How Bayesian Persuasion can Help Reduce Illegal Parking and Other Socially Undesirable Behavior

Penélope Hernández and Zvika Neeman

November 5, 2017

1 Introduction

This proposal addresses the question of how best to allocate law enforcement resources across different locations. This question is typically analyzed in the context of what is known as a "security game." A security game is a two-player, possibly zero-sum, simultanous-move game in which an attacker has to decide where to strike while a defender has to decide where to allocate its limited defense resources. Analysis of such games has been applied to the question of how to defend against terrorist attacks (Powell, 2007), as well as to a host of related issues (see Tambe, 2011, and the references therein).

The novelty in our approach is that we allow the defender to employ the techniques of "Bayesian persuasion," namely the use of carefully disseminated truthful communication, to maximize its advantage.[1] As we show, Bayesian persuasion can significantly increase the defender's payoff and may imply a completely different allocation of resources.

The question addressed here of how to allocate a given amount of law enforcement resources is different from, and complementary to, the questions famously posed by Becker (1968) about how much resources should be allocated to law enforcement and how to divide these resources between enforcement effort that increases the probability that the offender is caught and the penalty imposed on the offender if caught. Polinsky and Shavell (2000) provide a survey of the theoretical literature on the optimal form of enforcement, and Chalfin and McCrary (forthcoming) provide a survey of the relevant empirical literature.

The two papers that are most closely related to this proposal are by Lando and Shavell (2004) and Eeckhout et al. (2010) who both consider the question of how to allocate enforcement resources. Both papers show that it may be beneficial to concentrate enforcement on a subset of the population. The following example illustrates their idea. Suppose that deterrence of the entire population requires 10 units of resources, but only 5 units are available. In this case, allocation of the 5 units of resources across the entire population fails to achieve deterrence, but concentrating the 5 units on half of the population successfully deters this half.

Our paper is more general in that it considers any number of neighborhoods and adds uncertainty, and in that we consider the question of how to further improve deterrence

[1]The fact that in our formulation, the attacker responds only after observing the defender's signal turns our game into a sequential rather than a simultanous move game.

through Bayesian persuasion, or communication. However, our paper is not a generalization of their work because we consider the set of locations as exogenously given, while focusing enforcement on a sub-population as in Lando and Shavell (2004) and Eeckhout et al. (2010) may be interpreted as endogenously generating locations.

A small literature, see especially Aumann and Maschler (1995), Kamenica and Gentzkow (2011), and the survey by Sobel (2013), studies how a sender of information can affect a receiver's beliefs and thereby induce it to act in a way that benefits the sender. We use similar ideas to study how the communication of information about enforcement can improve deterrence. However, unlike the aforementioned literature on Bayesian persuasion, we consider signaling to several receivers simultaneously, which complicates the analysis. And, of course, we consider the joint problem of how best to allocate and communicate while the Bayesian persuasion literature has focused on communication only.

To fix ideas and simplify the presentation, we focus on the problem of how to allocate resources in order to reduce the extent of illegal parking. However, the same model can also be applied to many other types of socially undesirable behavior.

We show how "Bayesian Persuasion" or in other words carefully disseminated information about parking enforcement in different locations can help, on its own and without strengthening enforcement, to reduce the extent of illegal parking. Moreover, the dissemination of information affects the optimal allocation of enforcement efforts. We show that for Bayesian persuasion to be effective, it is crucial that the amount of parking enforcement resources be uncertain. Bayesian persuasion provides no advantage if the amount of resources is commonly known. We show that the optimal level of resources allocated to any given location depends both on the state of the world (the total amount of resources available) and the signal that is sent. Not conditioning the allocation on the signals sent implies a loss of generality and limits deterrence.

Our ultimate goal is to develop an algorithm that would describe the optimal way to "persuade" (i.e., send signals as a function of the state of the world) and allocate enforcement resources across different locations depending on the state of the world and the message sent. Even though the problem we consider is non-linear, we show that it can be transformed into a linear programming problem.

While a complete description of the optimal algorithm still eludes us, we show by example that a policy that imposes a moratorium on parking enforcement in some clearly defined situations can be an important part of an optimal policy. Intuitively, such a moratorium improves overall deterrence because it achieves stronger deterrence when it is not applied. Indeed, casual empiricism suggests that local governments occasionally experiment with such moratoriums. For example, it is supposedly well known and certainly widely believed among residents of Tel Aviv that the city does not enforce parking violations from Friday to Saturday evenings as well as from the evening before to the evening of state holidays.

We believe and hope that our results can ultimately be used to decrease the extent of illegal parking, as well as other socially undesirable behavior.

2 Model

Consider a city with $n \geq 1$ different neighborhoods. The set of neighborhoods or locations is denoted $\mathbf{N} = \{1, ..., n\}$. Illegal parking is a problem in all of these neighborhoods. The city determines the amount of resources devoted to enforcement in each neighborhood out of the total amount of available resources, denoted r. The amount of available resources is uncertain. We assume that $r = r_k$, $k \in \{1, ..., K\}$, with probability π_k, respectively, where $0 \leq r_1 < \cdots < r_K$ and $\sum_{k=1}^{K} \pi_k = 1$. We treat the distribution of resources as exogenously given, but it may obviously depend on the city's decisions, and provides another dimension on which to optimize the allocation of resources.

