Existence of a Stable Outcome under Observable Substitutability across Doctors in Many-to-Many Matching with Contracts*

Keisuke Bando† Toshiyuki Hirai‡

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Abstract

We consider a many-to-many matching problem with contracts between hospitals and doctors. We examine the existence of a stable outcome under observable substitutability across doctors that is a relaxed notion of substitutability. We propose an appropriate extension of the cumulative offer process and show that it generates a stable outcome under observable substitutability across doctors of every hospital's preferences together with some additional conditions; Preferences of every agent (doctor and hospital) are unitary, and preferences of every doctor is substitutable and satisfy size monotonicity. We also show the essentiality of the additional conditions via examples. Namely, we show a stable outcome may fail to exist if one of the three additional conditions is violated. These counterexamples are valid even if we consider a weaker stability notion instead of a stable outcome and stronger relaxed notions of substitutability instead of observable substitutability across doctors.

Keywords: many-to-many matching with contracts, observable substitutability across doctors, stable outcome, cumulative offer process.
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†Shinshu University, 3-1-1 Asahi, Matsumoto City 390-8621, Japan; E-mail: k_bando@shinshu-u.ac.jp

‡University of Toyama, 3190 Gofuku, Toyama 930-8555 Japan; E-mail: thirai@eco.u-toyama.ac.jp

1 Introduction

Since the seminal paper of Gale and Shapley (1962), matching problems have been extensively analyzed by many researchers. This paper considers many-to-many matching with contracts between hospitals and doctors, which was first formulated by Roth (1984a). We investigate the existence of a stable outcome, which was formulated by Hatfield and Kominers (2017).[1] Substitutability introduced by Kelso and Crawford (1982) has also been playing an important role to guarantee the existence of a stable outcome in many-to-many matching with contracts. For example, Roth (1984a), Blair (1988), Chambers and Yenmez (2017), and Hatfield and Kominers (2017) showed the existence of a stable outcome under substitutability.[2] The purpose of this paper is to relax the condition of substitutability in many-to-many matching with contracts for an existence result.

In the context of many-to-one matching with contracts, many authors have examined weaker notions of substitutability to obtain existence results of stable matchings.[3] Hatfield and Kojima (2010) proposed unilateral and bilateral substitutability; Hatfield and Kominers (2016) proposed substitutes completability. Kadam (2017) showed a clear relationship between these three weak conditions of substitutability in many-to-one matching with contracts. Under these conditions, a cumulative offer process originated with Hatfield and Milgrom (2005) was employed for finding a stable matching. Other relaxed notions of substitutability are proposed by paying attention to contracts appearing along a cumulative offer process only. Flanagan (2014) proposed cumulative offer revealed bilateral substitutability, which is another sufficient condition for the existence of a stable matching. Recently, Hatfield, *et al.* (2017b) proposed observable substitutability and observable substitutability across doctors. They showed that the latter condition is weakest among previously reviewed conditions.[4] Hatfield, *et al.* (2017a) showed that observable substitutability, and hence observable substitutability across doctors, are satisfied by a choice function naturally derived from preferences of a hospital that has multiple divisions and flexible allotments for these divisions. The relaxed notions of substitutability

[1] A similar definition was also proposed by Roth (1984a).

[2] Substitutability is important even in many-to-many matching without contracts, see for example, Sotomayor (1999), Echenique and Oviedo (2006), and Konishi and Ünver (2006), among others.

[3] They also studied properties of stable matchings under the relaxed notions of substitutability, which we do not address here.

[4] For a more general model, which we do not consider in this paper, Zhang (2016) proposed further weaker conditions called weakly observable substitutability and weakly observable substitutability across doctors.

are proposed not only for theoretical concerns, but also for practical applicability. See, for example, Sönmez and Switzer (2013) and Kominers and Sönmez (2016).

We examine the existence of a stable outcome under observable substitutability across doctors in a many-to-many matching with contracts. More precisely, we propose a variation of the cumulative offer process and show that it generates a stable outcome under observable substitutability across doctors for hospitals' preferences and some additional conditions. Such additional conditions are essential because Hatfield and Kominers (2017) showed that substitutability for all agents is a necessary and sufficient condition for the existence of a stable outcome in the sense of a maximal domain. We assume that every agent's preferences satisfy unitarity, and every doctor's preferences satisfy substitutability and size monotonicity (Hatfield and Milgrom, 2005)[5], in addition to observable substitutability across doctors for hospitals' preferences. We show the existence of a stable outcome under these assumptions. Unitarity requires that no agent prefers to choose multiple contracts between a same doctor-hospital pair. Size monotonicity requires that an agent prefers to choose more contracts when the set of available contracts is expanded.

Yenmez (2018) also showed the existence of a stable outcome with a relaxed notion of substitutability. Precisely, he showed that a stable outcome exists if preferences of hospitals are substitutes completable, preferences of doctors are substitutable, and every agent's preferences satisfy unitarity.[6] Our result is independent of Yenmez (2018) because we further relaxed substitutes completability to observable substitutability across doctors, while we additionally need size monotonicity for doctors' preferences.

We will also discuss the essentiality of unitarity of every agent's preferences, and substitutability and size monotonicity of every doctor's preferences. In those examples, we show that a stable outcome fails to exist without one of these conditions even if we consider stronger conditions than observable substitutability across doctors for hospitals' preferences and a weaker notion of stability called weak setwise stability (Klaus and Walzl, 2009). It is worth noting that size monotonicity is essential for guaranteeing the existence of a stable outcome in contrast to the literature. In many-to many-matching with contracts, Hatfield and Kominers (2017) showed that size monotonicity together

[5] Hatfield and Milgrom (2005) called this condition the law of aggregate demand. Later, Hatfield, et al. (2017) called it size monotonicity.

[6] Yenmez (2018) considered a problem in the context of college admission. Moreover, unitarity is implicitly assumed in his model.

with substitutability restricts the structure of the set of stable matchings; Every agent signs the same number of contracts in every stable matching, which is called the rural hospitals theorem.[7] In many-to-one matching with contracts, Hatfield and Milgrom (2005) showed that size monotonicity together with substitutability implies that the cumulative offer process is strategy-proof for doctors. We show that a stable matching may not exist without size monotonicity even if every hospital's preferences satisfy bilateral substitutability, which is stronger than observable substitutability across doctors, and every doctor's preferences satisfy substitutability.

Despite its essentiality, at least one of unitarity and size monotonicity is violated in some practical situations. For example, unitarity and size monotonicity are incompatible in matching with couples (Roth, 1984b). Unitarity is violated when a couple of doctors prefers to be hired at a same hospital. To avoid this situation, we may bundle contracts for a couple to one contract like Hatfield and Kominers (2017). However, this makes the couple's preferences violate size monotonicity. Nevertheless, there are some applications where our result may apply. For example, Yenmez (2018) considered a college admission problem as a many-to-many matching with contracts[8]. In this formulation, preferences of colleges may violate substitutability due to their admission policies. Of course, our result may also apply to classical problems of a labor market for consultants who are hired from multiple firms.

The remaining of this paper is organized as follows. In section 2, we introduce the model of many-to-many matching with contracts and the definition of a stable outcome. Conditions of preferences, including observable substitutability across doctors, are also introduced. In section 3, we propose an extension of the cumulative offer process. We also state and prove that the cumulative offer process generates a stable outcome under observable substitutability across doctors and additional conditions. Moreover, we show some examples that show the essentiality of these additional conditions. In section 4, we conclude with some remarks.

[7] In many-to-many matching without contracts, Alkan (2002) showed that size monotonicity together with substitutability implies a strong lattice property of the set of stable matchings.

[8] Students may be matched with multiple colleges because a "match" between a college-student pair means not enrollment, but an admission.

2 Preliminaries

Let F be a finite set of agents. The set of agents F is divided into two nonempty and disjoint sets of doctors D and hospitals H, that is, $F = D \cup H$ and $D \cap H = \emptyset$. There is a finite set of contracts X. Each $x \in X$ involves a doctor $x_D \in D$ and a hospital $x_H \in H$. For each $x \in X$, denote $x_F = \{x_D, x_H\}$ a doctor-hospital pair involved in x. There are possibly multiple contracts between a pair of a doctor and a hospital, that is, there may exist distinct $x, x' \in X$ such that $x_D = x'_D$ and $x_H = x'_H$. For each $Y \subseteq X$, denote $Y_D = \cup_{y \in Y}\{y_D\}$, $Y_H = \cup_{y \in Y}\{y_H\}$ and $Y_F = Y_D \cup Y_H$. For each $Y \subseteq X$ and each $i \in F$, let $Y_i = \{y \in Y | i \in y_F\}$ be the set of contracts in Y that involve i. We consider a many-to-many matching problem, that is, every agent can have multiple contracts. Therefore, each agent $i \in F$ has a strict preference ordering $\succ_i$ over 2^{X_i}. For each $Y \subseteq X$ and each $i \in F$, let $C^i(Y)$ be the subset of Y_i such that $C^i(Y) \succeq_i Y^i$ for all $Y^i \subseteq Y_i$. For each $Y \subseteq X$, denote $C^D(Y) = \bigcup_{d \in D} C^d(Y)$ and $C^H(Y) = \bigcup_{h \in H} C^h(Y)$. A tuple $(F, X, (\succ_i)_{i \in F})$ is called a many-to-many matching problem with contracts or simply a matching problem.

We introduce a solution concept called a stable outcome by Hatfield and Kominers (2017). A set of contracts $A \subseteq X$ is called an outcome. An outcome A is individually rational if $C^i(A) = A_i$ for all $i \in F$.

Definition 1

- *An outcome $A \subseteq X$ is blocked via $Z \subseteq X$ if $\emptyset \neq Z \subseteq X \setminus A$ and $Z_i \subseteq C^i(A \cup Z)$ for all $i \in Z_F$.*
- *An outcome is stable if it is individually rational and not blocked.*

We sometimes refer Z in Definition 1 a blocking set to A.

We introduce conditions on preferences. Indeed, these conditions are imposed on choice functions rather than preferences.

Definition 2 *For each $i \in F$, choice function C^i satisfies consistency if for any $Y, Y' \subseteq X$, $C^i(Y) \subseteq Y' \subseteq Y$ implies $C^i(Y') = C^i(Y)$.*

Consistency was introduced to matching theory by Blair (1988).[9] Note that consistency is always satisfied in our model because preferences are primitive in our model.[10]

[9] In Aygün and Sönmez (2013), this condition is called irrelevance of rejected contracts.

[10] In general, consistency may fail if we consider choice function rather than preferences as a primitive of a model.

