INTEGRATING SCHOOL DISTRICTS: BALANCE, DIVERSITY, AND WELFARE† (PRELIMINARY AND INCOMPLETE)

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Abstract. Inter-district school choice programs—where a student can be matched with a school outside of her district—is widespread in the US, yet the market-design literature has not considered such programs. We introduce a model of district integration to study inter-district school choice and present two mechanisms that produce *stable* or *efficient* matchings. We consider three categories of policy goals on matching outcomes and identify when the mechanisms can achieve them. By introducing a novel framework of district integration, we provide a new avenue of research in market design.

1. Introduction

School choice is a program that uses preferences of children and their parents over public schools to determine placement. It has expanded rapidly in the United States and many other countries. Growing popularity and interest in school choice stimulated research in market design, which has not only studied this problem in the abstract but also contributed to designing specific placement mechanisms.[1]

Existing market-design research about school choice is, however, limited to *intra-district* choice, where each student can be matched with a school only in her own district. In other words, the literature has not studied *inter-district* choice, where a student can be matched with a school outside of her district. This is a severe limitation for at least two reasons. First, inter-district school choice is widespread; in the U.S., 43 states have some form of inter-district school choice.[2] Second, as we illustrate in detail below, many policy goals

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[1]See Abdulkadiroğlu et al. (2005a,b, 2009) for details of the implementation of these new school choice procedures in New York and Boston.

[2]See http://ecs.force.com/mbdata/mbquest4e?rep=OE1705, accessed on July 14, 2017. Among these, notable examples include the following: *Interdistrict Public School Choice Program* in New Jersey facilitates school districts to enroll students who do not reside within their districts (http://www.state.nj.us/education/choice/, accessed on July 14, 2017). Omaha Public Schools has *School Transfer Request* where students can apply to a non-neighborhood school (https://district.ops.org/DEPARTMENTS/SchoolSupportandSupervision/Community,SchoolsFamilyEngagement/StudentPlacement.aspx, accessed on July 14, 2017). Lastly, Wisconsin Department of Public Instruction has the *Integration Aid* program

in inter-district school choice impose constraints across districts in reality, but the existing literature assumes away such constraints. This omission limits our ability to analyze policies of interest in the context of inter-district school choice.

In this paper, we propose a model of district integration to study inter-district school choice. Our paper builds upon matching models in the tradition of Gale and Shapley (1962). We study algorithms and inter-district admissions rules to assign students to schools under which a variety of policy goals can be established, an approach similar to the standard school choice literature (Abdulkadiroğlu and Sönmez, 2003). In our setting, however, policy goals are defined on the district level—or sometimes even on multiple districts—rather than the individual school level, making our model outside of the standard setting. To facilitate the analysis in this setting, we model the problem as matching with contracts (Hatfield and Milgrom, 2005) between students and districts in which a contract specifies the particular school within the district that the student attends.

Following the school choice literature, we begin our analysis by considering *stability* (we also consider efficiency, as explained later). To define stability in our framework, we assume that each district is endowed with an admissions rule represented by a choice function over sets of contracts. A matching is stable if it satisfies the following two properties. First, every district's admissions rule chooses all of the students assigned to it. Second, there exists no student who prefers to transfer to a school to which she will be admitted according to the district's admissions rule. We focus our attention on the student-proposing deferred-acceptance algorithm (SPDA), a generalization of Gale and Shapley (1962) to our setting. This mechanism is not only stable but also strategy-proof, i.e., it renders truthtelling a weakly dominant strategy for each student.

In this context, we formalize a number of important policy goals. The first is *individual rationality* in the sense that no student should be hurt compared to the outcome in the absence of the inter-district school choice mechanism. This is an important requirement because, if a district integration harms students, then a public opposition is expected and district integration may not be sustainable. The second policy is what we call *the balanced-exchange policy*, that the number of students who each district receives from the other districts must be the same as the number of students that it sends to the others. Balanced exchange is also highly desired by school districts in practice because each district's funding depends on the number of students that it serves. Therefore, if the balanced-exchange policy is not satisfied then some districts lose funding, which may make district integration impossible. For each of these policy goals, we identify sufficient conditions for achieving that goal under SPDA as restrictions on district admissions rules. Moreover, we show that

that financially supports school districts transferring students from other districts (`https://dpi.wi.gov/sfs/aid/general/integration-220/overview`, accessed on July 14, 2017). We refer to Wells et al. (2009) for a review and discussion of inter-district school integration programs.

each of these sufficient conditions is also necessary for the corresponding policy goal in a "maximal domain" sense, that is, if the admissions rule of even one district violates the condition, the policy goal is violated at some student preference profile.

Last, but not least, we also consider a requirement that there be enough student diversity across districts. In fact, diversity appears to be the main motivation for many district integration programs. To put this into context, we note that segregation is prevalent under intra-district school choice programs even though they often seek diversity by controlled-choice constraints.[3] This is perhaps unsurprising given that only residents of the district can participate in intra-district school choice and there is often a severe residential segregation. In fact, a number of studies such as Rivkin (1994) and Clotfelter (1999, 2011) attribute the majority—as high as 80 percent for some data and measure—of racial and ethnic segregation in public schools to disparities between school districts rather than within school districts. Given this concern, many inter-district choice programs explicitly list achieving diversity as their main goals.

A case in point is the *Achievement and Integration (AI) Program* of the Minnesota Department of Education (MDE). Introduced in 2013, the AI program incentivizes school districts for integration. A district is required to participate in this program if the proportion of a minority group in the district is considerably higher than that of a neighboring district. In particular, every year the MDE commissioner analyzes fall enrollment data from every district, and when a district and one of its adjoining districts have difference of 20 percent or higher in the proportion of any group of enrolled *protected students* (American Indian, Asian or Pacific Islander, Hispanic, Black, not of Hispanic origin, and White, not of Hispanic origin), the district with the higher percentage is considered to be *racially isolated*. Racially isolated districts are required to be in the AI program.[4] In the 2015-16 school year, more than 120 school districts participated in this program. Figure 1, taken from MDE's website, shows school districts in the Minneapolis-Saint Paul metro area that take part in an AI program. In this figure, districts with the same color are the adjoining districts that work together in the same AI program.

Motivated by Minnesota's AI program, we consider a policy goal requiring that the difference of the proportions of each student type across districts be within a given bound. Then, we provide a necessary and sufficient condition (in the maximal domain sense as before) for SPDA to satisfy the diversity policy. The condition is provided as a condition

[3]Examples include Boston before 1999, Cambridge, Columbus, and Minneapolis. See Abdulkadiroğlu and Sönmez (2003) for details of these programs as well as analysis of controlled school choice.

[4]In Minnesota's AI program, if the difference in the proportion of protected students at a school is 20 percent or higher than a school in the same district, the school with the higher percentage is considered a *racially identifiable school* (RIS) and districts with RIS schools also need to participate in the AI program. In this paper, we focus on diversity issues across districts rather than within districts. Diversity problems within districts are studied in the controlled school choice literature that we discuss below.

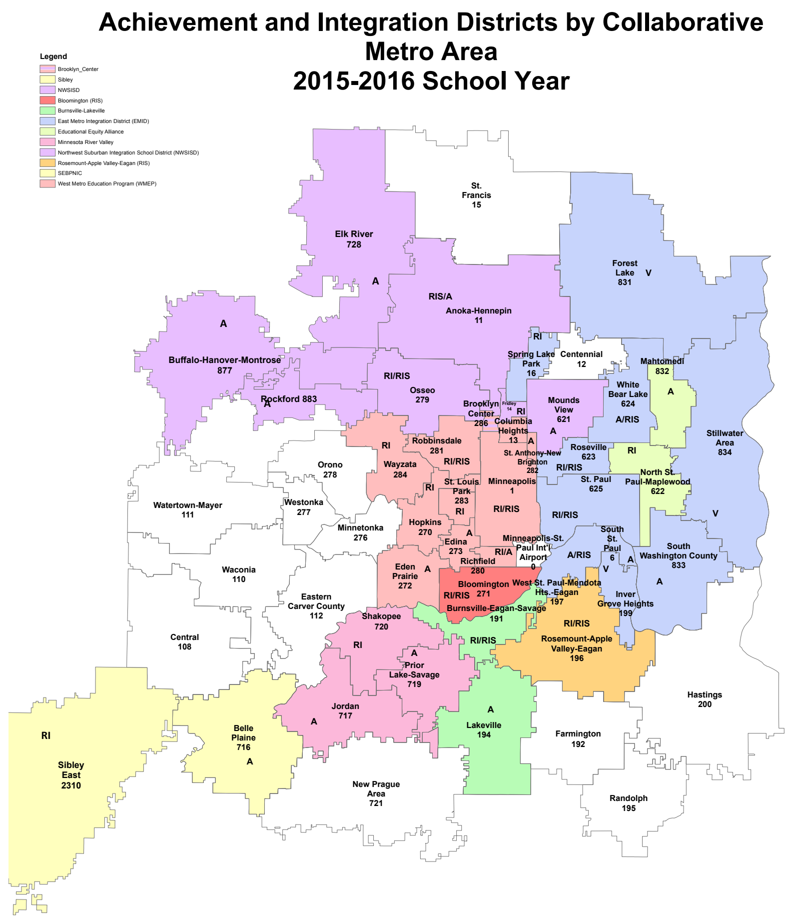


Figure 1. Minnesota Metro Area Participating School Districts

on district admissions rules that have a structure of type-specific ceilings, an analogue of the class of choice rules analyzed by Ehlers et al. (2014) in the context of a more standard intra-district school-choice problem.

Next we turn our attention to efficiency. Given that the policy goals work as constraints on matchings, we use the concept of *constrained efficiency*. We say that a matching is constrained efficient if it satisfies the policy goal and is not Pareto dominated by any matching that satisfies the same policy goal. In addition, we require individual rationality and strategy-proofness. We first demonstrate an impossibility result; when the diversity policy is given as type-specific ceilings at the district level, there is no mechanism that satisfies constrained efficiency, individual rationality, strategy-proofness, and the policy goal. By contrast, a version of top trading cycles (TTC) mechanism satisfies these three properties as well as the policy goal when the policy goal satisfies M-convexity , a concept in discrete mathematics (Murota, 2003). We proceed to show that the balanced-exchange policy and an alternative form of diversity policy—type specific ceilings at the individual school level instead of at the district level—are M-convex , so TTC satisfies the desired properties for these policies. The same conclusion holds even when both of these policy goals are imposed simultaneously.

Related Literature. Our paper is closely related to the controlled school choice literature that studies student diversity in schools in a given district. Abdulkadiroğlu and Sönmez (2003) introduce a policy that imposes type-specific ceilings on each school. This policy has been analyzed by Abdulkadiroğlu (2005), Ergin and Sönmez (2006), and Kojima (2012), among others. More accommodating policies using reserves rather than type-specific ceilings have been proposed and analyzed by Hafalir et al. (2013) and Ehlers et al. (2014). The latter paper finds difficulties associated with hard floor constraints, an issue further analyzed by Fragiadakis et al. (2015) and Fragiadakis and Troyan (2017).[5] In addition to sharing the motivation of achieving diversity, our paper is related to this literature in that we extend the type-specific reserve and ceiling constraints to district admissions rules. In contrast to this literature, however, our policy goals are imposed on districts rather than individual schools, which makes our model and analysis different from the existing ones.

The feature of our paper that constraints are imposed on sets of schools (i.e., districts), rather than individual schools, is shared by several recent studies in matching with constraints. Kamada and Kojima (2015) study a model where the number of doctors who can be matched with hospitals in each region has an upper bound constraint. Variations and generalizations of this problem are studied by Goto et al. (2014, 2017), Biro et al. (2010), and Kamada and Kojima (2017, 2018), among others. While sharing the broad interest in constraints, these papers are different from ours in at least two major respects. First, they do not assume a set of hospitals is endowed with a well-defined choice function, while each school district has a choice function in our model. Second, the policy issues studied in these papers and ours are different given differences in the intended applications. These differences render our analysis distinct from those of the other papers, with none of their results implying ours nor vice versa.

One of the notable features of our model is that district admission rules do not necessarily satisfy the standard assumptions in the literature such as *substitutability*, which guarantees the existence of a stable matching.[6] In fact, even a seemingly very reasonable district admissions rule may violate substitutability because a district can choose at most

[5]In addition to works discussed above, recent studies on the controlled school choice and other two-sided matching problems with diversity concerns include Westkamp (2013), Echenique and Yenmez (2015), Bó (2016), Doğan (2016), Sönmez (2013), Kominers and Sönmez (2016), Erdil and Kumano (2012), Dur et al. (2014), Aygün and Bó (2016), Aygün and Turhan (2016), Dur et al. (2016), and Nguyen and Vohra (2017).

[6]Substitutability is a condition on choice functions. It states that whenever a contract is chosen from a set, then it must be chosen from any subset containing that contract.

one contract associated with the same student, namely just one contract representing one school that the student can attend. Rather, we make weaker assumptions following the approach of Hatfield and Kominers (2014). This issue is playing an increasingly prominent role in matching with contracts literature, for example, in matching with constraints (Kamada and Kojima, 2015), college admissions (Aygün and Turhan, 2016; Yenmez, 2018), and postgraduate admissions (Hassidim et al., 2017), just to name a few.

Our analysis of Pareto efficient mechanisms is related to a small but rapidly growing literature that uses discrete optimization techniques for matching problems. Closest to ours is Suzuki et al. (2017) who show that a version of the TTC mechanism satisfies desirable properties if the constraint satisfies M-convexity.[7] Our analysis builds upon and generalizes theirs. While the use of discrete convex analysis for efficient object allocation is still rare, it has been utilized in an increasing number of matching problems such as two-sided matching with possibly bounded transfer (Fujishige and Tamura, 2006, 2007), matching with substitutable choice functions (Murota and Yokoi, 2015), matching with constraints (Kojima et al., 2018), and trading networks (Candogan et al., 2016).

At a high level, the present paper is part of research in resource allocation under constraints. Real-life auction problems often feature constraints (Milgrom, 2009), and a great deal of attention was paid to cope with complex constraints in a recent FCC auction for spectrum allocation (Milgrom and Segal, 2014). Handling constraints is also a subject of a series of papers on probabilistic assignment mechanisms (Budish et al., 2013; Che et al., 2013; Pycia and Ünver, 2015; Akbarpour and Nikzad, 2017; Nguyen et al., 2016). Closer to ours are Dur and Ünver (2015) and Dur et al. (2015). They consider the balance of incoming and outgoing members—a requirement that we also analyze–while modelling exchanges of members of different institutions under constraints. Although the differences in the model primitives and exact constraints make it impossible to directly compare their studies with ours, these papers and ours clearly share broad interests in designing mechanisms under constraints.

The rest of the paper is organized as follows. In Section 2, we introduce the model. In Sections 3 and 4, we study when the policy goals can be satisfied by SPDA and TTC, respectively. Section 5 concludes. Additional results, examples, and omitted proofs are presented in the Appendix.

2. Model

In this section, we introduce our concepts and notation.

[7]See Kurata et al. (2016) for an earlier work on TTC in a more specialized setting involving floor constraints at individual schools.

2.1. Preliminary Definitions. There exist finite sets of students $\mathcal{S}$, districts $\mathcal{D}$, and schools $\mathcal{C}$. Each student s and school c has a home district represented by $d(s)$ and $d(c)$, respectively. Each student s has a type $\tau(s)$ that can represent different aspects of a student such as gender, race, socioeconomic status, etc. The set of all types is finite and denoted by $\mathcal{T}$. Each school c has a capacity q_c, which is the maximum number of students that the school can enroll. For each district d, k_d is the number of students whose home district is d".In each district, schools have sufficiently large capacities to accommodate all students from the district, i.e., for every district d, $k_d \leq \sum_{c:d(c)=d} q_c$. For each type t, k^t is the number of type-t students.

Throughout this paper, we model district integration as a matching problem between students and districts. However, merely identifying the district with which a student is matched leaves the specific school she is enrolled in unspecified. To specify which school within the district the student is matched with, we use the notion of contracts: A contract $x = (s, d, c)$ specifies a student s, a district d, and a school c within this district, i.e., $d(c) = d$.[8] For any contract x, let $s(x)$, $d(x)$, and $c(x)$ denote the student, district, and school associated with this contract, respectively. Let $\mathcal{X} \subseteq \mathcal{S} \times \mathcal{D} \times \mathcal{C}$ denote the set of all contracts. For any set of contracts X, let X_s denote the set of contracts in X associated with student s, i.e., $X_s = \{x \in X | s(x) = s\}$. Similarly, let X_d and X_c denote the sets of contracts in X associated with district d and school c, respectively.

Each district d has an ***admissions rule*** that is represented by a choice function Ch_d. Given a set of contracts X, the district chooses a subset of contracts associated with itself, i.e., $Ch_d(X) = Ch_d(X_d) \subseteq X_d$.

Each student s has a strict preference order P_s over all schools and the outside option of being unmatched, which is denoted by $\emptyset$. Likewise, P_s is also used to rank contracts associated with s. Furthermore, we assume that the outside option is the least preferred outcome, so for every contract x associated with s, $x \; P_s \; \emptyset$. The corresponding weak order is denoted by R_s. More precisely, for any two contracts x, y associated with s, $x \; R_s \; y$ if $x \; P_s \; y$ or $x = y$.

A ***matching*** is a set of contracts. A matching X is ***feasible for students*** if there exists at most one contract associated with every student in X. A matching X is ***feasible*** if it is feasible for students and the number of contracts associated with every school in X is at most its capacity (i.e., for any $c \in \mathcal{C}$, $|X_c| \leq q_c$). We assume that there exists a feasible ***initial matching*** $\tilde{X}$.[9] For any student s, we call $\tilde{X}_s$ the ***initial match*** of student s. Whenever $\tilde{X}_s$ is nonempty, we call it the ***initial school*** of student s.

[8]For ease of exposition, a contract will sometimes be denoted by a pair (s, c) with the understanding that the district associated with the contract is the home district of school c.

[9]In Appendix A, we also consider the case when the initial matching for each district is constructed using the student preferences and district admissions rules.