We refer to k as the state of the world. The city knows the realization of the state of the world k and hence also the realization r_k, but drivers only know the distribution $\pi = (\pi_1, ..., \pi_K)$.

As explained above, we assume that the city may disseminate information about its enforcement effort. We model this possibility by assuming that the city may send a signal $m \in \{1, ..., M\}$ about the state of the world k.[2] The probability that the city sends signal m in state k is denoted by $p_k(m) = \Pr(m|k)$. It follows that

$$p_k(m) \geq 0 \text{ for every } k \text{ and } m, \text{ and} \sum_{m=1}^{M} p_k(m) = 1 \text{ for every } k. \tag{1}$$

The posterior belief that drivers have over the state of the world k upon receiving the signal m is denoted

$$\Pr(k|m) = \frac{p_k(m)\,\pi(k)}{\sum_{k'=1}^{K} p_{k'}(m)\,\pi(k')}.$$

Denote the amount of resources allocated to enforcement in neighborhood i in state k when the city sends the message m by $a_k^i(m)$.[3] If signal m is sent with probability zero, then $a_k^i(m) \equiv 0$ for every location i and state k.

The city chooses the amounts $a_k^i(m)$ subject to its resource constraint. In every state $k \in \{1, ..., K\}$,

$$\sum_{i=1}^{n} a_k^i(m) \leq r_k \tag{2}$$

for every signal $m \in \{1, ..., M\}$.[4]

The objective of the city is to allocate the amounts of enforcement resources $\{a_k^i(m)\}$ and send the signals $m \in \{1, ..., M\}$ with probabilities $\{p_k(m)\}$ so as to minimize the

[2] "No signal" is also a signal.

[3] We show below that conditioning the level of enforcement on the signal on top of just the state of the world may contribute to deterrence.

[4] Observe that there is no need to also sum over the signals in the resource constraint because the constraint only requires that resources add up to no more than what is available given a state of the world and the fact that a specific given signal has been sent.

For example, if there are just two locations, just two signals m and m', and r_k units are available in state k, then we need to require that $a_k^1(m) + a_k^2(m) \leq r_k$ and $a_k^1(m') + a_k^2(m') \leq r_k$ rather than the weaker requirement that $p_k(m)\left(a_k^1(m) + a_k^2(m)\right) + p_k(m')\left(a_k^1(m') + a_k^2(m')\right) \leq r_k$ because the city may allocate the entire amount of available resources available r_k upon sending any signal m.

extent of illegal parking. The measure of illegal parking in each neighborhood i is given by a function $q^i(a^i(m))$ that is decreasing in the expected amount of enforcement resources $a^i(m) \equiv \sum_{k=1}^{K} a_k^i(m) \Pr(k|m)$ in that neighborhood given message m. For simplicity, we focus on the special case where each q^i is given by a threshold function. Namely, there exists some threshold τ^i such that

$$q^i(a^i(m)) = \begin{cases} 1 & \text{if } a^i(m) < \tau^i \\ 0 & \text{if } \tau^i \leq a^i(m) \end{cases} .$$

Hence, the city's objective is to allocate the amounts of enforcement resources $\{a_k^i(m)\}$ and send signals with probabilities $\{p_k(m)\}$ so as to minimize the expected social cost of illegal parking as given by

$$\sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{i=1}^{n} q^i(a^i(m)) s^i p_k(m) \pi_k \tag{3}$$

where s^i, $i \in \{1, ..., n\}$, denotes the social disutility generated by illegal parking in neighborhood i, subject to the resource constraint (2) and the constaints imposed by the fact that the $p_k(m)$'s are probabilities (1).

Observe that the constraints (1) and (2) are linear in resources $\{a_k^i(m)\}$ and probabilities $\{p_k(m)\}$, but the objective function (3) is non-linear both because $q^i(a^i(m))$ is a non-linear function of $a^i(m)$ and because $a^i(m)$ itself is a non-linear function of the probabilities $\{p_k(m)\}$.

Alternatively, it is also useful to consider the city's problem as how to allocate the amounts of enforcement resources $\{a_k^i(m)\}$ and send signals with probabilities $\{p_k(m)\}$ so as to maximize expected weighted deterrence as given by

$$\sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{i=1}^{n} d^i(a^i(m)) s^i p_k(m) \pi_k \tag{4}$$

where the function $d^i(a^i(m)) = 1 - q^i(a^i(m))$ describes the strength of deterrence and s^i is interpreted as the benefit of deterrence in neighborhood i (which is equal to the decrease in social distutility). Again, the constraints (1) and (2) are linear in $\{a_k^i(m)\}$ and $\{p_k(m)\}$, but the objective function (4) is not.

2.1 Examples

The analysis yields several general insights that we illustrate by examples below.

