Shephard's Lemma as a Partial Differential Equation

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Abstract

This paper studies a partial differential equation that is called Shephard's lemma in economics. It is known that if the demand function is continuously differentiable, then the local existence of this equation is equivalent to the symmetry of the Slutsky matrix. We extend this result to the class of locally Lipschitz function. Furthermore, we show by using this result that in locally Lipschitz environments with some additional requirements, the symmetry and negative semi-definiteness of the Slutsky matrix is a necessary and sufficient condition for reverse calculation to a utility function.

Keywords: demand function, integrability, income-Lipschitzian, expenditure function, Shephard's lemma.

JEL Classification Numbers: D11, C61, C65.

1 Introduction

Hurwicz and Uzawa (1971) showed that if a continuously differentiable function $f(p, m)$ satisfies the Walras' law and some additional assumption, then it is a demand function of some preference relation if and only if the Slutsky

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matrix is symmetric and negative semi-definite. This result is strengthened by Hosoya (2017), which showed that 'some additional assumption' is not needed. Moreover, Hosoya (2017) presented a concrete reverse calculation procedure for utility function from this demand function. This result expected to be able to use in microeconometric theory.

In their results, a partial differential equation plays an important roll. This equation is called the Shephard's lemma. In classical consumer theory, it is known that if $f(p,m)$ is a demand function of some preference relation and $E^x(p)$ is the expenditure function of the same preference relation, then

$$DE^x(p) = f(p, E^x(p))$$

for every p. This equation is called the Shephard's lemma. Moreover, if the given preference relation is continuous, then x is preferred to y if and only if $E^x(p) \geq E^y(p)$. This result is crucial for the reverse calculation procedure from a demand function to a preference relation.

We shall explain this reverse calculation method. The main idea is separated into three steps.

- *The first step.* Suppose that $f(p,m)$ is a demand function of u, and we have already known u. Then, we can calculate the expenditure function $E^x(p)$. Fix any $\bar{p}$ and consider the function $x \mapsto E^x(\bar{p})$. Then, this function repair the information of u.

- *The second step.* Suppose that we know that $f(p,m)$ is a demand function of some u, but there is no known information on u. In this case, we use the Shephard's lemma to calculate the function E^x. That is, we can obtain the expenditure function by solving the above partial differential equation. If we can solve this equation, then $x \mapsto E^x(\bar{p})$ have all information about u.

- *The third step.* In general, we do not know whether the given function $f(p,m)$ is a demand function of some preference relation or not. Hosoya (2017) showed that if f is continuously differentiable, then the global existence of the **concave** solution of the Shephard's lemma is the necessary and sufficient condition of the existence of the corresponding preference relation to f. Hosoya (2018) showed that even if f is not continuously differentiable but only differentiable and locally Lipschitz, this result still holds.

However, in economics, sometimes a demand function f is not necessarily differentiable. For example, if $u(x) = x_1x_2$, then the corresponding demand function is differentiable. However, $u(x) = (x_1 + \varepsilon)(x_2 + \varepsilon)$ for some $\varepsilon > 0$, then the corresponding demand function is not necessarily differentiable. That is, the differentiability of the demand function is a very sensitive property to perturbation, and thus we remove the differentiability assumption of the demand function from theory.

In this study, we treat the Shephard's lemma, and show that even in locally Lipschitz environments, almost the same results as in differential case hold. That is, 1) the local existence of the solution is equivalent to the symmetry of the Slutsky matrix, and 2) the global existence of the concave solution is equivalent to the symmetry and negative semi-definiteness of the Slutsky matrix. (Theorems 1-2, Corollary 1)

However, the existence of the concave solution of the Shephard's lemma does not assure the effectiveness of our reverse calculation procedure in locally Lipschitz environments. To solve this problem, we need an additional requirement: that is, pseudo continuously differentiable requirement is needed. This assumption is weaker than the differentiability, and under this assumption, we can assure the effectiveness of our procedure. (Theorem 3, Corollaries 2-3)

There is an important class of pseudo continuosly differentiable functions: that is, patchily smooth function must be pseudo continuously differentiable. (Theorem 4) In consumer theory, almost all demand functions that admit corner solutions are patchily smooth, and thus our results is applicable for almost all cases.

In section 2, we introduce several definitions, and results of ordinary differential equations that is needed in this paper. Moreover, we interpret our main idea. In section 3, we exhibit main results. The proofs of all results are in section 4.

2 Preliminary

2.1 Definitions of Notations

We consider that the notation Ω denotes the consumption space, and assume that Ω is a subset of $\mathbb{R}^n_+$, where $n \ge 2$ be given. We write $x \gg y$ if $x_i > y_i$ for any i.

Choose any binary relation $\succsim$ on Ω, that is, $\succsim \subset \Omega^2$. We write $x \succsim y$ if $(x, y) \in \succsim$ and $x \not\succsim y$ if $(x, y) \notin \succsim$. We say that $\succsim$ is

- **complete** if for any $x, y \in \Omega$, either $x \succsim y$ or $y \succsim x$,
- **transitive** if for any $x, y, z \in \Omega$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$,
- **continuous** if $\succsim$ is closed in Ω^2,

- **upper semi-continuous** if for any $x \in \Omega$, the set $\{y \in \Omega | y \succsim x\}$ is closed in Ω,
- **monotone** if for any $x, y \in \Omega$, $x \succsim y$ and $y \not\succsim x$ when $x \gg y$,
- **strictly convex** if for any $x, y \in \Omega$ with $x \succsim y$ and $x \neq y$, and $t \in]0,1[$, $(1-t)x + ty \succsim y$ and $y \not\succsim (1-t)x + ty$.

We call a binary relation $\succsim$ on Ω a **preference relation** if it is complete and transitive. If $\succsim$ is a preference relation, then we write $x \succ y$ if $x \succsim y$ and $y \not\succsim x$, and $x \sim y$ if $x \succsim y$ and $y \succsim x$.

Suppose that $u : \Omega \to \mathbb{R}$ satisfies the following condition:

$$u(x) \geq u(y) \Leftrightarrow x \succsim y.$$

Then, we say that u represents $\succsim$, or u is a **utility function** of $\succsim$. Note that if some function u represents $\succsim$, then $\succsim$ is a preference relation, and $\succsim$ is continuous (resp. upper semi-continuous) if u is continuous. (resp. upper semi-continuous.)[1]

Next, we call a continuous function $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \Omega$ a *candidate of demand* (CoD) function if it satisfies budget inequality: that is,

$$p \cdot f(p, m) \leq m,$$

for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. If

$$p \cdot f(p, m) = m$$

for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, then this CoD function is said to satisfy the Walras' law.

Now, let $\succsim$ be a binary relation on Ω and define

$$f^{\succsim}(p, m) = \{x \in \Omega | \forall y, p \cdot y \leq m \Rightarrow x \succsim y\}.$$

If $\succsim$ is strictly convex, then $f^{\succsim}$ is a CoD function. In this case, we call $f^{\succsim}$ a *demand function of* $\succsim$ and say that $\succsim$ corresponds with f (or, f corresponds with $\succsim$) if $f = f^{\succsim}$. If u represents $\succsim$, then $f^{\succsim}$ is sometimes written as f^u, and we say that u corresponds with f (or, f corresponds with u) if $f^u = f$. Note that if $\succsim$ is monotone, then $f^{\succsim}$ satisfies the Walras' law.

Suppose that $f : P \to \mathbb{R}^n$ and $P \subset \mathbb{R}^m \times \mathbb{R}^\ell$. Note that f is possibly not a CoD function. This function $f(x, y)$ is said to be **locally Lipschitz in** x

[1] Conversely, if a preference relation $\succsim$ is continuous, (resp. upper semi-continuous,) then there is a continuous (resp. upper semi-continuous) function u that represents $\succsim$. This result is obtained by the second countability of Ω. See Debreu (1954).

if and only if for every compact set $C \subset P$, there exists $L > 0$ such that for any $y \in \mathbb{R}^\ell$ and $x_1, x_2 \in \mathbb{R}^m$ with $(x_i, y) \in C$,

$$\|f(x_1, y) - f(x_2, y)\| \le L\|x_1 - x_2\|.$$

Similarly, f is said to be **locally Lipschitz** if for any compact set $C \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there exists $L > 0$ such that for any $(x_1, y_1), (x_2, y_2) \in C$,

$$\|f(x_1, y_1) - f(x_2, y_2)\| \le L\|(x_1, y_1) - (x_2, y_2)\|.$$

Note that if f is a CoD function, the local Lipschitz condition in m is called **income-Lipschitzian**.[2]

Finally, suppose that $f : P \to \mathbb{R}^n$ and $P \subset \mathbb{R}^n \times \mathbb{R}$ is open, and is differentiable at (p, m). We define

$$S_f(p, m) = D_p f(p, m) + D_m f(p, m) f^T(p, m).$$

That is, the (i, j)-th element $s_{ij}(p, m)$ of $S_f(p, m)$ is

$$\frac{\partial f_i}{\partial p_j}(p, m) + \frac{\partial f_i}{\partial m}(p, m) f_j(p, m).$$

This matrix-valued function $S_f(p, m)$ is called the **Slutsky matrix**. We say that f satisfies (S) (resp. (NSD)) if and only if f is differentiable at everywhere and $S_f(p, m)$ is always symmetric (resp. negative semi-definite). Similarly, we say that f satisfies (S)-a.e. (resp. (NSD)-a.e.) if and only if f is differentiable at almost everywhere and $S_f(p, m)$ is symmetric (resp. negative semi-definite) at almost everywhere.