An ***integration problem*** is a tuple $(\mathcal{S}, \mathcal{D}, \mathcal{C}, \mathcal{T}, \{d(a)\}_{a \in \mathcal{S} \cup \mathcal{C}}, \{\tau(s), P_s\}_{s \in \mathcal{S}}, \{Ch_d\}_{d \in \mathcal{D}}, \{q_c\}_{c \in \mathcal{C}}, \tilde{X})$. In what follows, we assume that all the components of an integration problem are publicly known except for student preferences. Therefore, we sometimes refer to an integration problem by the student preference profile which we denote as $P_{\mathcal{S}}$. The preference profile of a subset of students $S \subset \mathcal{S}$ is denoted by P_S.

2.2. Properties of Admissions Rules. A district admissions rule Ch_d is ***feasible*** if it always chooses a feasible matching. It is ***acceptant*** if, for any contract x associated with district d and matching X that is feasible for students, if x is rejected from X, then at $Ch_d(X)$, either

- the number of students assigned to school $c(x)$ is equal to $q_{c(x)}$, or
- the number of students assigned to district d is at least k_d.

In words, when a district admissions rule is acceptant, a contract $x = (s, d, c)$ can be rejected by district d from a set which is feasible for students only if either the capacity of school c is filled or district d has accepted at least k_d students. Equivalently, if neither of these two conditions is satisfied, then the district has to accept the student. Throughout the paper, we assume that admissions rules are feasible and acceptant.[10]

A district admissions rule satisfies ***substitutability*** if, whenever a contract is chosen from a set, then it is also chosen from any subset containing that contract (Kelso and Crawford, 1982; Roth, 1984). More formally, a district admissions rule Ch_d satisfies substitutability if, for every $x \in X \subseteq Y \subseteq \mathcal{X}$ with $x \in Ch_d(Y)$, it must be that $x \in Ch_d(X)$. A district admissions rule satisfies ***the law of aggregate demand*** (LAD) if the number of contracts chosen from a set is weakly greater than that of a subset (Hatfield and Milgrom, 2005). Mathematically, a district admissions rule Ch_d satisfies LAD if, for every $X \subseteq Y \subseteq \mathcal{X}$, $|Ch_d(X)| \leq |Ch_d(Y)|$.[11] A ***completion*** of a district admissions rule Ch_d is another admissions rule Ch'_d such that for every matching X either $Ch'_d(X)$ is equal to $Ch_d(X)$ or it is not feasible for students (Hatfield and Kominers, 2014). Throughout the paper, we assume that district admissions rules have completions that satisfy substitutability and LAD.[12] In Appendix B, we provide classes of district admisstions rule that satisfy our assumptions.

2.3. Matching Properties, Policy Goals, and Mechanisms. A feasible matching X satisfies ***individual rationality*** if every student weakly prefers the outcome in X to her initial match, i.e., for every student s, $X_s \; R_s \; \tilde{X}_s$.

A ***distribution*** $\xi \in (\mathbb{Z}_+)^{|\mathcal{C}| \times |\mathcal{T}|}$ is a vector such that the entry for school c and type t is denoted by ξ_c^t. The entry ξ_c^t is interpreted as the number of type-t students in school c.

[10]In Section 3.3, we assume a weaker notion of acceptance when the admissions rules limit the number of students of each type.

[11]Alkan (2002) and Alkan and Gale (2003) introduce related monotonicity conditions.

[12]Hatfield and Kojima (2010) study other notions of weak substitutability.

Furthermore, $\xi^t_d \equiv \sum_{c:d(c)=d} \xi^t_c$ denotes the number of type-t students in district d at ξ. Likewise, for any feasible matching X, the ***distribution associated with*** X is $\xi(X)$ whose c, t entry $\xi^t_c(X)$ is the number of type-t students assigned to school c at X. Similarly, $\xi^t_d(X)$ denotes the number of type-t students assigned to district d at X.

We represent a policy goal as a set of distributions. Let Ξ denote a generic set of distributions. The policy that each student is matched without assigning any school more students than its capacity is denoted by Ξ^0, i.e., $\Xi^0 \equiv \{\xi | \sum_{c,t} \xi^t_c = \sum_d k_d \text{ and } \forall c,\ q_c \geq \sum_t \xi^t_c\}$. A matching X ***satisfies the policy goal*** Ξ if the distribution associated with X is in Ξ.

A feasible matching X ***Pareto dominates*** another feasible matching Y if every student weakly prefers the outcome in X to the outcome in Y and at least one student strictly prefers the former to the latter. Given a policy goal, a feasible matching X that satisfies the policy goal satisfies ***constrained efficiency*** if there exists no feasible matching that satisfies the policy goal and Pareto dominates X.

A matching X is ***stable*** if it is feasible and

- districts would choose all contracts assigned to them, i.e., $Ch_d(X) = X_d$ for every district d, and
- there exist no student s and district d who would like to match with each other, i.e., there exists no contract $x = (s, d, c) \notin X$ such that $x\ P_s\ X_s$ and $x \in Ch_d(X \cup \{x\})$.

Stability was introduced by Gale and Shapley (1962) for the college admissions problem. In the context of assigning students to public schools, it is viewed as a fairness notion (Abdulkadiroğlu and Sönmez, 2003).

A ***mechanism*** ϕ takes a profile of student preferences as input and produces a feasible matching. The outcome for student s at the reported preference profile $P_{\mathcal{S}}$ under mechanism ϕ is denoted as $\phi_s(P_{\mathcal{S}})$. A mechanism ϕ satisfies ***strategy-proofness*** if no student can misreport her preferences and get a more preferred contract. More formally, for every student s and preference profile $P_{\mathcal{S}}$, there exists no preference P'_s such that $\phi_s(P'_s, P_{\mathcal{S}\setminus\{s\}})\ P_s\ \phi_s(P_{\mathcal{S}})$. For any property on matchings, a mechanism satisfies the property if, for every preference profile, the matching produced by the mechanism satisfies the property.

3. Achieving Policy Goals with Stable Outcomes

To achieve stable matchings with desirable properties, we use a generalization of the deferred-acceptance algorithm of Gale and Shapley (1962).

Student-Proposing Deferred Acceptance Algorithm (SPDA).

Step 1: Each student s proposes a contract (s, d, c) to district d where c is her most preferred school. Suppose that X^1_d is the set of contracts proposed to district d.

District d tentatively accepts contracts in $Ch_d(X_d^1)$ and permanently rejects the rest. If there are no rejections, then stop and return $\cup_{d\in\mathcal{D}} Ch_d(X_d^1)$ as the outcome.

Step n (n > 1): Each student s whose contract was rejected in Step $n-1$ proposes a contract (s,d,c) to district d where c is her next preferred school. If there is no such school, then the student does not make any proposals. Suppose that X_d^n is the set of contracts that were tentatively accepted by district d in Step $n-1$ and contracts that were proposed to district d in this step. District d tentatively accepts contracts in $Ch_d(X_d^n)$ and permanently rejects the rest. If there are no rejections, then stop and return $\cup_{d\in\mathcal{D}} Ch_d(X_d^n)$.

When district admissions rules have completions that satisfy substitutability and LAD, SPDA is stable and strategy-proof (Hatfield and Kominers, 2014). Therefore, when we analyze SPDA, we assume that students report their preferences truthfully.

We illustrate SPDA using the following example. We come back to this example later to study the effects of district integration.

Example 1. Consider an integration problem with two school districts, d_1 and d_2. District d_1 has school c_1 with capacity one and school c_2 with capacity two. District d_2 has school c_3 with capacity two. There are four students: students s_1 and s_2 are from district d_1, whereas students s_3 and s_4 are from district d_2. The initial matching is $\{(s_1,c_1),(s_2,c_2),(s_3,c_3),(s_4,c_3)\}$.

Given any set of contacts, district d_1 chooses students who have contracts with school c_1 first and then chooses from the remaining students who have contracts with school c_2. For school c_1, the district prioritizes students according to order $s_3 \succ s_4 \succ s_1 \succ s_2$ and chooses one applicant if there is any. For school c_2, the district prioritizes students according to order $s_1 \succ s_2 \succ s_3 \succ s_4$ and chooses as many applicants as possible without going over the school's capacity while ignoring the contracts of the students who have already been accepted at c_1. Likewise, district d_2 prioritizes students according to order $s_3 \succ s_4 \succ s_1 \succ s_2$ and chooses as many applicants as possible without going over the capacity of school c_3. These admissions rules are feasible and acceptant, and they have completions that satisfy substitutability and LAD.[13] In addition, student preferences are given by the following table,

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}
c_1	c_3	c_1	c_2
c_2	c_1	c_2	c_1
c_3	c_2	c_3	c_3

[13]See Appendix B.1 for a general class of admissions rules including this one that satisfy our assumptions. In Appendix B.1, we also prove that those admission rules are feasible and acceptant, and they have completions that satisfy substitutability and LAD.

which means that, for instance, student s_1 prefers c_1 to c_2 to c_3.

In this integration problem, SPDA runs as follows. At the first step, student s_1 proposes to district d_1 with contract (s_1, c_1), student s_2 proposes to district d_2 with contract (s_2, c_3), student s_3 proposes to district d_1 with contract (s_3, c_1), and student s_4 proposes to district d_1 with contract (s_4, c_2). District d_1 first considers contracts associated with school c_1, (s_1, c_1) and (s_3, c_1), and tentatively accepts (s_3, c_1) while rejecting (s_1, c_1) because student s_3 has a higher priority than student s_1 at school c_1. Then district d_1 considers contracts of the remaining students associated with school c_2. In this case, there is only one such contract, (s_4, c_2), which is tentatively accepted. District d_2 considers contract (s_2, c_3) and tentatively accepts it. The tentative matching is $\{(s_2, c_3), (s_3, c_1), (s_4, c_2)\}$. Since there is a rejection, the algorithm proceeds to the next step.

At the second step, student s_1 proposes to district d_1 with contract (s_1, c_2). District d_1 first considers contract (s_3, c_1) and tentatively accepts it. Then district d_1 considers contracts (s_1, c_2) and (s_4, c_2) and tentatively accepts them both. District d_2 does not have any new contracts, so tentatively accepts (s_2, c_3). Since there is no rejection, the algorithm stops. The outcome of SPDA is $\{(s_1, c_2), (s_2, c_3), (s_3, c_1), (s_4, c_2)\}$. □

In the rest of this section, we formalize three policy goals and characterize conditions under which SPDA satisfies them.

3.1. Individual Rationality. In our context, individual rationality requires that every student is matched with a weakly more preferred school than her initial school. As a result, SPDA does not necessarily satisfy individual rationality even though each student is either unmatched or matched with a school that is more preferred than being unmatched.

If individual rationality is violated so that some students prefer their initial schools to the outcome of SPDA, then there may be public opposition which may harm integration efforts. For this reason, individual rationality is a desirable property for policymakers. The following condition proves to play a crucial role to achieve this property.

Definition 1. *A district admissions rule Ch_d **respects the initial matching** if, for any student s whose initial matching is $x = (s, d, c)$ for some school c in district d and matching X that is feasible for students, $x \in X$ implies $x \in Ch_d(X)$.*

When a district's admissions rule respects the initial matching, it has to admit those contracts in which students apply to their initial schools. The following theorem shows that this is exactly the condition for SPDA to satisfy individual rationality.

Theorem 1. *If each district's admissions rule respects the initial matching, then SPDA satisfies individual rationality. Moreover, if at least one district's admissions rule fails to respect the initial matching, then SPDA does not satisfy individual rationality.*

The intuition for the first part of this theorem is simple; When district admissions rules respect the initial matching, no student is matched with a school which is less preferred to her initial school under SPDA because she is guaranteed to be accepted by that school if she applies to it. For the second part of the theorem, we construct a specific student preference profile that makes one student strictly worse off whenever there exists one district with an admissions rule that does not respect the initial matching.

In the next example, we illustrate SPDA with district admissions rules that respect the initial matching.

Example 2. Consider the integration problem in Example 1. Recall that in this problem, the outcome of SPDA is $\{(s_1, c_2), (s_2, c_3), (s_3, c_1), (s_4, c_2)\}$. This matching is not individually rational because student s_1 prefers her initial school c_1 to school c_2 that she is matched with. This observation is consistent with Theorem 1 because the admissions rule of district d_1 does not respect the initial matching. In particular, $Ch_{d_1}(\{(s_1, c_1), (s_3, c_1)\}) = \{(s_3, c_1)\}$, so student s_1 is rejected from a matching that is feasible for students and includes the contract with her initial school.

Now modify the priority ranking of district d_1 at school c_1 so that $s_1 \succ s_2 \succ s_3 \succ s_4$ but, otherwise, keep the construction of the district admissions rules and students preferences the same as before. With this change, district admissions rules respect the initial matching because each student is accepted when she applies to the district with her initial school.[14] In particular, the proposal of student s_1 to district d_1 with her initial school c_1 is always accepted. With this modification, it is easy to check that the outcome of SPDA is $\{(s_1, c_1), (s_2, c_3), (s_3, c_2), (s_4, c_2)\}$. This matching satisfies individual rationality. □

In some school districts, such as Boston, each student gets a priority at her neighborhood school as in this example. In the absence of other types of priorities, neighborhood priority guarantees that SPDA satisfies individual rationality.

3.2. Balanced Exchange. When school districts are integrated, maintaining a balance of students incoming from and outgoing to the other districts is important. To formalize this idea, we say that a mechanism satisfies the ***balanced-exchange*** policy if the number of students that a district gets from the other districts and the number of students that the district sends to the others are the same for every district and for every profile of student preferences. Equivalently, the number of students assigned to a district must be equal to the number of students from that district.

This is an important policy because the funding that a district gets depends on the number of students it serves. Therefore, integration may not be sustainable if SPDA does not

[14] In Appendix B.2, we construct a class of district admissions rules that includes this admissions rule as a special case. These admissions rules are feasible and acceptant, and have completions that satisfy substitutability and LAD. Furthermore, they also respect the initial matching.

satisfy the balanced-exchange policy. For achieving this policy goal, the following condition on admissions rules proves important.

Definition 2. *A matching X is* ***rationed*** *if, for every district, it does not assign more students to the district than the number of students from there. A district admissions rule is* ***rationed*** *if it chooses a rationed matching from any matching that is feasible for students.*

When a district admissions rule is rationed, then the district does not accept more students than the number of students from the district at any matching that is feasible for students. The result below establishes that this property is exactly the condition to guarantee that SPDA satisfies the balanced-exchange policy.

Theorem 2. *If each district's admissions rule is rationed, then SPDA satisfies the balanced-exchange policy. Moreover, if at least one district's admissions rule fails to be rationed, then SPDA does not satisfy the balanced-exchange policy.*

To obtain intuition for this theorem, consider a student. Acceptance requires that a district can reject all contracts of this student only when the number of students assigned to the district is at least as large as the number of students from that district. As a result, all students are guaranteed to be matched with some school district. In addition, when district admissions rules are rationed, a district cannot accept more students than the number of students from the district. These two facts together imply that the number of students assigned to a district in SPDA is equal to the number of students from that district. Therefore, SPDA satisfies the balanced-exchange policy when each district's admissions rule is rationed. Conversely, when there exists one district with an admissions rule that fails to be rationed, then we can construct student preferences such that this district is matched with more students than the number of students from there in SPDA, which means that the outcome does not satisfy the balanced-exchange policy.

Now we illustrate SPDA when district admissions rules are rationed.

Example 3. Consider the integration problem in Example 1. Recall that in this problem, the SPDA outcome is $\{(s_1, c_2), (s_2, c_3), (s_3, c_1), (s_4, c_2)\}$. Since there are three students matched with district d_1 and that there are only two students from that district, SPDA does not satisfy the balanced-exchange policy. This is consistent with Theorem 2 because admissions rule of district d_1 is not rationed. In particular, $Ch_{d_1}(\{(s_1, c_2), (s_3, c_1), (s_4, c_2)\}) = \{(s_1, c_2), (s_3, c_1), (s_4, c_2)\}$, so district d_1 accepts more students than the number of students from there.

Suppose that we modify admissions rule of district d_1 as follows. If the district chooses a contract associated with school c_1, then at most one student is admitted to school c_2. Therefore, the district never chooses more than two contracts, which is the number of students from there. Therefore, the updated admissions rule is rationed.[15] With this change, it is easy to check that the SPDA outcome is $\{(s_1, c_2), (s_2, c_3), (s_3, c_1), (s_4, c_3)\}$, which satisfies the balanced-exchange policy. $\square$

3.3. Diversity. The third policy goal we consider is that of diversity. More specifically, we are interested in how to ensure that there is enough diversity across districts so that the student composition in terms of demographics do not vary too much from district to district.

We are mainly motivated by a program that is used in the state of Minnesota. The state law in Minnesota identifies racially isolated (relative to one of its neighbors) school districts and requires them to be in the *Achievement and Integration (AI) Program*. The goal is to increase the racial parity between neighboring school districts. We first introduce a diversity policy in the spirit of the *AI* Program: Given a constant $\alpha \in [0, 1]$, we say that a mechanism satisfies the ***α-diversity policy*** if for all submitted preferences, for each type t and districts d and d', the difference between the ratios of type-t students in districts d and d' is not more than α. We interpret α to be the maximum ratio difference tolerated under the diversity policy; for instance, $\alpha = 0.2$ for Minnesota.

We are interested in admissions rules that satisfy the α-diversity policy when school districts are integrated. A common method to achieve diversity is to use type-specific ceilings. Formally:

Definition 3. *A district admissions rule Ch_d has a **ceiling** of q_d^t for type-t students if the number of type-t students admitted cannot exceed this ceiling. More formally, for any matching X that is feasible for students,*

$$|\{x \in Ch_d(X) | \tau(s(x)) = t\}| \leq q_d^t.$$

A type-t ceiling limits the number of type-t students that the district can admit. It has two immediate implications. First, the district can allocate the rest of the school seats to students of other types. Second, type-t students who cannot be admitted to the district can be matched with the other districts.

Note that district admissions rules do not necessarily satisfy acceptance once type-specific ceilings are imposed. A weaker version of the acceptance assumption, however, can still be satisfied.

Definition 4. *A district admissions rule Ch_d is **weakly acceptant** if, for any contract x associated with a type-t student and district d and matching X that is feasible for students, if x is rejected from X, then at $Ch_d(X)$,*

[15] In Appendix B.3, we construct a class of rationed district admissions rules that includes this admissions rule as a special case. These admissions rules are feasible and acceptant, and they have completions that satisfy substitutability and LAD.

