What Drives Monetary Policy Shifts?: A New Approach to Regime Switching in DSGE Models

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Main References

- A New Approach to Regime Switching

- Policy Rules with Endogenous Regime Changes

- Endogenous Policy Shifts in a Simple DSGE Model
  - Chang, Maih and Tan (2018) State Space Models with Endogenous Regime Switching
Introduction

Background: New Approach to Regime Switching

Endogenous Policy Shifts in a Simple DSGE Model
Introduction

Monetary policy behavior is purposeful and responds endogenously to the state of the economy.


Subsequently, Markov switching processes is introduced to DSGE models to explore these empirical findings.

- Policy regime shifts assumed to be *exogenous*, inconsistent with the central tenet of Taylor rule

Calls for a model that makes the policy change a purposeful response to the state of the economy.

- Davig and Leeper (2006) build a New Keynesian model with monetary policy rule that switches when past inflation crosses a threshold value.

- Is inflation true or only source of monetary policy shifts?
This Work

Address why have monetary policy regimes shifted and what are the driving forces. We investigate macroeconomic sources of monetary policy shifts.

We introduce the new endogenous switching by Chang, Choi and Park (2017) into state space models

- an autoregressive latent factor determines regimes, and generates endogenous feedback that links current monetary policy stance to historical fundamental shocks
- time-varying transition probabilities
- endogenous-switching Kalman filter
- application to monetary DSGE model

Greater scope for understanding complex interaction between regime switching and economic behavior
Panel A: effective federal funds rate (blue solid) vs. inertial version of Taylor rule (red dashed); Panel B: differential

 Loose policy in late 70s vs. tight policy in early 80s
Introduction

Background: New Approach to Regime Switching
  Basic Switching Models
  Relationship with Conventional Markov Switching Model
  MLE and Modified Markov Switching Filter
  Illustrations

Endogenous Policy Shifts in a Simple DSGE Model
The basic mean model with regime switching is given by

\[(y_t - \mu_t) = \gamma(y_{t-1} - \mu_{t-1}) + u_t\]

with

\[\mu_t = \mu(s_t),\]

where

- \((s_t)\) is a state process specifying a binary state of regime
- \(s_t = 0\) and \(1\) are referred respectively to as low and high mean regimes

in the model.
Volatility Switching Model

The basic volatility model with regime switching is given by

\[ y_t = \sigma_t u_t \]

with

\[ \sigma_t = \sigma(s_t), \]

where

- \((s_t)\) is a state process specifying a binary state of regime
- \(s_t = 0\) and \(1\) are referred respectively to as low and high volatility regimes

in the model.
Conventional Regime Switching Model

The state process \((s_t)\) is assumed to be entirely independent of other parts of the underlying model, and specified as a two state Markov chain.

Therefore, the two transition probabilities

\[
\begin{align*}
    a &= \mathbb{P}\{s_t = 0 | s_{t-1} = 0\}, \\
    b &= \mathbb{P}\{s_t = 1 | s_{t-1} = 1\},
\end{align*}
\]

completely specify the state process \((s_t)\).
Chang, Choi and Park (2017) specify a model

\[ y_t = m_t + \sigma_t u_t \]
\[ = m(x_t, y_{t-1}, \ldots, y_{t-k}, s_t, \ldots, s_{t-k}) + \sigma(x_t, s_t, \ldots, s_{t-k})u_t \]
\[ = m(x_t, y_{t-1}, \ldots, y_{t-k}, w_t, \ldots, w_{t-k}) + \sigma(x_t, w_t, \ldots, w_{t-k})u_t \]

where

- **covariate** \( (x_t) \) is exogenous,
- **state process** \( (s_t) \) is driven by \( s_t = 1\{w_t \geq \tau\} \),
- **latent factor** \( (w_t) \) is specified as \( w_t = \alpha w_{t-1} + v_t \),

and

\[
\begin{pmatrix}
  u_t \\
  v_{t+1}
\end{pmatrix}
\overset{d}{\sim}
\mathcal{N}
\left(\begin{pmatrix}
  0 \\
  0
\end{pmatrix},
\begin{pmatrix}
  1 & \rho \\
  \rho & 1
\end{pmatrix}\right)
\]

with endogeneity parameter \( \rho \).
New Mean Switching Model

The mean model with autoregressive latent factor is given by

$$\gamma(L)(y_t - \mu_t) = u_t$$

where $\gamma(z) = 1 - \gamma_1 z - \cdots - \gamma_k z^k$ is a $k$-th order polynomial, $\mu_t = \mu(s_t)$, $s_t = 1\{w_t \geq \tau\}$, $w_t = \alpha w_{t-1} + v_t$ and

$$\begin{pmatrix} u_t \\ v_{t+1} \end{pmatrix} = d \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

Again a shock $(u_t)$ at time $t$ affects the regime at time $t + 1$, and the regime switching becomes endogenous. The endogeneity parameter $\rho$ represents the reversion of mean in our mean model.
The volatility model with autoregressive latent factor is given by

\[ y_t = \sigma_t u_t \]

where \( \sigma_t = \sigma(s_t) = \sigma(w_t) \), \( s_t = 1 \{ w_t \geq \tau \} \), \( w_t = \alpha w_{t-1} + v_t \) and

\[
\begin{pmatrix} u_t \\ v_{t+1} \end{pmatrix} = d N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)
\]

A shock \( (u_t) \) at time \( t \) affects the regime at time \( t + 1 \), and the regime switching becomes endogenous. The endogeneity parameter \( \rho \), which is expected to be negative, represents the leverage effect in our volatility model.
The new model reduces to the conventional Markov switching model when the underlying autoregressive latent factor is stationary with $|\alpha| < 1$, and exogenous with $\rho = 0$, i.e., independent of the model innovation.

Assume $\rho = 0$, and obtain transition probabilities of $(s_t)$. In our approach, they are given as functions of the autoregressive coefficient $\alpha$ of the latent factor and the level $\tau$ of threshold.

