
Mean	0.3954	0.0051	0.0204	78.642	206.996	$1,064.71$
Median	0.3895	0.0051	0.0203	78.376	198.686	$1,017.68$
S.E.	0.0012	0.0001	0.0002	1.336	1.301	43.13
			$T = 1000$			
Mean	0.3943	0.0051	0.0202	78.762	207.016	$1,090.52$
Median	0.3957	0.0051	0.0201	78.847	203.016	$1,062.82$
S.E.	0.0007	0.0001	0.0002	0.869	0.749	26.59
			$T = 4000$			
Mean	0.3996	0.0050	0.0201	79.763	200.828	$1,024.73$
Median	0.3998	0.0050	0.0201	79.775	200.311	998.60
S.E.	0.0002	0.0001	0.0001	0.3137	0.250	8.15

Notes: See notes to table 1A.

Table 1C. Monte Carlo Results:

GEV Innovations and Linear Auxiliary Model

Parameter	ρ	η	ψ	γ	ϕ	κ	ϑ
True value	0.4000	0.0050	0.0200	80.000	200.000	1000.00	0.0950
				$T = 100$			
Mean	0.3969	0.0050	0.0203	80.652	212.420	915.12	0.0935
Median	0.3967	0.0049	0.0201	81.268	211.688	913.41	0.0935
S.E.	0.0029	0.0002	0.0004	3.677	6.142	134.690	0.0258
				$T = 200$			
Mean	0.3938	0.0050	0.0199	79.813	194.692	984.19	0.0970
Median	0.3962	0.0050	0.0199	80.393	199.768	978.08	0.0953
S.E.	0.0027	0.0002	0.0003	5.611	4.997	240.53	0.0416
				$T = 400$			
Mean	0.4002	0.0050	0.0200	79.516	200.061	1,051.48	0.0957
Median	0.4021	0.0050	0.0200	79.769	201.147	995.43	0.0960
S.E.	0.0017	0.0001	0.0002	3.038	3.445	142.72	0.0260
				$T = 1000$			
Mean	0.4077	0.0050	0.0200	78.146	182.710	1,139.51	0.0973
Median	0.4054	0.0050	0.0200	78.552	185.599	1,100.36	0.0971
S.E.	0.0012	0.0001	0.0001	2.821	2.151	134.01	0.0231
				$T = 4000$			
Mean	0.4010	0.0050	0.0201	79.715	193.954	941.97	0.0956
Median	0.3994	0.0050	0.0201	79.967	194.056	940.71	0.0951
S.E.	0.0003	0.0001	0.0001	1.136	1.057	69.45	0.01049

Notes: See notes to table 1A.

Table 2: Descriptive Statistics

Variable	Mean	Standard Deviation	Skewness	Kurtosis	Jarque-Bera Test	Lilliefors Test
Productivity innovations	0.000	0.763	-0.059	3.970	0.018	0.005
Productivity growth	0.262	0.781	-0.018	3.674	0.087	0.013
Consumption growth	0.366	0.561	-0.455	5.393	< 0.001	0.004
Investment growth	0.340	1.984	-0.648	5.078	< 0.001	< 0.001
Stock returns	1.744	8.399	-0.740	4.179	< 0.001	0.002
Bond returns	0.197	0.656	0.457	3.654	0.009	0.074

Notes: The table reports descriptive statistics of the data and p-values of the Jarque-Bera and Lilliefors tests. Productivity innovations are the residuals of an ordinary least squares (OLS) regression of productivity growth on a constant and its lagged value. The rates of growth (productivity, consumption,and investment) and return (bond and stocks) are quarterly and expressed as a percent. The sample consists of 232 quarterly observations between 1960Q1 and 2017Q4.