- *the number of students assigned to school $c(x)$ is equal to $q_{c(x)}$, or*
- *the number of students assigned to district d is at least k_d, or*
- *the number of type-t students assigned to district d is at least q_d^t.*

Weak acceptance requires that a student can be rejected only for one of the reasons listed above. In other words, if a district d considers a set of contracts that is feasible for students, no contract of a type-t student can be rejected as long as the number of contracts associated with type-t students is no more than its type-t ceiling, the number of contracts associated with the school is no more than its capacity, and there are at most k_d contracts for district d.

In SPDA, a student may be left unassigned because of type-specific ceilings even when district admissions rules are weakly acceptant. To make sure that every student is matched, we make the following assumption.

Definition 5. *District admissions rules $(Ch_d)_{d\in\mathcal{D}}$ **accommodate unmatched students** if for any student s and feasible matching X in which student s is unmatched, there exists $x = (s, d, c) \in \mathcal{X}$ such that $x \in Ch_d(X \cup \{x\})$.*

When district admissions rules accommodate unmatched students, for any feasible matching in which a student is unmatched, there exists a school such that the district associated with the school would admit that student if she applies. For example, when admissions rules respect the initial matching, they also accommodate unmatched students. Lemma 1 in Appendix C shows that when district admissions rules accommodate unmatched students, every student is matched to a school in SPDA.

In general, accommodation of unmatched students may be in conflict with type-specific ceilings because there may not be enough space for a student type when ceilings are small for this type. We assume that type-specific ceilings are high enough so that there exists a feasible matching that matches each student with a school.[16]

In Appendix B.4, we provide a class of admissions rules that accommodate unmatched students and have type-specific ceilings. These admissions rules satisfy all of the assumptions that we make in this section. Furthermore, they generalize the concept of reserves in the context of schools to districts.[17] Specifically, as we formally define in the Appendix, we say that a district d has a ***reserve*** r_d^t for type-t students if district d accepts a type-t student at *some* school in its district whenever the number of type-t students currently matched in

[16]For instance, ignoring integer problems, $q_d^t \geq k_d \frac{k_t}{\sum_{t'\in\mathcal{T}} k_{t'}}$ for all t, d, would make ceilings compatible with this property as it would be possible to assign the same percentage of students of each type to all districts.

[17]See Hafalir et al. (2013) and Ehlers et al. (2014) for the concept of reserves and Echenique and Yenmez (2015) for an axiomatic characterization of choice rules of schools with reserves.

the district is less than r_d^t. Note that using (high enough) reserves is one way to guarantee that district admissions rules accommodate unmatched students.

We focus on choice rules that would result in rationed matchings where every student is matched under SPDA.[18] Note that with these choice rules, a type-t ceiling of district d may result in a floor of another type t' in district d in the sense that the number of type-t' students in the district should be at least a certain number. Moreover, this may further impose a ceiling for type t' in another district d'. To see this suppose, for example, that (i) there are two districts d and d', (ii) in each district, there is one school and 100 students, (iii) 100 students are of type t and 100 students are of another type t', and (iv) each district has type-t ceiling set to 60 and type-t' ceiling set to 70. In this environment, for all rationed matchings that match every student, each district has to have at least 40 type-t' students (because otherwise the number of type-t students in that district would have to be more than 60). Moreover, this would mean that there cannot be more than 60 type-t' students in any district (because otherwise there would need to be more than 40 type-t' students in the other district, contradicting the floor we just calculated). Hence, in this example, in effect we have a floor of 40 and a (further restricted) ceiling of 60 for type-t' students for each district.

Faced with this complication, our approach is to find the "tightest" lower and upper bounds induced by type-specific ceilings For this purpose, a certain optimization problem proves useful. More specifically, consider a linear-programming problem where for each type t and district d, we seek the minimum and maximum values of y_d^t subject to (i) $\sum_{t' \in \mathcal{T}} y_{d'}^{t'} = k_{d'}$ for all $d' \in \mathcal{D}$, (ii) $\sum_{d' \in \mathcal{D}} y_{d'}^{t'} = k^{t'}$ for all $t' \in \mathcal{T}$, and (iii) $y_{d'}^{t'} \leq q_{d'}^{t'}$ for all $t' \in \mathcal{T}$ and $d' \in \mathcal{D}$. Let $\hat{p}_d^t$ and $\hat{q}_d^t$ be the solutions y_d^t to the minimization and maximization problems, respectively.

Both of these optimization problems belong to a special class of linear-programming problems called a minimum-cost flow problem, and many computationally efficient algorithms to solve it are known in the literature.[19] A straightforward but important observation is that $\hat{p}_d^t$ (resp. $\hat{q}_d^t$) is exactly the lowest (resp. highest) number of type-t students who can be matched to district d in a rationed matching that match every student (Lemma 2

[18]SPDA produces a rationed matching if district choice rules are rationed, and matchings where every student is matched when choice rules accommodate unmatched students. We formalize matchings that (i) are rationed, (ii) match each student, and (iii) satisfies type-specific ceilings as *legitimate matchings* in the Appendix.

[19]To see that our problem is a minimum-cost flow problem, note that we can take vector $(k_d)_d$ as the "supply," vector $(k^t)_t$ as the "demand," and matrix $(q_d^t)_{d,t}$ as the "arc capacity bounds," and the objective functions for $\hat{p}_d^t$ and $\hat{q}_d^t$ to be $\min y_d^t$ and $\min -y_d^t$, respectively. These problems have an "integrality property" that if the supply, demand and bounds are integers, then all the solutions are integers as well. As already mentioned, many algorithms have been proposed to solve different objective functions for these problems. For instance, the capacity scaling algorithm of Edmonds and Karp (1972) gives the solutions in polynomial time. For more information, see Chapter 10 of Ahuja (2017). We are grateful to Fatma Kilinc-Karzan for helpful discussions.

in Appendix C). Given this observation, we call $\hat{p}_d^t$ the ***implied floor*** and $\hat{q}_d^t$ the ***implied ceiling***.

Now we are ready to state the main result of this section.

Theorem 3. *Suppose that each district admissions rule is rationed and weakly acceptant, and has type-specific ceilings. Moreover, suppose that the district admissions rules accommodate unmatched students. If $\hat{q}_d^t/k_d - \hat{p}_{d'}^t/k_{d'} \leq \alpha$ for every type t and districts d, d' such that $d \neq d'$, then SPDA satisfies the α-diversity policy. Moreover, if $\hat{q}_d^t/k_d - \hat{p}_{d'}^t/k_{d'} > \alpha$ for some type t and districts d, d' with $d \neq d'$, then SPDA does not satisfy the α-diversity policy.*

The proof of this theorem, given in Appendix C, is based on a number of steps. First, as mentioned above, we note that $\hat{p}_d^t$ and $\hat{q}_d^t$ are lower and upper bounds, respectively, of the numbers of type-t students who can be matched in district d in any matching that satisfies type-specific ceilings. This observation immediately establishes the first part of the theorem. Then, we further establish that the implied floors and ceilings are not arbitrary lower and upper bounds, but "achievable" bounds in the sense that, for any pair of districts d and d', there exists a matching that satisfies type-specific ceilings and assigns exactly $\hat{q}_d^t$ type-t students in district d and exactly $\hat{p}_{d'}^t$ type-t students in district d' (Lemma 3). In other words, we establish that the implied ceiling and floor are achieved in two different districts, and they are achieved at *one* matching at the same time. We complete the proof of the theorem by constructing preferences such that the outcome of SPDA achieves these bounds.

Let us now consider an example in which the conditions on the admissions rules stated in Theorem 3 are satisfied and, therefore, districts get a diverse student body as required by the law.

Example 4. Consider an integration problem with two school districts, d_1 and d_2. District d_1 has school c_1 with capacity three and school c_2 with capacity two. District d_2 has school c_3 with capacity two and school c_4 with capacity one. There are seven students: students s_1, s_2, s_3, and s_4 are from district d_1 and have type t_1, whereas students s_5, s_6, and s_7 are from district d_2 and have type t_2.

To construct district admissions rules that satisfy the properties stated in Theorem 3, let us first specify type-specific ceilings and calculate implied floors and implied ceilings. Suppose that

$$q_{d_1}^{t_1} = 2,\ q_{d_1}^{t_2} = 3,\ q_{d_2}^{t_1} = 3,\ q_{d_2}^{t_2} = 2.$$

These yield,

$$\hat{p}_{d_1}^{t_1} = 1,\ \hat{p}_{d_1}^{t_2} = 2,\ \hat{p}_{d_2}^{t_1} = 2,\ \hat{p}_{d_2}^{t_2} = 0,$$

and

$$\hat{q}^{t_1}_{d_1} = 2,\ \hat{q}^{t_2}_{d_1} = 3,\ \hat{q}^{t_1}_{d_2} = 3,\ \hat{q}^{t_2}_{d_2} = 1.$$

For any two districts d and d', denote $\hat{q}^t_d/k_d - \hat{p}^t_{d'}/k_{d'}$ by $\Delta^t_{d,d'}$. Using the implied floors and ceilings above we get:

$$\begin{aligned}
\Delta^{t_1}_{d_1,d_2} &= 2/4 - 2/3 = -1/6,\\
\Delta^{t_1}_{d_2,d_1} &= 3/3 - 1/4 = 3/4,\\
\Delta^{t_2}_{d_1,d_2} &= 3/4 - 0/3 = 3/4, \text{ and}\\
\Delta^{t_2}_{d_2,d_1} &= 1/3 - 2/4 = -1/6.
\end{aligned}$$

Hence, these type-specific ceilings satisfy the condition stated in Theorem 3 that $\Delta^t_{d,d'} \leq \alpha$ for $\alpha = 0.75$.

We construct district admissions rules that have type-specific ceilings, accommodate unmatched students, and are rationed and weakly acceptant. As in Appendix B.4, we consider type-specific reserves. Let us consider the reserves for schools as follows:

$$r^{t_2}_{c_4} = 0, \text{ and } r^t_c = 1 \text{ for all other } c, t$$

Consider the following district admissions rule. Suppose each district has a master priority list over students and schools are ordered. First, schools choose contracts for its reserved seats till the reserves are filled or all the applicants of the relevant type are processed. Then schools choose from the remaining contracts to fill the rest of their seats until the school capacity is filled or the district has k_d contracts or district type-specific ceilings are filled or there are no more remaining contracts.

To give a more concrete example, suppose that the master priority list for all schools is as follows: $s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \succ s_6 \succ s_7$ and schools are ordered from the lowest index to the highest. Then, for example, we have the following:

$$Ch_{d_1}(\{(s_1,c_1),(s_2,c_1),(s_3,c_1),(s_4,c_2),(s_5,c_2)\}) = \{(s_1,c_1),(s_2,c_1),(s_4,c_2),(s_5,c_2)\}.$$

Let us elaborate on how we determine the chosen set of contracts in the above case. School c_1 considers contracts with students s_1, s_2, s_3. Among these students student s_1 has the highest priority, so she is admitted to school c_1 and fills the reserve of c_1. Next, student s_2 has the highest priority and school c_1 still has two empty seats, so student s_2 is admitted to school c_1. The type-t_1 ceiling for district d_1 is filled at this point. Therefore c_1 rejects s_3. Next, school c_2 considers contracts with students s_4, s_5. Among these students student s_4 has the highest priority, so she is admitted to school c_2 and fills the type-t_1 reserve of c_2. Next, student s_5 has the highest priority. When the school admits s_2, neither its school capacity nor its type-t_2 ceiling is violated, so student s_5 is admitted to school c_2, resulting in the chosen set of contracts presented above.

To illustrate the SPDA outcomes with and without integration, consider student preferences given by the following table,

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_{s_5}	P_{s_6}	P_{s_7}
c_2	c_3	c_4	c_2	c_1	c_1	c_2
$\vdots$	c_1	c_2	c_3	c_2	c_4	c_3
	$\vdots$	$\vdots$	c_1	c_3	$\vdots$	$\vdots$
			c_4	c_4		

where the dots in the table mean that the corresponding parts of the preferences are arbitrary.

Without integration, each district runs its own SPDA, and the algorithm produces the following outcome:

$$\{(s_1, c_2), (s_2, c_1), (s_3, c_2), (s_4, c_1), (s_5, c_3), (s_6, c_4), (s_7, c_3)\}.$$

With integration, we run SPDA with both districts on one side of the market and all students on the other.

It results in the following outcome:

$$\{(s_1, c_2), (s_2, c_3), (s_3, c_4), (s_4, c_2), (s_5, c_1), (s_6, c_1), (s_7, c_3)\}.$$

Without integration, district d_1 had four type-t_1 students and district d_2 had three type-t_2 students. With integration, district d_1 gets two students of both types and district d_2 gets two type-t_1 students and one type-t_2 student. As a result, the ratio difference for type-t_1 students between these districts decreased from 1 to roughly 0.17 and the ratio difference for type-t_2 students decreased from 1 to roughly 0.17.

This example illustrates that the actual ratio differences change as the student preferences change. Theorem 3 guarantees that ratio differences are never more than $\alpha = 0.75$, yet for the preference profile we considered it is much lower–only 0.17. □

4. Achieving Policy Goals with Efficient Outcomes

In this section, we study the existence of a mechanism that satisfies constrained efficiency together with strategy-proofness, individual rationality, and a given policy goal on the distribution of agents. We first consider a policy with type-specific ceilings in districts and establish an impossibility result.

Theorem 4. *Suppose that, for every type t and district d, there is a ceiling q_t^d on the number of type-t students in district d so that the policy goal is given by $\Xi \equiv \{\xi | \forall d\ \forall t\ q_d^t \geq \xi_d^t$ and $\forall c\ q_c \geq \sum_t \xi_c^t\}$.*

Then there exist an integration problem and ceilings $(q_t^d)_{t\in\mathcal{T},d\in\mathcal{D}}$ for which no mechanism satisfies constrained efficiency, individual rationality, strategy-proofness, and the policy goal Ξ.

We show this result using the following example.

Example 5. Consider the following integration problem with districts d_1 and d_2. District d_1 has schools c_1, c_2, and c_3 and district d_2 has schools c_4, c_5, and c_6. All schools have a capacity of one. There are six students: students s_1 and s_4 have type t_1, students s_2 and s_5 have type t_2, and students s_3 and s_6 have type t_3. Both districts have a ceiling of one for types t_1 and t_2: $q_{d_1}^{t_1} = q_{d_1}^{t_2} = 1$ and $q_{d_2}^{t_1} = q_{d_2}^{t_2} = 1$. Initially, student s_i is matched with school c_i, for $i = 1, \ldots, 6$. Student preferences are as follows.

s_1	s_2	s_3	s_4	s_5	s_6
c_6	c_6	c_5	c_3	c_3	c_1
c_1	c_2	c_4	c_4	c_5	c_2
$\vdots$	$\vdots$	c_3	$\vdots$	$\vdots$	c_6
		$\vdots$			$\vdots$

In this example, there are two matchings that are constrained efficient and individually rational:

$$X = \{(s_1, c_6), (s_2, c_2), (s_3, c_4), (s_4, c_3), (s_5, c_5), (s_6, c_1)\}, \text{ and}$$
$$Y = \{(s_1, c_1), (s_2, c_6), (s_3, c_5), (s_4, c_4), (s_5, c_3), (s_6, c_2)\}.$$

If a mechanism satisfies constrained efficiency, individual rationality, and the policy goal Ξ, then its outcome at the above student preference profile must produce either matching X or Y.

Consider the case where the mechanism produces matching X at the above student preference profile. Suppose student s_3 misreports her preference by ranking c_4 below c_3 while leaving c_5 as the first choice. Under the new report, the mechanism produces matching Y because it is the only constrained-efficient and individually rational matching. Since student s_3 strictly prefers her school in Y to her school in X, she has a profitable deviation.

Similarly, consider the case where the mechanism produces matching Y at the above student preference profile. Suppose student s_6 misreports her preference by ranking c_2 below c_6 while leaving c_1 as the first choice. In this case, the mechanism produces matching X because it is the only constrained-efficient and individually-rational matching. Since student s_6 strictly prefers her school in X to her school in Y, she has a profitable deviation.

In both cases, there exists a student with a profitable misreporting, so the desired conclusion follows. □

This example also shows that there is no mechanism that satisfies constrained efficiency, individual rationality, strategy-proofness, and the α-diversity policy goal for $\alpha = 0$ introduced in Section 3.3. Consequently, without any assumptions, a policy goal may not be implemented with the desirable properties. To establish a positive result, we consider policy goals that satisfy the following notion of discrete convexity, which is studied by the mathematics and operations research literatures (Murota, 2003).

Definition 6. *Let $\chi_{c,t}$ denote the distribution where there is one type-t student at school c and there are no other students. A set of distributions Ξ is **M-convex** if whenever $\xi, \tilde{\xi} \in \Xi$ and $\xi_c^t > \tilde{\xi}_c^t$ for some school c and type t then there exist school c' and type t' with $\xi_{c'}^{t'} < \tilde{\xi}_{c'}^{t'}$ such that $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi$ and $\tilde{\xi} + \chi_{c,t} - \chi_{c',t'} \in \Xi$.*[20]

To illustrate this concept, suppose that a set of distributions Ξ is M-convex. Consider two distributions ξ and $\tilde{\xi}$ in the set such that there are more type-t students in school c at ξ than at $\tilde{\xi}$. Then there exist school c' and type t' such that there are more type-t' students in school c' at $\tilde{\xi}$ than ξ with the following two properties. First, removing one type-t student from school c and adding one type-t' student to school c' in ξ produces a distribution in Ξ. Second, removing one type-t' student from school c' and adding one type-t student to school c in $\tilde{\xi}$ gives a distribution in Ξ (see Figure 2). Intuitively, from each of these two distributions we can move closer to the other distribution in an incremental manner, a property analogous to the standard convexity notion but adapted to a discrete setting. We illustrate this concept with the following example.

Example 6. Consider the integration problem and the set of distributions Ξ defined in Example 5. We show that Ξ is not M-convex. Recall matchings X and Y in that example. By construction, both X and Y satisfy the policy goal Ξ. Furthermore, $\xi_{c_3}^{t_1}(X) = 1 > 0 = \xi_{c_3}^{t_1}(Y)$ because (i) school c_3 is matched with student s_4 at X, whose type is t_1, while (ii) school c_3 is matched with student s_5 at Y, whose type is $t_2 \neq t_1$. If the set of distributions Ξ is M-convex, there exist a school c and a type t such that $\xi_c^t(X) < \xi_c^t(Y)$ and $\xi(X) - \chi_{c_3,t_1} + \chi_{c,t}$ is in Ξ. Because each school's capacity is one, and at matching X all schools have filled their capacities, this means that the only candidate for (c, t) satisfying the above condition is such that $c = c_3$. But the only nonzero $\xi_{c_3}^t(Y)$ is for $t = t_2$ (corresponding to s_5 matched with c_3 at Y), but $\xi(X) - \chi_{c_3,t_1} + \chi_{c_3,t_2}$ does not satisfy the policy goal because district d_1's ceiling for type t_2 is violated (note $\xi_{c_2}^{t_2}(X) = 1$ because student s_2 is matched with c_2 at X).