Next condition was originally introduced by Kominers (2012).[11]

Definition 3 • *For each $h \in H$, choice function C^h is unitary if $x, x' \in C^h(Y)$ and $x \neq x'$ imply $x_D \neq x'_D$.*

- *For each $d \in D$, choice function C^d is unitary if $x, x' \in C^d(Y)$ and $x \neq x'$ imply $x_H \neq x'_H$.*

Note that the unitarity condition is naturally satisfied in a many-to-one matching problem. Note also that Klaus and Walzl (2009) and Yenmez (2018) imposed the unitarity condition in a many-to-many matching problem with contracts, while Hatfield and Kominers (2017) did not.

Definition 4 *For each $i \in F$, choice function C^i satisfies size monotonicity if for any $Y, Y' \subseteq X$, $Y \subseteq Y'$ implies $|C^i(Y)| \leq |C^i(Y')|$.*

This condition is introduced by Hatfield and Milgrom (2005) and called the law of aggregate demand in their paper.

Definition 5 *For each $i \in F$, choice function C^i is substitutable if for any $Y \subseteq X$ and any $x, z \in X$ with $x, z \notin Y$, $z \notin C^i(Y \cup \{z\})$ implies $z \notin C^i(Y \cup \{z, x\})$.*

Substitutability is originally introduced by Kelso and Crawford (1982) and reformulated by Roth (1984a) and Hatfield and Milgrom (2005) to more general models. In this model, Hatfield and Kominers (2017) showed that a stable outcome exists when every agent's preferences satisfy substitutability. We will examine the existence of a stable outcome under a weaker notion of substitutability together with the conditions defined above.

Remark 1 For each $i \in F$, choice function C^i is said to satisfy path-independence if for any $Y, Y' \subseteq X$, $C^i(Y \cup Y') = C^i(Y \cup C^i(Y'))$. Aizerman and Malishevski (1981) showed that a choice function satisfies consistency and substitutability if and only if the choice function satisfies path-independence. We will use this property in our proof.

[11] This condition was originally imposed on a matching problem rather than choice functions.

Path-independence was introduced to matching theory by Blair (1988). In the literature on choice theory, Aizerman and Malishevski (1981) showed that a path-independece choice is characterized by a collection of strict preference orderings over the set of single contracts. Based on this result, Chambers and Yenmez (2017) analyzed many-to-many matcing problelms with contracts.

We turn to the definition of observable substitutability across doctors by Hatfield, *et al.* (2017b). A finite sequence of contracts $(x^1, ..., x^M)(M \geq 1)$ is said to be an offer process for $h \in H$ if $x^m_H = h$ for all $m = 1, ..., M$. An offer process $(x^1, ..., x^M)$ for h is observable if $M = 1$ or $x^m_D \notin C^h(\{x^1, ..., x^{m-1}\})_D$ for all $m = 2, ..., M$. Now, we define observable substitutability across doctors.

Definition 6 *For each $h \in H$, choice function C^h is observably substitutable across doctors if for any observable offer process $(x^1, ..., x^M)$ for h and any $x \in \{x^1, ..., x^{M-1}\}$, $x \in C^h(\{x^1, ..., x^M\}) \setminus C^h(\{x^1, ..., x^{M-1}\})$ implies $x_D \in C^h(\{x_1, ..., x^{M-1}\})_D$.*

In many-to-one matching with contracts, Hatfield, *et al.* (2017b) showed the existence of a stable outcome under observable substitutability across doctors by a cumulative offer process. Their cumulative offer process repeats the following procedure until it finds a stable outcome: an unmatched doctor offers the best contract to the corresponding hospital among contracts that have not been rejected by any hospital, and the hospital chooses the best set of contracts from the cumulated set of contracts that have ever been offered. An offer process is observable in the sense that such a offer process may be appeared along the cumulative offer process because a doctor does not offer a new contract to a hospital if the hospital is currently choosing a contract with that doctor. Observable substitutability across doctors requires that no doctor is newly accepted by a hospital when an observable offer process adds another contract.

3 An existence result of a stable outcome

In this section, we first propose an extension of the cumulative offer process that is an extension of Hatfield and Milgrom (2005). Then, we show that the cumulative offer process generates a stable outcome under certain conditions.

3.1 Main result

We begin with introducing an extension of the cumulative offer process.

For each $k = 0, 1, ...$, $A^H(k)$ represents the set of available contracts for hospitals at step k, while $A^D(k)$ represents the set of available contracts for doctors at step k. The following cumulative offer process specifies how to revise these sets. Later, we will show that a resulting outcome is stable.

- Set $A^D(0) = X$ and $A^H(0) = C^D(A^D(0))$ and proceed to step 0.
- Step $k(\geq 0)$: If $[C^H(A^H(k))_d]_H = [C^d(A^D(k))]_H$ for all $d \in D$, then the algorithm terminates at this step. Otherwise, define

  $$R(k) = \{x \in A^H(k) | x_D \notin C^{x_H}(A^H(k))_D\}$$

  and

  $$A^D(k+1) = A^D(k) \setminus R(k) \text{ and } A^H(k+1) = A^H(k) \cup C^D(A^D(k+1)).$$

  If $A^D(k+1) = \emptyset$, the algorithm terminates at this step. Otherwise, proceed to Step $k+1$.

For each step k, $R(k)$ is the set of contracts that are not available for doctors in later steps. The definition of $R(k)$ requires that a contract x offered to a hospital h be left available for x_D as long as h chooses a contract with x_D, even if x itself is not chosen by h.

Now, we are ready to state the main result.

Theorem 1 *Suppose that (i) every agent's choice function is unitary, (ii) every hospital's choice function is observably substitutable across doctors, (iii) every doctor's choice function is substitutable, and (iv) every doctor's choice function satisfies size monotonicity. Then, the cumulative offer process generates a stable outcome.*

Before turning to the proof of Theorem 1, we state that our cumulative offer process terminates in finite steps under assumptions (i)-(iii) in Theorem 1.

Proposition 1 *Suppose that (i) every agent's choice function is unitary, (ii) every hospital's choice function is observably substitutable across doctors, and (iii) every doctor's choice function is substitutable. Then, the cumulative offer process terminates in finite steps.*

Proof. See the Appendix. ■

We now give a proof of Theorem 1.

Proof of Theorem 1. We assume that (i) every agent's choice function is unitary, (ii) every hospital's choice function is observably substitutable across doctors, (iii) every doctor's choice function is substitutable, and (iv) every doctor's choice function satisfies size monotonicity. Then, the cumulative offer process terminates at a finite step t^* by Proposition 1. There are two cases to consider; (a) $[C^H(A^H(t^*))_{d'}]_H \neq [C^{d'}(A^D(t^*))]_H$ for some $d' \in D$ and $A^D(t^*+1) = \emptyset$, and (b) $[C^H(A^H(t^*)_d]_H = [C^d(A^D(t^*)]_H$ for all $d \in D$.

Claim 1 *If (a) holds, then $\emptyset$ is a stable outcome.*

Proof of Claim 1. See the Appendix. □

Assume that (b) $[C^H(A^H(t^*))_d]_H = [C^d(A^D(t^*))]_H$ for all $d \in D$. Define $X^* = C^H(A^H(t^*))$ and $Y^* = C^D(A^D(t^*))$. Note that $X^*_d = C^H(A^H(t^*))_d$ and $Y^*_d = C^d(A^D(t^*))$ hold for all $d \in D$. Therefore, $[X^*_d]_H = [Y^*_d]_H$ holds for all $d \in D$. Note also that for all $d \in D$, $x, x' \in X^*_d$ and $x \neq x'$ imply $x_H \neq x'_H$. To see this, suppose that there exist $\hat{d} \in D$ and $x, x' \in X^*_{\hat{d}}$ such that $x \neq x'$ and $x_H = x'_H$. By $x, x' \in X^*_{\hat{d}} = C^H(A^H(t^*))_{\hat{d}}$, we have that $x, x' \in C^{x_H}(A^H(t^*))$ and $x_D = x'_D = \hat{d}$, contradicting that C^{x_H} is unitary.

We will show that X^* is stable.

Claim 2 *For all $d \in D$, $C^d(X^*_d \cup Y^*_d) = X^*_d$.*

Proof of Claim 2. Fix any $d \in D$. We first claim that $C^d(X^*_d \cup Y^*_d)_H = [X^*_d]_H$. Suppose that $C^d(X^*_d \cup Y^*_d)_H \neq [X^*_d]_H$. By $[X^*_d]_H = [Y^*_d]_H$, we have $C^d(X^*_d \cup Y^*_d)_H \neq [Y^*_d]_H$. Note that $[X^*_d \cup Y^*_d]_H = [Y^*_d]_H$ by $[X^*_d]_H = [Y^*_d]_H$. Therefore, $C^d(X^*_d \cup Y^*_d)_H \subseteq [Y^*_d]_H$ holds. By $C^d(X^*_d \cup Y^*_d)_H \neq [Y^*_d]_H$, $C^d(X^*_d \cup Y^*_d)_H \subsetneq [Y^*_d]_H$. By consistency of C^d and $Y^*_d = C^d(A^D(t^*))$, $C^d(Y^*_d) = Y^*_d$. Therefore, $C^d(X^*_d \cup Y^*_d)_H \subsetneq [C^d(Y^*_d)]_H$. By unitarity of C^d, this implies $|C^d(X^*_d \cup Y^*_d)| < |C^d(Y^*_d)|$. However, this contradicts that C^d satisfies size monotonicity.

We next show that $X^*_d \subseteq C^d(X^*_d \cup Y^*_d)$. Take any $x \in X^*_d$. Suppose that $x \notin C^d(X^*_d \cup Y^*_d)$. Let $x_H = h$. By $h \in [X^*_d]_H$, we have $h \in C^d(X^*_d \cup Y^*_d)_H$. Therefore, there exists $y \in C^d(X^*_d \cup Y^*_d)$ such that $y_H = h$. By $x \neq y$, $x \in X^*_d$, $x_H = y_H = h$ and unitarity of C^h, we have $y \in Y_d^*$. By substitutability of C^d, we have that $y \in C^d(\{x\} \cup Y_d^*)$. By $x \in X_d^* = C^H(A^H(t^*))_d$, we have $x \in A^H(t^*)$. It follows that there exists some $t' \leq t^*$ such that $x \in C^D(A^D(t'))$ from the definition of the cumulative offer process. Thus, $x \in A^D(t')$. We also have $x \in C^d(A^D(t'))$ by $x_D = d$. Note that $Y_d^* \subseteq A^D(t^*)$ holds by $Y_d^* = C^d(A^D(t^*))$. By $A^D(t^*) \subseteq A^D(t')$, we have $Y_d^* \subseteq A^D(t')$. By substitutability of C^d and $\{x\} \cup Y_d^* \subseteq A^D(t')$, $x \in C^d(A^D(t'))$ implies $x \in C^d(\{x\} \cup Y_d^*)$. Therefore, we have $x, y \in C^d(\{x\} \cup Y_d^*)$, $x \neq y$, and $x_H = y_H = h$, contradicting that C^d is unitary.