2.2 The Basic Knowledge on Ordinary Differential Equations

In this section, we introduce several basic results on ordinary differential equations (ODE). The proof of FACTs 1-3 are in Hosoya (2018). In proof section, we provide proofs of FACTs 4-5.

Consider the following ODE:

$$\dot{x}(t) = h(t, x(t)),\ x(t_0) = x_0,$$

where $\dot{x}$ denotes the derivative of x with respect to t. A **solution** of this equation is a C^1-class function x defined on an interval I containing t_0 such

[2]This name is in Mas-Colell (1977).

that $x(t_0) = x_0$ and $\dot{x}(t) = h(t, x(t))$ for any $t \in I$.[3] We assume that $h : P \to \mathbb{R}^m$, where $P \subset \mathbb{R} \times \mathbb{R}^m$ is open and $(t_0, x_0) \in P$, and that h is continuous in (t, x) and locally Lipschitz in x. Then, the following fact holds.

FACT 1. There is a solution of the above equation defined on some open interval including t_0. Moreover, for any two solutions x, y, $x(t) = y(t)$ for all t included in the intersection of the domains of x and y. (This fact is known as the Picard-Lindelöf's theorem.)

A solution y is called an **extension** of a solution x if and only if the domain of y includes the domain of x. A solution x is **nonextendable** if and only if there is no extension of x except for x itself. Two facts on nonextendable solutions hold.

FACT 2. There uniquely exists a nonextendable solution defined on an open interval.

FACT 3. If x is a nonextendable solution defined on $]a, b[$, then for any compact set $C \subset P$, there exist $\hat{t}, \bar{t} \in]a, b[$ such that $(t, x(t)) \notin C$ if either $a < t \le \hat{t}$ or $\bar{t} \le t < b$.

Second, consider the following parametrized ODE:

$$\dot{x}(t) = h(t, x(t); y),\ x(t_0) = x_0.$$

We assume that $h : \tilde{P} \to \mathbb{R}^m$, where $\tilde{P} \subset \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^\ell$ is open, and that h is continuous in (t, x, y) and locally Lipschitz in x. Then, the following fact holds.

FACT 4. There uniquely exists a function $x(t; y)$ defined on some open set in $\mathbb{R} \times \mathbb{R}^\ell$ such that if y is fixed and $(t_0, x_0, y) \in \tilde{P}$, then $x(t; y)$ is a nonextendable solution of the above problem. Moreover, x is continuous in (t, y), and if h is locally Lipschitz in (x, y), then x is also locally Lipschitz.

We also call this function $x(t; y)$ a **nonextendable solution**.

We can easily extend this result. Consider the following parametrized ODE:

$$\dot{x}(t) = h(t, x(t); y),\ x(t_0) = z.$$

The assumption of h is the same as above. Then, the following fact holds.

[3]In this paper, we call a subset I of $\mathbb{R}$ an **interval** if it is a convex set and includes at least two elements.

FACT 5. There uniquely exists a function $x(t; y, z)$ defined on some open set in $\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}^m$ such that if $(t_0, y, z) \in \tilde{P}$, then $x(t; y, z)$ is a nonextendable solution of the above problem. Moreover, x is continuous, and if h is locally Lipschitz in (x, y), then x is also locally Lipschitz.

Again, we call this function $x(t; y, z)$ a **nonextendable solution**.

2.3 The Basic Knowledges on Shephard's Lemma

Let f be a continuous CoD function that satisfies Walras' law. Consider the following two properties.

(I) For every $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there exists a concave solution $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ of the following partial differential equation (PDE):

$$DE(q) = f(q, E(q)), \tag{1}$$

with initial value condition $E(p) = m$.

(II) If $x \neq y,\ x = f(p, m),\ y = f(q, w)$, and $w \geq E(q)$ for a solution E of PDE (1) with $E(p) = m$, then $p \cdot y > m$.

Suppose that f is a demand function of some preference relation $\succsim$. Fix $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, and define

$$E^x(q) = \inf\{q \cdot y | y \succsim x\},$$

where $x = f(p, m)$. Then, it is known that E^x is a concave solution of (1) with $E^x(p) = m$. This result is called the **Shephard's lemma**.[4] Therefore, f satisfies (I).

Next, suppose that $f = f^{\succsim}$ is income-Lipschitzian, and choose any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$ and $q \in \mathbb{R}^n_{++}$. Let $x = f(p, m)$ and $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ be a solution of (1) with $E(p) = m$, and define $c_1(t) = E^x((1 - t)p + tq)$ and $c_2(t) = E((1 - t)p + tq)$. Then,

$$\dot{c}_i(t) = f((1 - t)p + tq, c_i(t)) \cdot (q - p),\ c_i(0) = m.$$

By Picard-Lindelöf's uniqueness theorem (FACT 1) of the solution of ODE,[5] we have $c_1 \equiv c_2$, and especially,

$$E(q) = c_2(1) = c_1(1) = E^x(q).$$

[4] A modern proof of this result is in Hosoya (Lemma 7, 2017).

[5] To apply this theorem, we need the income-Lipschitzian property of f.

Therefore, the solution of (1) is unique.

Thirdly, in addition to above assumptions, suppose that $\succsim$ is upper semi-continuous, and let $x \neq y$, $x = f(p, m)$, $y = f(q, w)$ and $w \geq E^x(q)$. By definition of $E^x(q)$, for every $\varepsilon > 0$, there exists $z \in \Omega$ such that $q \cdot z < E^x(q) + \varepsilon$ and $z \succsim x$. Because $f(q, E^x(q) + \varepsilon) \succsim z$, we have

$$f(q, E^x(q) + \varepsilon) \succsim x$$

by transitivity. Because f is continuous and $\succsim$ is upper semi-continuous, we have

$$f(q, E^x(q)) \succsim x,$$

and because $w \geq E^x(q)$, we have

$$y = f(q, w) \succsim f(q, E^x(q)).$$

Therefore, by transitivity,

$$y \succsim x,$$

and thus, we must have

$$p \cdot y > m.$$

Hence, f satisfies (II).

The above arguments shows that under some mild requirements, f is a demand function only if (I) and (II) hold. Conversely, suppose that f is a continuous CoD function that satisfies income-Lipschitzian property and Walras' law. Actually, the statement (I) (resp. the statement (II)) is the claim of Lemma 1 (resp. Lemma 4) of Hurwicz and Uzawa (1971). Suppose these statements hold, and choose any $\bar{p} \in \mathbb{R}^n_{++}$. If x is not in the range of f, then define $u_{f,\bar{p}}(x) = 0$. If $x = f(p, m)$, choose a solution $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ of (1) with $E(p) = m$, and define $u_{f,\bar{p}}(x) = E(\bar{p})$. Then, we can show, by using the same arguments as Hurwicz-Uzawa, that $f = f^{u_{f,\bar{p}}}$, and thus f is a demand function.

Therefore, (I) and (II) are the crucial condition for a CoD function to be able to calculate the corresponding utility function. Particularly, an existence result of the solution of PDE (1) has very important role in integrability theory.

2.4 The Basic Idea

To solve (1) directly is very difficult because this PDE has a serious nonlinearity. However, there is a method to reduce the problem in a simple ODE. First, let f be an arbitrary continuous income-Lipschitzian CoD function that satisfies the Walras' law. Suppose that $E : U \to \mathbb{R}$ is a solution of (1) with $E(p) = m$, where U is open and convex. Consider the following ODE:

$$\dot{c}(t) = f((1-t)p + tq, c(t)) \cdot (q - p), \tag{2}$$

with $c(0) = w$. Let $c(t; q, w)$ be the nonextendable solution of (2) with $c(0) = w$. If we define $c(t) = E((1-t)p + tq)$, then this is a solution of (2), and thus by Picard-Lindelöf's theorem, we have $c(1; q, m) = E(q)$.

Therefore, if U is an open and convex neighborhood of p, there exists a solution $E : U \to \mathbb{R}$ with $E(p) = m$ only if the domain of $(t, q) \mapsto c(t; q, m)$ includes $[0, 1] \times U$. Moreover, in this case $E(q) = c(1; q, m)$. Clearly, $c(t; p, m) \equiv m$, and thus the domain of $t \mapsto c(t; p, m)$ is $\mathbb{R}$. By FACT 4, the domain of the nonextendable solution $c(t; q, w)$ of ODE (2) is open, and c is continuous in (t, q, w). Therefore, if q is sufficiently near to p, the domain of $t \mapsto c(t; q, m)$ includes $[0, 1]$.