The above argument implies that $\Xi \cap \Xi^0$ is not M-convex either. To see this, note that both $\xi(X)$ and $\xi(Y)$ are in $\Xi \cap \Xi^0$ because all students are matched. Because we have shown that

[20]The letter M in the term M-convex set comes from the word *matroid*, a closely related and well-studied concept in discrete mathematics.

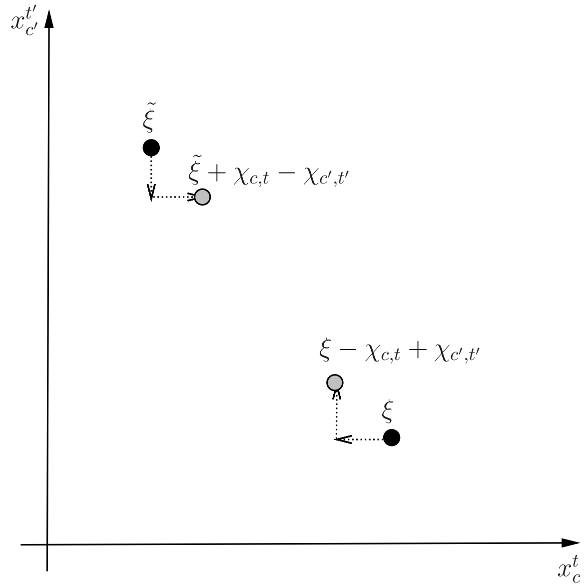


FIGURE 2. Illustration of M-convexity

no distribution of the form $\xi(X)-\chi_{c_3,t_1}+\chi_{c,t}$ is in Ξ, by set inclusion relation $\Xi\cap\Xi^0\subseteq\Xi$, there is no distribution of the form $\xi(X)-\chi_{c_3,t_1}+\chi_{c,t}$ in $\Xi\cap\Xi^0$ either.

Now we introduce an algorithm that achieves the desirable properties whenever the policy goal is M-convex. To do this, we first create a hypothetical matching market. On one side of the market, there are school-type pairs (c,t) where $c\in\mathcal{C}$ and $t\in\mathcal{T}$. On the other side, there are students from the original market, $\mathcal{S}$. Given any student $s\in\mathcal{S}$ and a preference order P_s of s in the original problem, define preference order $\tilde{P}_s$ over school-type pairs in the hypothetical market as follows: letting t be the type of student s and c_0 be her initial matching in the original problem, $(s,c)\ P_s\ (s,c') \iff (c,t)\ \tilde{P}_s\ (c',t)$ for any $c,c'\in\mathcal{C}$, and $(c_0,t)\ \tilde{P}_s\ (c,t')$ for any $c\in\mathcal{C}$ and $t'\neq t$. That is, $\tilde{P}_s$ is a preference order over school-type pairs that ranks the school-type pairs in which the type is t in the same order as in P_s, while finding all school-type pairs specifying a different type as less preferred than the pair corresponding to her initial matching. Furthermore, let (c_0,t) be the initial matching for s in the hypothetical market.

Next we define a priority ordering of students that school-type pairs use to rank students. For school-type pair (c,t), students initially matched with (c,t) havethe highest priority, and then all other students have the second highest priority. This gives us two priority classes for students. Then ties are broken according to a master priority list that every school-type pair uses.

We say that a type-t student s with initial matching school-type pair (c,t) is ***permissible*** to school-type pair (c',t') at matching X if $\xi(X)+\chi_{c',t'}-\chi_{c,t}$ is in Ξ. Note that a type-t

student with initial matching school-type pair (c, t) is always permissible to pair (c, t) at matching X whenever $\xi(X)$ is in Ξ.

The following is a generalization of Gale's top trading cycles algorithm (Shapley and Scarf, 1974), building on its recent extension by Suzuki et al. (2017).

Top Trading Cycles Algorithm (TTC).

Step 1: Let $X^1 \equiv \tilde{X}$. Each school-type pair points to the permissible student at matching X^1 with the highest priority. If there exists no such student, remove the school-type pair from the market. Each student s points to the highest ranked remaining school-type pair with respect to $\tilde{P}_s$. Identify and execute cycles. Any student who is part of an executed cycle is matched with the school-type pair she is pointing to and is removed from the market.

Step n (n $>$ 1): Let X^n denote the matching consisting of all students assigned in the previous steps, and initial matchings for all students who have not been processed in the previous steps. Each remaining school-type pair points to the unassigned student who is permissible at matching X^n with the highest priority. If there exists no such student, remove the school-type pair from the market. Each unassigned student s points to the highest ranked remaining school-type pair with respect to $\tilde{P}_s$. Identify and execute cycles. Any student who is part of an executed cycle is matched with the school-type pair she is pointing to and is removed from the market.

This algorithm terminates in the first step such that no student remains to be processed. The TTC outcome is defined as the matching at this step.

Our main result of this section is as follows.

Theorem 5. *Suppose that the initial matching satisfies the policy goal Ξ. If $\Xi \cap \Xi^0$ is M-convex, then TTC satisfies constrained efficiency, individual rationality, strategy-proofness, and the policy goal Ξ.*

A corollary of this result is that when the policy goal Ξ is such that no school is matched with more students than its capacity and it is M-convex, then TTC satisfies the desirable properties.

Corollary 1. *Suppose that the policy goal Ξ is such that for every $\xi \in \Xi$ and $c \in \mathcal{C}$, $\sum_t \xi_c^t \leq q_c$. Furthermore, suppose that the initial matching satisfies Ξ. If Ξ is M-convex, then TTC satisfies constrained efficiency, individual rationality, strategy-proofness, and the policy goal Ξ.*

In the proof of this corollary,we show that when Ξ is M-convex and no distribution in Ξ assigns more students to a school than its capacity, then $\Xi \cap \Xi^0$ is also M-convex. Therefore, the corollary follows directly from Theorem 5.

Next we illustrate TTC with an example.

Example 7. Consider the integration problem introduced in Example 4. We modify the preferences of students s_1 and s_6, so that the student preferences are as follows.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_{s_5}	P_{s_6}	P_{s_7}
c_2	c_3	c_4	c_2	c_1	c_4	c_2
c_3	c_1	c_2	c_3	c_2	c_1	c_3
$\vdots$	$\vdots$	$\vdots$	c_1	c_3	c_3	c_1
			c_4	c_4	c_2	c_4

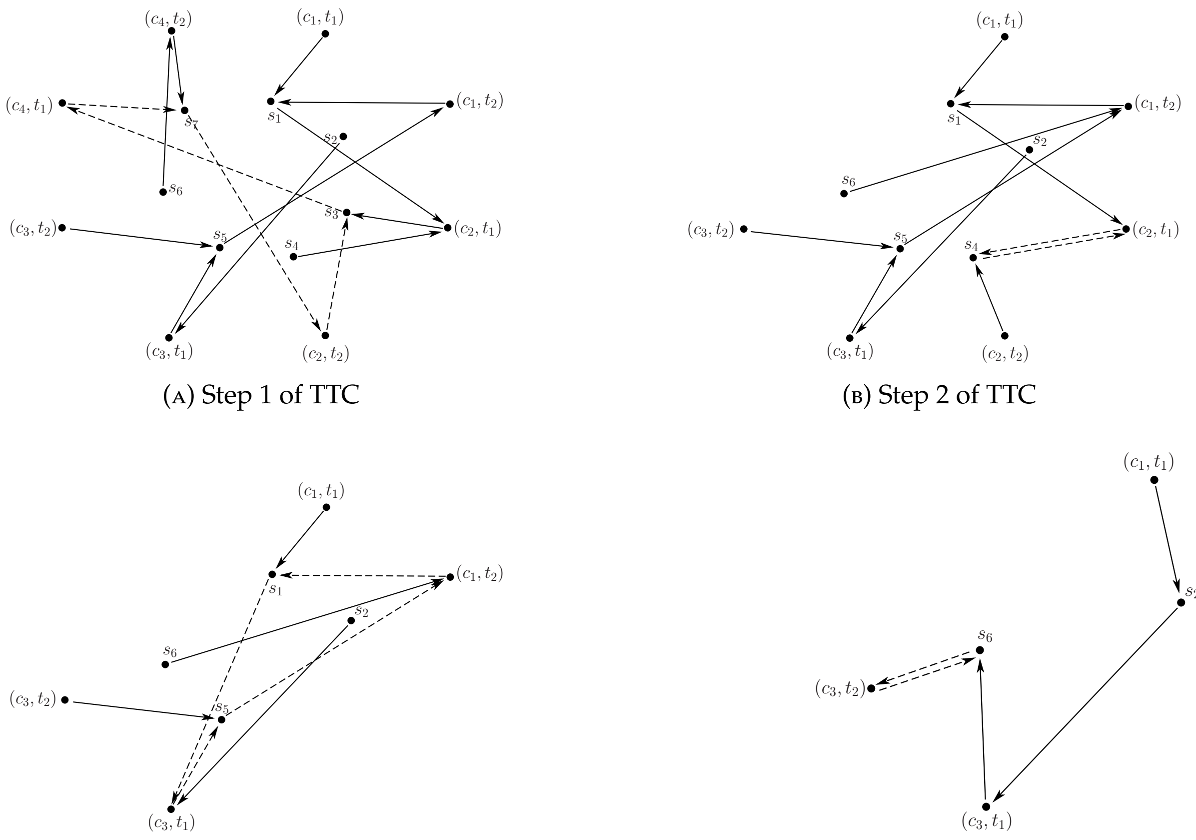
The initial matching is $\{(s_1, c_1), (s_2, c_1), (s_3, c_2), (s_4, c_2), (s_5, c_3), (s_6, c_3), (s_7, c_4)\}$.

In addition to the school capacities, there is only one additional constraint that school c_1 cannot have more than one type-t_2 student. As we show in the proof of Corollary 2, the set of distributions that satisfy this policy goal and the requirement that every student is matched is an M-convex set. Therefore, TTC satisfies constrained efficiency, individual rationality, strategy-proofness, and the policy goal.

To run TTC, we use a master priority list. Suppose that the master priority list ranks students as follows: $s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \succ s_6 \succ s_7$.

At Step 1 of TTC, there are eight school-type pairs. Consider (c_1, t_1). Initially, students s_1 and s_2 are matched with it, so they are both permissible to this pair. We use the master priority list to rank them, so s_1 gets the highest priority at (c_1, t_1). Therefore, (c_1, t_1) points to s_1. Now consider (c_1, t_2). Initially, it does not have any students because there is no type-t_2 student assigned to c_1 in the original market. Furthermore, s_1 is permissible to (c_1, t_2) because she can be removed from (c_1, t_1) and a type-t_2 student can be assigned to (c_1, t_2) without violating the school quotas or the policy goal. Therefore, (c_1, t_2) points to s_1 as well, who gets a higher priority than the other permissible students because of the master priority list. The rest of the pairs also point to the highest-priority permissible students. Each student points to the highest ranked school-type pair of the same type as shown in Figure 3A. There is only one cycle: $s_7 \to (c_2, t_2) \to s_3 \to (c_4, t_1) \to s_7$. Therefore, s_7 is matched with (c_2, t_2) and s_3 is matched with (c_4, t_1).

At Step 2, there are six remaining school-type pairs: There are no permissible students for (c_4, t_1) and (c_4, t_2) because c_4 has a capacity of one and it already is assigned to s_3. Each remaining school-type pair points to the highest-ranked remaining permissible student. Each student points to the highest-ranked remaining school-type pair (see Figure 3B). There is only one cycle: $s_4 \to (c_2, t_1) \to s_4$. Hence, s_4 is assigned to (c_2, t_1).



(A) Step 1 of TTC

(B) Step 2 of TTC

(C) Step 3 of TTC

(D) Step 4 of TTC

FIGURE 3. The first four steps of TTC. In each step, there is only one cycle, which is represented by the dashed lines.

The algorithm ends in five steps. Steps 3 and 4 are also shown in Figure 3. In Step 5, s_2 points to (c_1, t_1), which points back to the student. The outcome of the algorithm is

$$\{(s_1, c_3), (s_2, c_1), (s_3, c_4), (s_4, c_2), (s_5, c_1), (s_6, c_3), (s_7, c_2)\}.$$

It is easy to see that the distribution associated with this matching satisfies the policy goal because no school has more students than its capacity and c_1 has only one type-t_2 student. $\square$

Now that we have established a general result based on M-convexity of the policy goal in Theorem 5, we proceed to apply it to a variety of situations. To begin, consider the set Ξ of distributions of feasible matchings. In other words, consider a situation in which no policy goal is imposed other than feasibility. Then it is rather straightforward to show that the set $\Xi \cap \Xi^0$ is an M-convex set. This implies that when there is no policy goal, TTC is efficient, individually rational, and strategy-proof, a standard result in the literature (Abdulkadiroğlu and Sönmez, 2003).[21]

Now we are ready to apply Theorem 5 to a variety of policy goals. This result turns out to be applicable to many specific cases, as a wide variety of policy goals induce distributions that satisfy M-convexity. To be more specific, first suppose that the policy goal Ξ sets type-specific floors and ceilings at each school, i.e., $\Xi \equiv \{\xi | \forall c, t \; q_c^t \geq \xi_c^t \geq p_c^t, \; \forall c \; q_c \geq \sum_t \xi_c^t\}$ where q_c^t is the ceiling and p_c^t is the floor for type t at school c. Therefore, for each school, the number of students of a given type must be within the ceiling and floor of this type at the school. We call Ξ the ***school-level diversity policy*** and show that $\Xi \cap \Xi^0$ is an M-convex set. This finding, together with Theorem 5, implies the following positive result.

Corollary 2. *Suppose that the initial matching satisfies the school-level diversity policy. Then TTC satisfies constrained efficiency, individual rationality, strategy-proofness, and the school-level diversity policy.*

We note a sharp contrast between this result and Theorem 4. The latter result demonstrates that no mechanism is guaranteed to satisfy the policy goal and other desiderata such as constrained efficiency, individual rationality, and strategy-proofness if the floors or ceilings are imposed at the district level. Corollary 2, by contrast, shows that a mechanism with the desirable properties exists if the floors and ceilings are imposed at the school level. Taken together, these results inform policy makers about what kinds of diversity policies are compatible with the other desiderata.

Next, we study the balanced-exchange policy introduced in Section 3.2. We establish that the balanced-exchange policy induces a distribution that satisfies M-convexity. This implies the following result.

Corollary 3. *TTC satisfies constrained efficiency, individual rationality, strategy-proofness, and the balanced-exchange policy.*

One of the advantages of Theorem 5 is that M-convexity is so general that a wide variety of policy goals satisfy it, and that it is likely to be applicable for policy goals that one may encounter in the future. To highlight this point, we consider imposing the diversity and balanced-exchange policies at the same time. More specifically, define a set of distributions $\Xi = \{\xi | \forall c, t \; q_c^t \geq \xi_c^t \geq p_c^t, \; \forall c \; q_c \geq \sum_t \xi_c^t$ and $\forall d \; \sum_t \sum_{c:d(c)=d} \xi_c^t = k_d\}$ and call it the ***combination of balanced-exchange and school-level diversity policies***. This is the set of distributions that satisfy both the (school-level) floors and ceilings and the balanced-exchange requirement. We can establish this set is M-convex, implying the following result.

[21]This result and its proof are formally presented as Corollary 5 in Appendix C.

Corollary 4. *Suppose that the initial matching satisfies the combination of balanced exchange and school-level diversity policies. Then TTC satisfies constrained efficiency, individual rationality, strategy-proofness, and the combination of balanced exchange and school-level diversity policies.*

5. Conclusion

Despite increasing interest in inter-district school choice in the US, the scope of matching theory has been limited to intra-district choice. In this paper, we proposed a new framework to study district integration that allows for inter-district admissions, both from stability and efficiency perspectives. For stable mechanisms, we characterized conditions on district admissions rules that achieve a variety of important policy goals such as student diversity across districts. For efficient mechanisms, we showed that certain types of diversity policies are incompatible with desirable properties such as strategy-proofness, while alternative forms of diversity policies can be achieved by a strategy-proof mechanism: a variation of the top trading cycles algorithm. Overall, our analysis suggests that district integration may help achieve desirable policy goals such as student diversity, but only with an appropriate design of constraints, admission rules, and placement mechanisms.

We regard this paper as a first step toward formal analysis of school district integration based on tools of market design. As such, we envision a variety of directions of future research. For example, it may be interesting to study cases in which the conditions for our results are violated. Although we already know the policy goals are not guaranteed to be satisfied for our stability results (our results provide necessary and sufficient conditions), how serious the failure of the policy goals studied in the present paper is an open question. Quantitative measures or an approximation argument like those used in "large matching market" studies (e.g., Roth and Peranson (1999), Kojima and Pathak (2009), Kojima et al. (2013), and Ashlagi et al. (2014)) may prove useful, although this is speculative at this point and beyond the scope of the present paper.

We studied policy goals that we regarded as among the most important ones, but they are far from being exhaustive. Other important policy goals may include a diversity policy requiring certain proportions of different student types in each district (see Nguyen and Vohra (2017) for a related policy at the level of schools), as well as a balanced exchange policy requiring a certain bound on the difference in the numbers of students received from and sent to other districts (see Dur and Ünver (2015) for a related policy at the level of schools). Given that the existing literature has not studied district integration, we envision that many policy goals await to be studied within our framework.

While our paper is primarily theoretical and aimed at proposing a general framework to study school district integration, the main motivation comes from applications to actual integration programs such as Minnesota's AI program. Given this motivation, it would be interesting to study district integration empirically. For instance, evaluating how well the existing integration programs are doing in terms of balanced exchange, student welfare, and diversity, and how much improvement could be made by a conscious design based on theories such as the ones suggested in the present paper are important questions left for future work. In addition, implementation of our designs in practice would be interesting. For instance, doing so may shed new light on the tradeoff between SPDA and TTC—that has been studied in the intra-district school choice from a practical perspective (e.g., Abdulkadiroglu et al. (2006), Abdulkadiroğlu et al. (2017)). We are only beginning to learn about the district integration problem, and thus we expect that these and other questions could be answered as more researchers analyze it.