By unitarity of C^d together with $X_d^* \subseteq C^d(X_d^* \cup Y_d^*)$ and $[X_d^*]_H = [C^d(X_d^* \cup Y_d^*)]_H$, we have that $X_d^* = C^d(X_d^* \cup Y_d^*)$. □

We now show that X^* is stable. Recall that every doctor's choice function satisfies path-independence by its consistency and substitutability. (See Remark 1.) By $C^H(A^H(t^*)) = X^*$ and consistency, we have $C^h(X^*) = X_h^*$ for all $h \in H$ and hence X^* is individually rational for all hospitals. By Claim 2 together with consistency, we have $C^d(X^*) = X_d^*$ for all $d \in D$ and hence X^* is individually rational. Finally we show that X^* is not blocked. Suppose not. Then, there exists a blocking set Z to X^*. Note that $Z \cap X^* = \emptyset$. We claim that $Z \subseteq A^H(t^*)$. To obtain this, it is sufficient to show that $Z_d \subseteq A^H(t^*)$ for all $d \in Z_D$. Pick any $d \in Z_D$. Then, $Z_d \subseteq C^d(Z_d \cup X_d^*)$ because Z is a blocking set to X^*. Note that

$$C^d(Z_d \cup X_d^*) = C^d(Z_d \cup C^d(X_d^* \cup Y_d^*)) = C^d(Z_d \cup X_d^* \cup Y_d^*)$$

holds where the first equality follows from Claim 2 and the second equality follows from path-independence of C^d. By $Z_d \subseteq C^d(Z_d \cup X_d^*)$, we have that $Z_d \subseteq C^d(Z_d \cup X_d^* \cup Y_d^*)$. By substitutability of C^d, we have that $Z_d \subseteq C^d(Z_d \cup Y_d^*)$. Note that

$$C^d(Z_d \cup Y_d^*) = C^d(Z_d \cup C^d(A^D(t^*))) = C^d(Z_d \cup A^D(t^*))$$

holds where the first equality follows from $Y_d^* = C^d(A^D(t^*))$ and the second equality follows from path-independence of C^d. By $Z_d \subseteq C^d(Z_d \cup Y_d^*)$, we obtain $Z_d \subseteq C^d(Z_d \cup A^D(t^*))$. We now show that $Z_d \subseteq A^H(t^*)$. Pick any $z \in Z_d$. If $z \notin A^D(t^*)$, then $z \in R(t')$ for some $t' < t^*$ and hence $z \in A^H(t') \subseteq A^H(t^*)$. Suppose that $z \in A^D(t^*)$. Then, by substitutability of C^d and $z \in C^d(Z_d \cup A^D(t^*))$, we have that $z \in C^d(A^D(t^*))$ and hence $z \in A^H(t^*)$. Therefore, we obtain $Z_d \subseteq A^H(t^*)$.

Pick any $h \in Z_H$. By $C^h(A^H(t^*)) = X_h^*$ and $Z \subseteq A^H(t^*)$, we have $C^h(A^H(t^*)) \subseteq X_h^* \cup Z \subseteq A^H(t^*)$. By consistency of C^h, we have $C^h(A^H(t^*)) = C^h(X_h^* \cup Z)$. This implies $C^h(X_h^* \cup Z) = X_h^*$ contradicting that Z is a blocking set to X^*. Therefore, X^* is stable. ■

3.2 Tightness of Theorem 1

In this subsection, we show tightness of Theorem 1 by showing counterexamples in each of which one of the conditions in Theorem 1 is violated. Moreover, we introduce stronger conditions than observable substitutability across doctors and a weaker solution concept than the stable outcome and show that the counterexamples are valid even if we consider those stronger conditions and/or a weaker solution concept.

First, we introduce two conditions on preferences by Hatfield and Kojima (2010).

Definition 7 *For each $h \in H$, choice function C^h is unilaterally substitutable if for any $Y \subseteq X$ and any $x, z \in X$ with $z_D \notin Y_D$, $z \notin C^h(Y \cup \{z\})$ implies $z \notin C^h(Y \cup \{z, x\})$.*

Definition 8 *For each $h \in H$, choice function C^h is bilaterally substitutable if for any $Y \subseteq X$ and any $x, z \in X$ with $x_D, z_D \notin Y_D$, $z \notin C^h(Y \cup \{z\})$ implies $z \notin C^h(Y \cup \{z, x\})$.*

The next condition is substitutable completability by Hatfield and Kominers (2016).

Definition 9 *Let $h \in H$.*

- *A function $\bar{C}^h$ is a completion of choice function C^h if for any $Y \subseteq X$, either $\bar{C}^h(Y) = C^h(Y)$ or there exist $x, x' \in \bar{C}^h(Y)$ such that $x_D = x'_D$.*
- *A choice function C^h is substitutes completable if there exists a completion $\bar{C}^h$ that satisfies substitutability and consistecy.*

These three conditions for a doctor's preferences can be defined in the same way. Therefore, we omit it. Yenmez (2018) showed that if every agent's choice function is unitary, every hospital's choice function is substitutes completable and every doctor's choice function is substitutable, then there exists a stable outcome.

Remark 2 We briefly survey the relationship between the relaxed notions of substitutability, where preferences of agents are given as primitives so that consistency is automatically satisfied[12]. These conditions are originally proposed in the context of many-to-one matching with contracts. Therefore, unitarity of every agent's choice function

[12]The relationship in a model where consistency is unnecessarily satisfied was clearly summarized by Zhang (2016).

is necessarily assumed. Hatfield and Kojima (2010) proposed unilateral substitutability as a stronger notion of bilateral substitutability. Kadam (2017) showed that unilateral substitutability implies substitutes completability. Hatfield, *et al.* (2017b) showed that observable substitutability across doctors is implied from each of substitutes completability, bilateral substitutability, cumulative offer revealed bilateral substitutability (Flanagan, 2014), and observable substitutability (not across doctors) (Hatfield, *et al.*, 2017), though we do not introduce the latter two conditions explicitly. Therefore, observable substitutability across doctors is weakest among these known conditions. It should be remarked that these relationships are valid under the unitarity condition.

Finally, we introduce a weaker solution concept than a stable outcome originally proposed by Klaus and Walzl (2009).

Definition 10

- *An outcome $A \subseteq X$ is weakly setwise blocked via $Z \subseteq X$ if $\emptyset \neq Z \subseteq X \setminus A$ and there exists Y^* satisfying (i) $Z \subseteq Y^*$, (ii) $Y^* \subseteq A \cup Z$, (iii) $Y_i^* = C^i(A \cup Z)$ for all $i \in Z_F$.*
- *An outcome is weakly setwise stable if it is individually rational and not weakly setwise blocked.*

It is straightforward to see that any stable outcome is a weakly setwise stable outcome because a weakly setwise block requires that the blocking agents agree on a resulting outcome, in addition to the definition of a block. On the other hand, there may be a weakly setwise stable outcome that is not stable.

For a simple example, consider a case where $H = \{h\}$, $D = \{d\}$, and $X = \{x, y\}$. Preferences of each agent is given by the following list.

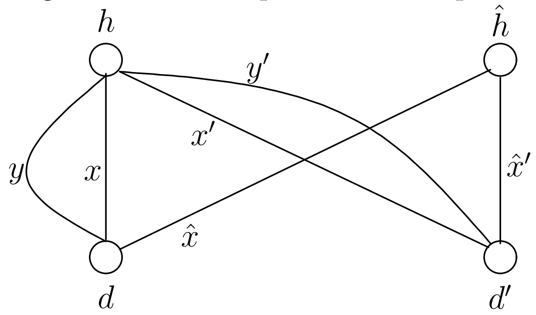
$\succ_h$: $\{x\}$ $\{y\}$ $\emptyset$.

$\succ_d$: $\{x, y\}$ $\{y\}$ $\{x\}$ $\emptyset$,

Such a list as $\succ_h$ means $\{x\} \succ_h \{y\} \succ_h \emptyset$, and $\emptyset \succ_h Y$ for any nonempty $Y \subseteq X$ with $Y \neq \{x\}, \{y\}$. It is easy to see that $\{y\}$ is weakly setwise stable, but not stable because it is blocked via $\{x\}$. In this example, C^d is not unitary. The following example shows that the set of weakly setwise stable outcomes may be strictly larger than that of stable outcomes even if every agent's choice function is unitary. In this example, every agent except for a hospital h and a doctor d has substitutable choice function, while choice functions of h and d are bilaterally substitutable.

Example 1 Let $H = \{h, \hat{h}\}$ and $D = \{d, d'\}$. The set of contracts is given as $X = \{x, y, x', y', \hat{x}, \hat{x}'\}$. Agents involved in each contract are described in Figure 1.

Figure 1: A description of Example 1.



The preferences of the agents are given as follows.

$\succ_h$: $\{x, x'\}$ $\{y\}$ $\{y'\}$ $\{x\}$ $\{x'\}$ $\emptyset$.

$\succ_{\hat{h}}$: $\{\hat{x}\}$ $\{\hat{x}'\}$ $\emptyset$.

$\succ_d$: $\{x, \hat{x}\}$ $\{y\}$ $\{x\}$ $\{\hat{x}\}$ $\emptyset$.

$\succ_{d'}$: $\{y', \hat{x}'\}$ $\{x', \hat{x}'\}$ $\{y'\}$ $\{\hat{x}'\}$ $\{x'\}$ $\emptyset$.

We proceed with the choice functions C^h, $C^{\hat{h}}$, C^d, and $C^{d'}$ that are derived from these preferences where all of them satisfy unitarity. Note that $C^{\hat{h}}$ and $C^{d'}$ are substitutable. On the other hand, C^h and C^d are not substitutable, while they are bilaterally substitutable.