Table 3. Parameter Estimates

Description	Notation	GEV Nonlinear	GEV Linear	Normal Nonlinear
			Model	
IES	ψ	0.0121*	0.0137*	0.0124*
		(0.0006)	(0.0024)	(0.0006)
Risk aversion	γ	103.68*	83.08*	155.10*
		(21.00)	(24.27)	(40.39)
Capital-adjustment cost	ϕ	481.18*	0.545	12.07
		(122.22)	(16.795)	(18.660)
Asymmetry parameter	κ	$26,838.7^*$	$1,616.7$	$-1,633.8$
		$(5,627.7)$	$(28,954.4)$	(991.9)
Autoregressive coefficient	ρ	0.0726*	0.2614*	0.2716*
		(0.0056)	(0.0382)	(0.0679)
Scale parameter	η	0.0054*	0.0062*	0.0049*
		(0.0012)	(0.0021)	(0.0012)
Shape parameter	ϑ	0.0935	0.00003*	$-$
		(0.0693)	(< 0.00001)	

Notes: The table reports indirect inference estimates of the model parameters. The figures in parenthesis are standard errors. The superscript $*$ denotes statistical significance at the 5% level. For the versions where the auxiliary model is nonlinear, the binding function is impulse responses to productivity growth innovations in the 5th and 95th percentiles of the respective distributions. For the version where the auxiliary model is linear, the binding function is impulse responses to productivity growth innovations in the 95th percentile of the distribution. In the case of the Normal distribution, the scale parameter is the standard deviation.

Table 4. Composition of the Equity Risk Premium (in %)

	Model		
	GEV Nonlinear	GEV Linear	Normal Nonlinear
Variance risk	75.59	77.99	100
Skewness risk	24.42	22.01	0

Note: The tables reports the decomposition of the equity risk premia between variance and skewness risk in percent.

Table 5. Test of No Conditional Heteroskedasticity

	U.S.	Model GEV	GEV	Normal
Description	Data	Nonlinear	Linear	Nonlinear
Productivity growth	0.143	0.665	0.647	0.420
Consumption growth	0.009	0.193	0.659	0.480
Investment growth	0.116	< 0.001	< 0.001	< 0.001
Stock returns	0.079	< 0.001	< 0.001	< 0.001
Bond returns	0.072	< 0.001	< 0.001	< 0.001

Note: The table report p-values of the test of the hypothesis of no conditional heteroskedasticity (Engle, 1982). Tests on data generated from the model were carried out on artificial series with 4640 observations.

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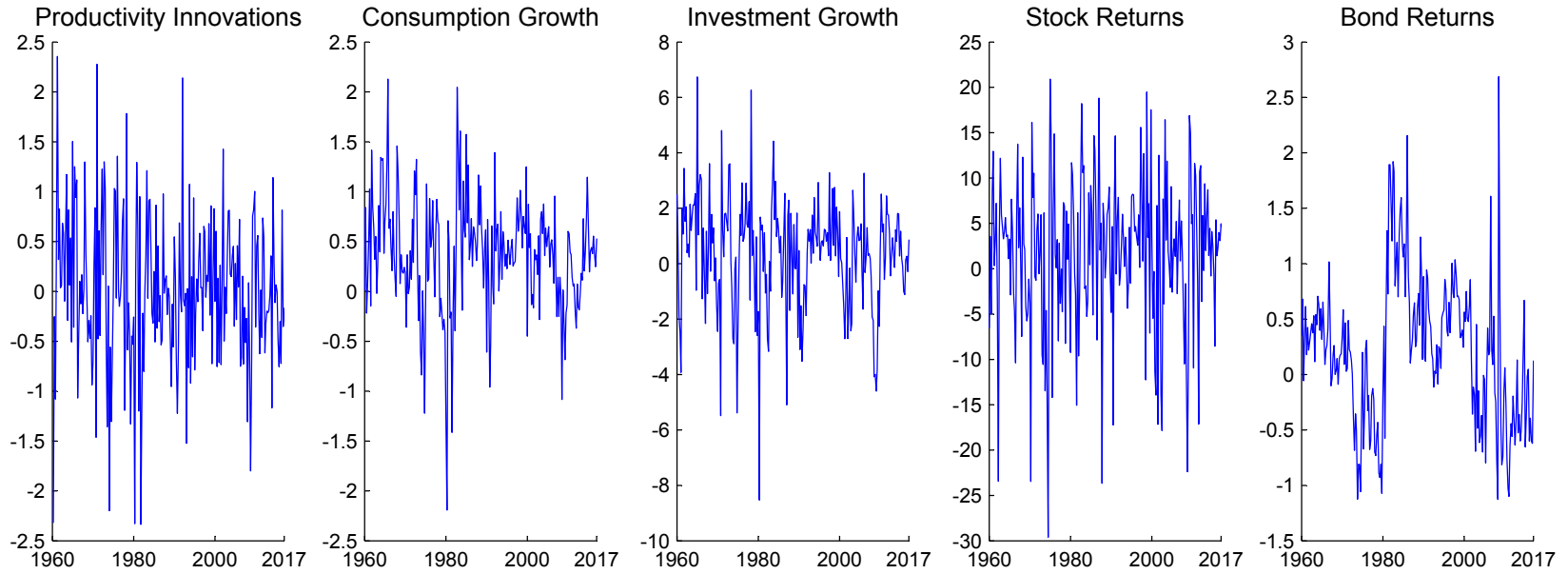
Figure 1: U.S. Data