Note that

$$
\mathbb{P}\{s_t = 0 \mid w_{t-1}\} = \mathbb{P}\{w_t < \tau \mid w_{t-1}\} = \Phi(\tau - \alpha w_{t-1})
$$

$$
\mathbb{P}\{s_t = 1 \mid w_{t-1}\} = \mathbb{P}\{w_t \geq \tau \mid w_{t-1}\} = 1 - \Phi(\tau - \alpha w_{t-1})
$$

from $w_t = \alpha w_{t-1} + v_t$ and $v_t \sim \mathcal{N}(0, 1)$. 

Transition of Stationary State Process

Transition probabilities of state process \((s_t)\) from low state to low state \(a(\alpha, \tau)\) and high state to high state \(b(\alpha, \tau)\) are given by

\[
a(\alpha, \tau) = \mathbb{P}\{s_t = 0|s_{t-1} = 0\} = \frac{\int_{-\infty}^{\tau \sqrt{1-\alpha^2}} \Phi\left(\tau - \frac{\alpha x}{\sqrt{1-\alpha^2}}\right) \varphi(x)dx}{\Phi\left(\tau \sqrt{1-\alpha^2}\right)}
\]

\[
b(\alpha, \tau) = \mathbb{P}\{s_t = 1|s_{t-1} = 1\} = 1 - \frac{\int_{\tau \sqrt{1-\alpha^2}}^{\infty} \Phi\left(\tau - \frac{\alpha x}{\sqrt{1-\alpha^2}}\right) \varphi(x)dx}{1 - \Phi\left(\tau \sqrt{1-\alpha^2}\right)}
\]

- One-to-one correspondence between \((\alpha, \tau)\) and \((a, b)\).
- An important by-product from the new approach: regime factor \(w_t\) determining regime and regime strength.
MLE and New Markov Switching Filter

The new endogenous model can be estimated by ML method.

\[
\ell(y_1, \ldots, y_n) = \log p(y_1) + \sum_{t=2}^{n} \log p(y_t|\mathcal{F}_{t-1})
\]

where \( \mathcal{F}_t = \sigma(x_t, (y_s)_{s \leq t}) \), for \( t = 1, \ldots, n \).

To compute the log-likelihood function and estimate the new switching model, we need to develop a new filter. Write

\[
p(y_t|\mathcal{F}_{t-1}) = \sum_{s_t} \cdots \sum_{s_{t-k}} p(y_t|s_t, \ldots, s_{t-k}, \mathcal{F}_{t-1}) p(s_t, \ldots, s_{t-k}|\mathcal{F}_{t-1}).
\]

Since \( p(y_t|s_t, \ldots, s_{t-k}, y_{t-1}, \ldots, y_{t-k}) = \mathcal{N}(m_t, \sigma_t^2) \), it suffices to compute \( p(s_t, \ldots, s_{t-k}|\mathcal{F}_{t-1}) \). This is done by repeated implementations of the prediction and updating steps, as in the usual Kalman filter.
Prediction Step

For the prediction step, note

$$p(s_t, \ldots, s_{t-k} | \mathcal{F}_{t-1})$$

$$= \sum_{s_{t-k-1}} p(s_t | s_{t-1}, \ldots, s_{t-k-1}, \mathcal{F}_{t-1}) p(s_{t-1}, \ldots, s_{t-k-1} | \mathcal{F}_{t-1}),$$

and

$$p(s_t | s_{t-1}, \ldots, s_{t-k-1}, \mathcal{F}_{t-1})$$

$$= p(s_t | s_{t-1}, \ldots, s_{t-k-1}, y_{t-1}, \ldots, y_{t-k-1}).$$
The transition probability of state is given as

\[ p(s_t|s_{t-1}, \ldots, s_{t-k-1}, y_{t-1}, \ldots, y_{t-k-1}) \]

\[ = (1 - s_t) \omega(\rho(s_{t-1}, \ldots, s_{t-k-1}, y_{t-1}, \ldots, y_{t-k-1})) \]

\[ + s_t \left[ 1 - \omega(\rho(s_{t-1}, \ldots, s_{t-k-1}, y_{t-1}, \ldots, y_{t-k-1})) \right], \]

where, in turn, if \(|\alpha| < 1\),

\[ \omega(\rho(s_{t-1}, \ldots, s_{t-k-1}, y_{t-1}, \ldots, y_{t-k-1})) \]

\[ = \frac{\left[ (1-s_{t-1}) \int_{-\infty}^{\tau \sqrt{1-\alpha^2}} + s_{t-1} \int_{\tau \sqrt{1-\alpha^2}}^{\infty} \right] \Phi\left( \tau - \rho \frac{y_{t-1} - m_{t-1}}{\sigma_{t-1}} - \frac{\alpha x}{\sqrt{1-\alpha^2}} \right) \varphi(x) \, dx}{(1 - s_{t-1}) \Phi(\tau \sqrt{1-\alpha^2}) + s_{t-1} \left[ 1 - \Phi(\tau \sqrt{1-\alpha^2}) \right]}, \]

Therefore, \( p(s_t, \ldots, s_{t-k}|\mathcal{F}_{t-1}) \) can be readily computed, once \( p(s_{t-1}, \ldots, s_{t-k-1}|\mathcal{F}_{t-1}) \) obtained from the previous updating step.
For the updating step, we have

\[ p(s_t, \ldots, s_{t-k} | \mathcal{F}_t) = p(s_t, \ldots, s_{t-k} | y_t, \mathcal{F}_{t-1}) \]

\[ = \frac{p(y_t | s_t, \ldots, s_{t-k}, \mathcal{F}_{t-1}) p(s_t, \ldots, s_{t-k} | \mathcal{F}_{t-1})}{p(y_t | \mathcal{F}_{t-1})}, \]

where

\[ p(y_t | s_t, \ldots, s_{t-k}, \mathcal{F}_{t-1}) = p(y_t | s_t, \ldots, s_{t-k}, y_{t-1}, \ldots, y_{t-k}) \]

We may now readily obtain \( p(s_t, \ldots, s_{t-k} | \mathcal{F}_t) \) from \( p(s_t, \ldots, s_{t-k} | \mathcal{F}_{t-1}) \) and \( p(y_t | \mathcal{F}_{t-1}) \) from previous prediction step.
Extraction of Latent Factor

From prediction and updating steps, we compute $p(w_t, s_{t-1}, \ldots, s_{t-k-1}, \mathcal{F}_{t-1})$ and $p(w_t, s_{t-1}, \ldots, s_{t-k-1}, \mathcal{F}_t)$.