The first example shows that the city may be able to decrease the extent of illegal parking by disseminating information about the realizations of the amount of enforcement effort $\{a_k^i(m)\}$. For simplicity, the amounts of enforcement efforts in this example are independent of the signals, so the index m is ommited, and they are denoted by $\{a_k^i\}$.

Example 1. Consider a city with one neighborhood. Suppose that (a mass one of) drivers park illegally if they perceive the expected amount of enforcement to be smaller than $\tau_1 =$

2/5. Suppose that resources are given by $(r_1, r_2) = (0, 1)$ with probabilities $(\pi_1, \pi_2) = (\frac{2}{3}, \frac{1}{3})$, respectively, and that the social cost of illegal parking is $s_1 = 1$. The fact that there is only one neighborhood greatly simplifies the problem of how to allocate the amount of enforcement efforts $\{a_k^i\}_{i,k}$. The city cannot do better than simply allocate its entire enforcement resources in every state of the world to this single neighborhood, so that $a_1^1 = 0$ and $a_2^1 = 1$.

If the city disseminates no information about the state of the world, then drivers park illegally because the expected amount of enforcement is only

$$\frac{2}{3} \cdot a_1^1 + \frac{1}{3} \cdot a_2^1 = \frac{1}{3},$$

which is smaller than the critical threshold $\tau_1 = 2/5$. The expected social cost of illegal parking in this case is 1.

The city can do better by fully revealing the state of the world to the drivers. In this case, when the state of the world is $k = 1$, drivers would realize that there is no enforcement because $a_1^1 = 0$ and would park illegally, but when the state of the world is $k = 2$, drivers would be deterred from parking illegally because $a_2^1 = 1$, which implies that the expected social cost of illegal parking in this case is

$$\frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 0 = \frac{2}{3}.$$

The city can do even better by providing partial information about the state of the world as follows: when $k = 2$ it sends the signal H, and when $k = 1$, it sends signals H and L with probability 1/2 each. When drivers receive the signal L they know that $k = 1$ and so the amount of enforcement is $a_1^1 = 0$ and so they park illegally. However, when they receive the signal H, their posterior belief about the amount of enforcement is

$$\begin{aligned}
E\left[a^1 \,|m = H\right] &= \tfrac{\pi(H|1)\pi_1}{\pi(H|1)\pi_1+\pi(H|2)\pi_2} \cdot a_1^1 + \tfrac{\pi(H|2)\pi_2}{\pi(H|1)\pi_1+\pi(H|2)\pi_2} \cdot a_2^1 \\
&= \tfrac{\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{2}{3}+1\cdot\frac{1}{3}} \cdot a_1^1 + \tfrac{1\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{2}{3}+1\cdot\frac{1}{3}} \cdot a_2^1 \\
&= \tfrac{1}{2} \cdot a_1^1 + \tfrac{1}{2} \cdot a_2^1 \\
&= \tfrac{1}{2}.
\end{aligned}$$

The fact that this posterior belief is larger than the critical threshold $\tau_1 = 2/5$ implies that drivers don't park illegally. This signaling strategy further decreases the expected social cost of illegal parking from $\frac{2}{3}$ to the probability that the city sends the signal L, or to[5]

$$\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{3}.$$

■

It is important to emphasize that the dissemination of information provides no advantage if there is no uncertainty about the state of the world. This is because when the state of the

[5]The city can decrease the expected social cost of illegal parking even further to $\frac{1}{6}$ by sending the signals L and H with probabilities $\frac{1}{4}$ and $\frac{3}{4}$, respectively, when $k = 1$ and just the signal H when $k = 2$. This is the lowest possible value of the expected social cost in this example.

world is known any induced beliefs about the allocation of enforcement effort that maximize deterrence can already be induced by the allocation itself.

Example 2. Consider a city with two neighborhoods. Suppose that (a mass one of) drivers park illegally if they perceive the expected amount of enforcement in the two neighborhoods to be smaller than $\tau_1 = 2/5$ and $\tau_2 = 4/5$, respectively. Suppose that resources are given by $r_1 = 1$, and that the social cost of illegal parking in the two neighborhoods is $s_1 = 1$ and $s_2 = 3$. The city does not have enough resources to deter illegal parking in more than one neighborhood, and so would deter parking in the second, more important, neighborhood by allocating at least 4/5 of the resources there.

There are two polar ways in which the city can allocate the amount 4/5 to enforcement in the second neighborhood: it can either allocate the amount 4/5 with certainty, or it can allocate the amount 1 with probability 4/5. It is obvious that the city cannot manipulate the beliefs of the drivers if the allocation of enforcement resources involves no uncertainty. So, suppose that the city allocates all its enfrocement resources to the second neighborhood with probability 4/5 and to the first neighborhood with probability 1/5. This deters illegal parking in the second neighborhood but not in the first. Any additional information that the city provides about its enforcement effort can only undermine it. If for example the city announces that it enforces in the first neighborhood, then it would deter illegal parking there, but at the price of allowing illegal parking in the second, more important, neighborhood. And an announcement that the city enforces in the second neighborhood does improve deterrence there, which is already maximal, and if anything can only undermine deterrence in the first neighborhood because it implies that there is no enforcement there. ■

The next example shows that the optimal allocation of enforcement resources depends on whether the city is able to disseminate information or not. It shows that a city that disseminates information about its enforcement allocates its resources differently than a city that does not.