Figure 2: Impulse Responses to a Productivity Shock
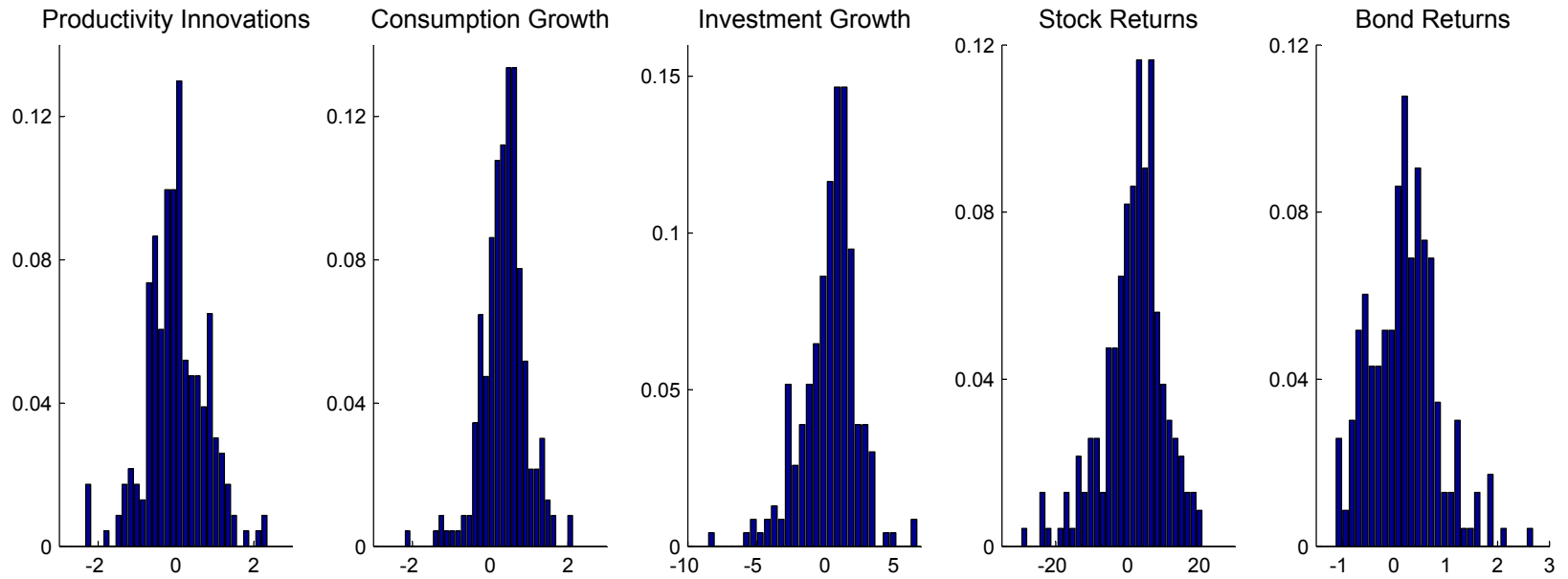