By marginalizing obtain

$$p(w_t | \mathcal{F}_t) = \sum_{s_{t-1}} \cdots \sum_{s_{t-k}} p(w_t, s_{t-1}, \ldots, s_{t-k} | \mathcal{F}_t).$$

which yields the inferred factor

$$\mathbb{E}(w_t | \mathcal{F}_t) = \int w_t p(w_t | \mathcal{F}_t) dw_t.$$ 

We may easily extract the inferred factor, once the maximum likelihood estimates of $p(w_t | \mathcal{F}_t)$, $1 \leq t \leq n$, are available.
GDP Growth Rates

We use

- GDP growth rates are obtained as the first differences of their logs

to fit the mean model

$$\gamma(L) (y_t - \mu(s_t)) = \sigma u_t$$

where $$\gamma(z) = 1 - \gamma_1 z - \gamma_2 z^2 - \gamma_3 z^3 - \gamma_4 z^4.$$
US Real GDP Growth Rates
<table>
<thead>
<tr>
<th>Sample Periods</th>
<th>1952-1984</th>
<th>1984-2012</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ignored</td>
<td>Allowed</td>
</tr>
<tr>
<td>$\mu$</td>
<td>-0.165</td>
<td>-0.083</td>
</tr>
<tr>
<td></td>
<td>(0.219)</td>
<td>(0.161)</td>
</tr>
<tr>
<td>$\bar{\mu}$</td>
<td>1.144</td>
<td>1.212</td>
</tr>
<tr>
<td></td>
<td>(0.113)</td>
<td>(0.095)</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>0.068</td>
<td>0.147</td>
</tr>
<tr>
<td></td>
<td>(0.123)</td>
<td>(0.104)</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>-0.015</td>
<td>0.044</td>
</tr>
<tr>
<td></td>
<td>(0.112)</td>
<td>(0.096)</td>
</tr>
<tr>
<td>$\gamma_3$</td>
<td>-0.175</td>
<td>-0.260</td>
</tr>
<tr>
<td></td>
<td>(0.108)</td>
<td>(0.090)</td>
</tr>
<tr>
<td>$\gamma_4$</td>
<td>-0.097</td>
<td>-0.067</td>
</tr>
<tr>
<td></td>
<td>(0.104)</td>
<td>(0.095)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.794</td>
<td>0.784</td>
</tr>
<tr>
<td></td>
<td>(0.065)</td>
<td>(0.057)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\mathbf{-0.923}$</td>
<td>$\mathbf{1.000}$</td>
</tr>
<tr>
<td></td>
<td>(0.151)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>log-likelihood</td>
<td>-173.420</td>
<td>-169.824</td>
</tr>
<tr>
<td>$p$-value</td>
<td>0.007</td>
<td>0.004</td>
</tr>
</tbody>
</table>
Transition Probability Comparison

GDP Growth Rate and NBER Recession Period

NER announced recession

Jun-07 Dec-07 Jun-08 Dec-08 Jun-09 Dec-09
Transition Probability Comparison

![Graph showing transition probability comparison between June 2007 to December 2009, with a particular focus on the NBER announced recession period. The graph highlights the transition probability from low mean regime to low mean regime, using an exogenous RS model.](image)
Transition Probability Comparison

Transition Probability from Low Mean Regime to Low Mean Regime: Endogenous RS Model

Transition Probability from Low Mean Regime to Low Mean Regime: Exogenous RS Model

NBER announced recession
NBER Recession Period and Latent Factor: 1952-1984
Recession Probabilities: 1952-1984

Endogenous Regime Switching

Exogenous Regime Switching
Stock Return Volatility

We use

- Monthly CRSP returns for 1926/01 - 2012/12 (1,044 obs.)
- One-month T-bill rates used to obtain excess returns
- Demeaned excess returns

to fit the volatility model

\[ y_t = \sigma(s_t)u_t, \]

where

\[ \sigma(s_t) = \sigma(1 - s_t) + \bar{\sigma}s_t \]

and

\[ s_t = 1\{w_t \geq \tau\}. \]
## Estimation Result: Monthly Volatility Model

<table>
<thead>
<tr>
<th>Sample Periods</th>
<th>1926-2012</th>
<th>1990-2012</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Endogeneity</strong></td>
<td>Ignored</td>
<td>Allowed</td>
</tr>
<tr>
<td>$\sigma = \sigma(s_t)$ when $s_t = 0$</td>
<td>0.0385</td>
<td>0.0380</td>
</tr>
<tr>
<td></td>
<td>(0.0010)</td>
<td>(0.0011)</td>
</tr>
<tr>
<td>$\sigma = \sigma(s_t)$ when $s_t = 1$</td>
<td>0.1154</td>
<td>0.1153</td>
</tr>
<tr>
<td></td>
<td>(0.0087)</td>
<td>(0.0090)</td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-0.9698$</td>
<td>$-1.0000$</td>
</tr>
<tr>
<td></td>
<td>(0.0847)</td>
<td>(0.0059)</td>
</tr>
<tr>
<td><strong>log-likelihood</strong></td>
<td>1742.28</td>
<td>1747.98</td>
</tr>
<tr>
<td>$p$-value (LR test for $\rho = 0$)</td>
<td>$0.001$</td>
<td></td>
</tr>
</tbody>
</table>
Transition Probability Comparison