Let U be some open and convex neighborhood of p such that for every $q \in U$, $c(t; q, m)$ is defined for every $t \in [0, 1]$. The following results are well-known:

1. If f is continuously differentiable, then $E : q \mapsto c(1; q, m)$ is a solution of (1) if and only if f satisfies (S). (Theorem 10.9.4, Dieudonne (2006))

2. If f is differentiable and locally Lipschitz, then $E : q \mapsto c(1; q, m)$ is a solution of (1) if and only if f satisfies (S). (Nikliborc (1929), Hosoya (2018))

However, we want to treat some nondifferentiable f, and thus the above results cannot be used. Fortunately, if f is locally Lipschitz, then by Rademacher's theorem, f is differentiable at almost everywhere, and hence the Slutsky matrix $S_f(p, m)$ can be defined at almost everywhere. The result in which we want to obtain is as follows: $E : q \mapsto c(1; q, m)$ is a solution of (1) if and only if f satisfies (S)-a.e..

However, there is a serious difficulty. Because the image of $q \mapsto (q, c(1; q, m))$ is just n-dimensional, this set is probably Lebesgue measure zero, and thus maybe f is not differentiable at every $(q, E(q))$! This is a serious problem. Thus, we should use the parameter w and consider $E(q, w) = c(1; q, w)$ instead of $E(q) = c(1; q, m)$. This is the main idea for our main theorem.

3 Results

3.1 Local Existence Theorem

Our first result is as follows.

Theorem 1. Suppose that $f : P \to \mathbb{R}^n$, where $P \subset \mathbb{R}^n \times \mathbb{R}$ is open and f is locally Lipschitz. Then, the following two statements are equivalent.

1. For every $(p, m) \in P$, there exists an open and convex neighborhood U of p such that the PDE (1) has a solution $E : U \to \mathbb{R}$ with $E(p) = m$.

2. f satisfies (S)-a.e..

3.2 Global Existence Theorem

By Theorem 1, we have that the local existence of the solution of (1) is equivalent to (S)-a.e.. However, our statement (I) requires the global existence of the solution: that is, the domain U of E should be the same as $\mathbb{R}^n_{++}$ itself. The following result gives such a result.

Theorem 2. Suppose that $f : P \to \mathbb{R}^n$, where $P \subset \mathbb{R}^n \times \mathbb{R}$ is open and f is locally Lipschitz. Moreover, suppose that f satisfies (S)-a.e.. Choose any $(p, m) \in P$. Then, for any convex neighborhood C of p, the following two statements are equivalent.

1. There uniquely exists a solution $E : C \to \mathbb{R}$ of PDE (1) with $E(p) = m$.[6]

2. For every $q \in C$, the domain of the mapping $t \mapsto c(t; q, m)$ includes $[0, 1]$, where c is the nonextendable solution of ODE (2).

Moreover, in this case, $E(q) = c(1; q, m)$ for every $q \in C$.

As its corollary, we can show the following result.

Corollary 1. Suppose that f is a CoD function that is locally Lipschitz and satisfies Walras' law. Then, the following statements are equivalent.

1. For every $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there uniquely exists a concave solution $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ of the PDE (1) with $E(p) = m$.

2. f satisfies (S)-a.e. and (NSD)-a.e..

[6] If C is open, then E is a solution of (1) if and only if $DE(q) = f(q, E(q))$ for every $q \in C$. If C is not necessarily open, then E is a solution of (1) if and only if for every $q \in C$, there is a local extension $\tilde{E}$ of E that is defined on some neighborhood of q and $D\tilde{E}(q) = f(q, \tilde{E}(q))$.

3.3 Pseudo Continuous Differentiability and Patchy Smoothness

By Corollary 1, we have that for a locally Lipschitz CoD with Walras' law, (I) is equivalent to (S)-a.e. and (NSD)-a.e.. However, for (II), these requirements are not sufficient.[7] Thus, the following notion is needed.

Definition 2. Suppose that $P \subset \mathbb{R}^n \times \mathbb{R}$ is open and $f : P \to \mathbb{R}^n$ is a locally Lipschitz function. Let $(q, w) \in \mathbb{R}^n \times \mathbb{R}$, and define

$$df_{q,w}(p, m) = \left\{ \lim_{k\to\infty} \frac{f(p + t_k q, m + t_k w) - f(p, m)}{t_k} \middle| t_k \downarrow 0 \right\},$$

for every $(p, m) \in P$. We say that f is **pseudo continuously differentiable** if for every $(p, m) \in P$ and $(q, w) \in \mathbb{R}^n \times \mathbb{R}$, there exists $v \in df_{q,w}(p, m)$ and a convergent sequence $((p_k, m_k))$ to (p, m) such that f is differentiable at (p_k, m_k), and

$$v = \lim_{k\to\infty} Df(p_k, m_k)(q, w).$$

Theorem 3. If f is a Locally Lipschitz and pseudo continuously differentiable CoD function that satisfies the Walras' law. Then, (I) implies (II).

Clearly, if f is differentiable, then it is pseudo continuously differentiable. (Choose $(p_k, m_k) \equiv (p, m)$.) Therefore, the following corollary is immediately obtained.

Corollary 2. If f is a Locally Lipschitz and differentiable CoD function that satisfies the Walras' law, then (I) implies that (II).

This result has a surprising corollary. In Hosoya (2018), the following result is proved. If f is a Locally Lipschitz and differentiable CoD function, then it is a demand function of some utility function if and only if (S) and (NSD) hold. Meanwhile, we have already argued that (I) and (II) is a sufficient condition for such CoD to be a demand function. Therefore, we can obtain the following corollary.[8]

Corollary 3. If f is a Locally Lipschitz and differentiable CoD function that

[7] Actually, we do not obtain any counterexample. Thus, there may be no CoD that satisfies (I) and violates (II). However, at least, we could not show that (I) implies (II) under locally Lipschitz environment.

[8] If f is continuously differentiable, this result is trivial. However, we assume that f is only differentiable.

satisfies the Walras' law, then (S) and (NSD) are equivalent to (S)-a.e. and (NSD)-a.e..

There is a class of CoDs that is not differentiable but pseudo continuously differentiable. This is the class of patchily smooth CoDs.

Definition 3. Suppose that f is a CoD function that is locally Lipschitz and satisfies the Walras' law. We say that f is patchily smooth if there exists functions $f^1, ..., f^N : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}^n$ such that 1) all f^i are continuously differentiable and satisfies the Walras' law, 2) there exists $A_1, ..., A_N$ such that $f(p,m) = f^i(p,m)$ for every $(p,m) \in A_i$, and $\cup_{i=1}^N A_i = \mathbb{R}^n_{++} \times \mathbb{R}_{++}$.

Theorem 4. Suppose that a CoD f is patchily smooth. Then, it is pseudo continuously differentiable.

4 Proofs

4.1 Proof of FACT 4

Recall the ODE

$$\dot{x}(t) = h(t, x(t); y),\ x(t_0) = x_0,$$

where $h : \tilde{P} \to \mathbb{R}^m$ and $\tilde{P} \subset \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^\ell$. If $(t_0, x_0, y) \in \tilde{P}$, then by FACT 2, there exists a nonextendable solution $x_y : I_y \to \mathbb{R}^m$. Therefore, $x(t; y)$ can be defined on the set $U = \{(t,y)|t \in I_y\}$.

Next, we introduce the following lemma.

Lemma 1. Suppose that a continuous function $u : [t_0, \bar{t}] \to \mathbb{R}$ satisfies

$$u(t_0) = 0,\ u(t) \le \int_{t_0}^t [Au(s) + B]ds,$$

for some $A, B > 0$. Then,

$$u(t) \le \frac{B}{A}(e^{A(t-t_0)} - 1).$$

Proof of lemma 1. Let $u_0(t) = u(t)$, and when $u_k(t)$ is already defined, define

$$u_{k+1}(t) = \int_{t_0}^t [Au_k(t) + B]ds.$$

Then, $u_k(t)$ is increasing in k, and

$$\begin{aligned}
u_2(t) - u_1(t) &\leq A \int_{t_0}^{t} (u_1(s) - u_0(s))ds \\
&\leq A(t - t_0)\|u_1 - u_0\|_\infty, \\
u_3(t) - u_2(t) &\leq A \int_{t_0}^{t} (u_2(s) - u_1(s))ds \\
&\leq \frac{A^2(t - t_0)^2}{2}\|u_1 - u_0\|_\infty, \\
&\dots \\
u_{k+1}(t) - u_k(t) &\leq \frac{A^k(t - t_0)^k}{k!}\|u_1 - u_0\|_\infty.
\end{aligned}$$

Thus, the sequence (u_k) is a Cauchy sequence. Therefore, it converges uniformly to some function v and, clearly,

$$v(t) = \int_{t_0}^{t} [Av(s) + B]ds.$$

This integral equation has a unique solution $v(t) = \frac{B}{A}(e^{A(t-t_0)} - 1)$, and obviously $u(t) \leq v(t)$. This completes the proof. ■

Choose any y^* such that $(t_0, x_0, y^*) \in \tilde{P}$, and define $x_y : t \mapsto x(t; y)$ and $I_y =]a_y, b_y[$ as its domain. Choose any r_1, r_2 such that $a_{y^*} < r_1 \leq t_0 \leq r_2 < b_{y^*}$. To prove that U is open and x is continuous, it suffices to show that there exists a neighborhood V of y^* such that if $y \in V$, then $a_y < r_1$ and $r_2 < b_y$, and on $[r_1, r_2] \times V$, x is continuous.