Example 3. Consider a city with three neighborhoods with $(s_1, s_2, s_3) = (\frac{1}{2}, 2, 4)$. Suppose that (a mass one of) drivers park illegally in each neighborhood if they perceive the expeced amount of enforcement is to be smaller than $(\tau_1, \tau_2, \tau_3) = (\frac{1}{7}, \frac{2}{7}, \frac{4}{7})$, respectively. Suppose that resources are given by $(r_1, r_2) = (\frac{1}{3}, \frac{2}{3})$ with probabilities $(\pi_1, \pi_2) = (\frac{1}{2}, \frac{1}{2})$, respectively. Without information dissemination, the best that the city can do is to allocate its parking resources to the first neighborhood when $k = 1$ and to the second neighborhood when $k = 2$. This would deter illegal parking in neighborhoods 1 and 2, but not in neighborhood 3. The expected payoff to the city under this policy is 4. Notice that allocating all the resources to enforcement in neighborhood 3 cannot improve upon this allocation because without information dissemination, the expected enforcement in the third neighborhood under this alternative policy is $\frac{1}{2}$, which is not enough to deter illegal parking there.

If on the other hand the city disseminates information, then it can allocate its entire resources to the second neighborhood when $k = 1$ and to the third neighborhood when $k = 2$. If the city reveals the state of the world k to drivers, then drivers would always park illegally in the first neighborhood, but would be deterred from parking in the second and third neighborhoods with probability 1/2 each. Consequently, the expected payoff to the city in this case is $\frac{1}{2} + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 4 = 3\frac{1}{2} < 4$.

The allocation of resources is obviously different in these two cases, which implies that in general the optimal allocation of enforcement resources depends on whether the city may disseminate information about it or not. ■

The next example demonstrates that conditioning the amounts of enforcement $\{a_k^i(m)\}$ on the signal may strictly improve the value of the city's objective function.

Example 4. Consider a city with two neighborhoods and two equally likely states of the world. Suppose that (a mass one of) drivers park illegally if they perceive the expected amount of enforcement in the two neighborhoods to be smaller than $\tau_1 = \tau_2 = 2$. Suppose that resources are given by $r_1 = 3$ and $r_2 = \frac{9}{2}$, and that the social cost of illegal parking in the two neighborhoods is equal. Optimal partial deterrence that does not split allocations according to messages allocates resources equally in the two neighborhoods in the two states $\left(\frac{3}{2}, \frac{3}{2}\right)$ and $\left(\frac{9}{4}, \frac{9}{4}\right)$, respectively. The optimal signal that is associated with this allocation is in the second state send signal H and in the first state send signals H and L with probability $\left(\frac{1}{2}, \frac{1}{2}\right)$. This generates overall deterrence of

$$2\left[\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right] = \frac{3}{2}.$$

Splitting the allocation can improve upon this solution. In the second state allocate $\left(\frac{9}{4}, \frac{9}{4}\right)$ as before. In the first state, with probability $\frac{1}{6}$ send the signal H and allocate $\left(\frac{1}{2}, \frac{1}{2}\right)$, and with probability $\frac{5}{6}$ send the signal L and allocate $(2, 0)$. This generates overall deterrence of

$$2\left[\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{6}\right] + \frac{1}{2} \cdot \frac{5}{6} = \frac{19}{12} > \frac{3}{2}.$$

The reason this allocation and signalling structure is more efficient is that it does not waste resources when signal L is transmitted like in the first solution. ■

Resources are "wasted" when they are allocated in such a way that does not contribute to deterrence given the signal that is sent. Intuitively, waste should be avoided or at least minimized. However, the next example illustrates that allocating resources so as to minimize "waste" need not be optimal.

Example 5. Consider a city with three neighbourhoods. The parameters of the associated thresholds are $\tau^1 = 2$, $\tau^2 = 3$ and $\tau^3 = 4$. The set of resources is $r_1 = 1$, $r_2 = 8$ and $r_3 = 14$, each one with equal probability, $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$ for $k \in \{1, 2, 3\}$. Suppose that the city does not send any message. Under this situation there is no way to get the total deterrence of illegal parking. Table 1 conveys the problem and the solution under the above circumstances. The first column refers the probability $\pi = (\pi_1, \ldots, \pi_K)$. The last column stands the resources for each state of the world. The last row corresponds with the thresholds $\{\tau^i\}$ for each neighbour and finally the matrix of size $I \times K$ shows the allocations $\{a_k^i\}$:

$\frac{1}{3}$	1	0	0	1
$\frac{1}{3}$	2	3	3	8
$\frac{1}{3}$	3	6	5	14
	2	3	4	

If the city allocates the amounts of enforcement resources $\{a_k^i\}$, nobody will park at both neighbours 1 and 2, since for neighbour 1, it takes $\frac{1}{3} \times 1 + \frac{1}{3} \times 2 + \frac{1}{3} \times 3 = 2$ and for neighbour 2, $\frac{1}{3} \times 3 + \frac{1}{3} \times 6 = 3$. Nevertheless at neighbour 3 it is not possible to deter illegal behaviour since $\frac{1}{3} \times 3 + \frac{1}{3} \times 5 < 4$. Therefore this disposition of enforcement resources causes a waste of $\frac{1}{3}(3+5) = \frac{8}{3}$ units.