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Appendix A. Improving Student Welfare for Centralized Districts

In Section 3.1, we studied when SPDA satisfies individual rationality, which requires that under integration, every student is matched with a school that is weakly more preferred than her initial matching. In this appendix, we consider an alternative setting where each district uses SPDA to assign its students to schools when districts are not integrated. More explicitly, each student ranks schools in their home districts (or contracts associated with their home districts) and SPDA is used between a district and students from that district. Note that each student's ranking over contracts associated with the home district is the same as the relative ranking in the original preferences. Thus, the initial matching is not fixed but is determined by student preferences and district admissions rules. In this setting, we characterize district admissions rules which guarantee that no student is hurt from integration.

The next property of district admissions rules proves to play a crucial role to achieve this policy.

Definition 7. *A district admissions rule Ch_d* ***favors own students*** *if for any matching that is feasible for students,*

$$Ch_d(X) \supseteq Ch_d(\{x \in X | d(s(x)) = d\}).$$

When a district admissions rule favors own students any contract that is chosen from a set of contracts associated with students from a district is also chosen from a superset that includes additional contracts associated with students from the other districts. Roughly, the intuition is that a district should prioritize its own students that it used to admit over students from the other districts even though an out-of-district student can still be admitted when a student from the district is rejected.

The following theorem shows that this is exactly the condition which guarantees that district integration weakly improves student welfare.

Theorem 6. *If each district's admissions rule favors own students, then every student is weakly better off when school districts integrate under SPDA. Moreover, if at least one district's admissions rule fails to favor own students, then there exists a student preference profile such that at least one student is strictly worse off when school districts integrate under SPDA.*

In the proof, we show that in the no-integration case the SPDA outcome can alternatively be produced when students rank contracts with all districts and districts have modified admissions rules: For any set of contracts X, each district d chooses the following contracts: $Ch_d(\{x \in X | d(s(x)) = d\})$. Since district admissions rules favor own students, the chosen set is a subset of $Ch_d(X)$ when X is feasible for students. Then the conclusion that students are weakly better off when school districts integrate follows from a comparative statics property of SPDA that we show (Lemma 4).[22] To show the second statement, when there exists a district admissions rule that fails to favor own students, we construct preferences of students such that integration makes at least one student strictly worse off.

Appendix B. Examples of District Admissions Rules

In this appendix, we first provide a class of district admissions rules that are feasible and acceptant (or weakly acceptant) and, furthermore, they have completions that satisfy substitutability and LAD. Then, based on this class, we identify admissions rules that also satisfy the properties stated in Theorems 1, 2, 3, and 6.

Before we proceed, we introduce another admissions rule property. An admissions rule Ch satisfies ***path independence*** if for every $X, Y \subseteq \mathcal{X}$, $Ch(X \cup Y) = Ch(X \cup Ch(Y))$. Path independence states that a set can be divided into not-necessarily disjoint subsets and the admissions rule can be applied to the subsets in any order so that the chosen set of contracts is always the same. Path independence is equivalent to substitutability and a consistency condition (Aizerman and Malishevski, 1981). Furthermore, an admissions rule satisfies substitutability and LAD if, and only if, it satisfies path independence and LAD.[23]

B.1. An Example of District Admissions Rule. Consider a district d with schools $c_1, \ldots, c_n$. Each school c_i has an admissions rule Ch_{c_i} such that, for any set of contracts X, $Ch_{c_i}(X) = Ch_{c_i}(X_{c_i}) \subseteq X_{c_i}$. District d's admissions rule Ch_d is defined as follows. For any set of contracts X,

$$Ch_d(X) = Ch_{c_1}(X) \cup Ch_{c_2}(X \setminus Y_1) \cup \ldots \cup Ch_{c_n}(X \setminus Y_{n-1}),$$

where Y_i for $i = 1, \ldots, n-1$ is the set of all contracts in X associated with students who have contracts in $Ch_{c_1}(X) \cup \ldots \cup Ch_{c_i}(X \setminus Y_{i-1})$. In words, we order the schools and let schools choose in that order. Furthermore, if a student is chosen by some school, we remove all contracts associated with this student for the remaining schools.

We study when district admissions rule Ch_d satisfies our assumptions.

Claim 1. *Suppose that for every school c_i and matching X, $|Ch_{c_i}(X)| \leq q_{c_i}$. Then district admissions rule Ch_d is feasible.*

Proof. Since every student-school pair uniquely defines a contract, for every X, every school c_i, and every student s, there is at most one contract associated with s in $Ch_{c_i}(X)$. In addition, whenever a student's contract with a school c_i is chosen, her contracts with the remaining schools are included in Y_j for every $j \geq i$ by the construction of Ch_d. Hence,

[22] We cannot use the comparative statics result of Yenmez (2018) because in our setting $Ch_d(X) \supseteq Ch'_d(X)$ only when X is feasible for students whereas Yenmez (2018) requires this property for all X.

[23] See Aygün and Sönmez (2013) for a study of the consistency condition and Chambers and Yenmez (2017) for a study path independence in a matching context.

for every X, $Ch_d(X)$ is feasible for students. Furthermore, by assumption, $|Ch_{c_i}(X)| \leq q_{c_i}$ for each c_i. Therefore, Ch_d is feasible. □

Claim 2. *Suppose that for every school c_i and matching X, $|Ch_{c_i}(X)| = \min\{q_{c_i}, |X_{c_i}|\}$. Then district admissions rule Ch_d is acceptant.*

Proof. Suppose that matching X is feasible for students and $x \in X_d \setminus Ch_d(X)$. There exists $i \leq n$ such that $c_i = c(x)$. Since X is feasible for students, $x \in X \setminus Y_{i-1}$ where Y_{i-1} is as defined in the construction of Ch_d. Because $x \in X_d \setminus Ch_d(X)$, $x \notin Ch_{c_i}(X \setminus Y_{i-1})$. Then $|Ch_{c_i}(X \setminus Y_{i-1})| = q_{c_i}$ by assumption, which implies that district admissions rule Ch_d is acceptant. □

Next we study when district admissions rule Ch_d has a completion that satisfies path independence and LAD. Consider the following district admissions rule Ch'_d: For any set of contracts X,

$$Ch'_d(X) = Ch_{c_1}(X) \cup \ldots \cup Ch_{c_n}(X).$$

Claim 3. *Suppose that for every school c_i, Ch_{c_i} satisfies path independence and LAD. Then district admissions rule Ch'_d is a completion of Ch_d and it satisfies path independence and LAD.*

Proof. To show that Ch'_d is a completion of Ch_d, suppose that X is a set of contracts such that $Ch'_d(X)$ is feasible for students. By mathematical induction, we show that $Ch_{c_i}(X) = Ch_{c_i}(X \setminus Y_{i-1})$ for $i = 1, \ldots, n$ where Y_i is defined as above for $i > 1$ and $Y_0 = \emptyset$. The claim trivially holds for $i = 1$. Suppose that it also holds for $1, \ldots, i-1$. We show the claim for i. Since $Ch'_d(X)$ is feasible for students, $Ch_{c_i}(X)$ and $Ch_{c_1}(X) \cup \ldots \cup Ch_{c_{i-1}}(X)$ do not have any contracts associated with the same student. Therefore, $Ch_{c_i}(X) \cap Y_{i-1} = \emptyset$. Since Ch_{c_i} is path independent, $Ch_{c_i}(X) = Ch_{c_i}(X \setminus Y_{i-1})$.[24] As a result, $Ch_d(X) = Ch'_d(X)$, which completes the proof that Ch'_d is a completion of Ch_d.

Since all school admissions rules satisfy path independence and LAD, so does Ch'_d. □

All of the assumptions on school admissions rules stated in Claims 1, 2, and 3 are satisfied when school admissions rules are ***responsive***: Each school has a ranking of contracts associated with itself. From any given set of contracts, each school chooses contracts with the highest rank until the capacity of the school is full or there are no more contracts left. Responsive admissions rules satisfy path independence and LAD. Furthermore, for every school c_i, $|Ch_{c_i}(X)| = \min\{q_{c_i}, |X_{c_i}|\}$.[25] By the claims stated above, when school admissions rules are responsive, district admissions rule Ch_d is feasible and acceptant, and it has a completion that satisfies path independence and LAD.

[24]More precisely, this follows from the consistency condition that removing rejected contracts does not change the chosen set.

[25]See Chambers and Yenmez (2018) for a characterization of responsive admissions rules using this property.

Based on these results, we provide examples of district admissions rules that rules that further satisfy additional assumptions considered in different parts of our paper.

B.2. District Admissions Rules Satisfying the Assumptions in Theorem 1. We use the district admissions rule construction above and we further specify each school's admissions rule. Each school has a responsive admissions rule. If a student is initially matched with a school, then her contract with this school is ranked higher than contracts of students who are not initially matched with the school. As before, district admissions rule Ch_d is feasible and acceptant, and it has a completion that satisfies path independence and LAD.

Claim 4. *District admissions rule Ch_d respects the initial matching.*

Proof. Let $x = (s, d, c)$ be the initial matching of student s. By construction, for any matching X that is feasible for students, $x \in X$ implies $x \in Ch_d(X)$ because c chooses x from any set of contracts and s does not have any other contract in X. Therefore, Ch_d respects the initial matching. $\square$

B.3. District Admissions Rules Satisfying the Assumptions in Theorem 2. We modify the district admissions rule construction in Appendix B.1. Each school has a ranking of contracts associated with itself. When it is the turn of a school, it accepts contracts that have the highest rank until the capacity of the school is full or the number of contracts chosen by the district is k_d or there are no more contracts left. The remaining contracts of a chosen student are removed.

District admissions rule Ch_d is feasible because no school admits more students than its capacity and no student is admitted to more than one school.

Claim 5. *District admissions rule Ch_d is acceptant.*

Proof. To show acceptance, suppose that matching X is feasible for students and $x \in X_d \setminus Ch_d(X)$. There exists $i \leq n$ such that $c_i = c(x)$. Since X is feasible for students, $x \in X \setminus Y_{i-1}$ where Y_{i-1} is the set of all contracts in X associated with students who are chosen by schools $c_1, \ldots, c_{i-1}$. Because $x \in X_d \setminus Ch_d(X)$, x is not chosen by c_i. Then, by construction, either c_i fills its capacity or the district admits k_d students, both of which imply that Ch_d is acceptant. $\square$

Claim 6. *District admissions rule Ch_d has a completion that satisfies substitutability and LAD.*

Proof. First, we construct a completion of Ch_d. Define the following district admissions rule: Given a set of contracts X, when it is the turn of a school, it chooses from all the contracts in X. Each school chooses contracts using the same priority order until the school capacity is full or the district has k_d contracts or there are no more contracts left. Denote this admissions rule by Ch'_d. Suppose that $Ch'_d(X)$ is feasible for students. Then, by construction, $Ch'_d(X) = Ch_d(X)$. Therefore, Ch'_d is a completion of Ch_d.

Next, we show that Ch'_d satisfies LAD. Suppose that $Y \supseteq X$. Every school c_i chooses weakly more contracts from Y than X unless the number of contracts chosen from Y by the district reaches k_d. Since the number of chosen contracts from X cannot exceed k_d by construction, Ch'_d satisfies LAD.

Finally, we show that Ch'_d satisfies substitutability. Suppose that $x \in X \subseteq Y$ and $x \in Ch'_d(Y)$. Therefore, the number of contracts chosen from Y by schools preceding $c(x)$ is strictly less than k_d. This implies that the number of contracts chosen from X by schools preceding $c(x)$ is weakly less than this number as weakly more contracts are chosen by schools preceding school $c(x)$ in Y than X. As a result, for school $c(x)$, weakly more contracts can be chosen from X than Y.

The ranking of contract x among Y in the ranking of school $c(x)$ is high enough that it is chosen from set Y. Therefore, the ranking of contract x among X in the ranking of school $c(x)$ must be high enough to be chosen from set X because weakly more contracts can be chosen from X than Y for school $c(x)$. $\square$

Furthermore, by construction, district admissions rule Ch_d never chooses more than k_d students. Therefore, it is also rationed.

B.4. District Admissions Rules Satisfying the Assumptions in Theorem 3. District admissions rules can accommodate unmatched students by *reserving* seats for different types of students:

Definition 8. *A district admissions rule Ch_d has a* ***reserve*** *of r_d^t for type-t students if, for any feasible matching X that does not have any contract associated with type-t student s, if $|\{x \in X_d | \tau(s(x)) = t\}| < r_d^t$, then there exists $x = (s, d, c) \in \mathcal{X}$ such that $x \in Ch_d(X \cup \{x\})$.*

A reserve for a student type guarantees space for this type at some school in the district.[26] Therefore, when a student is unmatched and the reserve for her type is not yet filled in the district, the district will accept this student at some school if she applies with the corresponding contract. Note that this definition does not imply that the reserves will always be filled when there are enough applicants of the corresponding type. This may not be the case, for example, when all students apply to the same school. The condition guarantees that an unmatched student will be accepted at *some* school if she applies there.

[26]District admissions rules with type-specific ceilings and reserves are similar to existing concepts in the school choice setting. See Abdulkadiroğlu and Sönmez (2003) for ceilings, Hafalir et al. (2013) and Ehlers et al. (2014) for reserves, and Echenique and Yenmez (2015) for an axiomatic characterization of admission rules with ceilings and reserves.

Claim 7. *Suppose that districts have admissions rules with reserves such that $\sum_t r_d^t = k_d$ for every district d and $\sum_d r_d^t = k^t$ for every type t. Then district admissions rules accommodate unmatched students.*

Proof. Suppose that student s is unmatched at a feasible matching X. Let $t \equiv \tau(s)$. Then there exists a district d such that the number of type-t students in d at X is strictly less than r_d^t. By definition of reserves, there exists a contract $x = (s, d, c)$ such that $x \in Ch_d(X \cup \{x\})$. □

A district can reserve its seats for types in different ways. In the rest of this example, we use school admissions rules with reserves introduced by Hafalir et al. (2013) to construct a general example in which a district has type-specific reserves. Each school reserves some of its seats for every type of student. Let $r_{c_i}^t$ be the number of seats reserved by school c_i for type-t students. Suppose that the type-specific ceilings are given and they satisfy the assumptions in Section 3.3. Furthermore, for every district d, assume that $\sum_t r_d^t = k_d$, $\sum_d r_d^t = k^t$ and, for every type t, $r_d^t \leq q_d^t$. We set the reserves at each school so that, for every district d,

$$\begin{aligned}\sum_{d(c_i)=d} r_{c_i}^t &= r_d^t, \text{ for every type } t, \text{ and} \\ \sum_t r_{c_i}^t &\leq q_{c_i} \text{ for every school } c_i.\end{aligned}$$

This is possible because the sum of capacities of schools in d is weakly greater than k_d.

Consider the following district admissions rule. Each school has a ranking of contracts associated with itself. Schools are ordered. When it is the turn of a school all contracts of students chosen previously are removed. First, schools choose contracts for its reserved seats in the specified order so that, for every type, either reserved seats are filled or there are no more contracts of students of that type remaining. Then schools choose from the remaining contracts to fill the rest of their seats in the specified order until the school capacity is filled or the district has k_d contracts or district ceilings are filled or there are no more remaining contracts. Denote this district admissions rule by Ch_d.

District admissions rule Ch_d is feasible because a student cannot have more than one contract and a school cannot have more contracts than its capacity at any chosen set. It is also weakly acceptant and rationed by construction. Furthermore, for every type t, it cannot admit more than q_d^t students, so it has a ceiling of q_d^t for type-t students.

Claim 8. *District admissions rule Ch_d has a completion that satisfies substitutability and LAD.*

Proof. Use the same construction as above, i.e., the construction of Ch_d, but do not remove contracts of students who are chosen previously. Denote this district admissions rule by

Ch'_d. To show that Ch'_d is a completion of Ch_d, suppose that $Ch'_d(X)$ is feasible for students for some X. Since the only difference in the constructions of Ch_d and Ch'_d is the removal of contracts of previously chosen students, it must be that $Ch'_d(X) = Ch_d(X)$. Therefore, Ch'_d is a completion of Ch_d.

To show substitutability, let $x \in X \subseteq Y$ and $x \in Ch'_d(Y)$. Let $c \equiv c(x)$. If contract x was chosen from Y at the first stage because of reserves, then it is also chosen from X at the first stage because of reserves since X is a subset of Y. Likewise, if x is chosen from Y at the second stage, then x is chosen from X either at the first or second stage. Thus, Ch'_d is substitutable.

To show LAD, let $X \subseteq Y$. If $|Ch'_d(X)| = k_d$, then $|Ch'_d(Y)| = k_d$ as well. If $|Ch'_d(Y)| = k_d$, then LAD holds. Consider the case when $|Ch'_d(X)|, |Ch'_d(Y)| < k_d$. In this case, Ch'_d can be written as the union of choice rules of schools where each school c_i in the district has a reserve of $r^t_{c_i}$ for type t and where the schools stop accepting a type of a student as soon as the ceiling of the type is filled in the district. We show by mathematical induction that the number of contracts chosen by the first i schools from Y is weakly greater than that of X. LAD proof to be completed...

□

Claim 9. *District admissions rule Ch_d accommodates unmatched students.*

Proof. If X is a feasible matching that does not have any contract associated with type-t student s and $|\{x \in X_d | \tau(s(x)) = t\}| < r^t_d$ there exists a school c_i such that the number of type-t students in school c_i at X is less than $r^t_{c_i}$. Let $x = (s, d, c_i)$. Then $x \in Ch_d(X \cup \{x\})$ by the construction of Ch_d. Therefore, Ch_d has a reserve of r^t_d for type-t students. As a result, district admissions rules accommodate unmatched students. □

B.5. District Admissions Rules Satisfying the Assumptions in Theorem 6. Consider the district admissions rule construction in Appendix B.1. In this example, let each school use a priority ranking in such a way that all contracts of students from district d are ranked higher than the other contracts.