Choose $a > 0, b > 0$ such that

$$\tilde{\Pi} = \{(t, x, y) | r_1 \leq t \leq r_2, \|x - x(t; y^*)\| \leq a, \|y - y^*\| \leq b\} \subset \tilde{P}.$$

Because $\tilde{\Pi}$ is compact, there exists $L > 0$ such that if $(t, x_1, y), (t, x_2, y) \in \tilde{\Pi}$, then

$$\|h(t, x_1, y) - h(t, x_2, y)\| \leq L\|x_1 - x_2\|.$$

Moreover, because h is continuous on $\tilde{P}$, it is uniformly continuous on $\tilde{\Pi}$, and thus there exists a nondecreasing nonnegative function $\beta(e)$ such that $\lim_{e \downarrow 0} \beta(e) = 0$ and, if $(t, x, y_1), (t, x, y_2) \in \tilde{\Pi}$, then

$$\|h(t, x, y_1) - h(t, x, y_2)\| \leq \beta(\|y_1 - y_2\|).$$

Choose any $t \in [t_0, r_2]$. For y_1, y_2, if $x(t; y_i)$ is defined and $(s, x(s; y_i), y_i) \in \tilde{\Pi}$ for all $s \in [t_0, t]$, then

$$\begin{aligned}
\|x(t; y_1) - x(t; y_2)\| &\leq \int_{t_0}^{t} \|h(s, x(s; y_1), y_1) - h(s, x(s; y_2), y_2)\| ds \\
&\leq \int_{t_0}^{t} [\|h(s, x(s; y_1), y_1) - h(s, x(s; y_2), y_1)\| \\
&\quad + \|h(s, x(s; y_2), y_1) - h(s, x(s; y_2), y_2)\|] ds \\
&\leq \int_{t_0}^{t} [L\|x(s; y_1) - x(s; y_2)\| + \beta(\|y_1 - y_2\|)] ds.
\end{aligned}$$

By Lemma 1,

$$\|x(t; y_1) - x(t; y_2)\| \leq \frac{\beta(\|y_1 - y_2\|)}{L}(e^{L(r_2 - t_0)} - 1) \equiv C_2 \beta(\|y_1 - y_2\|),$$

for some constant $C_2 > 0$. Choose any $\rho_2 > 0$ such that

$$\rho_2 \leq b, \; C_2 \beta(\rho_2) < a.$$

Choose any y with $\|y - y^*\| \leq \rho_2$ and define $\bar{t} = \sup\{t \in [t_0, r_2] | (t, y) \in U, (t, x(t; y), y) \in \tilde{\Pi}\}$. For any $t \in [t_0, \bar{t}[$,

$$t \in [r_1, r_2], \|x(t; y) - x(t; y^*)\| \leq C_2 \beta(\rho_2) < a, \|y - y^*\| \leq b,$$

and thus, $(t, x(t; y), y) \in \tilde{\Pi}$. By FACT 3 and the continuity of $x_y(t)$, we have the mapping $t \mapsto x(t; y)$ is defined at $\bar{t}$, $(\bar{t}, x(\bar{t}; y), y) \in \tilde{\Pi}$, and $\|x(\bar{t}; y) - x(\bar{t}; y^*)\| \leq C_2 \beta(\rho_2) < a$. If $\bar{t} < r_2$, then we have that for all $t > \bar{t}$ such that $t - \bar{t}$ is sufficiently small, $x(t; y)$ is defined and

$$\|x(t; y) - x(t; y^*)\| < a,$$

which contradicts the definition of $\bar{t}$. Therefore, $\bar{t} = r_2$ and $x_y(\cdot)$ is defined on $[t_0, r_2]$. Moreover, if $\|y_1 - y^*\| \leq \rho_2$ and $\|y_2 - y^*\| \leq \rho_2$, then

$$\|x(t; y_1) - x(t; y_2)\| \leq C_2 \beta(\|y_1 - y_2\|).$$

By symmetrical arguments, we can show that there exists $\rho_1 > 0$ such that if $\|y - y^*\| \leq \rho_1$, then $x_y(\cdot)$ is defined on $[r_1, t_0]$, and if $\|y_1 - y^*\| \leq \rho_1$ and $\|y_2 - y^*\| \leq \rho_1$, then

$$\|x(t; y_1) - x(t; y_2)\| \leq C_1 \beta(\|y_1 - y_2\|).$$

Define

$$V = \{y | \|y - y^*\| < \min\{\rho_1, \rho_2\}\}.$$

If $y \in V$, then $(t, y) \in U$ for all $t \in [r_1, r_2]$. Moreover, if $(t_1, y_1), (t_2, y_2) \in [r_1, r_2] \times V$, then

$$\begin{aligned}\|x(t_1; y_1) - x(t_2; y_2)\| &\le \|x(t_1; y_1) - x(t_1; y_2)\| + \|x(t_1; y_2) - x(t_2; y_2)\| \\ &\le \max\{C_1, C_2\}\beta(\|y_1 - y_2\|) + M|t_1 - t_2|,\end{aligned}$$

where $M = \max_{(t,x,y)\in\tilde{\Pi}} \|h(t, x, y)\|$. Therefore, x is continuous on $[r_1, r_2] \times V$.

Finally, if h is locally Lipschitz in (x, y), then we can choose $\beta(e) = Le$. Therefore,

$$\|x(t_1; y_1) - x(t_2; y_2)\| \le (\max\{C_1, C_2\}L + M)\|(t_1, y_1) - (t_2, y_2)\|$$

on $[r_1, r_2] \times V$. Now, suppose that $x(t; y)$ is not locally Lipschitz. Then, there exists a compact set $C \subset U$ and there exists a sequence $(t_k, y_k), (s_k, z_k)$ in C such that

$$\|x(t_k; y_k) - x(s_k; z_k)\| \ge k\|(t_k, y_k) - (s_k, z_k)\|.$$

Because C is compact, we can assume without loss of generality that $(t_k, y_k) \to (t^*, y^*)$ and $(s_k, z_k) \to (s^*, z^*)$. If $(t^*, y^*) = (s^*, z^*)$, then we can choose r_1, r_2 and $\varepsilon > 0$ such that $r_1 < t^* < r_2$ and x is Lipschitz on $W = [r_1, r_2] \times \{y | \|y - y^*\| < \varepsilon\}$. Then, for every sufficiently large k, we have $(t_k, y_k), (s_k, z_k) \in W$, a contradiction. Thus,$(t^*, y^*) \ne (s^*, z^*)$. However, this implies that

$$\|x(t^*; y^*) - x(s^*; z^*)\| = +\infty,$$

a contradiction. Therefore, x is locally Lipschitz. This completes the proof. ■

4.2 Proof of FACT 5

This is just a corollary of FACT 4. Choose any y_0, z_0 such that $(t_0, z_0, y_0) \in \tilde{P}$, and define the following equation.

$$\tilde{h}(t, x; y, z) = h(t, x + z - z_0; y).$$

Consider the following ODE:

$$\dot{x}(t) = \tilde{h}(t, x(t); y, z),\ x(t_0) = z_0.$$

Then, by FACT 4, the nonextendable solution $\tilde{x}(t; y, z)$ is defined on some open set U and continuous. Moreover, if h is locally Lipschitz in (x, y), then $\tilde{h}$ is locally Lipschitz in (x, y, z), and thus $\tilde{x}$ is locally Lipschitz. However, this is the same as $x(t; y, z) - (z - z_0)$. ■

4.3 Proof of Theorem 1

First, we will prove 1. implies 2.. Assume that 1. holds. Because of the Rademacher's theorem, we have f is differentiable at almost everywhere. Suppose that f is differentiable at (p, m). By 1., there exist an open and convex neighborhood U of p and $E : U \to \mathbb{R}$ such that

$$DE(q) = f(q, E(q)),\ \ E(p) = m.$$

Because f is differentiable at (p, m), E is twice differentiable at p, and

$$D^2E(p) = S_f(p, m).$$

We introduce the following theorem.