Suppose now the city sends two messages, either L or H depending on the amount of resources. Specifically the city sends the message L if the realization of the nature is 1 and consequently it has 1 unit of resources and H when the amount or resources is either 8 or 14. This can be designed as a thick line between the first and the second row in Table 2.

$\frac{1}{3}$	1	0	0	1
$\frac{1}{3}$	2	3	3	8
$\frac{1}{3}$	3	6	5	14
	2	3	4	

In order to compute the deterrence level we have to compute both $a^i|L = \sum_{k=1}^{K} \pi(k|L) a_k^i$ and $a^i|H = \sum_{k=1}^{K} \pi(k|H) a_k^i$. In particular, $a^1|L = 1 < \tau^1 = 2$; $a^2|L = 0 < \tau^2 = 3$ and $a^3|L = 1 < \tau^3 = 3$. Therefore there is no deterrence when the message L is sent. But for the message H we get $a^1|H = \frac{5}{2} > \tau^1 = 2$; $a^2|H = \frac{9}{2} > \tau^2 = 3$ and $a^3|H = 4 = \tau^3 = 4$ reaching total deterrence. If we compute the wasted resources in this setting, we get $\frac{1}{3} \times 1 + \frac{2}{3}(\frac{1}{2} + \frac{3}{2}) = \frac{5}{2}$. Even when the wasted resources is greater than the above case, the deterrence is complete in all neighbours with probability $\frac{2}{3}$. These examples illustrate that it may not be optimal to use all the resources since message may help to deter in a better way with less resources. Actually, it could be more efficient to deter instead with probability 1 some of the neighborhoods, than to deter with high probability all neighborhoods in the city. ■

The examples show that:

1. When there is no uncertainty about the state of the world, the dissemination of information about enforcement is unhelpful (Example 2).[6]

2. When there is uncertainty about the state of the world, the dissemination of information about enforcement effort, by itself and without changing the enforcement effort, can improve deterrence (Example 1).

3. Moreover, the optimal allocation of enforcement effort depends on whether it is possible to disseminate information about it. The optimal allocation of enforcement effort when it is possible to disseminate information is different from the optimal allocation of enforcement effort when it is impossible to disseminate information about it (Example 3).

[6] This insight may appear to contradict the finding of Eeckhout, Persico, and Todd (2010) in which announcements do improve deterrence in a model with no uncertainty about enforcement resources. However, in that paper the announcements serve to create endogenous new neighborhoods (that may be defined by non-geographic characteristics) whereas in this paper we take the structure of neighborhoods as given.

4. Conditioning the level of enforcement on both the state of the world and the signal may strictly improve the value of the city's objective function because it helps the city achieve more flexibility in the way it signals (Example 4).

5. Allocation of resources so as to minimize "waste" can be sub-optimal (Example 5).

6. Finally, note that full revelation of the state of the world can be either better or worse than no information. Example 1 above shows it can be better. It can also be worse when there are enough resources to achieve deterrence with no information, and full revelation implies no deterrence in states of the world with low resources.

2.2 Analysis

Each signal m achieves deterrence on some set of locations $S(m)$. We may thus identify the set of signals M with the set of subsets of the set of neighborhoods $\mathbf{N} = \{1, ..., n\}$, or $M \equiv 2^{\mathbf{N}}$. The set of signals includes a signal that achieves no deterrence (or that achieves deterrence on the empty set, $\varnothing \in 2^{\mathbf{N}}$). And no loss of generality is implied by the assumption that exactly one signal deters on any given set of locations. This is because if two different signals m and m' deter on the same set of locations then they can be merged into one signal $m \cup m'$.

The identification of signals with the set of locations on which they achieve deterrence clarifies that signaling can only be useful if there is some underlying uncertainty.

Proposition 1. *Signaling is ineffective without true underlying uncertainty. In other words, if there is only one state of the world, then there exists an optimal solution that does not involve (non-trivial) signaling.*

Proof. Suppose that there is only one state of the world. Optimality requires that in this state a signal m that is such that $S(m)$ maximizes the value of deterrence is sent with probability one. Sending another signal that induces less deterrence is strictly dominated. ■

The next result shows that no loss of generality is implied by restricting attention to a specific class of allocations of resources.