Claim 10. *District admissions rule Ch_d favors own students.*

Proof. Suppose that X is feasible for students. When it is the turn of school c_i, it considers X_{c_i}. Therefore, $Ch_d(X) = Ch_{c_1}(X_{c_1}) \cup \ldots \cup Ch_{c_k}(X_{c_k})$. Furthermore, $Ch_{c_i}(X_{c_i}) \supseteq Ch_{c_i}(\{x \in X_{c_i} | d(s(x)) = d\})$ by construction. Taking the union of all subset inclusions yields $Ch_d(X) \supseteq Ch_d(\{x \in X_d | d(s(x)) = d\})$. Therefore, Ch_d favors own students. □

Appendix C. Omitted Proofs

In this appendix, we include the omitted proofs.

Proof of Theorem 1. First, suppose that all district admissions rules respect the initial matching. In SPDA, each student s goes down in her preference order, and either SPDA ends before student s reaches her initial school (which is a preferred outcome than the initial school), or student s reaches her initial school. In the latter case, she is matched with her initial school because the district's admissions rule respects the initial matching and the district always considers a set of contracts that is feasible for students at any step of SPDA. From this step on, the district accepts this contract, so student s is matched with her initial school. Therefore, SPDA satisfies individual rationality.

To prove the second statement, suppose that there exists a district d with an admissions rule that fails to respect the initial assignment. Hence, there exists a matching X, which is feasible for students, that includes $x = (s, d, c)$ where school c is the initial school of student s and $x \notin Ch_d(X)$. Now, consider student preferences such that every student associated with a contract in X_d prefers that contract the most and all other students prefer a contract associated with a different district the most. Then, at the first step of SPDA, district d considers matching X_d and tentatively accepts $Ch_d(X_d)$. Since $x \notin Ch_d(X_d)$, contract x is rejected at the first step. Therefore, student s is matched with a less preferred school than her initial matching school, which implies that SPDA does not satisfy individual rationality. □

Proof of Theorem 2. We first prove that when each district admissions rule is rationed, then SPDA satisfies the balanced-exchange policy. Let μ be the matching produced by SPDA for a given preference profile. We show that each student must be matched with a school in μ using acceptance.

Suppose, for contradiction, that student s is unmatched. Since μ is a stable matching, every contract $x = (s, d, c)$ associated with the student is rejected by the corresponding district, i.e., $x \notin Ch_d(\mu \cup \{x\})$. Otherwise, student s and district d would like to match with each other using contract x contradicting stability of matching μ. Since $\mu \cup \{x\}$ is feasible for students, acceptance implies that, for every district d, either every school in the district is full or that the district has at least k_d students at matching μ. Both of them imply that the district has at least k_d students in matching μ since the sum of the school capacities in district d is at least k_d. But this is a contradiction to the assumption that student s is unmatched since the existence of an unmatched student implies that there is at least one district d such that the number of students in μ_d is less than k_d. Therefore, all students are matched in μ.

Because μ is the outcome of SPDA, it is feasible for students. Therefore, because district admissions rules are rationed, the number of students in district d cannot be more than k_d for every district d. Furthermore, since every student is matched, the number of students in district d must be exactly k_d because, otherwise, at least one student would have been unmatched. As a result, SPDA satisfies the balanced-exchange policy.

Next, we prove that if at least one district's admissions rule fails to be rationed, then there exists a student preference profile under which SPDA does not satisfy the balanced-exchange policy. Suppose that there exist district d and a matching X, which is feasible for students, such that $|Ch_d(X)| > k_d$. Consider a feasible matching X' where (i) all students are matched, (ii) $X'_d = Ch_d(X)$, and (iii) for every district $d' \neq d$, $|X'_{d'}| \leq k_{d'}$. The existence of such X' is guaranteed since every district has enough capacity to serve its students (i.e., for every district d', $\sum_{c:d(c)=d'} q_c \geq k_{d'}$), and $|Ch_d(X)| > k_d$. Now, consider any student preferences such that every student likes her contract in X' the most.

We show that SPDA stops in the first step. For district $d' \neq d$, $X'_{d'}$ is feasible and the number of students is weakly less than $k_{d'}$. Since $Ch_{d'}$ is acceptant, $Ch_{d'}(X'_{d'}) = X'_{d'}$. For district d, we need to show that $Ch_d(X'_d) = X'_d$, which is equivalent to $Ch_d(Ch_d(X)) = Ch_d(X)$. Let Ch'_d be a completion of Ch_d that satisfies path independence. Because X and $Ch_d(X)$ are feasible for students, $Ch'_d(X) = Ch_d(X)$ and $Ch'_d(Ch'_d(X)) = Ch_d(Ch_d(X))$. Furthermore, since Ch'_d is path independent, $Ch'_d(Ch'_d(X)) = Ch'_d(X)$, which implies $Ch_d(Ch_d(X)) = Ch_d(X)$. As a result, $Ch_d(X'_d) = X'_d$. Therefore, SPDA stops at the first step since no contract is rejected.

Since SPDA stops at the first step, the outcome is matching X'. But matching X' fails the balanced-exchange policy because $|X'_d| = |Ch_d(X)| > k_d$. □

Proof of Theorem 3. Recall that for any matching X, the number of type-t students in district d is denoted by $\xi^t_d(X)$. We say that a feasible matching X is ***legitimate*** if (i) every student is matched at X, (ii) X is rationed, and (iii) for each type t and district d, we have $\xi^t_d(X) \leq q^t_d$. We use this definition while proving this Theorem.

To prove this result, we provide the following lemmas.

Lemma 1. *If district admissions rules accommodate unmatched students, every student is matched to a school in SPDA.*

Proof of Lemma 1. Let μ be the outcome of SPDA for some preference profile. Suppose, for contradiction, that student s is unmatched. Since μ is a stable matching and student s prefers any contract $x = (s, d, c)$ to being unmatched, $x \notin Ch_d(\mu \cup \{x\})$. But this is a contradiction to the assumption that district admissions rules accommodate unmatched students. □

Lemma 2. *For each $t \in \mathcal{T}$, $d \in \mathcal{D}$, and for all legitimate matchings X, we have $\xi^t_d(X) \geq \hat{p}^t_d$, and $\xi^t_d(X) \leq \hat{q}^t_d$. Moreover, for each $t \in \mathcal{T}$, $d \in \mathcal{D}$, there exists a legitimate matching X, where $\xi^t_d(X) = \hat{p}^t_d$, and there exists a legitimate matching X, where $\xi^t_d(X) = \hat{q}^t_d$.*

Proof of Lemma 2. This simply follows from capacity scaling algorithm of Edmonds and Karp (1972).

□

Lemma 3. *For each $t \in \mathcal{T}$ and $d, d' \in \mathcal{D}$ with $d \neq d'$, there exists a legitimate matching X where $\xi^t_d(X) = \hat{q}^t_d$ and $\xi^t_{d'}(X) = \hat{p}^t_{d'}$.*

Proof of Lemma 3. Let $\hat{X}$ be some legitimate matching such that $\xi^t_d(\hat{X}) = \hat{q}^t_d$, and $\mathcal{M}_0$ be the set of all legitimate matchings. Let

$$\mathcal{M}_1 = \{X \in \mathcal{M}_0 | \xi^t_{d'}(X) = \hat{p}^t_{d'}\},$$
$$\mathcal{M}_2 = \{X \in \mathcal{M}_1 | \xi^t_d(X) \geq \xi^t_d(X') \text{ for every } X' \in \mathcal{M}_1\},$$
$$\mathcal{M}_3 = \{X \in \mathcal{M}_2 | \sum_{t,d} \mid \xi^t_d(X) - \xi^t_d(\hat{X}) \mid \leq \sum_{t,d} \mid \xi^t_d(X') - \xi^t_d(\hat{X}) \mid \text{ for every } X' \in \mathcal{M}_2\}.$$

First, note that all these sets are well-defined and nonempty. We wish to show that for any $X \in \mathcal{M}_3$, $\xi^t_d(X) = \xi^t_d(\hat{X}) = \hat{q}^t_d$.

For the sake of contradiction, assume that for some $X \in \mathcal{M}_3$, $\xi^t_d(X) \neq \xi^t_d(\hat{X})$. By Lemma 2, this means that $\xi^t_d(X) < \xi^t_d(\hat{X})$. Since X and $\hat{X}$ are both legitimate—thus row- and column- sums for X and $\hat{X}$ are identical to each other for each row and column—this means there exists t_1 such that $\xi^{t_1}_d(X) > \xi^{t_1}_d(\hat{X})$. Similarly, this means there exists d_1 such that $\xi^{t_1}_{d_1}(X) < \xi^{t_1}_{d_1}(\hat{X})$. Recursively we can define t_2, d_2, t_3, d_3 and so on. While defining t_i's and d_i's, since we have finitely many t's and d's, we would have to have either $d_i = d_j$ or $t_i = t_j$ for some $i < j$. When $d_i = d_j$, we choose $t_{i+1} = t_{j+1}$.

Consider the first time $t_i = t_j$ for $i < j$. Then we have,
$\xi^{t_i}_{d_i}(X) < \xi^{t_i}_{d_i}(\hat{X})$,
$\xi^{t_{i+1}}_{d_i}(X) > \xi^{t_{i+1}}_{d_i}(\hat{X})$,
...
$\xi^{t_j}_{d_{j-1}}(X) > \xi^{t_j}_{d_{j-1}}(\hat{X})$.

Then we argue that there exists a quota abiding matching $\tilde{X}$ where
$\xi^{t_i}_{d_i}(\tilde{X}) = \xi^{t_i}_{d_i}(X) + 1$,
$\xi^{t_{i+1}}_{d_i}(\tilde{X}) = \xi^{t_{i+1}}_{d_i}(X) - 1$,
...
$\xi^{t_j}_{d_{j-1}}(\tilde{X}) = \xi^{t_j}_{d_{j-1}}(X) - 1$,
and
$\xi^{t'}_{d'}(\tilde{X}) = \xi^{t'}_{d'}(X)$ for all $(d', t') \neq (d_i, t_i), \ldots, (d_{j-1}, t_{j-1})$.

This is true because (i) between $\tilde{X}$ and X, for each row (or column), the sum of the entries over that row is unchanged because either none of the entries of that row has been changed from X or exactly two entries have been changed from X, with one of the entries increasing by one and the other decreasing by one, and (ii) for each t, d, $0 \leq \min(\xi^t_d(\tilde{X}), \xi^t_d(X)) \leq$

$\xi_d^t(\tilde{X})$, and $\xi_d^t(\tilde{X}) \le \max(\xi_d^t(\tilde{X}), \xi_d^t(X)) \le q_d^t$. Moreover, $\tilde{X} \in \mathcal{M}_1$ since by construction the (t, d') entry is not increased from X when constructing $\tilde{X}$, and $\xi_d^t(\tilde{X}) \ge \xi_d^t(X)$ since by construction of $\tilde{X}$, either $\xi_d^t(\tilde{X}) = \xi_d^t(X)$, or $\xi_d^t(\tilde{X}) = \xi_d^t(X) + 1$. This implies that $\tilde{X} \in \mathcal{M}_2$. Lastly, $\sum_{t,d} \mid \xi_d^t(\tilde{X}) - \xi_d^t(\hat{X}) \mid < \sum_{t,d} \mid \xi_d^t(X) - \xi_d^t(\hat{X}) \mid$ since for every t, d, we have either $\mid \xi_d^t(\tilde{X}) - \xi_d^t(\hat{X}) \mid = \mid \xi_d^t(X) - \xi_d^t(\hat{X}) \mid$, or $\mid \xi_d^t(\tilde{X}) - \xi_d^t(\hat{X}) \mid = \mid \xi_d^t(X) - \xi_d^t(\hat{X}) \mid -1$ (where the latter is true for at least one t, d) by construction. This contradicts the fact that $X \in \mathcal{M}_3$.

Thus, there exists a legitimate matching X where $\xi_d^t(X) = \hat{q}_d^t$ and $\xi_{d'}^t(X) = \hat{p}_{d'}^t$.

□

Now we are ready to prove the theorem. The first part follows from Lemmas 1 and 2. Specifically, by Lemma 1, SPDA produces a legitimate matching. Therefore, by Lemma 2, we have $\hat{p}_d^t \le \xi_d^t(X) \le \hat{q}_d^t$ for all $t \in \mathcal{T}$ and $d \in \mathcal{D}$. For each school district d, hence, the maximum proportion of type-t students that can be admitted is $\hat{q}_d^t/k_d$ and the minimum proportion of type t students that can be admitted is $\hat{p}_d^t/k_d$. Therefore, the ratio difference of type-t students in any two districts is at most $\max\limits_{d \ne d'}\{(\hat{q}_d^t/k_d - \hat{p}_{d'}^t/k_{d'})\}$. We conclude that the α-diversity policy is achieved when $\hat{q}_d^t/k_d - \hat{p}_{d'}^t/k_{d'} \le \alpha$ for every t, d, and d'.

The second part of the theorem follows from Lemma 3. Suppose that $\hat{q}_d^t/k_d - \hat{p}_{d'}^t/k_{d'} > \alpha$ for some t, d, and d'. From Lemma 3, we know the existence of a legitimate matching X where $\xi_d^t(X) = \hat{q}_d^t$ and $\xi_{d'}^t(X) = \hat{p}_{d'}^t$. Consider a student preference profile where each student prefers her contract in X the most. Then, since the admissions rules are weakly acceptant, SPDA ends at the first step. Thus matching X is the outcome and, therefore, the α-diversity is not satisfied. □

Proof of Theorem 5. Suzuki et al. (2017) study a setting in which each student is initially endowed with a school and there are no constraints associated with student types, that is, when there is just one type. In that setting, they show that if the distribution is M-convex, then their mechanism, called TTC-M, satisfies constrained efficiency, individual rationality, strategy-proofness, and the policy goal. To adapt their result to our setting, consider the hypothetical matching problem that we have introduced before the definition of TTC in which each student is endowed with a school-type pair and each student has strict preferences over all school-type pairs. It is straightforward to verify that this hypothetical market satisfies all the conditions assumed by Suzuki et al. (2017). In particular, M-convexity of $\Xi \cap \Xi^0$ holds by assumption. Therefore, TTC-M in this market satisfies constrained efficiency, individual rationality, strategy-proofness, and the policy goal.

We note that the outcome of our TTC is isomorphic to the outcome of TTC-M in the hypothetical market in the following sense. Student s is allocated to contract (s, c) under preference profile $P = (P_s)_{s \in \mathcal{S}}$ at the outcome of the TTC mechanism if and only if student

s is allocated to the school-type pair (c,t) under preference profile $\tilde{P} = (\tilde{P}_s)_{s\in\mathcal{S}}$ at TTC-M in the hypothetical market.

To show constrained efficiency, let X be the outcome of TTC and, for each student $s \in \mathcal{S}$, let (s, c_s) be the contract associated with student s at matching X. Suppose, for contradiction, that there exists a feasible matching X' with $\xi(X') \in \Xi$ that Pareto dominates matching X. Denoting $X'_s = (s, c'_s)$ for each student $s \in \mathcal{S}$, this implies $(s, c'_s)\ R_s\ (s, c_s)$ for every student $s \in \mathcal{S}$, with at least one relation being strict. Then, by the construction of preferences $\tilde{R}_s$ in the hypothetical market, we have $(c'_s, \tau(s))\ \tilde{R}_s\ (c_s, \tau(s))$ for every student $s \in \mathcal{S}$, with at least one relation being strict. Moreover, because matching X' is feasible in the original problem, $Y' = \{(c'_s, \tau(s)) | (s, c'_s) \in X'\}$ is feasible in the hypothetical problem, and $Y = \{(c_s, \tau(s)) | (s, c_s) \in X\}$ is the result of TTC-M. This is a contradiction to the result in Suzuki et al. (2017) that TTC-M is constrained efficient.

To show individual rationality, let matching X be the outcome of TTC and, for each student $s \in \mathcal{S}$, let $X_s = (s, c_s)$ be the contract associated with student s at matching X. Additionally, let $Y = \{(c_s, \tau(s)) | (s, c_s) \in X\}$ be the result of TTC-M in the hypothetical market. Suzuki et al. (2017) establish that TTC-M is individually rational, so $(c_s, \tau(s))\ \tilde{R}_s\ (c_0(s), \tau(s))$ for every $s \in \mathcal{S}$, where $c_0(s)$ denotes the initial match for student s. By the construction of $\tilde{R}_s$, this relation implies $(s, c_s)\ R_s\ (s, c_0(s))$ for every student $s \in \mathcal{S}$, which means X is individually rational in the original problem.

To show strategy-proofness, in the original market, let s be a student, t her type, P_{-s} the preference profile of students other than student s, P_s the true preference of student s, and P'_s a misreported preference of student s. Furthermore, let c and c' be schools assigned to student s under (P_s, P_{-s}) and (P'_s, P_{-s}) for TTC, respectively. Note that the previous argument establishes that, in the hypothetical market, student s is allocated to (c,t) and (c',t) under $(\tilde{P}_s, \tilde{P}_{-s})$ and $(\tilde{P}'_s, \tilde{P}'_{-s})$, respectively. Because TTC-M is strategy-proof, it follows that $(c,t)\ \tilde{P}_s\ (c',t)$ or $c = c'$. By the construction of $\tilde{P}_s$, this relation implies $(s,c)\ P_s\ (s,c')$ or $(s,c) = (s,c')$, establishing strategy-proofness of TTC in the original market.

The result that TTC satisfies the policy goal follows from the result in Suzuki et al. (2017) that the distribution corresponding to the TTC-M outcome is in $\Xi \cap \Xi^0$. □

Proof of Corollary 1. We show that when the policy goal Ξ is M-convex, so is $\Xi \cap \Xi^0$. Then the result follows immediatelfy from Theorem 5.

Suppose that $\xi, \tilde{\xi} \in \Xi \cap \Xi^0$ such that $\xi^t_c > \tilde{\xi}^t_c$ for some school c and type t. Since Ξ is M-convex and $\xi, \tilde{\xi} \in \Xi$ there exist school c' and type t' such that $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi$ and $\tilde{\xi} + \chi_{c,t} - \chi_{c',t'} \in \Xi$. We need to show that $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi^0$ and $\tilde{\xi} + \chi_{c,t} - \chi_{c',t'} \in \Xi^0$ as well.