Consider an outcome $\{y, \hat{x}'\}$. We show that this outcome is blocked via $\{x, x', \hat{x}\}$ and hence not stable. We have that

$$\begin{aligned}
C^h(\{y, \hat{x}', x, x', \hat{x}\}) &= \{x, x'\} = \{x, x', \hat{x}\}_h; \\
C^{\hat{h}}(\{y, \hat{x}', x, x', \hat{x}\}) &= \{\hat{x}\} = \{x, x', \hat{x}\}_{\hat{h}}; \\
C^d(\{y, \hat{x}', x, x', \hat{x}\}) &= \{x, \hat{x}\} = \{x, x', \hat{x}\}_d; \\
C^{d'}(\{y, \hat{x}', x, x', \hat{x}\}) &= \{x', \hat{x}'\} \supseteq \{x'\} = \{x, x', \hat{x}\}_{d'}
\end{aligned}$$

because each of h, $\hat{h}$, and d chooses own best choice, and d' chooses own second best choice while her best choice $\{y', \hat{x}'\}$ is not available from $\{y, \hat{x}', x, x', \hat{x}\} (\not\ni y')$. Note that A is not weakly setwise blocked via $\{x, x', \hat{x}\}$ because $\hat{x}' \notin C^{\hat{h}}(\{y, \hat{x}', x, x', \hat{x}\})$ while $\hat{x}' \in C^{d'}(\{y, \hat{x}', x, x', \hat{x}\})$.

On the other hand, $\{y, \hat{x}'\}$ is weakly setwise stable. Suppose that there exists a weakly setwise blocking set Z to $\{y, \hat{x}'\}$. We claim that $d \notin Z_D$. Suppose that $d \in Z_D$. Then, $x, \hat{x} \in Z$ because $C^d(Y \cup \{y, \hat{x}'\}) = \{y\}$ for any $Y \subseteq X \setminus \{y, \hat{x}'\}$ unless $x, \hat{x} \in Y$. It follows that $h, \hat{h} \in Z_H$. Then, $x' \in Z$ because $C^h(Y \cup \{y, \hat{x}'\}) = \{y\}$ for any $Y \subseteq X \setminus \{y, \hat{x}'\}$ unless $x, x' \in Y$. On the other hand, $y' \notin Z$ because $y' \notin C^h(Y \cup \{y, \hat{x}'\})$ for any $Y \subseteq X \setminus \{y, \hat{x}'\}$. Since $Z \cap \{y, \hat{x}'\} = \emptyset$, $Z = \{x, x', \hat{x}\}$. However, $\{y, \hat{x}'\}$ is not weakly setwise blocked via $\{x, x', \hat{x}\}$ as we mentioned earlier. Hence, $d \notin a(Z)$.

By $d \notin a(Z)$ and $Z \cap \{y, \hat{x}'\} = \emptyset$, $Z \subseteq \{x', y'\}$. Since C^h and $C^{d'}$ are unitary, $Z = \{x'\}$ or $\{y'\}$. However, neither $\{x'\}$ nor $\{y'\}$ is a weakly setwise blocking set to $\{y, \hat{x}'\}$ by $\{y\} = C^h(\{y, \hat{x}'\} \cup \{x'\}) = C^h(\{y, \hat{x}'\} \cup \{y'\})$. Hence, $\{y, \hat{x}'\}$ is a weakly setwise stable outcome. □

We show that stable and weakly setwise stable outcomes coincide with each other under weaker conditions than Theorem 1. Nevertheless, it is still meaningful to confirm that even a weakly setwise stable outcome fails to exist when a condition in the following proposition is violated.

Proposition 2 *Suppose that (i) every agent's choice function is unitary and (ii) every doctor's choice function is substitutable. Then, the set of weakly setwise stable outcomes coincides with the set of stable outcomes.*

Proof. We show that any weakly setwise stable outcome is stable because the converse direction follows from the definition. To obtain this, it is sufficient to show that for any outcome A, if A is blocked, then it is weakly setwise blocked under (i) and (ii).

Let A be an outcome. Suppose that A is blocked. Let Z be a blocking set to A. Fix any hospital $h \in Z_H$ and define $\tilde{Z} = Z_h$. Note that $\tilde{Z} \neq \emptyset$, $\tilde{Z}_H = \{h\}$, and $\tilde{Z} \subseteq X \setminus A$. We will show that $\tilde{Z}$ is a weakly setwise blocking set to A.

We first show that $\tilde{Z}_i \subseteq C^i(A \cup \tilde{Z})$ for all $i \in \tilde{Z}_F$. This statement clearly holds for h. Pick any $d \in \tilde{Z}_F$ with $d \neq h$. By $\tilde{Z}_H = \{h\}$, $d \in D$ holds. By $d \in \tilde{Z}_F \subseteq Z_F$, we have $\tilde{Z}_d \subseteq C^d(A_d \cup Z_d)$. By substitutability of C^d, we have $\tilde{Z}_d \subseteq C^d(A_d \cup \tilde{Z}_d)$.

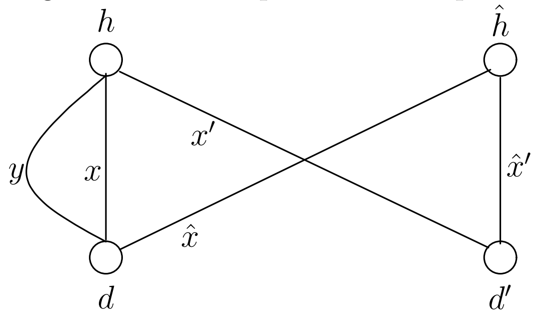
For each $i \in \tilde{Z}_F$, define $Y^i = C^i(A \cup \tilde{Z})$ and $Y^* = \bigcup_{i \in Z_F} Y^i$. Note that $Y^* \subseteq A \cup \tilde{Z}$ holds because $C^i(A \cup \tilde{Z}) \subseteq A_i \cup \tilde{Z}_i$ for all $i \in \tilde{Z}_F$. Note also that $\tilde{Z} \subseteq Y^*$ holds because

$\tilde{Z}_i \subseteq C^i(A \cup \tilde{Z})$ for all $i \in \tilde{Z}_F$. Therefore, to obtain that $\tilde{Z}$ is a weakly setwise blocking set to A, it is sufficient to show that $Y_i^* = Y^i (= C^i(A \cup \tilde{Z})\ )$ for all $i \in \tilde{Z}_F$. Let $i \in \tilde{Z}_F$. Suppose that $Y_i^* \neq Y^i$. Then, $Y^i \subsetneq Y_i^*$ because $Y^i \subseteq Y_i^*$ holds from the definition of Y^*. Consider any $y \in Y_i^*$ with $y \notin Y^i$. Note that $y \in A$ holds by $y \notin Y^i$. Then, by $y \in Y^*$ and $y \notin Y^i$, there exists $j \in \tilde{Z}_F \setminus \{i\}$ such that $y \in Y^j$. By $y_F = \{i, j\}$, either (i) $i \in D$ and $j \in H$ or (ii) $j \in D$ and $i \in H$. By $\{i, j\} \subseteq \tilde{Z}_F$ and $\tilde{Z}_H = \{h\}$, in either case, there exists a contract $z \in \tilde{Z}$ such that $z_F = \{i, j\}$. By $z \in \tilde{Z}$, we have $z \in Y^j$ and hence $z, y \in Y^j = C^j(A \cup \tilde{Z})$. By $z \in \tilde{Z}$ and $y \in A$, we have $y \neq z$ contradicting unitarity. ■

Now, we show three counterexamples, where one of the assumptions in Theorem 1 is violated, while some other conditions are possibly strengthened. First, we show an example without the unitarity condition. The following example shows that a weakly setwise stable outcome may fail to exist in general if the unitarity condition is violated even though one hospital's choice function is not only observably substitutable across doctors but also unilaterally substitutable and substitutes completable, and the remaining agents' choice functions are all substitutable. Moreover, every agent's choice function satisfies size monotonicity.

Example 2 Let $H = \{h, \hat{h}\}$ and $D = \{d, d'\}$. The set of contracts is given as $X = \{x, y, x', \hat{x}, \hat{x}'\}$. Agents involved in each contract are described in Figure 2.

Figure 2: A description of Example 2.



Preferences of the agents are given as follows.

$\succ_h$: $\{x, y, x'\}$ $\{x, x'\}$ $\{y, x'\}$ $\{x\}$ $\{y\}$ $\{x'\}$ $\emptyset$.

$\succ_{\hat{h}}$: $\{\hat{x}\}$ $\{\hat{x}'\}$ $\emptyset$.

Table 1: Comparison of C^h and $\bar{C}^h$.

Y	$\emptyset$	$\{x\}$	$\{y\}$	$\{x'\}$	$\{x,y\}$	$\{x,x'\}$	$\{y,x'\}$	$\{x,y,x'\}$
$C^h(Y)$	$\emptyset$	$\{x\}$	$\{y\}$	$\{x'\}$	$\{x\}$	$\{x,x'\}$	$\{y,x'\}$	$\{x,y,x'\}$
$\bar{C}^h(Y)$	$\emptyset$	$\{x\}$	$\{y\}$	$\{x'\}$	$\{x,y\}$	$\{x,x'\}$	$\{y,x'\}$	$\{x,y,x'\}$
					$\uparrow$ $x_D = y_D$			

$\succ_d$: $\{x,y\}$ $\{x,\hat{x}\}$ $\{y,\hat{x}\}$ $\{x\}$ $\{y\}$ $\{\hat{x}\}$ $\emptyset$.

$\succ_{d'}$: $\{\hat{x}'\}$ $\{x'\}$ $\emptyset$.

We proceed with the choice functions C^h, $C^{\hat{h}}$, C^d, and $C^{d'}$ that are derived from these preferences. Note that C^h and C^d are not unitary because $C^h(\{x,y,x'\}) = \{x,y,x'\}$ and $C^d(Y) = \{x,y\}$ for all $Y \subseteq X$ with $\{x,y\} \subseteq Y$, where $x_F = y_F$. Therefore, condition (i) of Theorem 1 is violated. On the other hand, any other condition in Theorem 1 is satisfied. Moreover, $C^{\hat{h}}$, C^d, and $C^{d'}$ are substitutable, and C^h satisfies all of unilateral substitutability, substitutes completability, and observable substitutability across doctors, though it is not substitutable. (Table 1 shows a completion $\bar{C}^h$ of C^h and compares these functions so that we can confirm substitutes completability of C^h.)

Note that this example violates condition (i) of Proposition 2. Therefore, the set of weakly setwise stable outcomes may be larger than that of stable outcomes. However, we show that even a weakly setwise stable outcome fails to exist in this example. Suppose that there exists a weakly setwise stable outcome A. We distinguish two cases.