Annualized Volatility Measured by the Sum of Squared Daily Return
Transition Probability Comparison

![Graph showing transition probability comparison between high volatility regime to low volatility regime using exogenous and endogenous RS models. The graph highlights significant changes in probability from Jan-08 to May-09.](image)
Extracted Latent Factor from Volatility Model and VIX
High Volatility Probabilities: 1990-2012

Endogenous Regime Switching

Exogenous Regime Switching
Introduction

Background: New Approach to Regime Switching

Endogenous Policy Shifts in a Simple DSGE Model
- A Simple Fisherian Model
- A Prototypical New Keynesian Model
- A New Filtering Algorithm for Estimation
A Simple Regime Switching Fisherian Model

Fisher equation:

\[ i_t = \mathbb{E}_t \pi_{t+1} + \mathbb{E}_t r_{t+1} \]

Real rate process:

\[ r_t = \rho_r r_{t-1} + \sigma_r \epsilon_t^r \]

Monetary policy with endogenous feedback:

\[ i_t = \alpha(s_t) \pi_t + \sigma_e \epsilon_t^e \]

\[ s_t = 1 \{ w_t \geq \tau \} \]

\[ w_{t+1} = \phi w_t + v_{t+1}, \]

\[ \left( \begin{array}{c} \epsilon_t^e \\ v_{t+1} \end{array} \right) = d \text{ iid } \mathbb{N} \left( 0, \left( \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right) \right) \]

as considered in Chang, Choi and Park (2017).
Information Structure

Agents do not observe the level of latent regime factor $w_t$, but observe whether or not it crosses the threshold, as reflected in $s_t = 1\{w_t \geq \tau\}$.

Agents form expectations on future inflation as

$$\mathbb{E}_t \pi_{t+1} = \mathbb{E}(\pi_{t+1}|\mathcal{F}_t)$$

using the information

$$\mathcal{F}_t = \{i_u, \pi_u, r_u, \epsilon^r_u, \epsilon^e_u, s_u\}_{u=0}^t$$

Monetary authority observes all information in $\mathcal{F}_t$ and also the history of policy regime factor $(w_t)$. 
Endogenous Feedback Mechanism

To see the endogenous feedback mechanism, rewrite

\[ w_{t+1} = \phi w_t + \rho e_t + \sqrt{1 - \rho^2} \eta_{t+1}, \quad \eta_{t+1} \sim \text{i.i.d.} \mathcal{N}(0, 1) \]

From variance decomposition, we see that \( \rho^2 \) is the contribution of past intervention to regime change

- \( \rho = 0 \) : fully driven by exogenous non-structural shock
  \[ w_{t+1} = \phi w_t + \eta_{t+1} \]

- \( |\rho| = 1 \) : fully driven by past monetary policy shock
  \[ w_{t+1} = \phi w_t + \epsilon_t \]
Agents infer transition probabilities by integrating out the latent factor $w_t$ using its invariant distribution, $\mathcal{N}(0, 1/(1 - \phi^2))$.

Transition probabilities of the state from $t$ to $t + 1$

$$p_{00}(\epsilon_t^e) = \frac{\int_{-\infty}^{\tau \sqrt{1-\phi^2}} \Phi_\rho \left( \tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho \epsilon_t^e \right) \varphi(x) dx}{\Phi(\tau \sqrt{1-\phi^2})}$$

$$p_{10}(\epsilon_t^e) = \frac{\int_{\tau \sqrt{1-\phi^2}}^{\infty} \Phi_\rho \left( \tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho \epsilon_t^e \right) \varphi(x) dx}{1 - \Phi(\tau \sqrt{1-\phi^2})}$$

where $\Phi_\rho(x) = \Phi(x/\sqrt{1-\rho^2})$. Time varying and depend on $\epsilon_t^e$. 

\[\]
If $\rho = 0$, reduces to exogenous switching model

$\rho$ governs the fluctuation of transition probabilities
We solve the system of expectational nonlinear difference equations using the guess and verify method.

Davig and Leeper (2006) show that the analytical solution for the model with fixed regime monetary policy process is

\[ \pi_{t+1} = a_1 r_{t+1} + a_2 \epsilon_{t+1} \]

with some constants \( a_1 \) and \( a_2 \).

Motivated by this, we start with the following guess

\[ \pi_{t+1} = a_1 (s_{t+1}, p_{s_{t+1}, 0}(\epsilon_{t+1}^{e})) r_{t+1} + a_2 (s_{t+1}) \epsilon_{t+1}^{e} \]
Analytical Solution

\[ \pi_{t+1} = \frac{\rho_r}{\alpha(s_{t+1})} (\alpha_1 - \alpha_0) p_{s_{t+1}, 0}(\epsilon^e_{t+1}) + \alpha_1 \left( \frac{\alpha_0}{\rho_r} - \mathbb{E}p_{00}(\epsilon^e_{t+1}) \right) + \alpha_0 \mathbb{E}p_{10}(\epsilon^e_{t+1}) r_{t+1} \]

\[-\frac{\sigma_e}{\alpha(s_{t+1})} \epsilon^e_{t+1} \]

\[ a_1(s_{t+1}, p_{s_{t+1}, 0}(\epsilon^e_{t+1})) \]

\[ a_2(s_{t+1}) \]