Because $\xi \in \Xi^0$, the number of students assigned in ξ is $\sum_d k_d$. Therefore, the number of students assigned in $\xi - \chi_{c,t} + \chi_{c',t'}$ is also $\sum_d k_d$. Furthermore, $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi$

implies that the number of students in a school at $\xi - \chi_{c,t} + \chi_{c',t'}$ is less than or equal to the capacity of the school. These two results imply that $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi^0$. Similarly, $\tilde{\xi} + \chi_{c,t} - \chi_{c',t'} \in \Xi^0$. $\square$

Proof of Corollary 2. Suppose that the diversity policy Ξ sets a floor and ceiling for each type at each school. We show that $\Xi \cap \Xi^0$ is an M-convex set. The set $\Xi \cap \Xi^0$ can be represented as $\{\xi | \forall c, t\ q_c^t \geq \xi_c^t \geq p_c^t,\ \forall c\ q_c \geq \sum_t \xi_c^t \text{ and } \sum_{c,t} \xi_c^t = \sum_d k_d\}$.

Suppose that there exist $\xi, \tilde{\xi} \in \Xi \cap \Xi^0$ such that $\xi_c^t > \tilde{\xi}_c^t$. To show M-convexity, we need to find school c' and type t' with $\xi_{c'}^{t'} < \tilde{\xi}_{c'}^{t'}$ such that (1) $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi \cap \Xi^0$ and (2) $\tilde{\xi} + \chi_{c,t} - \chi_{c',t'} \in \Xi \cap \Xi^0$. To show both conditions, we look at two possible cases depending on whether $c' = c$ or not.

Case 1: First consider the case when there exists type t' such that $\xi_c^{t'} < \tilde{\xi}_c^{t'}$. We prove (1) that $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi \cap \Xi^0$. Since $\xi - \chi_{c,t} + \chi_{c,t'}$ assigns the same total number of students at school c as ξ, the capacity constraint at school c is satisfied at $\xi - \chi_{c,t} + \chi_{c,t'}$. Furthermore, the number of students assigned to any school in $\xi - \chi_{c,t} + \chi_{c,t'}$ is the same as ξ, which means that all students are assigned. Next, because $\xi_c^t - 1 \geq \tilde{\xi}_c^t \geq p_c^t$ (the former inequality comes from the assumption $\xi_c^t > \tilde{\xi}_c^t$, and the latter comes from the fact that $\tilde{\xi} \in \Xi$, the floor for type t at school c is satisfied at $\xi - \chi_{c,t} + \chi_{c,t'}$. Next, $\xi \in \Xi$ and $\xi_c^t > \tilde{\xi}_c^t$ imply $q_c^t \geq \xi_c^t \geq \tilde{\xi}_c^t + 1$. Therefore, the ceiling for type t at school c in $\xi - \chi_{c,t} + \chi_{c,t'}$ is satisfied.

The floor for type t' at school c is satisfied for $\xi - \chi_{c,t} + \chi_{c,t'}$ because $\xi_c^{t'} + 1 \geq \xi_c^{t'} \geq p_c^{t'}$ (the former inequality is obvious, and the latter comes from the fact $\xi \in \Xi$). Similarly, the ceiling for type t' at school c is satisfied at $\xi - \chi_{c,t} + \chi_{c,t'}$ because $q_c^t \geq \tilde{\xi}_c^{t'} \geq \xi_c^{t'} + 1$.

No other coefficients changed between ξ and $\xi - \chi_{c,t} + \chi_{c,t'}$, so all other constraints are satisfied at the latter distribution. Therefore, (1) is satisfied.

The proof that (1) is satisfied follows from the facts that $\xi_c^t > \tilde{\xi}_c^t$ and $\xi_c^{t'} < \tilde{\xi}_c^{t'}$. By changing the roles of t with t' and ξ with $\tilde{\xi}$ in the preceding argument, we get the implication of (1) that $\tilde{\xi} - \chi_{c,t'} + \chi_{c,t} \in \Xi \cap \Xi^0$. But this is exactly (2).

Case 2: In this case, $c' \neq c$ for every (c', t') such that $\xi_{c'}^{t'} < \tilde{\xi}_{c'}^{t'}$. Then, $\xi_c^{t'} \geq \tilde{\xi}_c^{t'}$ for every $t' \neq t$. In particular, the total number of students assigned to school c at ξ is strictly larger than at $\tilde{\xi}$. Because everyone is matched with some school by assumption, these imply that, without loss of generality, there exists school c' such that the total number of students matched at c' is strictly larger at $\tilde{\xi}$ than at ξ. In addition, there exists type t' such that $\tilde{\xi}_{c'}^{t'} > \xi_{c'}^{t'}$.

Now we proceed to show condition (1) for this case. To do so, we first note that $\xi - \chi_{c,t} + \chi_{c',t'}$ assigns the same number of students as in ξ, so all students are assigned in $\xi - \chi_{c,t} + \chi_{c',t'}$. Furthermore, it assigns a smaller number of students at school c than ξ, so the capacity constraint at school c is satisfied at $\xi - \chi_{c,t} + \chi_{c',t'}$. Likewise, $\xi - \chi_{c,t} + \chi_{c',t'}$

assigns a weakly smaller number of students at c' than $\tilde{\xi}$ does, so the capacity constraint at school c' is satisfied at $\xi - \chi_{c,t} + \chi_{c',t'}$.

Next we check that the type-specific floors at schools are satisfied. Because $\xi_c^t - 1 \geq \tilde{\xi}_c^t \geq p_c^t$ (the first inequality follows from the assumption $\xi_c^t > \tilde{\xi}_c^t$ and the second from the fact that $\tilde{\xi} \in \Xi$), the floor for type t at school c is satisfied at $\xi - \chi_{c,t} + \chi_{c',t'}$. For type t' at school c', we have $\xi_{c'}^{t'} + 1 \geq \xi_{c'}^{t'} \geq p_{c'}^{t'}$ (the first inequality is obvious and the second follows from the fact that $\xi \in \Xi$), so the floor for type t' at school c' is satisfied for $\xi - \chi_{c,t} + \chi_{c',t'}$.

Now we check that the type-specific ceilings at schools are satisfied. Since $q_c^t \geq \xi_c^t > \xi_c^t - 1$ (because $\xi \in \Xi$), the ceiling for type t at school c is satisfied at $\xi - \chi_{c,t} + \chi_{c',t'}$. For type t' at school c', we have $q_{c'}^{t'} \geq \tilde{\xi}_{c'}^{t'} \geq \xi_{c'}^{t'} + 1$ (the first inequality follows from the fact that $\tilde{\xi} \in \Xi$ and the second one follows from $\tilde{\xi}_{c'}^{t'} > \xi_{c'}^{t'}$), so the ceiling for type t' at school c' is satisfied at $\xi - \chi_{c,t} + \chi_{c',t'}$.

No other coefficients changed between ξ and $\xi - \chi_{c,t} + \chi_{c',t'}$, so all other constraints are satisfied at the latter distribution.

The proof that (1) is satisfied follows from the facts that $\xi_c^t > \tilde{\xi}_c^t$, $\tilde{\xi}_{c'}^{t'} > \xi_{c'}^{t'}$, there are more students assigned to school c at ξ than $\tilde{\xi}$, and there are more students assigned to school c' at $\tilde{\xi}$ than ξ. If we change the roles of ξ with $\tilde{\xi}$, c with c', and t with t', then (1) would imply $\tilde{\xi} - \chi_{c',t'} + \chi_{c,t} \in \Xi \cap \Xi^0$. But this is exactly (2), so we are done. Therefore, $\Xi \cap \Xi^0$ is an M-convex set. The result then follows from Theorem 5. ☐

Proof of Corollary 3. Let the set of matchings that satisfy the balanced-exchange policy be denoted by Ξ. Mathematically, Ξ can be written as $\{\xi | \forall d \ \sum_t \xi_d^t = k_d \text{ and } \forall c \ q_c \geq \sum_t \xi_c^t\}$. We show that Ξ is M-convex.

Suppose that there exist $\xi, \tilde{\xi} \in \Xi$ such that $\xi_c^t > \tilde{\xi}_c^t$. If there exists t' such that $\tilde{\xi}_c^{t'} > \xi_c^{t'}$, then the number of students in each district and each school are the same in ξ, $\tilde{\xi}$, $\xi - \chi_{c,t} + \chi_{c',t'}$, and $\tilde{\xi} + \chi_{c,t} - \chi_{c',t'}$ so both (1) $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi$ and (2) $\tilde{\xi} + \chi_{c,t} - \chi_{c',t'} \in \Xi$ are satisfied.

Otherwise, suppose that, for every type $t' \neq t$, $\tilde{\xi}_c^{t'} \leq \xi_c^{t'}$. Therefore, there are more students assigned to school c at ξ than $\tilde{\xi}$. Since the number of students assigned to district $d \equiv d(c)$ in ξ and $\tilde{\xi}$ are the same, there exists another school c' in district d such that c' has more students in $\tilde{\xi}$ than ξ. Furthermore, there exists type t' such that $\tilde{\xi}_{c'}^{t'} > \xi_{c'}^{t'}$.

We first show (1). Since both schools c and c' are in district d, the number of students assigned to district d is the same at ξ and $\xi - \chi_{c,t} + \chi_{c',t'}$. Therefore, the number of students assigned to district d at $\xi - \chi_{c,t} + \chi_{c',t'}$ is k_d.

Next we check the school capacity constraints. The number of students assigned to school c at $\xi - \chi_{c,t} + \chi_{c',t'}$ is one less than the corresponding number at ξ, so the capacity constraint of school c at $\xi - \chi_{c,t} + \chi_{c',t'}$ is satisfied. Furthermore, the number of students assigned to school c' at $\xi - \chi_{c,t} + \chi_{c',t'}$ is weakly smaller than the corresponding number at $\tilde{\xi}$. Therefore, the capacity constraint of school c' at $\xi - \chi_{c,t} + \chi_{c',t'}$ is also satisfied.

Since the other constraints are the same at ξ and $\xi - \chi_{c,t} + \chi_{c',t'}$, (1) holds.

Note that the above argument relies on the facts $\xi_c^t > \tilde{\xi}_c^t$, $\xi_{c'}^{t'} < \tilde{\xi}_{c'}^{t'}$, and $d(c) = d(c')$. If we switch the roles of c with c' and ξ with $\tilde{\xi}$, the implication of (1) is $\tilde{\xi} - \chi_{c',t'} + \chi_{c,t} \in \Xi$, which is exactly (2). Therefore, Ξ is M-convex.

The result then follows from Corollary 1 because Ξ is M-convex and no school is assigned more students than its capacity in Ξ. □

Proof of Corollary 4. We first show that the set of distributions $\Xi = \{\xi | \forall c, t\ q_c^t \geq \xi_c^t \geq p_c^t,\ \forall c\ q_c \geq \sum_t \xi_c^t$ and $\forall d\ \sum_t \xi_d^t = k_d\}$ is an M-convex set.

Suppose that there exist $\xi, \tilde{\xi} \in \Xi$ such that $\xi_c^t > \tilde{\xi}_c^t$. To show M-convexity, we need to find school c' and type t' with $\xi_{c'}^{t'} < \tilde{\xi}_{c'}^{t'}$ such that (1) $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi$ and (2) $\tilde{\xi} + \chi_{c,t} - \chi_{c',t'} \in \Xi$. Let $d \equiv d(c)$. To show both conditions, we look at two possible cases depending on whether $c' = c$ or not.

Case 1: First consider the case when there exists type t' such that $\xi_c^{t'} < \tilde{\xi}_c^{t'}$. We prove (1) that $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi$. Since $\xi - \chi_{c,t} + \chi_{c,t'}$ assigns the same total number of students at school c as ξ, the capacity constraint at school c at $\xi - \chi_{c,t} + \chi_{c,t'}$ is satisfied. Furthermore, the number of students assigned to any district in $\xi - \chi_{c,t} + \chi_{c,t'}$ is the same as ξ, which means that the number of students in every district is equal to the number of students who are from there. Next, because $\xi_c^t - 1 \geq \tilde{\xi}_c^t \geq p_c^t$ (the former inequality comes from the assumption $\xi_c^t > \tilde{\xi}_c^t$, and the latter comes from the fact $\tilde{\xi} \in \Xi$), the floor for type t at school c is satisfied at $\xi - \chi_{c,t} + \chi_{c,t'}$. Next, the fact that $\xi \in \Xi$ and $\xi_c^t > \tilde{\xi}_c^t$ imply $q_c^t \geq \xi_c^t \geq \tilde{\xi}_c^t + 1$. Therefore, the ceiling for type t at school c in $\xi - \chi_{c,t} + \chi_{c,t'}$ is satisfied.

The floor for type t' at school c is satisfied for $\xi - \chi_{c,t} + \chi_{c,t'}$ because $\xi_c^{t'} + 1 \geq \xi_c^{t'} \geq p_c^{t'}$ (the former inequality is obvious, and the latter comes from the fact $\xi \in \Xi$). Similarly, the ceiling for type t' at school c is satisfied at $\xi - \chi_{c,t} + \chi_{c,t'}$ because $q_c^t \geq \tilde{\xi}_c^{t'} \geq \xi_c^{t'} + 1$.

No other coefficients changed between ξ and $\xi - \chi_{c,t} + \chi_{c,t'}$, so all other constraints are satisfied at the latter distribution. Therefore, (1) is satisfied.

The proof that (1) is satisfied follows from the facts that $\xi_c^t > \tilde{\xi}_c^t$ and $\xi_c^{t'} < \tilde{\xi}_c^{t'}$. By changing the roles of t with t' and ξ with $\tilde{\xi}$ in the preceding argument, we get the implication of (1) that $\tilde{\xi} - \chi_{c,t'} + \chi_{c,t} \in \Xi$. But this is exactly (2).

Case 2: In this case, $c' \neq c$ for every (c', t') such that $\xi_{c'}^{t'} < \tilde{\xi}_{c'}^{t'}$. Then, $\xi_c^{t'} \geq \tilde{\xi}_c^{t'}$ for every $t' \neq t$. In particular, the total number of students assigned to school c at ξ is strictly larger than at $\tilde{\xi}$. Because the number of students in district d are the same in ξ and $\tilde{\xi}$, there exist school c' in district d and type t' such that the total number of students matched with c' is strictly larger at $\tilde{\xi}$ than at ξ. In addition, there exists type t' such that $\tilde{\xi}_{c'}^{t'} > \xi_{c'}^{t'}$.

Now we proceed to show condition (1) for this case. To do so, we first note that $\xi - \chi_{c,t} + \chi_{c',t'}$ assigns the same number of students to each district as in ξ, so the number of students assigned to each district d is k_d in $\xi - \chi_{c,t} + \chi_{c',t'}$. Furthermore, it assigns a smaller number of students at school c than ξ, so the capacity constraint at school c at $\xi-\chi_{c,t}+\chi_{c',t'}$ is satisfied. Likewise, $\xi-\chi_{c,t}+\chi_{c',t'}$ assigns a weakly smaller number of students at c' than $\tilde{\xi}$ does, so the capacity constraint at school c' at $\xi-\chi_{c,t}+\chi_{c',t'}$ is satisfied.

Next we check that the type-specific floors at schools are satisfied. Because $\xi_c^t-1\geq\tilde{\xi}_c^t\geq p_c^t$ (the first inequality follows from the assumption $\xi_c^t>\tilde{\xi}_c^t$ and the second from the fact that $\tilde{\xi}\in\Xi$), the floor for type t at school c is satisfied at $\xi-\chi_{c,t}+\chi_{c',t'}$. For type t' at school c', we have $\xi_{c'}^{t'}+1\geq\xi_{c'}^{t'}\geq p_{c'}^{t'}$ (the first inequality is obvious and the second follows from the fact that $\xi\in\Xi$), so the floor for type t' at school c' is satisfied for $\xi-\chi_{c,t}+\chi_{c',t'}$.

Now we check that the type-specific ceilings at schools are satisfied. Since $q_c^t\geq\xi_c^t>\xi_c^t-1$ (the first inequality follows from the fact that $\xi\in\Xi$ and the second inequality is obvious), the ceiling for type t at school c is satisfied at $\xi-\chi_{c,t}+\chi_{c',t'}$. For type t' at school c', we have $q_{c'}^{t'}\geq\tilde{\xi}_{c'}^{t'}\geq\xi_{c'}^{t'}+1$ (the first inequality follows from the fact that $\tilde{\xi}\in\Xi$ and the second one follows from $\tilde{\xi}_{c'}^{t'}>\xi_{c'}^{t'}$), so the ceiling for type t' at school c' is satisfied at $\xi-\chi_{c,t}+\chi_{c',t'}$.

No other coefficients changed between ξ and $\xi-\chi_{c,t}+\chi_{c',t'}$, so all other constraints are satisfied at the latter distribution.

The proof that (1) is satisfied follows from the facts that $d(c)=d(c')$, $\xi_c^t>\tilde{\xi}_c^t$, $\tilde{\xi}_{c'}^{t'}>\xi_{c'}^{t'}$, there are more students assigned to school c at ξ than $\tilde{\xi}$, and there are more students assigned to school c' at $\tilde{\xi}$ than ξ. If we change the roles of ξ with $\tilde{\xi}$, c with c', and t with t', then (1) would imply $\tilde{\xi}-\chi_{c',t'}+\chi_{c,t}\in\Xi$. But this is exactly (2), so we are done.

The result then follows from Corollary 1 because Ξ is M-convex and no school is assigned more students than its capacity in Ξ. □

Corollary 5. *When there are no distributional constraints, TTC satisfies constrained efficiency, individual rationality, strategy-proofness, and it is feasible.*

Proof of Corollary 5. Let Ξ denote the set of distributions of feasible matchings. We prove that $\Xi\cap\Xi^0$ is an M-convex set. $\Xi\cap\Xi^0$ can be represented as $\{\xi|q_c\geq\sum_t\xi_c^t$ and $\sum_{c,t}\xi_c^t=\sum_d k_d\}$.

Suppose that there exist $\xi,\tilde{\xi}\in\Xi\cap\Xi^0$ such that $\xi_c^t>\tilde{\xi}_c^t$. To show M-convexity, we need to find school c' and type t' with $\xi_{c'}^{t'}<\tilde{\xi}_{c'}^{t'}$ such that (1) $\xi-\chi_{c,t}+\chi_{c',t'}\in\Xi\cap\Xi^0$ and (2) $\tilde{\xi}+\chi_{c,t}-\chi_{c',t'}\in\Xi\cap\Xi^0$.