First, assume that $x' \in A$. Then, $\hat{x}' \notin A$ by the individual rationality of d'. We claim that $\{x,y\} \subseteq A$. Suppose not. Let $Z = \{x,y\} \setminus A$. Then, A is weakly setwise blocked via Z because $C^h(Z \cup A) = \{x,y,x'\} = \{x,y,x'\}_h$ and $C^d(Z \cup A) = \{x,y\} = \{x,y,x'\}_d$. Therefore, assume that $\{x,y\} \subseteq A$. Note that $\{x,y,x'\} \subseteq A$. Then, $\hat{x} \notin A$ by the individual rationality of d. Hence, $\{x,y,x'\} = A$. However, A is weakly setwise blocked via $\{\hat{x}'\}$ because $C^{\hat{h}}(\{x,y,x',\hat{x}'\}) = \{\hat{x}'\} = \{\hat{x}'\}_{\hat{h}}$ and $C^{d'}(\{x,y,x',\hat{x}'\}) = \{\hat{x}'\} = \{\hat{x}'\}_{d'}$. This contradicts weak setwise stability of A. Hence, there is no weakly setwise stable outcome that includes x'.

Next, assume that $x' \notin A$. By the individual rationality of h, at most one of x and y is in A. We claim that $\hat{x} \in A$. Suppose not. Then, $C^d(A \cup \{\hat{x}\}) = A_d \cup \{\hat{x}\}$ because $|A_d| \leq 1$. We also have $C^{\hat{h}}(A \cup \{\hat{x}\}) = \{\hat{x}\} = (A_{\hat{h}} \cup \{\hat{x}\})_{\hat{h}}$. Therefore, A is weakly setwise blocked via $\{\hat{x}\}$, contradicting weak setwise stability of A. Hence, assume $\hat{x} \in A$.

Then, $\hat{x}' \notin A$ by the individual rationality of $\hat{h}$. Note that $A_{d'} = \emptyset$ since $x' \notin A$ is assumed. We claim that A is weakly setwise blocked via $\bar{Z} := (\{x, y\} \setminus A) \cup \{x'\}$. Note that $\{x, y, x'\} \subseteq A \cup \bar{Z}$. Then, $C^h(A \cup \bar{Z}) = \{x, y, x'\} = \{x, y, x'\}_h$; $C^{d'}(A \cup \bar{Z}) = \{x'\} = \{x, y, x'\}_{d'}$ by $x' \in A \cup \bar{Z}$ and $\hat{x}' \notin A \cup \bar{Z}$; $C^d(A \cup \bar{Z}) = \{x, y\} = \{x, y, x'\}_d$. Therefore, A is weakly setwise blocked via $\bar{Z}$, contradicting weak setwise stability of A. Hence, there is no weakly setwise stable outcome in this example.

Next, we show an example where there is a doctor without substitutable choice function. The following example shows that a weakly setwise stable outcome may fail to exist in general if there is a doctor d with unilateral substitutable but not substitutable choice function, though a choice function of every agent except for d and one hospital h is substitutable. Moreover, the choice function of h is bilateral substitutable, and every agent's choice function satisfies unitarity and size monotonicity.

Example 3 Let $H = \{h, \hat{h}\}$ and $D = \{d, d'\}$. The set of contracts is given as $X = \{x, y, x', y', \hat{x}, \hat{x}'\}$. Agents involved in each contract are described in the same way as Example 1. (See Figure 1 in Example 1.)

The preferences of the agents are given as follows.

$\succ_h$: $\{x, x'\}$ $\{y'\}$ $\{y\}$ $\{x'\}$ $\{x\}$ $\emptyset$.

$\succ_{\hat{h}}$: $\{\hat{x}'\}$ $\{\hat{x}\}$ $\emptyset$.

$\succ_d$: $\{\hat{x}, y\}$ $\{\hat{x}\}$ $\{x\}$ $\{y\}$ $\emptyset$.

$\succ_{d'}$: $\{x'\}$ $\{\hat{x}'\}$ $\{y'\}$ $\emptyset$.

We proceed with the choice functions C^h, $C^{\hat{h}}$, C^d, and $C^{d'}$ that are derived from these preferences where all of them satisfy unitarity. . It is obvious that $C^{\hat{h}}$ and $C^{d'}$ are substitutable, while C^h and C^d are not because $[x \notin C^h(\{x, y\})$ and $x \in C^h(\{x, y, x'\})]$ and $[y \notin C^d(\{x, y\})$ and $y \in C^h(\{x, y, \hat{x}\})]$. Therefore, condition (iii) of Theorem 1 is violated. On the other hand, any other condition in Theorem 1 is satisfied. Moreover, C^h is bilaterally substitutable, and C^d is unilaterally substitutable that are stronger than observably substitutable across doctors.

Note that this example violates condition (ii) of Proposition 2. Therefore, the set of weakly setwise stable outcomes may be larger than that of stable outcomes. However, we show that even a weakly setwise stable outcome fails to exist in this example. Suppose that there exists a weakly setwise stable outcome A.

First, assume that $\hat{x}' \in A$. Then, $\hat{x} \notin A$ by the individual rationality for $\hat{h}$, and $x', y' \notin A$ by the individual rationality for d'. Moreover, by unitarity of C^h as well as C^d, at most one of x and y is in A. To summarize, A is either $\{\hat{x}'\}$, $\{x, \hat{x}'\}$, or $\{y, \hat{x}'\}$. If $A = \{\hat{x}'\}$, then $\{\hat{x}'\}$ is weakly setwise blocked via $\{x'\}$ because $C^h(\{\hat{x}'\} \cup \{x'\}) = \{x'\} = \{x'\}_h$ and $C^{d'}(\{\hat{x}'\} \cup \{x'\}) = \{x'\} = \{x'\}_{d'}$. If $A = \{x, \hat{x}'\}$, then $\{x, \hat{x}'\}$ is weakly setwise blocked via $\{x'\}$ because $C^h(\{x, \hat{x}'\} \cup \{x'\}) = \{x, x'\} = \{x, x'\}_h$ and $C^{d'}(\{x, \hat{x}'\} \cup \{x'\}) = \{x'\} = \{x, x'\}_{d'}$. If $A = \{y, \hat{x}'\}$, then $\{y, \hat{x}'\}$ is weakly setwise blocked via $\{x, x'\}$ because $C^h(\{y, \hat{x}'\} \cup \{x, x'\}) = \{x, x'\} = \{x, x'\}_h$, $C^d(\{y, \hat{x}'\} \cup \{x, x'\}) = \{x\} = \{x, x'\}_d$, and $C^{d'}(\{y, \hat{x}'\} \cup \{x, x'\}) = \{x'\} = \{x, x'\}_{d'}$. Every case contradicts weak setwise stability of A.

Assume therefore that $\hat{x}' \notin A$. Suppose that $x' \notin A$. Then, A is weakly setwise blocked via $\{\hat{x}'\}$ because $C^{\hat{h}}(A \cup \{\hat{x}'\}) = \{\hat{x}'\} = \{\hat{x}'\}_{\hat{h}}$ and $C^{d'}(A \cup \{\hat{x}'\}) = \{\hat{x}'\} = \{\hat{x}'\}_{d'}$. This contradicts weakly setwise stability of A. Thus, $x' \in A$. Then, $y, y' \notin A$ by the individual rationality of h. By the individual rationality of d, at most one of x and $\hat{x}$ is in A. To summarize, A is either $\{x'\}$, $\{x, x'\}$, or $\{\hat{x}, x'\}$. If $A = \{x'\}$, then $\{x'\}$ is weakly setwise blocked via $\{x\}$ because $C^h(\{x'\} \cup \{x\}) = \{x, x'\} = \{x, x'\}_h$ and $C^d(\{x'\} \cup \{x\}) = \{x\} = \{x, x'\}_d$. If $A = \{x, x'\}$, then $\{x, x'\}$ is weakly setwise blocked via $\{\hat{x}\}$ because $C^{\hat{h}}(\{x, x'\} \cup \{\hat{x}\}) = \{\hat{x}\} = \{\hat{x}\}_{\hat{h}}$ and $C^d(\{x, x'\} \cup \{\hat{x}\}) = \{\hat{x}\} = \{\hat{x}\}_d$. If $A = \{\hat{x}, x'\}$, then $\{\hat{x}, x'\}$ is weakly setwise blocked via $\{y\}$ because $C^h(\{\hat{x}, x'\} \cup \{y\}) = \{y\} = \{y, \hat{x}\}_h$ and $C^d(\{\hat{x}, x'\} \cup \{y\}) = \{y, \hat{x}\} = \{y, \hat{x}\}_d$. Every case contradicts weak setwise stability of A. Hence, there is no weakly setwise stable outcome in this example.

Finally, we show an example, where there is a doctor without size monotonicity. The following example shows that a stable outcome may fail to exist in general if size monotonicity is violated even though every agent except for one hospital h is substitutable, while choice function of h is bilateral substitutable. Moreover, there is only one doctor violating size monotonicity, and every agent's choice funtion satisfies unitarity.

Example 4 Let $H = \{h, \hat{h}\}$ and $D = \{d, d'\}$. The set of contracts is given as $X = \{x, y, x', y', \hat{x}, \hat{x}'\}$. Agents involved in each contract is described in the same way as Example 1. (See Figure 1 in Example 1.)

$\succ_h$: $\{x, x'\}$ $\{y'\}$ $\{y\}$ $\{x'\}$ $\{x\}$ $\emptyset$.

$\succ_{\hat{h}}$: $\{\hat{x}'\}$ $\{\hat{x}\}$ $\emptyset$.

$\succ_d$: $\{y, \hat{x}\}$ $\{y\}, \{\hat{x}\}$ $\{x\}$ $\emptyset$.

$\succ_{d'}$: $\{x'\}$ $\{y', \hat{x}'\}$ $\{y'\}$ $\{\hat{x}'\}$ $\emptyset$.

We proceed with the choice functions C^h, $C^{\hat{h}}$, C^d, and $C^{d'}$ that are derived from these preferences where all of them satisfy unitarity. It is straightforward to see that $C^{\hat{h}}$, C^d, and $C^{d'}$ are substitutable. However, $C^{d'}$ do not satisfy size monotonicity because $C^{d'}(\{x', y', \hat{x}'\}) = \{x'\}$ and $C^{d'}(\{y', \hat{x}'\}) = \{y', \hat{x}'\}$. Therefore, condition (iv) of Theorem 1 is violated. On the other hand, any other condition in Theorem 1 is satisfied. Moreover, we can see that C^h is bilateral substitutable though it is not substitutable. Therefore, a stronger condition than (ii) of Theorem 1 is satisfied.