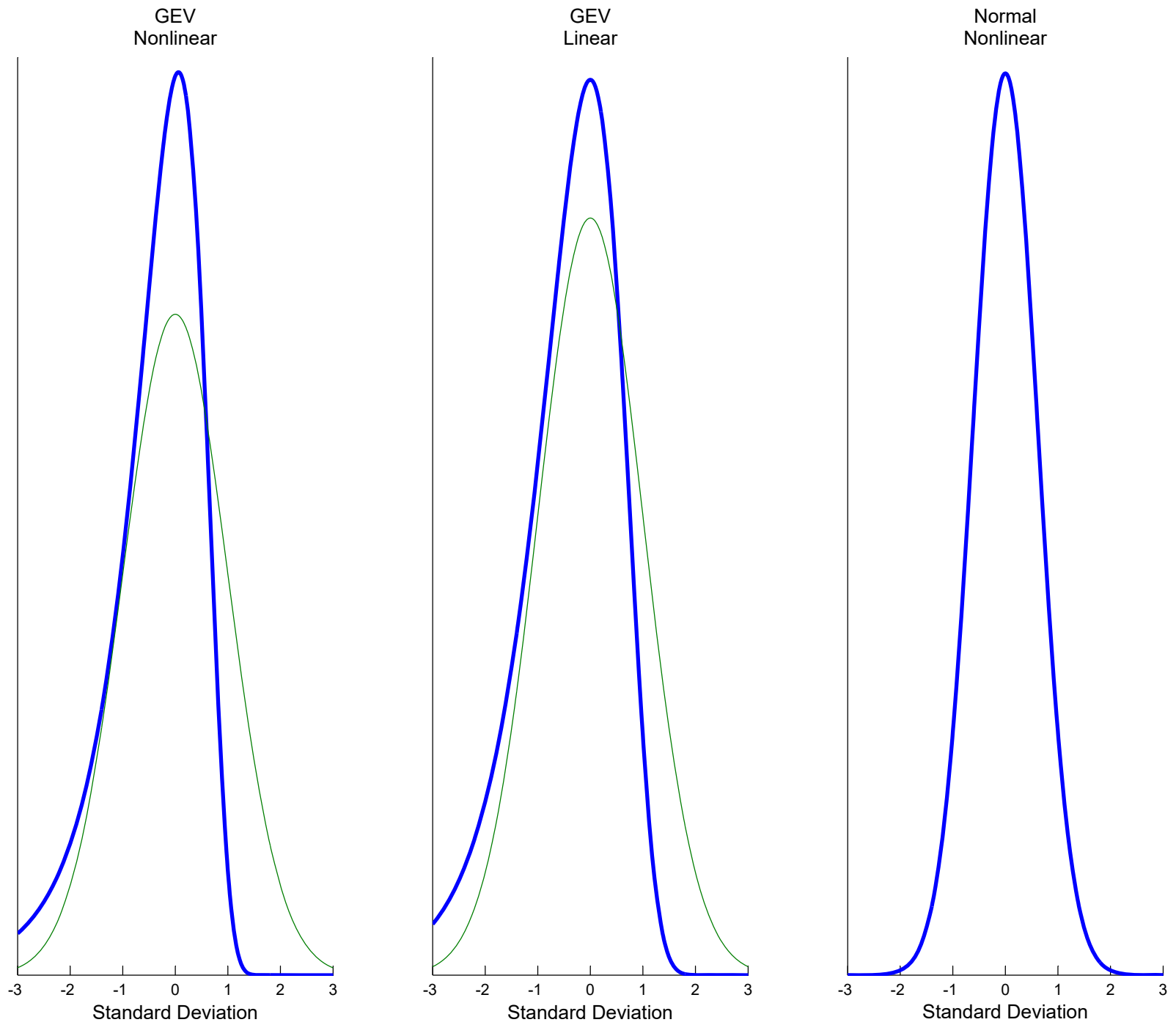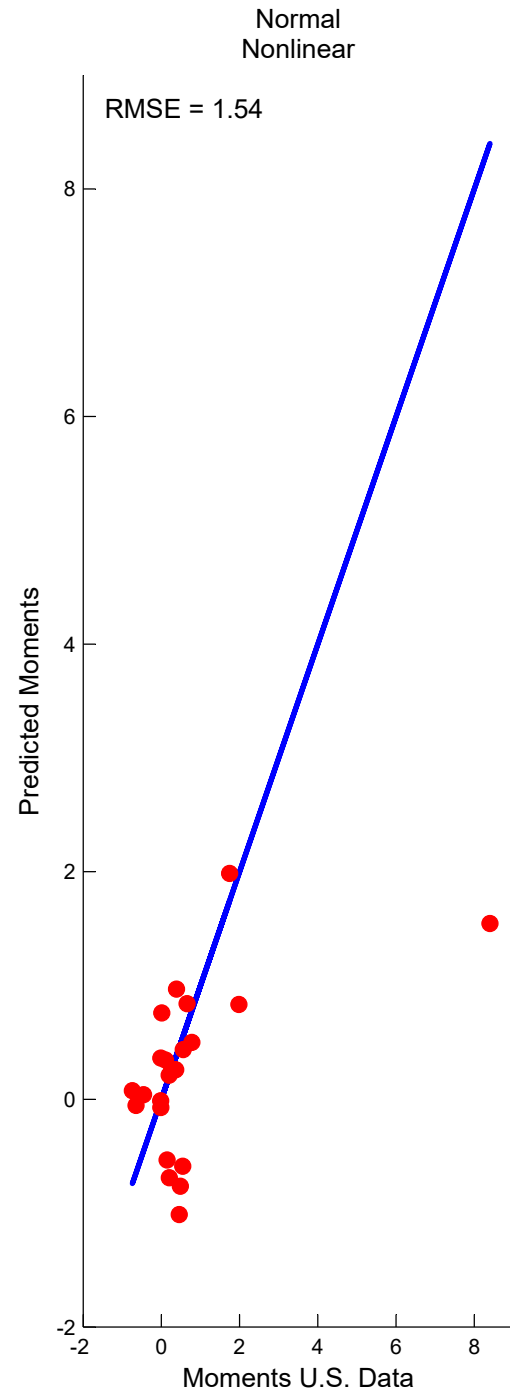