Case 1: If there exists a type $t'\neq t$ such that $\xi_c^{t'}<\tilde{\xi}_c^{t'}$ then we can take $c'=c$ because the number of students assigned to every school in $\xi-\chi_{c,t}+\chi_{c,t'}\in\Xi\cap\Xi^0$ and ξ are the same, so the school capacity constraints are sats=isfied. Furthermore, the number of students assigned in $\xi-\chi_{c,t}+\chi_{c,t'}\in\Xi\cap\Xi^0$ and ξ are also the same, so (1) is satisfied. We can repeat the same argument for $\tilde{\xi}+\chi_{c,t}-\chi_{c,t'}\in\Xi\cap\Xi^0$ and $\tilde{\xi}$, so (2) also holds.

Case 2: Suppose that $\xi_c^{t'}\geq\tilde{\xi}_c^{t'}$ for every $t'\neq t$. Then the number of students assigned to school c in ξ is more than in $\tilde{\xi}$. Since the total number of assigned students is the same in

ξ and $\tilde{\xi}$, there exists a school c' such that the number of students assigned to school c' in $\tilde{\xi}$ is more than in ξ. Therefore, there exists type t' such that $\tilde{\xi}_{c'}^{t'} > \xi_{c'}^{t'}$.

To show (1), note that the number of students assigned to a school in $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi \cap \Xi^0$ is the same as in ξ. Furthermore, school c has one less student in $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi \cap \Xi^0$ than in ξ, so the capacity constraint at school c is satisfied. Finally, the number of students assigned to school c' in $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi \cap \Xi^0$ is not greater than in $\tilde{\xi}$, so the capacity constraint at school c' is also satisfied. The number of students assigned to schools other than c and c' in $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi \cap \Xi^0$ and in ξ remain the same. Therefore, $\xi - \chi_{c,t} + \chi_{c',t'} \in \Xi \cap \Xi^0$.

To show (2), change the roles of ξ with $\tilde{\xi}$, t with t', and c with c' and repeat the same arguments as in the previous paragraph. Therefore, $\Xi \cap \Xi^0$ is an M-convex set. The result then follows from Theorem 5.

□

Proof of Theorem 6. Suppose that district admissions rules favor own students. Fix a student preference profile. Recall that when school districts are integrated, students are assigned to schools by SPDA where each student ranks all contracts associated with her and each district d has the admissions rule Ch_d. When districts are not integrated, students are assigned to schools by SPDA where students only rank the contracts associated with their home districts and each district d has the admissions rule Ch_d. We first show that the no-integration SPDA outcome can be produced by SPDA when all districts participate simultaneously and students rank all contracts including the ones associated with the other districts by modifying admissions rules for the districts. Let $Ch'_d(X) \equiv Ch_d(\{x \in X | d(s(x)) = d\})$ be the modified admissions rule.

In SPDA, if district admissions rules have completions that satisfy path independence, then SPDA outcomes are the same under the completions and the original admissions rules because in SPDA a district always considers a set of proposals which is feasible for students. Furthermore, SPDA does not depend on the order of proposals when district admissions rules are path independent. As a result, SPDA does not depend on the order of proposals when district admissions rules have completions that satisfy path independence. Therefore, the no-integration SPDA outcome can be produced by SPDA when all districts participate simultaneously and students rank all contracts including the ones associated with the other districts and each district d has the admissions rule Ch'_d. The reason is that when each district d has admissions rule Ch'_d, a student is not admitted to a school district other than her home district. Furthermore, because Ch_d favors own students, the set of chosen students under Ch'_d is the same with that under Ch_d for any set of contracts of the form $\{x \in X | d(s(x)) = d\}$ for any set X.

We next show that Ch'_d has a path-independent completion. By assumption, for every district d, there exists a path-independent completion $\widetilde{Ch}_d$ of Ch_d. Let $\widetilde{Ch}'_d(X) \equiv \widetilde{Ch}_d(\{x \in X|d(s(x)) = d\})$. We show that $\widetilde{Ch}'_d$ is a path-independent completion of Ch'_d. To show that $\widetilde{Ch}'_d(X)$ is a completion, consider a set X such that $\widetilde{Ch}'_d(X)$ is feasible for students. Let $X^* \equiv \{x \in X|d(s(x)) = d\}$. Then we have the following:

$$\widetilde{Ch}'_d(X^*) = \widetilde{Ch}_d(X^*) = Ch_d(X^*) = Ch'_d(X^*),$$

where the first equality follows from the definition of $\widetilde{Ch}'_d$, the second equality follows from the fact that $\widetilde{Ch}_d$ is a completion of Ch_d, and the third equality follows from the definition of Ch'_d. Furthermore, because $\widetilde{Ch}'_d(X) = \widetilde{Ch}'_d(X^*)$ and $Ch'_d(X^*) = Ch'_d(X)$, we get $\widetilde{Ch}'_d(X) = Ch'_d(X)$. Therefore, $\widetilde{Ch}'_d$ is a completion of Ch'_d.

To show that $\widetilde{Ch}'_d$ is path independent, consider two sets of contracts X and Y. Let $X^* \equiv \{x \in X|d(s(x)) = d\}$ and $Y^* \equiv \{x \in Y|d(s(x)) = d\}$. Then we have the following:

$$\begin{aligned}
\widetilde{Ch}'_d(X \cup \widetilde{Ch}'_d(Y)) &= \widetilde{Ch}'_d(X \cup \widetilde{Ch}_d(Y^*)) \\
&= \widetilde{Ch}_d(X^* \cup \widetilde{Ch}_d(Y^*)) \\
&= \widetilde{Ch}_d(X^* \cup Y^*) \\
&= \widetilde{Ch}'_d(X \cup Y),
\end{aligned}$$

where the first and second equalities follow from the definition of $\widetilde{Ch}'_d$, the third equality follows from path independence of $\widetilde{Ch}_d$, and the last equality follows from the definition of $\widetilde{Ch}'_d$. Therefore, $\widetilde{Ch}'_d$ is path independent.

Because Ch_d favors own students, we have $Ch_d(X) \supseteq Ch'_d(X)$ for every X which is feasible for students. Furthermore, for any such X, $\widetilde{Ch}_d(X) = Ch_d(X)$ and $\widetilde{Ch}'_d(X) = Ch'_d(X)$ because $\widetilde{Ch}_d$ is a completion of Ch_d and $\widetilde{Ch}'_d$ is a completion of Ch'_d, respectively. Therefore, for any X that is feasible for students, $\widetilde{Ch}_d(X) \supseteq \widetilde{Ch}'_d(X)$. We use this result to show the following lemma.[27]

Lemma 4. *Every student weakly prefers the SPDA outcome under $(\widetilde{Ch}_d)_{d\in\mathcal{D}}$ to the SPDA outcome under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$.*

Proof. Let μ be the SPDA outcome under $(\widetilde{Ch}_d)_{d\in\mathcal{D}}$ and μ' be the SPDA outcome under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$. If μ' is stable under $(\widetilde{Ch}_d)_{d\in\mathcal{D}}$, then the conclusion follows from the result that μ is the student-optimal stable matching under $(\widetilde{Ch}_d)_{d\in\mathcal{D}}$ because each $\widetilde{Ch}_d$ is path independent (Chambers and Yenmez, 2017).

[27]This proof follows the steps of the proof of Theorem E.1. in Echenique and Yenmez (2015).

Suppose that μ' is not stable under $(\widetilde{Ch}_d)_{d\in\mathcal{D}}$. Since μ' is stable under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$, $\widetilde{Ch}'_d(\mu'_d) = \mu'_d$ for every district d. Furthermore, μ'_d is feasible for students, so $\widetilde{Ch}_d(\mu'_d) \supseteq \widetilde{Ch}'_d(\mu'_d) = \mu'_d$. By definition of admissions rules, $\mu'_d \supseteq \widetilde{Ch}_d(\mu'_d)$, so $\widetilde{Ch}_d(\mu'_d) = \mu'_d$. As a result, there must exist a blocking contract for matching μ' so that it is not stable under $(\widetilde{Ch}_d)_{d\in\mathcal{D}}$. Whenever there exists a blocking pair, we consider the following algorithm to improve the student welfare. Let d_1 be the district associated with a blocking contract. Set $\mu^0 \equiv \mu'$.

Step n (n $\geq$ 1): Consider the following set of contracts associated with district d_n for which there exists an associated blocking contract: $X^n_{d_n} \equiv \{x = (s, d_n, c) | x \; P_s \; \mu^{n-1}_s\}$. District d_n accepts $\widetilde{Ch}_{d_n}(\mu^{n-1}_d \cup X^n_{d_n})$ and rejects the rest of the contracts. Let $\mu^n_{d_n} \equiv \widetilde{Ch}_{d_n}(\mu^{n-1}_{d_n} \cup X^n_{d_n})$ and $\mu^n_d \equiv \mu^{n-1}_d \setminus Y^n$ where $Y^n \equiv \{x \in \mu^{n-1} | \exists y \in \mu^n_{d_n} \text{ s.t. } s(x) = s(y)\}$ for $d \neq d_n$. If there are no blocking contracts for matching μ^n under $(\widetilde{Ch}_d)_{d\in\mathcal{D}}$, then stop and return μ^n, otherwise go to Step $n+1$.

We show that district d_n does not reject any contract in $\mu^{n-1}_{d_n}$ by mathematical induction on n, i.e., $\mu^n_{d_n} \supseteq \mu^{n-1}_{d_n}$ for every $n \geq 1$. Consider the base case for $n = 1$. Recall that $\mu^1_{d_1} = \widetilde{Ch}_{d_1}(\mu^0_{d_1} \cup X^1_{d_1}) = \widetilde{Ch}_{d_1}(\mu'_{d_1} \cup X^1_{d_1})$. By construction, $\mu^1_{d_1}$ is a feasible matching. We claim that $\mu'_{d_1} \cup \mu^1_{d_1}$ is feasible for students. Suppose, for contradiction, that it is not feasible for students. Then there exists a student s who has one contract in μ'_{d_1} and one in $\mu^1_{d_1} \setminus \mu'_{d_1}$. Call the latter contract z. By construction $z \; P_s \; \mu'_s$ and by path independence $z \in \widetilde{Ch}_{d_1}(\mu'_{d_1} \cup \{z\})$. Furthermore, since student s is matched with district d_1 in μ', $d(s) = d_1$. Therefore, $\widetilde{Ch}_{d_1}(\mu'_{d_1} \cup \{z\}) = \widetilde{Ch}'_{d_1}(\mu'_{d_1} \cup \{z\})$ by definition of $\widetilde{Ch}'_{d_1}$ and construction of μ'. Hence, $z \in \widetilde{Ch}'_{d_1}(\mu'_{d_1} \cup \{z\})$, which contradicts the fact that μ' is stable under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$. Hence, $\mu'_{d_1} \cup \mu^1_{d_1}$ is feasible for students. Feasibility for students implies that $\widetilde{Ch}_{d_1}(\mu'_{d_1} \cup \mu^1_{d_1}) \supseteq \widetilde{Ch}'_{d_1}(\mu'_{d_1} \cup \mu^1_{d_1})$. Path independence and construction of $\mu^1_{d_1}$ yield $\mu^1_{d_1} = \widetilde{Ch}_{d_1}(\mu'_{d_1} \cup \mu^1_{d_1})$. Furthermore, there exists no student s such that $d(s) = d_1$ who has a contract in $\mu^1_{d_1} \setminus \mu'$ as this would contradict stability of μ' under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$. This implies, by definition of $\widetilde{Ch}'_{d_1}$, that $\widetilde{Ch}'_{d_1}(\mu'_{d_1} \cup \mu^1_{d_1}) = \widetilde{Ch}'_{d_1}(\mu'_{d_1})$, and, by stability of μ' under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$, $\widetilde{Ch}'_{d_1}(\mu'_{d_1}) = \mu'_{d_1}$. Therefore, $\mu^1_{d_1} = \widetilde{Ch}_{d_1}(\mu'_{d_1} \cup \mu^1_{d_1}) \supseteq \widetilde{Ch}'_{d_1}(\mu'_{d_1} \cup \mu^1_{d_1}) = \mu'_{d_1} = \mu^0_{d_1}$, which means that district d_1 does not reject any contracts.

Now consider district d_n where $n > 1$. There are two cases to consider. First consider the case when $d_n \neq d_i$ for every $i < n$. In this case, $\mu^{n-1}_{d_n} \subseteq \mu^0_{d_n} = \mu'_{d_n}$. We repeat the same arguments in the previous paragraph. Stability of μ' under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$ and path independence of $\widetilde{Ch}'_{d_n}$ implies that $\mu^n_{d_n} \cup \mu^{n-1}_{d_n}$ is feasible for students. Therefore, $\widetilde{Ch}_{d_n}(\mu^{n-1}_{d_n} \cup \mu^n_{d_n}) \supseteq \widetilde{Ch}'_{d_n}(\mu^{n-1}_{d_n} \cup \mu^n_{d_n})$. Furthermore, there exists no student s such that $d(s) = d_n$ who has a contract in $\mu^n_{d_n} \setminus \mu^{n-1}_{d_n}$. As a result, by definition of $\widetilde{Ch}'_{d_n}$ and by path independence, $\widetilde{Ch}'_{d_n}(\mu^{n-1}_{d_n} \cup \mu^n_{d_n}) = \widetilde{Ch}'_{d_n}(\mu^{n-1}_{d_n}) = \mu^{n-1}_{d_n}$. As in the previous paragraph, we conclude that $\mu^n_{d_n} = \widetilde{Ch}_{d_n}(\mu^{n-1}_{d_n} \cup \mu^n_{d_n}) \supseteq \widetilde{Ch}'_{d_n}(\mu^{n-1}_{d_n} \cup \mu^n_{d_n}) = \mu^{n-1}_{d_n}$.

The second case is when there exists $i < n$ such that $d_i = d_n$. Let i^* be the last such step before n. Since the student welfare improves at every step before n by the mathematical induction hypothesis, $\mu^{i^*-1}_{d_n} \cup X^{i^*}_{d_n} \supseteq \mu^{n-1}_{d_n} \cup X^n_{d_n}$. By definition, $\mu^{i^*}_{d_n} = \widetilde{Ch}_{d_n}(\mu^{i^*-1}_{d_n} \cup X^{i^*}_{d_n})$, which implies by path independence that $\mu^{n-1}_{d_n} \subseteq \widetilde{Ch}_{d_n}(\mu^{n-1}_{d_n} \cup X^n_{d_n}) = \mu^n_{d_n}$ since $\mu^{n-1}_{d_n} \subseteq \mu^{i^*}_{d_n}$.

Finally, we need to show that the improvement algorithm terminates. We claim that $\mu^n_{d_n} \neq \mu^{n-1}_{d_n}$. Suppose, for contradiction, that these two matchings are the same. Then, by path independence of $\widetilde{Ch}_{d_n}$, for every $x \in X^n_{d_n}$, $\widetilde{Ch}_{d_n}(\mu^{n-1}_{d_n} \cup \{x\}) = \mu^{n-1}_{d_n}$. This is a contradiction because there exists at least one blocking contract associated with district d_n. Therefore, district d_n gets at least one new contract at Step n. Hence, at least one student gets a strictly more preferred contract at every step of the algorithm while every student gets a weakly more preferred contract. Since the number of contracts is finite, the algorithm has to end in a finite number of steps.

□

Because the SPDA outcome under $(Ch_d)_{d\in\mathcal{D}}$ is the same as the SPDA outcome under $(\widetilde{Ch}_d)_{d\in\mathcal{D}}$ and the SPDA outcome under $(Ch'_d)_{d\in\mathcal{D}}$ is the same as the SPDA outcome under $(\widetilde{Ch}')_{d\in\mathcal{D}}$, the lemma implies that every student weakly prefers the outcome of SPDA under $(Ch_d)_{d\in\mathcal{D}}$ to the outcome of SPDA under $(Ch'_d)_{d\in\mathcal{D}}$. This completes the proof of the first part.

To prove the second part of the theorem, we show that if at least one district's admissions rule fails to favor own students, then there exists a preference profile such that not every student is weakly better off when school districts integrate under SPDA. Suppose that for some district d, there exists a matching X, which is feasible for students, such that $Ch_d(X)$ is not a superset of $Ch_d(X^*)$, where $X^* \equiv \{x \in X | d(s(x)) = d\}$. Now, consider a matching Y where (i) all students from district d are matched with schools in district d, (ii) Y is feasible, and (iii) $Y \supseteq Ch_d(X^*)$. The existence of such Y follows from the fact that $Ch_d(X^*)$ is feasible and $k_{d'} \leq \sum_{c:d(c)=d'} q_c$, for every district d' (that is, there are enough seats in district d' to match all students from district d'.) Because Y is feasible and Ch_d is acceptant, $Ch_d(Y_d) = Y_d$.

Now consider the following student preferences. First we consider students from district d. Each student s who has a contract in X^* ranks X^*_s as her top choice. Note that doing so is well defined because X^* is feasible for students. Each student s who has a contract in $X^* \setminus Ch_d(X^*)$ ranks contract Y_s as her second top choice. Note that, in this case, Y_s cannot be the same as X^*_s because $Ch_d(Y_d) = Y_d$ and Ch_d is path independent. Each student s who has a contract in $Y \setminus X^*$ ranks that contract as her top choice. Next we consider students from the other districts. Each student s who has a contract in $X \setminus X^*$ ranks that contract as her top choice. Any other student ranks a contract not associated with district d as her top choice. Complete the rest of the student preferences arbitrarily.

Consider SPDA for district d when districts are not integrated. At the first step, students who have a contract in X^* propose that contract. The remaining students have contracts in $Y \setminus X^*$ propose the associated contracts. Because Y is feasible, Y contains $Ch_d(X^*)$, and Ch_d is acceptant, only contracts in $X^* \setminus Ch_d(X^*)$ are rejected. At the second step, these students propose their contracts in Y_d, the set of proposals that the district considers is Y_d. Because $Ch_d(Y_d) = Y_d$, no contract is rejected, and SPDA stops and returns Y_d. In particular, every student who has a contract in $Ch_d(X^*)$ has the corresponding contract at the outcome.

When districts are integrated, at the first step, each student who has a contract in X proposes that contract and every other student proposes a contract associated with a district different from d. District d considers X (or X_d), and tentatively accepts $Ch_d(X)$. Because $Ch_d(X) \not\supseteq Ch_d(X^*)$ by assumption, at least one student who has a contract in $Ch_d(X^*)$ is rejected. Therefore, this student is strictly worse off when districts are integrated. $\square$