Proposition 2 (the "Optimal Ratio Rule"). *Given probabilities $\{p_k(m)\}$ and an allocation $\{a_k^i(m)\}$, the same probabilities together with the allocation $\{a_k^{i*}(m)\}$ that is defined below achieves equal deterrence.*

For every state k, signal m, and for every location $i \in S(m)$, the allocation $a_k^{i*}(m)$ is defined such that:

$$a_k^{i*}(m) = \frac{\tau^i r_k}{\sum_{j \in S(m)} \tau^j};$$

and for every state k, signal m, and for every location $i \notin S(m)$, the allocation $a_k^{i*}(m)$ is defined such that

$$a_k^{i*}(m) = 0.$$

Proof. Fix probabilities $\{p_k(m)\}$ and an allocation $\{a_k^i(m)\}$. For every location $i \in S(m)$ that is deterred by signal m,

$$\sum_{k=1}^{K} \Pr(k\,|m)\, a_k^i(m) \geq \tau^i.$$

Summing over $i \in S(m)$ and changing the order of summation yields

$$\begin{array}{rcl} \sum_{i\in S(m)} \tau^i & \leq & \sum_{i\in S(m)} \sum_{k=1}^{K} \Pr(k\,|m)\, a_k^i(m) \\ & \leq & \sum_{k=1}^{K} \Pr(k\,|m) \sum_{i\in S(m)} a_k^i(m) \\ & \leq & \sum_{k=1}^{K} \Pr(k\,|m)\, r_k \end{array}$$

where the last inequality follows from feasibility (1).

It therefore follows that

$$\tau^i \leq \sum_{k=1}^{K} \Pr(k\,|m) \frac{\tau^i r_k}{\sum_{j\in S(m)} \tau^j}$$

and so the allocation $a_k^{i*}(m) = \frac{\tau^i r_k}{\sum_{j\in S(m)} \tau^j}$ for every $i \in S(m)$, state k, and signal m, and $a_k^{i*}(m) = 0$ for every $i \in \mathbf{N} \setminus S(m)$, state k, and signal m, also achieves deterrence of the set $S(m)$. ■

The next example illustrates the usefulness of the Optimal Ratio Rule.

Example 6. Consider the case in which the city has three neighborhoods with the corresponding thresholds $\tau^1 = 2$, $\tau^2 = 3$ and $\tau^3 = 4$. The Nature has three states with the resources $r_1 = 1$, $r_2 = 8$ and $r_3 = 14$, each one with equal probability, $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$ for $k \in \{1, 2, 3\}$. If there is no message, next table presents the amounts of enforcement resources reaching deterrence in neighbours 1 and 2 with a waste resources of $\frac{10}{3}$ from $a_2^3 = a_3^3 = 5$.

$\frac{1}{3}$	1	0	0	1
$\frac{1}{3}$	2	3	5	10
$\frac{1}{3}$	3	6	5	14
	2	3	4	

Now consider the message structure that the city sends two messages L or H with probabilities $\{p_k(m)\}$ and uses the enforcement resources $\{a_k^i(m)\}$. Next table displays the message structure by the dotted line. In this case this line divides the first row. This means that for the first state of the world $k = 1$ with $r_1 = 1$ the probability of message H is $p_1(H) = p$ and for message L is $p_1(L) = 1 - p$. For the other states $k = 2, 3$ it is only sent the message H, *i.e.*, $p_2(H) = p_3(H) = 1$ and $p_2(L) = p_3(L) = 0$.

$2^*\frac{1}{3}$				2*1
2-4	1			
$\frac{1}{3}$	2	3	5	10
$\frac{1}{3}$	3	6	5	14
	2	3	4	

The objective of the city is to maximize deterrence. In this example we look total deterrence when the message H is sent, or equivalently $a^i|H \geq \tau^i$. The three conditions in this case are:

$$
\begin{aligned}
p \times \frac{1}{3} \times 1 + \frac{1}{3} \times 2 + \frac{1}{3} \times 3 &\geq 2(\frac{1}{3}p + \frac{1}{3} + \frac{1}{3}) \\
p \times \frac{1}{3} \times 0 + \frac{1}{3} \times 3 + 6 \times \frac{1}{3} \times 6 &\geq 3(\frac{1}{3}p + \frac{1}{3} + \frac{1}{3}) \\
p \times \frac{1}{3} \times 0 + \frac{1}{3} \times 5 + \frac{1}{3} \times 5 &\geq 4(\frac{1}{3}p + \frac{1}{3} + \frac{1}{3})
\end{aligned}
$$

For $p = \frac{1}{2}$ we get a deterrence of the three neighbours with probability $(\frac{1}{3}\frac{1}{2} + \frac{1}{3} + \frac{1}{3})$.