This example satisfies all the conditions of Proposition 2. Therefore, it suffices to show that there exists no stable outcome. Let A be an individually rational outcome. We first show that there exists no stable outcome such that $|A_i| = 2$ for some $i \in F$. Suppose that $|A_h| = 2$. By the individual rationality, $A = \{x, x'\}$. Then, A is blocked via $\{\hat{x}\}$ because $C^{\hat{h}}(\{x, x'\} \cup \{\hat{x}\}) = \{\hat{x}\}$ and $C^d(\{x, x'\} \cup \{\hat{x}\}) = \{\hat{x}\}$. Suppose that $|A_d| = 2$. By the individual rationality, $A = \{y, \hat{x}\}$. Then, A is blocked via $\{y'\}$ because $C^h(\{y, \hat{x}\} \cup \{y'\}) = \{y'\}$ and $C^{d'}(\{y, \hat{x}\} \cup \{y'\}) = \{y'\}$. Suppose that $|A_{d'}| = 2$. By the individual rationality, $A = \{y', \hat{x}'\}$. Then, A is blocked via $\{x, x'\}$ because $C^h(\{y', \hat{x}'\} \cup \{x, x'\}) = \{x, x'\}$, $C^d(\{y', \hat{x}'\} \cup \{x, x'\}) = \{x\}$, and $C^{d'}(\{y', \hat{x}'\} \cup \{x, x'\}) = \{x'\}$. Clearly, there is no individually rational outcome A such that $|A_{\hat{h}}| = 2$.

We next show that A is not stable when $A_d = \emptyset$ or $A_{d'} = \emptyset$. Suppose $A_d = \emptyset$. Then, $x \notin A$. Thus, A is blocked via $\{x, x'\} \setminus A \neq \emptyset$ because $C^h(A_h \cup \{x, x'\}) = \{x, x'\}$, $C^d(A_d \cup \{x, x'\}) = C^d(\{x, x'\}) = \{x\}$, and $C^{d'}(A_{d'} \cup \{x, x'\}) = \{x'\}$. Suppose $A_{d'} = \emptyset$. Then, $x' \notin A$, and thus, $A_h \neq \{x, x'\}$. Moreover, $y' \notin A$. Then, A is blocked via $\{y'\}$ because $C^h(A_h \cup \{y'\}) = \{y'\}$ by $x' \notin A$, and $C^{d'}(A_{d'} \cup \{y'\}) = \{y'\}$.

From the above argument, it remains to consider the case with $|A_i| = 1$ for all $i \in F$. Then, there are the following four possibilities: $A = \{x, \hat{x}'\}$, $\{y, \hat{x}'\}$, $\{x', \hat{x}\}$, or $\{y', \hat{x}\}$.

However,

$\{x, \hat{x}'\}$ is blocked via $\{x'\}$ by $C^h(\{x, \hat{x}'\} \cup \{x'\}) = \{x, x'\}$ and $C^{d'}(\{x, \hat{x}'\} \cup \{x'\}) = \{x'\}$;
$\{y, \hat{x}'\}$ is blocked via $\{y'\}$ by $C^h(\{y, \hat{x}'\} \cup \{y'\}) = \{y'\}$ and $C^{d'}(\{y, \hat{x}'\} \cup \{y'\}) = \{y', \hat{x}'\}$;
$\{x', \hat{x}\}$ is blocked via $\{y\}$ by $C^h(\{x', \hat{x}\} \cup \{y\}) = \{y\}$ and $C^d(\{x', \hat{x}\} \cup \{y\}) = \{y, \hat{x}\}$;
$\{y', \hat{x}\}$ is blocked via $\{\hat{x}'\}$ by $C^{\hat{h}}(\{y', \hat{x}\} \cup \{\hat{x}'\}) = \{\hat{x}'\}$ and $C^{d'}(\{y', \hat{x}\} \cup \{\hat{x}'\}) = \{y', \hat{x}'\}$.

Therefore, there is no stable outcome in this example.

4 Concluding remarks

This paper showed the existence of a stable outcome in many-to-many matching with contracts under unitarity of all agents' choice functions, observable substitutability across doctors for hospitals' choice functions, and substitutability and size monotonicity for doctors' choice functions by employing the cumulative offer process. We also show the essentiality of the conditions via examples.

We finally discuss incentive problems under stable mechanisms. In the context of many-to-one matching with contracts, Hatfield, *et al.* (2017b) showed that the cumulative offer process is strategy-proof for doctors under observable substitutability (not across doctors) and some additional conditions. However, this property does not hold in many-to-many matching. In fact, Roth (1985)'s result implies that there exists no stable mechanism that is strategy-proof for doctors in many-to-many matching with contracts.[13]

On the other hand, non-revelation mechanisms for many-to-many matching markets have been studied by Sotomayor (2004), Echenique and Oviedo (2006), Klaus and Klijn (2017) and Romero-Medina and Triossi (2018).[14] These studies clarify relationship between subgame perfect Nash equilibrium outcomes and stable outcomes in certain extensive-form games induced by many-to-many matching markets. In particular, Romero-Medina and Triossi (2018) analyzed a take-it or leave-it offer game in many-to-many matching with contracts. They showed that the set of subgame perfect Nash

[13]Precisely, Roth (1985) considered a many-to-one matching problem between students and colleges where colleges have responsive preferences over subsets of students with some capacities. It was shown that there exists no stable mechanism that is strategy-proof for colleges.

[14]For many-to-one matching markets, see, for example, Alcalde and Romero-Medina (2000) and Romero-Medina and Triossi (2014).

equilibrium outcomes is a (possibly proper) subset of the set of pairwise stable outcomes and provided sufficient conditions for the existence of a subgame perfect equilibrium. We may analyze non-revelation mechanisms under observable substitutability across doctors. We leave this problem for future research.

Appendix

This appendix is devoted to the proof of Proposition 1 and Claim 1 in the proof of Theorem 1.

We begin with introducing some basic properties of the cumulative offer process that are used in both proofs of Proposition 1 and Claim 1. Suppose that the cumulative offer process proceeds until step $\hat{t}$ and $[C^H(A^H(\hat{t}))_{d'}]_H \neq [C^{d'}(A^D(\hat{t}))]_H$ for some $d' \in D$. Then, $A^D(t)$ and $A^H(t)$ are defined for all $t = 0, ..., \hat{t}+1$. Note that $A^D(t+1) \subseteq A^D(t)$ and $A^H(t+1) \supseteq A^H(t)$ for all $t = 0, ..., \hat{t}$ from the definition of the cumulative offer process. For each $t = 0, ..., \hat{t}+1$, we define the following two conditions named $P_1(t)$ and $P_{1'}(t)$;

- $P_1(t)$: For each $d \in D$ and $h \in H$, $d \in C^h(A^H(t))_D$ implies $h \in C^d(A^D(t))_H$.
- $P_{1'}(t)$: For each $h \in H$ with $A^H(t)_h \neq \emptyset$, there exists an observable offer process $(a^1, ..., a^M)$ for h such that $A^H(t)_h = \{a^1, ..., a^M\}$.

For each $t = 0, ..., \hat{t}$, we define the following condition named $P_2(t)$;

- $P_2(t)$: For each $x \in X$, $x \in A^H(t)$ and $x_D \notin C^{x_H}(A^H(t))_D$ imply $x \notin C^{x_H}(A^H(t+1))$.

Lemma 1 *Suppose that (i) every agent's choice function is unitary, (ii) every hospital's choice function is observably substitutable across doctors, and (iii) every doctor's choice function is substitutable. Suppose that the cumulative offer process proceeds until step $\hat{t}$ and $[C^H(A^H(\hat{t}))_{d'}]_H \neq [C^{d'}(A^D(\hat{t}))]_H$ for some $d' \in D$. Then, $P_1(t)$ and $P_{1'}(t)$ are satisfied for all $t = 0, ..., \hat{t}+1$ and $P_2(t)$ is satisfied for all $t = 0, ..., \hat{t}$*

Proof. We first show that $P_1(0)$ and $P_2(0)$ are satisfied as an induction base. Let $d \in D$ and $h \in H$. Suppose that $d \in C^h(A^H(0))_D$. Let $x \in C^h(A^H(0))$ with $x_D = d$. By $x \in A^H(0) = C^D(A^D(0))$, we have $x \in C^d(A^H(0))$ and hence $h \in C^d(A^H(0))_H$.

Thus, $P_1(0)$ is satisfied. To show $P_{1'}(0)$, let $h' \in H$ with $A^H(t)_{h'} \neq \emptyset$. We denote $A^H(0)_{h'} = \{\tilde{a}^1, ..., \tilde{a}^k\}$. Since every doctor's choice function is unitary, we have that $a_D \neq a'_D$ for any distinct $a, a' \in \{\tilde{a}^1, ..., \tilde{a}^k\}$. This implies that $(\tilde{a}^1, ..., \tilde{a}^k)$ is an observable offer process for h.

We next state and prove the following three claims.

Claim 3 *Let $t = 0, \cdots, \hat{t}$. Suppose that $P_1(t)$ is satisfied. Then, for all $x \in A^H(t+1) \setminus A^H(t)$, $x_D \notin C^{x_H}(A^H(t))_D$.*

Proof of Claim 3. Take any $t = 0, \cdots, \hat{t}$ and any $x \in A^H(t+1) \setminus A^H(t)$. We denote $x_D = d$ and $x_H = h$. We will show that $d \notin C^h(A^H(t))_D$. Suppose that $d \in C^h(A^H(t))_D$. By $P_1(t)$, $h \in C^d(A^D(t))_H$. Therefore, there exists $z \in C^d(A^D(t))$ with $z_H = h$. By $z \in C^d(A^D(t))$, $z \in A^H(t)$ holds. Then, $d \in C^h(A^H(t))_D$ implies $z \notin R(t)$ from the definition of the cumulative offer process. By $z \in C^d(A^D(t))$, $z \in A^D(t)$ holds. Then, $z \notin R(t)$ implies $z \in A^D(t+1)$. By substitutability of C^d and $A^D(t+1) \subseteq A^D(t)$, we have that $z \in C^d(A^D(t+1))$. By $x \in A^H(t+1) \setminus A^H(t)$, we also have $x \in C^d(A^D(t+1))$. Since C^d is unitary, we have $x = z$. Note that $z(= x) \in C^d(A^D(t))$ implies $z(= x) \in A^H(t)$. However, this contradicts $x \notin A^H(t)$. Hence, $d \notin C^h(A^H(t))_D$. □

Claim 4 *Let $t = 0, \cdots, \hat{t}$. Suppose that $P_1(t)$ and $P_{1'}(t)$ are satisfied. Then, $P_{1'}(t+1)$ and $P_2(t)$ are satisfied.*

Proof of Claim 4. Fix any $t = 0, \cdots, \hat{t}$. We assume that $P_1(t)$ and $P_{1'}(t)$ are satisfied throughout this proof. We first show that $P_{1'}(t+1)$ is satisfied. Note that $P_{1'}(t+1)$ follows from the following condition named $P_{1''}(t+1)$ together with $P_{1'}(t)$:

$P_{1''}(t+1)$**:** Let $h \in H$ with $A^H(t)_h \neq \emptyset$ and $A^H(t+1)_h \setminus A^H(t)_h \neq \emptyset$. Suppose that $(a^1, ..., a^M)$ is an observable offer process for h such that $A^H(t)_h = \{a^1, ..., a^M\}$. Then, for any sequence $(\tilde{a}^1, ..., \tilde{a}^{M'})$ such that $\{\tilde{a}^1, ..., \tilde{a}^{M'}\} \subseteq A_h(t+1) \setminus A_h(t)$, $(a^1, ..., a^M, \tilde{a}^1, ..., \tilde{a}^{M'})$ is observable for h.