Young's theorem. Suppose that $U \subset \mathbb{R}^2$ is open and $g : U \to \mathbb{R}$ is differentiable, and both $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are differentiable at $(x^*, y^*) \in U$. Then, $\frac{\partial^2 g}{\partial y \partial x}(x^*, y^*) = \frac{\partial^2 g}{\partial x \partial y}(x^*, y^*)$.[9]

Proof of Young's theorem. Because U is open, there exists $\varepsilon > 0$ such that if $|h| \le \varepsilon$, then $(x^* + h, y^* + h) \in U$. Define

$$\Delta(h) = g(x^* + h, y^* + h) - g(x^* + h, x^*) - g(x^*, y^* + h) + g(x^*, y^*).$$

Let

$$\varphi(x) = g(x, y^* + h) - g(x, y^*).$$

If $|h| \le \varepsilon$, then there exists $\theta \in [0, 1]$ such that

$$\begin{aligned}
\Delta(h) &= \varphi(x^* + h) - \varphi(x^*) \\
&= h\varphi'(x^* + \theta h) \\
&= h\left[\frac{\partial g}{\partial x}(x^* + \theta h, y^* + h) - \frac{\partial g}{\partial x}(x^* + \theta h, y^*)\right] \\
&= h\left[\frac{\partial g}{\partial x}(x^*, y^*) + \theta h \frac{\partial^2 g}{\partial^2 x}(x^*, y^*) + h\frac{\partial^2 g}{\partial y \partial x}(x^*, y^*) + o(h)\right. \\
&\quad \left. - \frac{\partial g}{\partial x}(x^*, y^*) - \theta h \frac{\partial^2 g}{\partial^2 x}(x^*, y^*) + o(h)\right] \\
&= h^2 \frac{\partial^2 g}{\partial y \partial x}(x^*, y^*) + o(h^2).
\end{aligned}$$

[9] We do not know any article or textbook that includes this result and is written in English. This result and its proof is in Takagi (1961). However, this textbook is written in Japanese. Thus, we decide to write the proof of this fact.

Thus,

$$\lim_{h\to 0}\frac{\Delta(h)}{h^2}=\frac{\partial^2 g}{\partial y\partial x}(x^*,y^*).$$

Because the assumption is symmetric, we can also show that

$$\lim_{h\to 0}\frac{\Delta(h)}{h^2}=\frac{\partial^2 g}{\partial x\partial y}(x^*,y^*),$$

and thus we have

$$\frac{\partial^2 g}{\partial y\partial x}(x^*,y^*)=\frac{\partial^2 g}{\partial x\partial y}(x^*,y^*).$$

This completes the proof. ■

By Young's theorem, we have that $S_f(p,m)=D^2E(p)$ is symmetric, and thus (S)-a.e. holds.

Thus, it suffices to show that 2. implies 1.. Assume that 2. holds. Let $(p,m)\in P$, and consider the following ODE:

$$\dot{c}(t)=f((1-t)p+tq,c(t))\cdot(q-p),\ c(0)=w.$$

Let $c(t;q,w)$ be the nonextendable solution of the above problem. Choose any $q\in\mathbb{R}^n$ such that $q_i\neq p_i$ for every i, and suppose that the domain of the mapping $t\mapsto c(t;q,m)$ includes $[0,t^*]$, where $t^*\in]0,1]$. We fix $i^*\in\{1,...,n\}$, and use the following notations. If $\hat{r}=(r_1,...,r_{i^*-1},r_{i^*+1},...,r_n)$ is given, then $r=(r_1,...,r_{i^*-1},q_{i^*},r_{i^*+1},...,r_n)$. Conversely, if $v=(v_1,...,v_n)$ is given, then $\hat{v}=(v_1,...,v_{i^*-1},v_{i^*+1},...,v_n)$.

By FACT 5, there exists an open and convex neighborhood $U\subset\mathbb{R}^{n-1}\times\mathbb{R}$ of $(\hat{q},m)$ such that the closure $\bar{U}$ of U is compact, if $(\hat{r},w)\in\bar{U}$, then $r_i\neq p_i$ for every i, and the domain of the mapping $t\mapsto c(t;r,w)$ includes $[0,t^*]$. Define $\xi:[0,t^*]\times\bar{U}\to\mathbb{R}^{n+1}$ as follows:

$$\xi(t,\hat{r},w)=((1-t)p+tr,c(t;r,w)).$$

Step 1. c is increasing in w and ξ is one-to-one on $]0,t^*[\times U$.

Proof of step 1. Suppose that $(t_i,\hat{r}_i,w_i)\in]0,t^*[\times U$ and $(t_1,\hat{r}_1,w_1)\neq(t_2,\hat{r}_2,w_2)$. First, suppose that $t_1\neq t_2$, then

$$(1-t_1)p_{i^*}+t_1q_{i^*}\neq(1-t_2)p_{i^*}+t_2q_{i^*},$$

and thus $\xi(t_1,\hat{r}_1,w_1)\neq\xi(t_2,\hat{r}_2,w_2)$. Second, suppose that $t_1=t_2$ and $\hat{r}_1\neq\hat{r}_2$. Then,

$$(1-t_1)\hat{p}+t_1\hat{r}_1\neq(1-t_2)\hat{p}+t_2\hat{r}_2,$$

and thus $\xi(t_1, \hat{r}_1, w_1) \neq \xi(t_2, \hat{r}_2, w_2)$. Third, suppose that c is increasing in w, and that $t_1 = t_2, \hat{r}_1 = \hat{r}_2$ and $w_1 < w_2$. Then,

$$c(t_1; r_1, w_1) < c(t_2; r_2, w_2),$$

and thus $\xi(t_1, \hat{r}_1, w_1) \neq \xi(t_2, \hat{r}_2, w_2)$. Therefore, it suffices to show that c is increasing in w. Suppose that for some r and w_1, w_2 with $w_1 < w_2$, there exists t^+ such that $c(t^+; r, w_1) \geq c(t^+; r, w_2)$. Because $c(0; r, w_1) = w_1 < w_2 = c(0; r, w_2)$, by intermediate value theorem, we can assume that $c(t^+; r, w_1) = c(t^+; r, w_2)$. Then, by FACT 1, we have $c(t; r, w_1) = c(t; r, w_2)$ for every t, which contradicts the fact $c(0; r, w_1) \neq c(0; r, w_2)$. This completes the proof. ■

Step 2. ξ is Lipschitz.

Proof of step 2. Because of FACT 5, c is Lipschitz on $[0, t^*] \times \bar{U}$. Thus, ξ is also Lipschitz.

Step 3. Define

$$V^\ell = \xi([\ell^{-1}t^*, t^*[\times U).$$

Then, ξ^{-1} is Lipschitz on V^ℓ.

Proof of step 3. Define

$$t(v) = \frac{v_{i^*} - p_{i^*}}{q_{i^*} - p_{i^*}},$$

$$\hat{r}(v) = \frac{1}{t(v)}[(1 - t(v))\hat{p} + \hat{v}].$$

Suppose that $(v_1, c_1), (v_2, c_2) \in V^\ell$ and $(v_i, c_i) = \xi(t_i, \hat{r}_i, w_i)$. Then, we have $t_i = t(v_i)$ and $\hat{r}_i = \hat{r}(v_i)$. Clearly, the functions $t(v)$ and $\hat{r}(v)$ are Lipschitz on $\xi([\ell^{-1}t^*, t^*] \times \bar{U})$, and thus it is Lipschitz on V^ℓ. Next, consider the following ODE:

$$\dot{d}(s) = f((1 - (s + t - t_2))p + (s + t - t_2)r(v), d(s)) \cdot (r(v) - p),\ d(t_2) = c.$$

Let $d(s; t, v, c)$ be the nonextendable solution of above ODE. If $(v, c) = \xi(t, \hat{r}, w)$ for some $(t, \hat{r}, w) \in [\ell^{-1}t^*, t^*] \times \bar{U}$, then $d(s; t, v, c) = c(s + t - t_2; r, v)$ by FACT 1. Moreover, the set

$$\{(t, v, c) | t \in [\ell^{-1}t^*, t^*],\ (v, c) = \xi(t, \hat{r}, w) \text{ for some } (\hat{r}, w) \in \bar{U}\}$$

is compact, and thus $(t, v, c) \mapsto d(t_2 - t; t, v, c)$ is Lipschitz on this set. Therefore,

$$\begin{aligned}|w_1 - w_2| &= |d(t_2 - t_1; t_1, v_1, c_1) - d(t_2 - t_2; t_2, v_2, c_2)| \\ &\le L[|t_1 - t_2| + \|(v_1, c_1) - (v_2, c_2)\|] \\ &= L[|t(v_1) - t(v_2)| + \|(v_1, c_1) - (v_2, c_2)\|] \\ &\le L(M + 1)\|(v_1, c_1) - (v_2, c_2)\|,\end{aligned}$$

where $L, M > 0$ are some constant. This completes the proof. ■

Step 4. For almost all $(\hat{r}, w) \in U$, f is differentiable at $\xi(t, \hat{r}, w)$ and the Slutsky matrix $S_f(\xi(t, \hat{r}, w))$ is symmetric for almost all $t \in]0, t^*[$.