$2^*\frac{1}{3}$				2*1
2-4	1			
$\frac{1}{3}$	2	3	5	10
$\frac{1}{3}$	3	6	5	14
	2	3	4	

The optimal ratio rule provides a way to distribute the resources. Consider in the above example with the same message structure ($p_1(H) = p$, $p_1(L) = 1 - p$; $p_2(H) = p_3(H) = 1$ and $p_2(L) = p_3(L) = 0$). Let $S(L) = \emptyset$ and $S(H) = \{1, 2, 3\}$. Given this information we can distribute the resources following the optimal location rule. Recall that

$$a_k^{i*}(m) = \frac{\tau^i r_k}{\sum_{j \in S(m)} \tau^j}$$

for every location $i \in S(m)$, state k, and signal m, and

$$a_k^{i*}(m) = 0$$

Next matrix shows the corresponding allocations:

$2^*\frac{1}{3}$				2*1
2-4	$\frac{2}{9} \times 1$	$\frac{3}{9} \times 1$	$\frac{4}{9} \times 1$	
$\frac{1}{3}$	$\frac{2}{9} \times 10$	$\frac{3}{9} \times 10$	$\frac{4}{9} \times 10$	10
$\frac{1}{3}$	$\frac{2}{9} \times 14$	$\frac{3}{9} \times 14$	$\frac{4}{9} \times 10$	14
	2	3	4	

Once the allocations have been fixed, we can compute the probability p and therefore the level of deterrence. Following the condition of deterrence when the message H is sent, the following expressions display such condition:

$$\begin{aligned}
p \times \frac{1}{3} \times \frac{2}{9} \times 1 + \frac{1}{3} \times \frac{2}{9} \times 10 + \frac{1}{3} \times \frac{2}{9} \times 14 &\geq 2(\frac{1}{3}p + \frac{1}{3} + \frac{1}{3}) \\
p \times \frac{1}{3} \times \frac{3}{9} \times 1 + \frac{1}{3} \times \frac{3}{9} \times 10 + \frac{1}{3} \times \frac{3}{9} \times 14 &\geq 3(\frac{1}{3}p + \frac{1}{3} + \frac{1}{3}) \\
p \times \frac{1}{3} \times \frac{4}{9} \times 1 + \frac{1}{3} \times \frac{4}{9} \times 10 + \frac{1}{3} \times \frac{4}{9} \times 14 &\geq 4(\frac{1}{3}p + \frac{1}{3} + \frac{1}{3})
\end{aligned}$$

The solution is $p \leq \frac{3}{4}$. Therefore the optimal for the city is $p = \frac{3}{4}$ getting a deterrence of all neighbours when H is sent with probability $(\frac{1}{3}\frac{3}{4} + \frac{1}{3} + \frac{1}{3})$.

Finally, we many consider other communication structures. Actually we may consider $S(L) = \{1, 2\}$ and $S(H) = \{2, 3\}$. With this information we can calculate the amounts of enforcement resources following the formula of the optimal ratio rule. Next matrix shows such amounts:

$\frac{1}{3}$	$\frac{2}{5} \times 1$	$\frac{3}{5} \times 1$		1
$2*\frac{1}{3}$	$\frac{2}{5} \times 10$	$\frac{3}{5} \times 10$		2*10
2-4	$\frac{2}{9} \times 10$	$\frac{3}{9} \times 10$	$\frac{4}{9} \times 10$	
$\frac{1}{3}$	$\frac{2}{9} \times 14$	$\frac{3}{9} \times 14$	$\frac{4}{9} \times 10$	14
	2	3	4	

Again, we should compute the optimal probability to send the message L or H when $k = 2$. Doing some algebra we obtain that $p \leq \frac{4}{5}$. Hence, for the neighbours 1and 2 we deterrence with probability $(\frac{1}{3} + \frac{1}{3}\frac{4}{5})$ and three of them with probability $(\frac{1}{3} + \frac{1}{3}\frac{1}{5})$. ■

The Optimal Ratio Rule also allows us to reformulate the city's problem as a linear programming problem. This is useful because the complexity of linear programming problems is polynomial in their input. We further discuss the subject of complexity in the next section below.

The Optimal Ratio Rule implies that the city's optimization problem is equivalent to the problem of choosing the probabilities $\{p_k(m)\}_{m \in M}$ of sending each signal in each state of the world to maximize expected deterrence subject to constraint (1) and the constraint that each message $m \in M$ indeed achieves deterrence on the set $S(m)$ through the allocation $\{a_k^{i*}(m)\}$. Namely, the city's problem is:

$$\max_{\{p_k(m)\}} \sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{i \in S(m)} p_k(m)\, s^i \pi_k \tag{5}$$

subject to constraint (1) and the deterrence constraint:

$$a^{i*}(m) = \sum_{k=1}^{K} a_k^{i*}(m) \Pr(k \,|m) \geq \tau^i \tag{6}$$

for every signal $m \in M$ that is sent with a positive probability, or that is such that $\sum_{k=1}^{K} p_k(m) > 0$.

Proposition 3. *The city's problem of choosing the probabilities* $\{p_k(m)\}_{m \in M}$ *to maximize the objective function* (5) *subject to constraints* (1) *and* (6) *can be transformed into the following linear programming problem: choose the probabilities* $\{p_k(m)\}_{m \in M}$ *to maximize the objective function* (5) *subject to constraint* (1) *and the following deterrence constraint:*

$$\sum_{k=1}^{K} p_k(m)\,\pi(k) a_k^{i*}(m) \geq \tau^i \sum_{k=1}^{K} p_k(m)\,\pi(k) \tag{7}$$

for every signal $m \in M$.