Limiting Case 1: Exogenous switching solution \((\rho = 0)\)

\[ \pi_{t+1} = \frac{\rho_r}{\alpha(s_{t+1})} (\alpha_1 - \alpha_0) \bar{p}_{s_{t+1}, 0} + \alpha_1 \left( \frac{\alpha_0}{\rho_r} - \bar{p}_{00} \right) + \alpha_0 \bar{p}_{10} r_{t+1} \]

\[-\frac{\sigma_e}{\alpha(s_{t+1})} \epsilon^e_{t+1} \]

\[ a_1(s_{t+1}) \]

\[ a_2(s_{t+1}) \]
Analytical Solution

\[
\pi_{t+1} = \frac{\rho_r}{\alpha(s_{t+1})} r_{t+1} - \frac{\sigma_e}{\alpha(s_{t+1})} \epsilon^{e}_{t+1}
\]

Limiting Case 2: Fixed-regime solution \((\alpha_1 = \alpha_0)\)

\[
\pi_{t+1} = \frac{\rho_r}{\alpha - \rho_r} r_{t+1} - \frac{\sigma_e}{\alpha} \epsilon^{e}_{t+1}
\]
Set a future policy intervention \( I_t = \{ \tilde{\epsilon}_{t+1}^e, \tilde{\epsilon}_{t+2}^e, \ldots, \tilde{\epsilon}_{t+K}^e \} \) and evaluate its effect on future inflation.

Consider the contractionary intervention

\[
I_T = \{4\%, \ldots, 4\%, 0, \ldots, 0\} \text{ with } K = 16, \quad s_T = 0
\]

with \( 8 \) periods \( 8 \) periods

As in Leeper and Zha (2003), define

- **Baseline** = \( \mathbb{E}(\pi_{T+K}|\mathcal{F}_T, s_t = s_T, t = T + 1, \ldots, T + K) \)
- **Direct Effects** = \( \mathbb{E}(\pi_{T+K}|I_T, \mathcal{F}_T, s_t = s_T, t = T + 1, \ldots, T + K) - \text{Baseline} \)
- **Total Effects** = \( \mathbb{E}(\pi_{T+K}|I_T, \mathcal{F}_T) - \text{Baseline} \)
- **Expectations Formation Effects** = Total Effects - Direct Effects
Expectations Formation Effect

\[ \epsilon_{T+1} > 0 \quad \rho > 0 \quad w_{T+2} \uparrow, \quad s_{T+2} \nearrow 1 \quad \rightarrow \text{more likely to switch} \]

\[ \rightarrow \text{price stabilized as agents adjust beliefs on future regimes} \]

\[ \rightarrow \text{black dot signifies period } T + 2 \text{ total effect;} \]
$\epsilon_{T+1} > 0 \quad \rho > 0 \quad w_{T+2} \uparrow, \quad s_{T+2} \nearrow 1 \rightarrow \text{more aggressive}$

endogenous mechanism helps explain price stabilization
Regime Switching Monetary DSGE Model

- **Benchmark Specification**
  - A regime-switching New Keynesian DSGE model, which has been a standard tool for monetary policy analysis.

- **Endogenous Switching in Monetary DSGE Model**
  - Links current monetary policy regime to past fundamental shocks by a policy regime factor.
  - Generates an endogenous feedback between monetary policy stance and observed economic behavior.
  - Solved by perturbation method in Maih and Waggoner (2018) up to first-order.
A Prototypical DSGE Model

We consider the small-scale new Keynesian DSGE model presented in An and Schorfheide (2007), whose essential elements include:

- a representative household
- a continuum of monopolistically competitive firms; each firm produces a differentiated good and faces nominal rigidity in terms of quadratic price adjustment cost
- a cashless economy with one-period nominal bonds
- a monetary authority that controls nominal interest rate as well as a fiscal authority that passively adjusts lump-sum taxes to ensure its budgetary solvency
- a labor-augmenting technology that induces a stochastic trend in consumption and output.
Notations

- $0 < \beta < 1$: the discount factor
- $\tau_c > 0$: the coefficient of relative risk aversion
- $c_t$: the detrended consumption
- $R_t$: the nominal interest rate
- $\pi_t$: the inflation between periods $t - 1$ and $t$
- $E_t$: expectation given information available at time $t$
- $1/\nu > 1$: elasticity of demand for each differentiated good
- $\phi$: the degree of price stickiness
- $\pi$: the steady state inflation
- $y_t$: the detrended output
- $0 \leq \rho_R < 1$: the degree of interest rate smoothing
- $r$: the steady state real interest rate,
- $\psi_{\pi} > 0$, $\psi_y > 0$: the policy rate responsive coefficients
- $y_t^* = (1 - \nu)^{1/\tau_c} g_t$: the detrended potential output
Shocks

$z_t$: an exogenous shock to the labor-augmenting technology
$g_t$: an exogenous government spending shock
$\epsilon_{R,t}$: an exogenous policy shock.

$\ln g_t$ and $\ln z_t$ evolve as autoregressive processes

$$\ln g_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t}$$

and

$$\ln z_t = \rho_z \ln z_{t-1} + \epsilon_{z,t}$$

where $0 \leq \rho_g, \rho_z < 1$ and $g$ is the steady state of $g_t$.

The model is driven by the three innovations $\epsilon_t = [\epsilon_{R,t}, \epsilon_{g,t}, \epsilon_{z,t}]'$ that are serially uncorrelated, independent of each other at all leads and lags, and normally distributed with means zero and standard deviations $(\sigma_R, \sigma_g, \sigma_z)$, respectively.
A Prototypical DSGE Model

Equilibrium conditions in fixed-regime benchmark:

**DIS:** \(1 = \beta^E_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\tau_c} \frac{R_t}{\gamma z_{t+1} \pi_{t+1}} \right]\)

**NKPC:** \(1 = \frac{1 - c_t^{\tau_c}}{\nu} + \phi(\pi_t - \pi) \left[ \left( 1 - \frac{1}{2\nu} \right) \pi_t + \frac{\pi}{2\nu} \right] - \phi \beta^E_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\tau_c} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \pi) \pi_{t+1} \right]\)