Proof of step 4. Define W as the set of all $(r, w) \in P$ such that f is differentiable at (r, w) and $S_f(r, w)$ is symmetric. By (S)-a.e., the Lebesgue measure of $P \setminus W$ is zero. Because ξ^{-1} is Lipschitz on V^ℓ, we have the Lebesgue measure of

$$\xi^{-1}(V^\ell \setminus W)$$

is zero. Therefore, the Lebesgue measure of

$$\cup_\ell \xi^{-1}(V^\ell \setminus W) = \{(t, \hat{r}, w) \in]0, t^*[\times U | \xi(t, \hat{r}, w) \notin W\}$$

is also zero.

Therefore, for almost every $(t, \hat{r}, w) \in]0, t^*[\times U$, f is differentiable and the Slutsky matrix is symmetric at $\xi(t, \hat{r}, w)$. The rest proof is just a simple application of Fubini's theorem. ■

Step 5. Let $U_{i^*} \subset U$ be the set of all $(\hat{r}, w) \in U$ such that for almost all $t \in]0, 1[$, f is differentiable and S_f is symmetric at $\xi(t, \hat{r}, w)$, and $c(t; r, w)$ is differentiable at $(t, \hat{r}, w)$. Then, for any $i \in \{1, ..., n\} \setminus \{i^*\}$ and $(\hat{r}, w) \in U_{i^*}$, $c(t; r, w)$ is partially differentiable w.r.t. r_i for all $t \in [0, t^*]$. Moreover, if we define

$$\varphi_i(t; \hat{r}, w) = \frac{\partial c}{\partial r_i}(t; r, w) - t f_i((1 - t)p + tr, c(t; r, w)),$$

then for every $(\hat{r}, w) \in U_{i^*}$,

$$\varphi_i(t; \hat{r}, w) \equiv 0.$$

Proof of step 5. First,[10]

$$
\begin{aligned}
&\lim_{h\to 0}\frac{c(t;r+he_i,w)-c(t;r,w)}{h}\\
=&\lim_{h\to 0}\frac{1}{h}\left[\int_0^t [f((1-s)p+s(r+he_i),c(s;\hat{r}+he_i,w))\cdot(r+he_i-p)]ds\right.\\
&\left.-\int_0^t [f((1-s)p+sr,c(s;\hat{r},w))\cdot(r-p)]ds\right]\\
=&\int_0^t\left[f_i+\sum_{j=1}^n\left[s\frac{\partial f_j}{\partial p_i}+\frac{\partial f_j}{\partial m}\frac{\partial c}{\partial r_i}\right](r_j-p_j)\right]ds,
\end{aligned}
$$

by the dominated convergence theorem. Therefore, c is partially differentiable w.r.t. r_i. Moreover, φ_i is absolutely continuous on $[0,t^*]$, and thus is differentiable at almost all $t\in[0,t^*]$. Further,

$$
\begin{aligned}
\dot{\varphi}_i=&\ f_i+\sum_{j=1}^n\left[t\frac{\partial f_j}{\partial p_i}+\frac{\partial f_j}{\partial m}\frac{\partial c}{\partial r_i}\right](r_j-p_j)\\
&-f_i-t\sum_{j=1}^n\left[\frac{\partial f_i}{\partial p_j}+\frac{\partial f_i}{\partial m}f_j\right](r_j-p_j)\\
=&\ t\sum_{j=1}^n\left[\frac{\partial f_j}{\partial p_i}-\frac{\partial f_i}{\partial p_j}-\frac{\partial f_i}{\partial m}f_j\right](r_j-p_j)\\
&+\sum_{j=1}^n\frac{\partial f_j}{\partial m}\frac{\partial c}{\partial r_i}(r_j-p_j)\\
=&\left(\frac{\partial c}{\partial r_i}-tf_i\right)\times\sum_{j=1}^n\frac{\partial f_j}{\partial m}(r_j-p_j)\\
=&\ a(t,\hat{r},w)\varphi_i.
\end{aligned}
$$

Therefore, φ_i is an absolutely continuous function that is a solution for some linear ODE, where $a(t,\hat{r},w)$ is bounded on $[0,t^*]\times\bar{U}$. Thus,

$$\varphi_i(t;\hat{r},w)=\varphi_i(0;\hat{r},w)e^{\int_0^t a(s,\hat{r},w)ds}.$$

However, it is obvious that $\varphi_i(0;\hat{r},w)=0$. Hence, we have $\varphi_i\equiv 0$. This completes the proof. ■

Step 6. If $t^*=1$, then $\frac{\partial c}{\partial q_i}(1;q,m)=f_i(q,c(1;q,m))$ for all $i\neq i^*$.

[10] Hereafter, we frequently abbreviate the variables of functions.

Proof of step 6. Choose any sufficiently small $\varepsilon > 0$ such that if we define

$$U' = \{(\hat{r}, w) || r_j - q_j| \le \varepsilon \text{ for } j \in \{1, ..., n\} \setminus \{i^*\}, |w - m| \le \varepsilon\},$$

$$V' = \{(\hat{r}, w) \in U' | r_i = q_i\},$$

then $(\hat{r} + h\hat{e}_i, w) \in U_{i^*}$ for almost all $(\hat{r}, w) \in V'$ and $h \in]-\varepsilon, \varepsilon[$, where e_i is the i-th unit vector. Hence,

$$c(1; r + he_i, w) - c(1; r, w) = \int_0^h f_i(r + se_i, c(1; r + se_i, w))ds,$$

and thus, by dominated convergence theorem, we have

$$c(1; q + he_i, m) - c(1; q, m) = \int_0^h f_i(q + se_i, c(1; q + se_i, m))ds.$$

Thus, by Newton-Leibniz formula, we have

$$\frac{\partial c}{\partial q_i}(1; q, m) = f_i(q, c(1; q, m)).$$

This completes the proof. ■

Because i^* is arbitrary, step 6 means that

$$D_q c(1; q, m) = f(q, c(1; q, m))$$

for all $q \in \mathbb{R}^n$ such that $t \mapsto c(t; q, m)$ is defined on $[0, 1]$ and $q_i \neq p_i$ for all i.

Step 7. $D_q c(1; q, m) = f(q, c(1; q, m))$ holds for all q such that the domain of $t \mapsto c(t; q, m)$ includes $[0, 1]$.

Proof of step 7. Let $q^k = (q_1 + k^{-1}, ..., q_n + k^{-1})$. Then, there exists $\varepsilon > 0$ and k_0 such that $q_i^k \neq p_i$ whenever $k \ge k_0$, and if $|h| \le \varepsilon$, then both $t \mapsto c(t; q + he_i, m)$ and $t \mapsto c(t; q^k + he_i, m)$ are defined on $[0, 1]$ for all $i \in \{1, ..., n\}$. Then, for such h,

$$c(1; q^k + he_i, m) - c(1; q^k, m) = \int_0^h f_i(q^k + se_i, c(1; q^k + se_i, m))ds,$$

and thus by dominated convergence theorem,

$$c(1; q + he_i, m) - c(1; q, m) = \int_0^h f_i(q + se_i, c(1; q + se_i, m))ds.$$

By Newton-Leibniz formula, we have

$$\frac{\partial c}{\partial q_i}(1; q, m) = f_i(q, c(1; q, m)),$$

which completes the proof. ■

Because $t \mapsto c(t; p, m) \equiv m$, we have its domain includes $[0, 1]$, and thus there exists an open and convex neighborhood U of p such that the domain of $t \mapsto c(t; q, m)$ includes $[0, 1]$. By step 7, we have

$$\frac{\partial c}{\partial q_i}(1; q, m) = f_i(q, c(1; q, m)),$$

and thus if we define $E(q) = c(1; q, m)$, then we have that 1. holds. This completes the proof of theorem 1. ■

4.4 Proof of Theorem 2

Suppose that 1. is correct. Then, for every $q \in C$, $c(t; q, m) = E((1-t)p+tq)$ is a solution of ODE (2) defined on $[0, 1]$. Thus, 2. holds.

Conversely, suppose that 2. holds. Let $c(t; q, w)$ be the nonextendable solution of (2) defined on U. Then, U includes $[0, 1] \times \tilde{C} \times \{m\}$, where $\tilde{C}$ is an open set including C. Now, choose any $q \in C$. Because of Theorem 1, we have that for any $r = (1 - t)p + tq$ with $t \in [0, 1]$, there exists a solution $E_r : U_r \to \mathbb{R}$ of (1), where U_r is an open ball $\{r' | \|r' - r\| < \varepsilon\}$ for some $\varepsilon > 0$ and $E_r(r) = c(t; q, m)$. We can assume that $U_r \subset \tilde{C}$.