We will show that $P_{1''}(t+1)$ is satisfied under $P_1(t)$ and $P_{1'}(t)$. Let $h \in H$ with $A^H(t)_h \neq \emptyset$ and $A^H(t+1)_h \setminus A^H(t)_h \neq \emptyset$. By $P_{1'}(t)$, there exists an observable offer process $(a^1, ..., a^M)$ for h such that $A^H(t)_h = \{a^1, ..., a^M\}$. Fix an arbitrary sequence $(\tilde{a}^1, ..., \tilde{a}^{M'})$ such that $\{\tilde{a}^1, ..., \tilde{a}^{M'}\} \subseteq A_h(t+1) \setminus A_h(t)$. Note that $a_D \neq a'_D$ for any distinct $a, a' \in \{\tilde{a}^1, ..., \tilde{a}^{M'}\}$ by unitarity of every doctor's choice function. Clearly, $(a^1, ..., a^M, \tilde{a}^1, ..., \tilde{a}^{M'})$ is an offer process for h.

We show that $(a^1, ..., a^M, \tilde{a}^1, ..., \tilde{a}^{M'})$ is observable for h. To this end, we show that for all $k = 1, ..., M'$, we have that (i) $\tilde{a}^k_D \notin C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^{k-1}\})_D$ and (ii) $\tilde{a}_D \notin C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^{k-1}, \tilde{a}^k\})_D$ for all $\tilde{a} \in \{\tilde{a}^1, ..., \tilde{a}^{M'}\} \setminus \{\tilde{a}^1, ..., \tilde{a}^{k-1}, \tilde{a}^k\}$, where $\{\tilde{a}^1, ..., \tilde{a}^0\} \equiv \emptyset$. We show this statement by a mathematical induction.

We first show that (i) and (ii) hold for $k = 1$. By $\tilde{a}^1 \in A^H(t+1)_h \setminus A^H(t)_h$ and Claim 3, we have $\tilde{a}^1_D \notin C^h(A^H(t)_h)_D = C^h(\{a^1, ..., a^M\})_D$ and hence (i) holds. Note that this implies $\{a^1, ..., a^M, \tilde{a}^1\}$ is observable for h. To show (ii), take any $\tilde{a} \in \{\tilde{a}^1, ..., \tilde{a}^{M'}\} \setminus \{\tilde{a}^1\}$. Suppose that $\tilde{a}_D \in C^h(\{a^1, ..., a^M, \tilde{a}^1\})_D$. Then, there exists $x \in C^h(\{a^1, ..., a^M, \tilde{a}^1\})$ such that $x_D = \tilde{a}_D$. By $\tilde{a} \neq \tilde{a}^1$, we have $\tilde{a}_D \neq \tilde{a}^1_D$ and hence $x_D \neq \tilde{a}^1_D$. This implies that $x \in \{a^1, ..., a^M\}$. When $x \notin C^h(\{a^1, ..., a^M\})$, by observable substitutability across doctors of C^h, we must have that $x_D = \tilde{a}_D \in C^h(\{a^1, ..., a^M\})_D = C^h(A^H(t)_h)_D$. When $x \in C^h(\{a^1, ..., a^M\})$, we have $x_D = \tilde{a}_D \in C^h(\{a^1, ..., a^M\})_D = C^h(A^H(t)_h)_D$. Therefore, we have $x_D = \tilde{a}_D \in C^h(A^H(t)_h)_D$ in every case. Note that $\tilde{a} \in A^H(t+1)_h \setminus A^H(t)_h$. However, this contradicts Claim 3. Hence, (i) and (ii) hold for $k = 1$.

Fix any k with $1 \leq k \leq M - 1$. We assume that for all $k' \leq k$, (i) $\tilde{a}^{k'}_D \notin C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}_1, ..., \tilde{a}^{k'-1}\})_D$ and (ii) $\tilde{a}_D \notin C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}_1, ..., \tilde{a}^{k'-1}, \tilde{a}^{k'}\})_D$ for all $\tilde{a} \in \{\tilde{a}^1, ..., \tilde{a}^{M'}\} \setminus \{\tilde{a}^1, ..., \tilde{a}^{k'-1}, \tilde{a}^{k'}\}$ where $\{\tilde{a}^1, ..., \tilde{a}^0\} \equiv \emptyset$ (Induction hypothesis). We show that (i) and (ii) hold for $k+1$. By the induction hypothesis for (ii), we have that $\tilde{a}^{k+1}_D \notin C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k\})$ and hence (i) holds for $k+1$. Note that this together with the induction hypothesis for (i) implies $(a^1, ..., a^M, \tilde{a}^1, ..., \tilde{a}^k, \tilde{a}^{k+1})$ is an observable offer process for h. To show (ii) for $k+1$, take any $\tilde{a}' \in \{\tilde{a}^1, ..., \tilde{a}^{M'}\} \setminus \{\tilde{a}_1, ..., \tilde{a}^k, \tilde{a}^{k+1}\}$. Suppose that $\tilde{a}'_D \in C^h(\{\tilde{a}^1, ..., \tilde{a}^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k, \tilde{a}^{k+1}\})_D$. Then, there exists $x' \in C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k, \tilde{a}^{k+1}\})$ such that $x'_D = \tilde{a}'_D$. By $\tilde{a}' \neq \tilde{a}^{k+1}$, we have $\tilde{a}'_D \neq \tilde{a}^{k+1}_D$ and hence $x'_D \neq \tilde{a}^{k+1}_D$. This implies $x' \in \{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k\}$. When $x' \notin C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k\})$, by observable substitutability across doctors of C^h, we must have $x'_D = \tilde{a}'_D \in C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k\})_D$. When $x' \in C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k\})$, we have $x'_D = \tilde{a}'_D \in C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k\})_D$. Therefore, we have $\tilde{a}'_D \in C^h(\{a^1, ..., a^M\} \cup \{\tilde{a}^1, ..., \tilde{a}^k\})_D$ in every case. Note that $\tilde{a}' \in \{\tilde{a}^1, ..., \tilde{a}^{M'}\} \setminus \{\tilde{a}^1, ..., \tilde{a}^k\}$ holds. However, this contradicts induction hypothesis for (ii). Therefore, (i) and (ii) hold for any $k = 1, ..., M$. In particular, (i) implies that

$(a^1, ..., a^M, \tilde{a}^1, ..., \tilde{a}^{M'})$ is an observable offer process for h. Hence $P_{1''}(t+1)$ is satisfied.

We next show that $P_2(t)$ is satisfied under $P_1(t)$ and $P_{1'}(t)$. Fix any $y \in X$ such that $y \in A^H(t)$ and $y_D \notin C^{y_H}(A^H(t))_D$. Let $y_H = \hat{h}$. We will show that $y \notin C^{\hat{h}}(A^H(t+1))$. To this end, it is sufficient to show that $y \notin C^{\hat{h}}(A^H(t+1)_{\hat{h}})$ by $C^{\hat{h}}(A^H(t+1)) = C^{\hat{h}}(A^H(t+1)_{\hat{h}})$. Let $\tilde{A} = A^H(t+1)_{\hat{h}} \setminus A^H(t)_{\hat{h}}$. Note that $a_D \neq a'_D$ for any distinct $a, a' \in \tilde{A}$ by unitarity of every doctor's choice function.

Let $\tilde{A}_{-y_D} = \{a \in \tilde{A} | a_D \neq y_D\}$. We show that for any $B \subseteq \tilde{A}_{-y_D}$, $y_D \notin C^{\hat{h}}(A^H(t)_{\hat{h}} \cup B)_D$ by a mathematical induction on $|B|$. When $|B| = 0$, this statement holds by $y_D \notin C^{\hat{h}}(A^H(t))_D$. Suppose that for any $B \subseteq \tilde{A}_{-y_D}$ with $|B| \leq k (\geq 0)$, $y_D \notin C^{\hat{h}}(A^H(t)_{\hat{h}} \cup B)_D$. We show this statement for any $B \subseteq \tilde{A}_{-y_D}$ with $|B| = k+1$. Consider any $\bar{B} \subseteq \tilde{A}_{-y_D}$ with $|\bar{B}| = k+1$. Note that $A^H(t)_{\hat{h}} \neq \emptyset$ by $y \in A^H(t)_{\hat{h}}$. By $P_{1'}(t)$, there exists an observable offer process $(b^1, ..., b^M)$ for $\hat{h}$ such that $A^H(t)_{\hat{h}} = \{b^1, ..., b^M\}$. We denote $\bar{B}$ by $\{\hat{b}^1, ..., \hat{b}^{k+1}\}$. Note also that $P_{1''}(t+1)$ is satisfied by the assumption that $P_1(t)$ and $P_{1'}(t)$ are satisfied. Therefore, $\{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^{k+1}\}$ is an observable offer process for $\hat{h}$. We now show that $y_D \notin C^{\hat{h}}(\{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^{k+1}\})_D = C^{\hat{h}}(A^H(t)_{\hat{h}} \cup \bar{B})_D$. Suppose that $y_D \in C^{\hat{h}}(\{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^{k+1}\})_D = C^{\hat{h}}(A^H(t)_{\hat{h}} \cup \bar{B})_D$. Then, there exists $b' \in C^{\hat{h}}(\{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^{k+1}\})_D$ such that $b'_D = y_D$. By $\hat{b}^{k+1} \in \bar{B} \subseteq \tilde{A}_{-y_D}$, we have that $b'_D = y_D \neq \hat{b}^{k+1}_D$. This implies $b' \in \{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^k\}$. When $b' \notin C^{\hat{h}}(\{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^k\})$, by observable substitutability across doctors of C^h, we must have that $b'_D = y_D \in C^{\hat{h}}(\{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^k\})_D = C^{\hat{h}}(A^H(t)_{\hat{h}} \cup (\bar{B} \setminus \{\hat{b}^{k+1}\}))_D$. When $b' \in C^{\hat{h}}(\{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^k\})$, we have $b'_D = y_D \in C^{\hat{h}}(\{b^1, ..., b^M, \hat{b}^1, ..., \hat{b}^k\})_D = C^{\hat{h}}(A^H(t)_{\hat{h}} \cup (\bar{B} \setminus \{\hat{b}^{k+1}\}))_D$. Therefore, we have $b'_D = y_D \in C^{\hat{h}}(A^H(t)_{\hat{h}} \cup (\bar{B} \setminus \{\hat{b}^{k+1}\}))_D$ in every case. However, this contradicts the induction hypothesis. Therefore, we have that for any $B \subseteq \tilde{A}_{-y_D}$ with $|B| = k+1$, $y_D \notin C^{\hat{h}}(A^H(t)_{\hat{h}} \cup B)_D$. Hence, this statement holds for any cardinality of $B \subseteq \tilde{A}_{-y_D}$.