Proof. The city's objective function (5) and constraint (1) are both linear in the probabilities $\{p_k(m)\}$. The deterrence constraint (6) is equivalent to the constraint:

$$\sum_{k=1}^{K} \Pr(k\,|m)\, a_k^{i*}(m) = \sum_{k=1}^{K} \frac{p_k(m)\,\pi(k) a_k^{i*}(m)}{\sum_{k'=1}^{K} p_{k'}(m)\,\pi(k')} \geq \tau^i,$$

which can be rewritten as

$$\sum_{k=1}^{K} p_k(m)\,\pi(k) a_k^{i*}(m) \geq \tau^i \sum_{k=1}^{K} p_k(m)\,\pi(k).$$

This last constraint is linear in the probabilities $\{p_k(m)\}$. Notice that if some signal m is sent with probability zero, then $p_k(m) = 0$ for every $k \in \{1, ..., K\}$, which implies that the deterrence constraint (7) is trivially satisfied. ■

2.3 Directions for Future Research

2.3.1 Complexity

Proposition 3 shows that the city's problem can be represented as a linear programming problem. As explained above, this is useful because linear programming problems are linear in their input. However, this usefulness is undermined by the fact that given K states of the world and n locations, there are $K \cdot 2^n$ different probabilities $p_k(m)$, so the city's problem does not appear to be polynomial in the parameters K and n.

Nevertheless, based on numerical analysis of several different examples, we have become convinced that the city never needs to send more than two signals. This is also in line with the results of Aumann and Maschler (1995) and Kamenica and Gentzkow (2011) who show that in admitedly more limited settings, there is no use for more than two signals in Bayesian persuasion problems.

We therefore conjecture that also in the problem that is described in this proposal there is no need for more than two signals. This implies that the set of possible signals can be reduced to only those signals that achieve deterrence in no more than two locations. There are only $\frac{n(n-1)}{2} + n + 1$ such signals.[7]. We therefore conjecture that the complexity of the city's problem is in fact proportional to $K \cdot n^2$

Importantly, if indeed it is enough to just send two signals, then it is possible to interpret the optimal policy as one that involves only two regimes: "high enforcement" and "low enforcement." The low enforcement regime can be interpreted as offering a moratorium on enforcement as practiced in some places. As explained in Section 2, intuitively, such a moratorium improves overall deterrence because it achieves stronger deterrence when it is not applied.

2.3.2 Dynamics

So far, the problem we have considered is static. One possible way to incorporate dynamics into our analysis is to consider the case in which the city is given a certain amount of weekly or monthly resources, and has to decide how to allocate these resources to specific days. Suppose that a unit of resources that is allocated to any given days "shows up" with probability p and "fails to show up" with the complimentary probability $1 - p$. This assumption generates a non-trivial distribution of resources over different days, and we can solve for the optimal endogenously determined distribution of resources along the same lines described in this proposal.

Another way to consider dynamics is to assume that the problem that is described in this proposal is repeated, and that the state of the world k evolves according to some Markov or another process. This approach is considerably harder than the first approach, however we are hopeful that the methods developed in Renault et al. (2016) and Ely (2017) can help us make progress on this front, or at least identify special cases that can be solved.

---

[7] $\frac{n(n-1)}{2}$ signals achieve deterrence on two different locations; n signals achieve detterence on just one locations; and an additional signal achieves deterrence on the empty set.

References

[1] Aumann, R. J., and M. Maschler (1995) *Repeated Games with Incomplete Information*, MIT Press.

[2] Becker, G. S. (1968) "Crime and Punishment: An Economic Approach," *Journal of Political Economy* 76, 169-217.

[3] Chalfin, A. and J. McCrary (2017) "Criminal Deterrence: A Review of the Literature." *Journal of Economic Literature*, forthcoming.

[4] Eeckhout, Jan, Nicola Persico, and Petra E. Todd. (2010). "A Theory of Optimal Random Crackdowns." *American Economic Review* 100, 1104-35.

[5] Ely, J. C. (2017) "Beeps," *American Economic Review* 107, pp. 31-53.

[6] Kamenica, E. and M. Gentzkow (2011) "Bayesian Persuasion," *American Economic Review* 101, 2590-2615.

[7] Lando, H. and S. Shavell (2004) "The advantage of focusing law enforcement effort," *International Review of Law and Economics* 24, 209-218.

[8] Polinsky M. A. and S. Shavell (2000) "The Economic Theory of Public Enforcement of Law," *Journal of Economic Literature* 38, 45-76.

[9] Powell, R. (2007) "Defending against Terrorist Attacks with Limited Resources," *The American Political Science Review* 101, pp. 527-541.

[10] Renault, J., E. Solan, and N. Vieille (2016) "Optimal Dynamic Information Provision," unpublished manuscript.

[11] Sobel, J. (2013) "Giving and receiving advice." Acemoglu, D., Arellano, M., Dekel, E., eds. *Advances in Economics and Econometrics, Tenth World Congress of the Econometric Society*, Cambridge University Press, 305-341.

[12] Tambe, M. (2011) *Security and Game Theory: Algorithms, Deployed Systems, Lessons Learned*, Cambridge University Press.