**MP:** \(R_t = R_{t-1}^{R^*} e^{\epsilon_{R,t}}, \quad R^*_t = r\pi \left( \frac{\pi_t}{\pi} \right)^{\psi_\pi} \left( \frac{y_t}{y^*_t} \right)^{\psi_y}\)

**ARC:** \(y_t = c_t + \left( 1 - \frac{1}{g_t} \right) y_t + \frac{\phi}{2}(\pi_t - \pi)^2 y_t\)

Regime switching process:

\(\psi_{\pi}(s_t) = \psi_{\pi}^0 (1 - s_t) + \psi_{\pi}^1 s_t, \quad 0 \leq \psi_{\pi}^0 < \psi_{\pi}^1\)
A Prototypical DSGE Model

Implied time-varying transition probabilities to regime-0 are an important part of the model solution

\[ p_{00}(\tilde{\epsilon}_t) = \mathbb{P}(s_{t+1} = 0|s_t = 0, \tilde{\epsilon}_t) = \frac{\int_{-\infty}^{\tau}\frac{1}{\sqrt{1-\alpha^2}} \Phi_{\rho}(\tau - \alpha x/\sqrt{1-\alpha^2} - \rho'\tilde{\epsilon}_t)p_N(x|0,1)dx}{\Phi(\tau\sqrt{1-\alpha^2})} \]

\[ p_{10}(\tilde{\epsilon}_t) = \mathbb{P}(s_{t+1} = 0|s_t = 1, \tilde{\epsilon}_t) = \frac{\int_{\tau}^{\infty}\frac{1}{\sqrt{1-\alpha^2}} \Phi_{\rho}(\tau - \alpha x/\sqrt{1-\alpha^2} - \rho'\tilde{\epsilon}_t)p_N(x|0,1)dx}{1 - \Phi(\tau\sqrt{1-\alpha^2})} \]

where \( \Phi_{\rho}(x) = \Phi(x/\sqrt{1-\rho'\rho}) \), \( \tilde{\epsilon}_t = [\epsilon_{R,t}/\sigma_R, \epsilon_{g,t}/\sigma_g, \epsilon_{z,t}/\sigma_z]' \), and \( \rho = [\rho_{Rv}, \rho_{gv}, \rho_{zv}]' = \text{corr}(\tilde{\epsilon}_t, \nu_{t+1}) \).

The presence of \( s_t \) poses keen computational challenges to solving the model. When \( \tilde{\epsilon}_t \) and \( \nu_{t+1} \) are orthogonal (i.e., \( \rho = 0_{3\times1} \)), our model reduces to the conventional Markov switching model.
Regime Switching DSGE Model

Switching in state space form
\[ y_t = D_{s_t} + Z_{s_t} x_t + F_{s_t} z_t + u_t, \quad u_t \sim \mathcal{N}(0, \Omega_{s_t}) \]
\[ x_t = C_{s_t} + G_{s_t} x_{t-1} + E_{s_t} z_t + M_{s_t} \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \Sigma_{s_t}) \]

New regime switching
- state process \( s_t \) driven by \( w_t \) as \( s_t = 1\{w_t \geq \tau\} \)
- latent factor \( w_t = \alpha w_{t-1} + v_t, \quad \tilde{\epsilon}_t = \Sigma_{s_t}^{-1/2} \), and
\[
\begin{pmatrix}
\tilde{\epsilon}_t \\
v_{t+1}
\end{pmatrix}
\sim \mathcal{N}
\begin{pmatrix}
\begin{pmatrix}0_{n \times 1} \\
0
\end{pmatrix}, \\
\begin{pmatrix}I_n & \rho \end{pmatrix}
\end{pmatrix},
\quad \rho' \rho < 1
\]

Advantage
- \( s_t \) is endogenous \( \rightarrow \) systematically affected by observables
- \( s_t \) can be nonstationary \( \rightarrow \) allow for regime persistency
- \( w_t \) is continuous \( \rightarrow \) directly related to other variables
Endogenous Feedback Mechanism

Latent factor

\[ w_{t+1} = \alpha w_t + \rho' \tilde{\epsilon}_t + \sqrt{1 - \rho' \rho} \eta_{t+1}, \]

where \( \eta_t \sim \mathbb{N}(0, 1) \), \( \rho = (\rho_{Rv}, \rho_{gv}, \rho_{zv})' \) and \( \rho' \rho < 1 \).

Forecast error variance decomposition:

\[
\text{Var}_t(w_{t+h}) = \sum_{k=1}^{3} \sum_{j=1}^{h} \rho_k^2 \alpha^{2(h-j)} + \sum_{j=1}^{h} \left(1 - \sum_{k=1}^{3} \rho_k^2 \right) \alpha^{2(h-j)}
\]

\( \rho_k^2 \): contribution of \( \tilde{\epsilon}_k \) to regime change

\( 1 - \rho' \rho \): contribution of \( \eta \) to regime change

Key to quantifying macro origins of monetary policy shifts
Taking Model to Data

- Quarterly observations from 1954:Q3 to 2007:Q4
  - YGR: per capita real output growth
  - INF: annualized inflation rate
  - INT: effective federal funds rate

- Measurement equations

\[
\begin{pmatrix}
YGR_t \\
\text{INF}_t \\
\text{INT}_t
\end{pmatrix}
= \begin{pmatrix}
\gamma^{(Q)} \\
\pi^{(A)} \\
\pi^{(A)} + r^{(A)} + 4\gamma^{(Q)}
\end{pmatrix}
+ 100
\begin{pmatrix}
\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t \\
4\hat{\pi}_t \\
4\hat{R}_t
\end{pmatrix}
\]

- Transition Equations
  - perturbation solution of Maih & Waggoner (2018)
  - RISE MATLAB toolbox developed by Maih
Endogenous-Switching Kalman Filter