Note that $y_D \notin C^{\hat{h}}(A^H(t)_{\hat{h}} \cup \tilde{A}_{-y_D})_D$ holds by setting $B = \tilde{A}_{-y_D}$ in the above property. Note also that $A^H(t+1)_{\hat{h}} = A^H(t)_{\hat{h}} \cup \tilde{A} = A^H(t)_{\hat{h}} \cup \tilde{A}_{-y_D} \cup \tilde{A}_{y_D}$. When $\tilde{A}_{y_D} = \emptyset$, $C^{\hat{h}}(A^H(t+1)_{\hat{h}}) = C^{\hat{h}}(A^H(t)_{\hat{h}} \cup \tilde{A}_{-y_D})$ and hence $y \notin C^{\hat{h}}(A^H(t+1)_{\hat{h}})$, the desired property. Therefore, suppose that $\tilde{A}_{y_D} \neq \emptyset$. This implies $|\tilde{A}_{y_D}| = 1$ by unitarity of C^{y_D}. Let $\tilde{A}_{y_D} = \{y'\}$. Then, $A^H(t+1)_{\hat{h}} = A^H(t)_{\hat{h}} \cup \tilde{A}_{-y_D} \cup \{y'\}$ holds. Note that $y' \neq y$ by $y' \in A^H(t+1) \setminus A^H(t)$ and $y \in A^H(t)$. When $y' \in C^{\hat{h}}(A(t+1)_{\hat{h}})$, we have $y \notin C^{\hat{h}}(A(t+1)_{\hat{h}})$ by unitarity of $C^{\hat{h}}$. When $y' \notin C^{\hat{h}}(A(t+1)_{\hat{h}})$, we have

$C^{\hat{h}}(A(t+1)_{\hat{h}}) = C^{\hat{h}}(A(t+1)_{\hat{h}} \setminus \{y'\}) = C^{\hat{h}}(A^H(t)_{\hat{h}} \cup \tilde{A}_{-y_D})$ by consistency of $C^{\hat{h}}$. Hence, $y \notin C^{\hat{h}}(A(t+1)_{\hat{h}})$ by $y_D \notin C^{\hat{h}}(A^H(t)_{\hat{h}} \cup \tilde{A}_{-y_D})_D$. Therefore, for every case, we have $y \notin C^{\hat{h}}(A(t+1)_{\hat{h}})$. Hence, $P_2(t)$ is satisfied. This completes the proof. □

Claim 5 *For each $t = 0, \cdots, \hat{t}$, if $P_1(t)$ and $P_2(t)$ are satisfied, then $P_1(t+1)$ is satisfied.*

Proof of Claim 5. Fix any $t = 0, \cdots, \hat{t}$. Suppose that $P_1(t)$ and $P_2(t)$ are satisfied. To obtain $P_1(t+1)$, fix any $d \in D$ and $h \in H$ such that $d \in C^h(A^H(t+1))_D$. We show that $h \in C^d(A^D(t+1))_H$. By $d \in C^h(A^H(t+1))_D$, there exists $x \in C^h(A^H(t+1))$ with $x_D = d$. By $x \in A^H(t+1)$, there exists $t' \leq t+1$ with $x \in C^d(A^D(t'))$ from the definition of the cumulative offer process.

We first suppose that $x \in A^D(t+1)$. By the definition of the cumulative offer process, $t' \leq t+1$ implies $A^D(t+1) \subseteq A^D(t')$. By substitutability of C^d, $x \in C^d(A^D(t'))$ implies $x \in C^d(A^D(t+1))$. Therefore, we have that $h \in C^d(A^D(t+1))_H$.

We next suppose that $x \notin A^D(t+1)$. This implies that $x \in R(s)$ for some $s \leq t$. Therefore, $x \in A^H(s)$ holds. This implies $x \in A^H(t)$ by $A^H(s) \subseteq A^H(t)$. Note that $x \in C^h(A^H(t+1))$ holds by the choice of x. Because $P_2(t)$ is satisfied, we have $x_D = d \in C^h(A^H(t))_D$. By $P_1(t)$, $h \in C^d(A^D(t))_H$ and hence there exists $z \in C^d(A^D(t))$ with $z_H = h$. By $z \in C^d(A^D(t))$, we have $z \in A^H(t)$. By $z_D = d \in C^h(A^H(t))_D$, we have $z \notin R(t)$ from the definition of the cumulative offer process. By $z \in C^d(A^D(t))$, $z \in A^D(t)$ holds. Then, $z \notin R(t)$ implies $z \in A^D(t+1)$. By substitutability of C^d and $A^D(t+1) \subseteq A^D(t)$, we have $z \in C^d(A^D(t+1))$. This implies $h \in C^d(A^D(t+1))_H$. Therefore, $P_1(t+1)$ is satisfied. □

Consider any $t = 0, \cdots, \hat{t}$. Suppose that $P_1(t)$ and $P_{1'}(t)$ are satisfied. Then $P_{1'}(t+1)$ and $P_2(t)$ are satisfied by Claim 4. Because $P_1(t)$ and $P_2(t)$ are satisfied, $P_1(t+1)$ is satisfied by Claim 5. Therefore, this lemma holds if $P_1(0)$ and $P_{1'}(0)$ are satisfied. We have already shown that $P_1(0)$ and $P_{1'}(0)$ are satisfied. This completes the proof. ■

The following lemma together with the finiteness of X proves Proposition 1.

Lemma 2 *Suppose that (i) every agent's choice function is unitary, (ii) every hospital's choice function is observably substitutable across doctors, (iii) every doctor's* *choice function is substitutable. Suppose that the cumulative offer process proceeds until step $\hat{t}$ and $[C^H(A^H(\hat{t}))_{d'}]_H \neq [C^{d'}(A^D(\hat{t}))]_H$ for some $d' \in D$. Then, we have that $A^D(\hat{t}+1) \subsetneq A^D(\hat{t})$.*

Proof. We first show that $[C^H(A^H(\hat{t}))_d]_H \subseteq [C^d(A^D(\hat{t}))]_H$ for all $d \in D$. Fix any $d \in D$. Take any $h \in [C^H(A^H(\hat{t}))_d]_H$. Then, there exists $x \in C^H(A^H(\hat{t}))_d$ such that $x_H = h$. Therefore, $x \in C^h(A^H(\hat{t}))$ and $x_D = d$ hold. This implies $d \in C^h(A^H(\hat{t}))_D$. Since we are assuming that $[C^H(A^H(\hat{t}))_{d'}]_H \neq [C^{d'}(A^D(\hat{t}))]_H$ for some $d' \in D$, $P_1(\hat{t})$ is satisfied from Lemma 1. Therefore, we have $h \in C^d(A^D(\hat{t}))_H$. Hence, $[C^H(A^H(\hat{t}))_d]_H \subseteq [C^d(A^D(\hat{t}))]_H$.

Let $d' \in D$ be a doctor such that $[C^H(A^H(\hat{t}))_{d'}]_H \neq [C^{d'}(A^D(\hat{t}))]_H$. Then, $[C^H(A^H(\hat{t}))_{d'}]_H \subsetneq [C^{d'}(A^D(\hat{t}))]_H$. Take any $h' \in [C^{d'}(A^D(\hat{t}))]_H$ with $h' \notin [C^H(A^H(\hat{t}))_{d'}]_H$. By $h' \in [C^{d'}(A^D(\hat{t}))]_H$, there exists $x' \in C^{d'}(A^D(\hat{t}))$ with $x'_H = h'$. Note that $x'_D = d'$. By $x' \in C^{d'}(A^D(\hat{t}))$, $x' \in A^H(\hat{t})$ holds. By $h' \notin [C^H(A^H(\hat{t}))_{d'}]_H$, we have $d' \notin C^{h'}(A^H(\hat{t}))_D$. This implies $x' \in R(\hat{t})$ from the definition of the cumulative offer process. Thus, $x' \notin A^D(\hat{t}+1)$. On the other hand, $x' \in C^{d'}(A^D(\hat{t}))$ implies $x' \in A^D(\hat{t})$. Hence $A^D(\hat{t}+1) \subsetneq A^D(\hat{t})$. ■

Proof of Claim 1. Assume that (a) $[C^H(A^H(t^*))_{d'}]_H \neq [C^{d'}(A^D(t^*))]_H$ for some $d' \in D$ and $A^D(t^*+1) = \emptyset$. We begin with showing $C^H(A^H(t^*+1)) = \emptyset$. Suppose that $C^H(A^H(t^*+1)) \neq \emptyset$. Let $x \in C^H(A^H(t^*+1))$. Then, we have $x_D \in C^{x_H}(A^H(t^*+1))_D$. Note that $P_1(t^*+1)$ is satisfied from Lemma 1 by (a). Therefore, we have $x_H \in C^{x_D}(A^D(t^*+1))_H$, contracting $A^D(t^*+1) = \emptyset$. Hence $C^H(A^H(t^*+1)) = \emptyset$.

We now show that $\emptyset$ is stable. Suppose that $\emptyset$ is not stable. Then, there exists a blocking set Z to $\emptyset$. We claim that $Z \subseteq A^H(t^*+1)$. Pick any $z \in Z$. By $A^D(t^*+1) = \emptyset$, there exists $t' \leq t^*$ such that $z \in R(t')$. By $R(t') \subseteq A^H(t')$ and $A^H(t') \subseteq A^H(t^*+1)$, we have $z \in A^H(t^*+1)$. Hence, $Z \subseteq A^H(t^*+1)$. Fix any $\hat{h} \in Z_H$. By $C^H(A^H(t^*+1)) = \emptyset$, $C^{\hat{h}}(A^H(t^*+1)) = \emptyset$. Note that $C^{\hat{h}}(A^H(t^*+1)) = \emptyset \subseteq Z \subseteq A^H(t^*+1)$ holds. By consistency of $C^{\hat{h}}$, we have $C^{\hat{h}}(Z) = C^{\hat{h}}(A^H(t^*+1)) = \emptyset$. This contradicts that Z is a blocking set to $\emptyset$. Therefore, $\emptyset$ is stable. □

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