Let $r_i = (1 - t_i)p + t_i q$ for $i \in \{1, 2\}$, $t_i \in [0, 1]$, and without loss of generality, assume $t_1 \le t_2$. Suppose that there exists $U_{r_1} \cap U_{r_2} \neq \emptyset$. Because both U_{r_i} are open balls, we have there exists $t_0 \in [t_1, t_2]$ such that $r = (1 - t_0)p + t_0 q \in U_{r_1} \cap U_{r_2}$. Consider the following ODE:

$$\dot{d}(t) = f((1 - t)p + tq, d(t)) \cdot (q - p),\ d(t_1) = c(t_1; q, m).$$

Then, both $d_1(t) = c(t; q, m)$ and $d_2(t) = E_{r_1}((1 - t)p + tq)$ are the solution of above ODE defined on $[t_1, t_0]$, and thus these are the same. Thus, we have $E_{r_1}(r) = c(t_0; q, m)$. By the same reason, we have $E_{r_2}(r) = c(t_0; q, m)$. Therefore,

$$E_{r_1}(r) = E_{r_2}(r).$$

Choose any $r' \in U_{r_1} \cap U_{r_2}$. Because U_{r_i} is convex for each i, $(1 - s)r + sr' \in U_{r_1} \cap U_{r_2}$ for every $s \in [0, 1]$. Define $\gamma_i(s) = E_{r_i}((1 - s)r + sr')$. Then,

$$\dot{\gamma}_i(s) = f((1 - s)r + sr', \gamma_i(s)) \cdot (r' - r),\ \gamma_i(0) = c(t_0; q, m),$$

and thus we have $E_{r_1}(r') = E_{r_2}(r')$.

Thus, $E_{r_1} \equiv E_{r_2}$ on $U_{r_1} \cap U_{r_2}$. Now, define $V = \cup_{r\in[p,q]} U_r$ and

$$v(r') = E_r(r')$$

if $r' \in U_r$. Then, $v : V \to \mathbb{R}$ is a solution of PDE (1) with $v(p) = m$. Because V is a neighborhood of $[p, q]$, there exists a open neighborhood W of q such that if $r \in W$ and $t \in [0, 1]$, then $(1 - t)p + tr \in V$. Then, the functions

$$t \mapsto c(t; r, m),\ v((1 - t)p + tr)$$

are solutions of the same ODE, and thus we have $v(r) = c(1; r, m)$. This implies that

$$D_q c(1; q, m) = Dv(q) = f(q, v(q)) = f(q, c(1; q, m)).$$

Therefore, if we define $E(q) = c(1; q, m)$, then $E : C \to \mathbb{R}$ is a solution of (1), and 1. holds.[11] This completes the proof. ■

4.5 Proof of Corollary 1

Suppose that 1. holds. Because f is locally Lipschitz, it is differentiable at almost everywhere in $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$. Suppose that f is differentiable at (p, m). By 1., there exists a concave solution $E : U \to \mathbb{R}_{++}$ of (1) with $E(p) = m$, where U is an open and convex neighborhood of p. By the same arguments as in the proof of Theorem 1, we have

$$D^2 E(p) = S_f(p, m).$$

Because of Young's theorem, we have $S_f(p, m)$ is symmetric. Also, because E is concave, we have $S_f(p, m)$ is negative semi-definite. Therefore, (S)-a.e. and (NSD)-a.e. hold.

Conversely, suppose that 2. holds. Fix any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. Let $c(t; q, w)$ be the nonextendable solution of ODE (2), and U be the domain of c. By Theorem 2, if $[0, 1] \times \mathbb{R}^n_{++} \times \{m\} \subset U$, then 1. holds.

Now, we introduce the following lemma.

Lemma 2. Suppose that $q \in \mathbb{R}^n_{++}, m > 0$, and the domain of $t \mapsto c(t; q, m)$ includes $[0, t^*]$, where $t^* > 0$. Define $p(t) = (1-t)p+tq$ and $x = f(p, m), y = f(p(t^*), c(t^*; q, m))$. Then, we have $p \cdot y \geq m$ and $p(t^*) \cdot x \geq c(t^*; q, m)$.

[11] The uniqueness of the solution immediately follows from FACT 4.

Proof. First, suppose that $q_n \neq p_n$. To choose $i^* = n$ and to apply almost the same arguments in steps 1-4 of the proof of Theorem 1, we have that there exists U^* and $V^* \subset U^*$ such that U^* is an open and convex neighborhood of $(\hat{q}, m)$ and the domain of the mapping $t \mapsto c(t; r, w)$ includes $[0, t^*]$ for all $(\hat{r}, w) \in U^*$, the Lebesgue measure of $U^* \setminus V^*$ is zero, and if $(\hat{r}, w) \in V^*$, then for almost all $t \in [0, t^*]$, f is differentiable at $\xi(t, \hat{r}, w)$ and $S_f(\xi(t, \hat{r}, w))$ is symmetric and negative semi-definite, where[12]

$$\xi(t, \hat{r}, w) = ((1-t)p + tr, c(t; r, w)).$$

Choose any sequence $((\hat{r}_k, w_k))$ on V^* such that $\lim_{k\to\infty}(\hat{r}_k, w_k) = (\hat{q}, m)$, and let $p_k(t) = (1-t)p + tr_k$ and $x_k = f(p, w_k),\ y_k = f(p_k(t^*), c(t^*; r_k, w_k))$. We will show that $p \cdot y_k \geq w_k$ and $p_k(t^*) \cdot x_k \geq c(t^*; r_k, w_k)$. Define $d(t) = p \cdot f(p_k(t), c(t; r_k, w_k))$. Then, d is an absolutely continuous function defined on $[0, t^*]$ and if f is differentiable at $\xi(t, \hat{r}_k, w_k)$, then

$$\dot{d}(t) = p^T S_f(p_k(t), c(t; r_k, w_k))(r - p).$$

Meanwhile, for such t, by Walras' law,

$$(p_k(t))^T S_f(p_k(t), c(t; r_k, w_k))(r - p) = 0.$$

Therefore, we have

$$\dot{d}(t) = -(r_k - p)^T S_f(p_k(t), c(t; r_k, w_k))(r_k - p) \geq 0,$$

for almost all $t \in [0, t^*]$. Thus, d is a nondecreasing function, and especially,

$$p \cdot y_k = d(t^*) \geq d(0) = p \cdot x_k = w_k.$$

The rest inequality can be verified symmetrically. If $k \to \infty$, then we have

$$p \cdot y \geq m,\ p(t^*) \cdot x \geq c(t^*; q, m),$$

and thus the claim of this lemma is correct.

In the general case, let $q_\varepsilon = (q_1, ..., q_{n-1}, q_n + \varepsilon)$, and define $p_\varepsilon(t) = (1-t)p + tq_\varepsilon$ and $y_\varepsilon = f(p_\varepsilon(t), c(t; q_\varepsilon, m))$. Then, for sufficiently small $\varepsilon > 0$, $q_{\varepsilon,n} \neq p_n$, and thus

$$p \cdot y_\varepsilon \geq m,\ p_\varepsilon(t^*) \cdot x \geq c(t^*; q_\varepsilon, m).$$

If $\varepsilon \to 0$, then we have

$$p \cdot y \geq m,\ p(t^*) \cdot x \geq c(t^*, q, m),$$

[12] Recall that if $\hat{r} = (r_1, ..., r_{n-1})$, then $r = (r_1, ..., r_{n-1}, q_n)$.

which completes the proof. ■

Choose any $q \in \mathbb{R}^n_{++}$, and suppose that the domain of $t \mapsto c(t; q, m)$ is $]\hat{t}, \bar{t}[$, where $\bar{t} \le 1$. Choose any sequence (t_k) of nonnegative numbers such that $t_k \uparrow \bar{t}$. Define $p(t) = (1-t)p + tq$ and $x = f(p, m)$, $y_k = f(p(t_k), c(t_k; q, m))$. Then, by Lemma 2, we have

$$p \cdot y_k \ge m = p \cdot x,\ p(t_k) \cdot x \ge c(t_k; q, m) = p(t_k) \cdot y_k.$$

Therefore, we must have

$$q \cdot x \ge q \cdot y_k$$

for every k. Thus, (y_k) is a sequence of a compact set $\{z \in \mathbb{R}^n_+ | q \cdot z \le q \cdot x\}$. By FACT 3, we have either $\limsup_{k\to\infty} c(t_k; q, m) = +\infty$ or $\liminf_{k\to\infty} c(t_k; q, m) = 0$. However, because $c(t_k; q, m) \le \max_{t\in[0,1]} p(t) \cdot x < +\infty$, we have that $\liminf_{k\to\infty} c(t_k; q, m) = 0$. Taking subsequences, we can assume that

$$\lim_{k\to\infty} c(t_k; q, m) = 0,\ \lim_{k\to\infty} y_k = y^* \in \mathbb{R}^n_+.$$

Because $p \cdot y^* \ge m$, we have $y^* \ne 0$. Thus,

$$0 = \lim_{k\to\infty} c(t_k; q, m) = \lim_{k\to\infty} p(t_k) \cdot y_k = p(t^*) \cdot y^* > 0,$$

a contradiction. Therefore, U includes $[0, 1] \times \{q, m\}$, and because q is arbitrary, we have 1. holds. This completes the proof. ■

4.6 Proof of Theorem 3

Because (I) holds, we have that $S_f(p, m)$ is symmetric and negative semi-definite whenever f is differentiable at (p, m).