Augmented state space form

\[ y_t = \begin{bmatrix} D_{st} + F_{st} z_t \end{bmatrix} + \begin{bmatrix} Z_{st} & 0_{l \times n} \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + u_t \]

\[ \begin{bmatrix} x_t \\ d_t \\ \varsigma_t \end{bmatrix} = \begin{bmatrix} C_{st} + E_{st} z_t \\ 0_{n \times 1} \\ \tilde{C}_{st} \end{bmatrix} + \begin{bmatrix} G_{st} & 0_{m \times n} \\ 0_{n \times m} & 0_{n \times n} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ d_{t-1} \end{bmatrix} + \begin{bmatrix} M_{st} \Sigma_{s_{st}}^{1/2} \\ \tilde{M}_{st} \end{bmatrix} \tilde{\epsilon}_t \]

- Key features of our filter
  - marginalization-collapsing procedure of Kim (1994)
  - time-varying transition probabilities
  - filtered/smoothed regime factor as by-product
Filtering Algorithm

Step 1: Forecasting

\[ s_{t|t-1}^{(i,j)} = \tilde{C}_j + \tilde{G}_j s_{t-1|t-1}^{i} \]
\[ P_{t|t-1}^{(i,j)} = \tilde{G}_j P_{t-1|t-1}^{i} \tilde{G}_j' + \tilde{M}_j \tilde{M}_j' \]
\[ p_{t|t-1}^{(i,j)} = \int_{\mathbb{R}} \mathbb{P}(s_t = j|s_{t-1} = i, \lambda_{t-1}) p_{t-1|t-1}^{i} p(\lambda_{t-1}|\mathcal{F}_{t-1}) d\lambda_{t-1} \]

- To compute e.g., \( p_{t|t-1}^{(0,0)} \)
  - since \( \lambda_{t-1} = \rho' \tilde{\epsilon}_{t-1} \) is latent, approximate \( p(\lambda_{t-1}|\mathcal{F}_{t-1}) \) by
    \[ p_N(\lambda_{t-1} | \rho' s_{d,t-1|t-1}^{0}, \rho' P_{d,t-1|t-1}^{0}) \]
  - time-varying transition probability \( \mathbb{P}(s_t = 0|s_{t-1} = 0, \lambda_{t-1}) \)
    \[ \int_{-\infty}^{\tau \sqrt{1-\alpha^2}} \Phi_\rho(\tau - \alpha x/\sqrt{1-\alpha^2} - \lambda_{t-1}) p_N(x|0, 1) dx \]
    \[ \Phi(\tau \sqrt{1-\alpha^2}) \]
Step 2: Likelihood evaluation

\[ y_{t|t-1}^{(i,j)} = \tilde{D}_j + \tilde{Z}_j s_{t|t-1}^{(i,j)} \]

\[ F_{t|t-1}^{(i,j)} = \tilde{Z}_j P_{t|t-1}^{(i,j)} \tilde{Z}'_j + \Omega_j \]

\[ p(y_t|F_{t-1}) = \sum_{j=0}^{1} \sum_{i=0}^{1} p_N(y_t|y_{t|t-1}^{(i,j)}, F_{t|t-1}^{(i,j)}) p_{t|t-1}^{(i,j)} \]
Filtering Algorithm (Cont’d)

Step 3: Updating

\[ s_{t|t}^{(i,j)} = s_{t|t-1}^{(i,j)} + P_{t|t-1}^{(i,j)} \tilde{Z}_j' (F_{t|t-1}^{(i,j)})^{-1} (y_t - y_{t|t-1}^{(i,j)}) \]
\[ P_{t|t}^{(i,j)} = P_{t|t-1}^{(i,j)} - P_{t|t-1}^{(i,j)} \tilde{Z}_j' (F_{t|t-1}^{(i,j)})^{-1} \tilde{Z}_j P_{t|t-1}^{(i,j)} \]
\[ p_{t|t}^{(i,j)} = p_N(y_t|y_{t|t-1}^{(i,j)}, F_{t|t-1}^{(i,j)}) P_{t|t-1}^{(i,j)} \]

- History truncation

  - collapse \((s_{t|t}^{(i,j)}, P_{t|t}^{(i,j)})\) into \((s_{t|t}^{j}, P_{t|t}^{j})\)
  - further collapse \((s_{t|t}^{j}, P_{t|t}^{j})\) into \((s_{t|t}, P_{t|t})\)
## Prior Distributions

### Fixed Regime

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Density</th>
<th>Para (1)</th>
<th>Para (2)</th>
<th>[5%, 95%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_c$, coefficient of relative risk aversion</td>
<td>G</td>
<td>2.00</td>
<td>0.50</td>
<td>[1.25, 2.89]</td>
</tr>
<tr>
<td>$\kappa$, slope of new Keynesian Phillips curve</td>
<td>G</td>
<td>0.20</td>
<td>0.10</td>
<td>[0.07, 0.39]</td>
</tr>
<tr>
<td>$\psi_{\pi}$, interest rate response to inflation</td>
<td>G</td>
<td>1.50</td>
<td>0.25</td>
<td>[1.11, 1.93]</td>
</tr>
<tr>
<td>$\psi_y$, interest rate response to output</td>
<td>G</td>
<td>0.50</td>
<td>0.25</td>
<td>[0.17, 0.97]</td>
</tr>
<tr>
<td>$\pi(A)$, s.s. annualized real interest rate</td>
<td>G</td>
<td>0.50</td>
<td>0.10</td>
<td>[0.35, 0.68]</td>
</tr>
<tr>
<td>$\pi(A)$, s.s. annualized inflation rate</td>
<td>G</td>
<td>3.84</td>
<td>2.00</td>
<td>[1.24, 7.61]</td>
</tr>
<tr>
<td>$\gamma(Q)$, s.s. technology growth rate</td>
<td>N</td>
<td>0.47</td>
<td>0.20</td>
<td>[0.14, 0.80]</td>
</tr>
<tr>
<td>$\rho_R$, persistency of monetary shock</td>
<td>B</td>
<td>0.50</td>
<td>0.10</td>
<td>[0.34, 0.66]</td>
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<tr>
<td>$\rho_g$, persistency of spending shock</td>
<td>B</td>
<td>0.50</td>
<td>0.10</td>
<td>[0.34, 0.66]</td>
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<tr>
<td>$\rho_z$, persistency of technology shock</td>
<td>B</td>
<td>0.50</td>
<td>0.10</td>
<td>[0.34, 0.66]</td>
</tr>
<tr>
<td>$100\sigma_R$, scaled s.d. of monetary shock</td>
<td>IG-1</td>
<td>0.40</td>
<td>4.00</td>
<td>[0.08, 1.12]</td>
</tr>
<tr>
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<td>IG-1</td>
<td>0.40</td>
<td>4.00</td>
<td>[0.08, 1.12]</td>
</tr>
<tr>
<td>$\nu$, inverse of demand elasticity</td>
<td>B</td>
<td>0.10</td>
<td>0.05</td>
<td>[0.03, 0.19]</td>
</tr>
<tr>
<td>$1/g$, s.s. consumption-to-output ratio</td>
<td>B</td>
<td>0.85</td>
<td>0.10</td>
<td>[0.66, 0.97]</td>
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### Threshold Switching