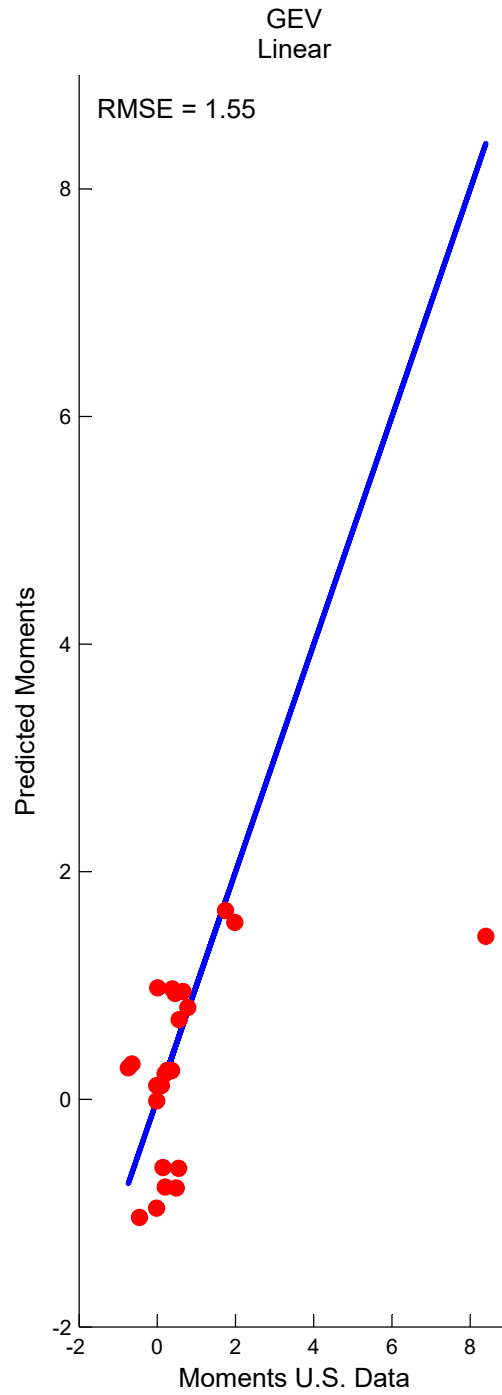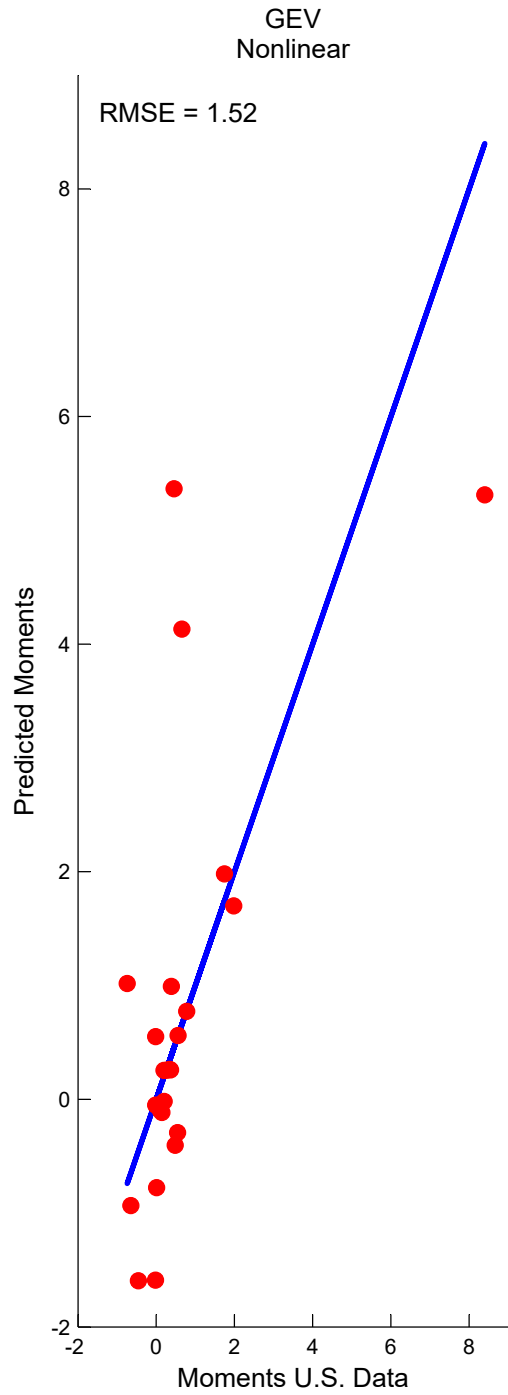
Figure 3: Estimated Probability Density Functions

Figure 4: Model Fit

Figure 5: Estimated Capital Adjustment-Cost Functions