Suppose that $x \ne y, x = f(p, m), y = f(q, w)$, and $w \ge E(q)$, where $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is the unique concave solution of the PDE (1) with $E(p) = m$. If $c(t; r, m)$ is the nonextendable solution of ODE (2), then clearly $E(r) = c(1; r, m)$ for every $r \in \mathbb{R}^n_{++}$. Let $F : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ be the unique concave solution of the PDE (1) with $F(q) = w$.

If $w = F(q) > E(q)$, then by the uniqueness of the solution of the PDE (1), we have $F(r) > E(r)$ for every $r \in \mathbb{R}^n_{++}$. Particularly, $F(p) > E(p) = m$. Consider the following ODE:

$$\dot{d}(t) = f((1-t)q + tp, d(t)) \cdot (p - q),\ d(0) = w.$$

Then, we have $d(t) = F((1-t)q + tp)$ is the unique solution of above ODE, and by Lemma 2, we have

$$p \cdot y = p \cdot f(q, w) \ge d(1) = F(p) > E(p),$$

and thus (II) holds in this case.

Therefore, we can assume that $w = E(q)$, and thus $F \equiv E$. Suppose $p \cdot y \leq m$. Let $d(t) = p \cdot f((1-t)p + tq, c(t; q, m))$. By Lemma 2, we have that $d(t)$ is nondecreasing, and $p \cdot y = d(1)$, $p \cdot x = d(0)$. Thus, we have that $\dot{d}(t) = 0$ for all $t \in]0,1[$. Define $p(t) = (1-t)p + tq$, $X(r) = f(r, E(r))$ and $Y(t) = X(p(t))$. Because Y is absolutely continuous and $Y(1) = y \neq x = Y(0)$, there exists $t^* \in]0,1[$ such that $\dot{Y}(t^*) \neq 0$. Let $w^* = \frac{d}{ds}E(p(s))\big|_{s=t^*}$, and choose a sequence (p_k, m_k) and $v \in df_{q-p,w^*}(p(t^*), E(p(t^*)))$ such that f is differentiable at (p_k, m_k), $(p_k, m_k) \to (p(t^*), E(p(t^*)))$, and

$$\lim_{k \to \infty} Df(p_k, m_k)(q-p, w^*) = v.$$

Let S_k denote $S_f(p_k, m_k)$. Because S_k is symmetric and negative semi-definite, there exists a symmetric and positive semi-definite matrix A_k such that $S_k = -A_k^2$.[13] Then,

$$-t(q-p)S_k(q-p) = t\|A_k(q-p)\|^2.$$

For $t > t^*$, we have

$$\begin{aligned}
&\left|\frac{d(t)-d(t^*)}{t-t^*} - p \cdot \frac{f(p(t), E(p(t^*)) + (t-t^*)w^*) - f(p(t^*), E(p(t^*)))}{t-t^*}\right| \\
&\leq \|p\| \left\|\frac{f(p(t), E(p(t))) - f(p(t), E(p(t^*)) + (t-t^*)w^*)}{t-t^*}\right\| \\
&\leq L\|p\| \left|\frac{E(p(t)) - E(p(t^*)) - (t-t^*)w^*}{t-t^*}\right| \\
&\to 0 \text{ as } t \downarrow t^*,
\end{aligned}$$

for some $L > 0$.[14] Because $\dot{d}(t^*) = 0$, we have that

$$p^T Df(p_k, m_k)(q-p, w^*) \to p \cdot v = 0.$$

[13] If

$$S_k = P^T \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_n \end{pmatrix} P$$

for some orthogonal matrix P, then

$$A_k = P^T \begin{pmatrix} \sqrt{|\lambda_1|} & 0 & ... & 0 \\ 0 & \sqrt{|\lambda_2|} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \sqrt{|\lambda_n|} \end{pmatrix} P.$$

[14] Recall that f is locally Lipschitz.

Meanwhile,

$$\begin{aligned}
&Df(p_k, m_k)(q-p, w^*) \\
&= [D_p f(p_k, m_k) + D_m f(p_k, m_k) f^T(p(t^*), E(p(t^*)))](q-p) \\
&= S_k(q-p) \\
&\quad + D_m f(p_k, m_k)(f^T(p(t^*), E(p(t^*))) - f^T(p_k, m_k))(q-p),
\end{aligned}$$

and $p_k^T S_k = 0^T$ by Walras' law. Therefore,

$$\begin{aligned}
& p^T Df(p_k, m_k)(q-p, w^*) \\
= & -t^*(q-p)^T S_k(q-p) + (p(t^*) - p_k)^T S_k(q-p) \\
& + p^T D_m f(p_k, m_k)(f^T(p(t^*), E(p(t^*))) - f^T(p_k, m_k))(q-p),
\end{aligned}$$

where the second and third terms of the right-hand side goes to zero as $k \to \infty$.[15] This implies that

$$\|A_k(q-p)\| \to 0$$

as $k \to \infty$. Meanwhile, for $t > t^*$, we have

$$\begin{aligned}
& \left| \frac{Y(t) - Y(t^*)}{t - t^*} - \frac{f(p(t), E(p(t^*)) + (t-t^*)w^*) - f(p(t^*), E(p(t^*)))}{t - t^*} \right| \\
& \le L \left| \frac{E(p(t)) - E(p(t^*)) + (t-t^*) \frac{d}{ds} E(p(s))\big|_{s=t^*}}{t - t^*} \right| \\
& \to 0 \text{ as } t \downarrow t^*,
\end{aligned}$$

and thus

$$\begin{aligned}
0 \neq \dot{Y}(t^*) = v = & \lim_{k\to\infty} Df(p_k, m_k)(q-p, w^*) \\
= & \lim_{k\to\infty} [S_k(q-p) \\
& + D_m f(p_k, m_k)[f^T(p(t^*), E(p(t^*))) - f^T(p_k, m_k)](q-p)] \\
= & \lim_{k\to\infty} A_k(A_k(q-p)) = 0,
\end{aligned}$$

a contradiction. This completes the proof. ■

[15] Again, recall that f is locally Lipschitz, and thus the operator norm of S_k is bounded.

4.7 Proof of Theorem 4

Fix $(q, w) \in \mathbb{R}^n \times \mathbb{R}$. Clearly, f is locally Lipschitz, and by Rademacher's theorem, it is differentiable at almost every point. If f is differentiable at (p, m), there exists i and a sequence (t_k) of positive real numbers such that $t_k \downarrow 0$ and $(p + t_k q, m + t_k w) \in A_i$. By continuity of f and f^i, we have $f(p, m) = f^i(p, m)$, and thus,

$$\begin{aligned} Df(p,m)(q,w) &= \lim_{k\to\infty} \frac{f(p+t_k q, m+t_k w) - f(p,m)}{t_k} \\ &= \lim_{k\to\infty} \frac{f^i(p+t_k q, m+t_k w) - f^i(p,m)}{t_k} \\ &= Df^i(p,m)(q,w). \end{aligned}$$

Therefore, if we define

$$B_i = \{(p,m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} | f(p,m) = f^i(p,m),\ Df(p,m)(q,w) = Df^i(p,m)(q,w)\},$$

then $\cup_i B_i$ is dense in $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$.

Choose any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$ and $v \in df_{q,w}(p, m)$, and a sequence (t_k) of positive real numbers such that $t_k \downarrow 0$ and

$$v = \lim_{k\to\infty} \frac{f(p+t_k q, m+t_k w) - f(p,m)}{t_k}.$$

Taking a subsequence, we can assume that there exists i such that for every k, $(p + t_k q, m + t_k w)$ is in the closure of B_i. Then, (p, m) is also in the closure of B_i, and by continuity of f and f^i, we have $f(p, m) = f^i(p, m)$ and $f(p + t_k q, m + t_k w) = f^i(p + t_k q, m + t_k w)$. Clearly,

$$v = Df^i(p,m)(q,w),$$

and thus, if we choose any sequence $((p_k, m_k))$ in B_i such that $(p_k, m_k) \to (p, m)$, then

$$Df(p_k,m_k)(q,w) = Df^i(p_k,m_k)(q,w) \to Df^i(p,m)(q,w) = v,$$

which completes the proof. ■

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