<table>
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<tr>
<th>Parameter</th>
<th>Density</th>
<th>Para (1)</th>
<th>Para (2)</th>
<th>[5%, 95%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0_{\pi}$, $\psi_{\pi}$ under regime-0</td>
<td>G</td>
<td>1.00</td>
<td>0.10</td>
<td>[0.84, 1.17]</td>
</tr>
<tr>
<td>$\psi^1_{\pi}$, $\psi_{\pi}$ under regime-1</td>
<td>G</td>
<td>2.00</td>
<td>0.25</td>
<td>[1.61, 2.43]</td>
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<tr>
<td>$\alpha$, persistency of latent factor</td>
<td>B</td>
<td>0.90</td>
<td>0.05</td>
<td>[0.81, 0.97]</td>
</tr>
<tr>
<td>$\tau$, threshold level</td>
<td>N</td>
<td>-1.00</td>
<td>0.50</td>
<td>[−1.82, −0.18]</td>
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<tr>
<td>$\rho_{Rv}$, endogeneity from monetary shock</td>
<td>U</td>
<td>-1.00</td>
<td>1.00</td>
<td>[−0.90, 0.90]</td>
</tr>
<tr>
<td>$\rho_{gv}$, endogeneity from spending shock</td>
<td>U</td>
<td>-1.00</td>
<td>1.00</td>
<td>[−0.90, 0.90]</td>
</tr>
<tr>
<td>$\rho_{zv}$, endogeneity from technology shock</td>
<td>U</td>
<td>-1.00</td>
<td>1.00</td>
<td>[−0.90, 0.90]</td>
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### Posterior Estimates

<table>
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<tr>
<th>Parameter</th>
<th>No Switching Model</th>
<th>Regime Switching Model</th>
</tr>
</thead>
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<tr>
<td></td>
<td>Mode</td>
<td>Median</td>
</tr>
<tr>
<td>Fixed Regime</td>
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</tr>
<tr>
<td>( \tau_c )</td>
<td>3.54</td>
<td>3.29</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>0.47</td>
<td>0.49</td>
</tr>
<tr>
<td>( \psi_\pi )</td>
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<td>1.01</td>
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<td>( \psi_y )</td>
<td>0.15</td>
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<tr>
<td>( r(A) )</td>
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<td>0.47</td>
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<tr>
<td>( \pi(A) )</td>
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<tr>
<td>( \gamma(Q) )</td>
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</tr>
<tr>
<td>( \rho_R )</td>
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<td>0.70</td>
</tr>
<tr>
<td>( \rho_g )</td>
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<tr>
<td>( \rho_z )</td>
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<td>0.95</td>
</tr>
<tr>
<td>100( \sigma_R )</td>
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<td>0.26</td>
</tr>
<tr>
<td>100( \sigma_g )</td>
<td>1.00</td>
<td>1.03</td>
</tr>
<tr>
<td>100( \sigma_z )</td>
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<td>0.07</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.04</td>
<td>0.09</td>
</tr>
<tr>
<td>1/( g )</td>
<td>0.87</td>
<td>0.87</td>
</tr>
<tr>
<td>Threshold Switching</td>
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<td></td>
</tr>
<tr>
<td>( \psi^0_\pi )</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( \psi^k_\pi )</td>
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<td>–</td>
</tr>
<tr>
<td>( \alpha )</td>
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<td>–</td>
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<tr>
<td>( \tau )</td>
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<tr>
<td>( \rho_{R\nu} )</td>
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<td>–</td>
</tr>
<tr>
<td>( \rho_{g\nu} )</td>
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<td>–</td>
</tr>
<tr>
<td>( \rho_{z\nu} )</td>
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<td>–</td>
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<td>Log marginal likelihood</td>
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<td>Bayes factor vs. no switching</td>
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</table>
Evidence of Endogeneity

- $\rho_{gv} > 0$: MP is ‘leaning against the wind’
- $\rho_{zv} < 0$: MP is promoting economic growth
Extracted Regime Factor

- Sluggish switching b/w more and less active regimes
- Timing and nature are consistent with narrative record
Main Findings

- Prima facie evidence of endogeneity in monetary policy shifts
  - MP is ‘leaning against the wind’—expansionary gvt spending shock increases the likelihood of more active regime.
  - MP is promoting long-term growth—favorable tech shock decreases the likelihood of shifting into more active regime.
  - overall, non-policy shocks have played a predominant role in driving regime changes during the post-World War II period.

- Estimated regime factor identifies MP as slowly fluctuating between more and less active regimes, in ways consistent with conventional view and narrative record.

- Endogenizing regime changes in monetary DSGE models provides a promising venue for understanding the purposeful nature of monetary policy.