On the possibility of deflationary equilibria with monetary expansion: A reconciliation between the fiscal theory of the price level and the quantity theory of money

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Abstract: In this paper, a deflationary economy with monetary expansion and growing public debts under near-zero interest rates is characterized as a continuum of equilibria. Given an unprecedented possibility of a large, one-off rise in the price level at switching from the non-*Ricardian* regime with the fiscal theory of the price level to the *Ricardian* regime with the quantity theory of money, mild deflations continue to balance such a big price jump at switching, and the nominal public bonds are gradually appreciated in a real term before they are heavily devaluated at switching. Thanks to the continuation of *ex post* deflations, which are always exceeding expected deflations in the non-*Ricardian* regime, the real valuation of the public bonds is supported beyond future fiscal surpluses by the stochastic bubbles in the government's intertemporal budget constraint, but the bubbles burst because of a heavy devaluation caused by a price jump at switching. As is implied by a calibration exercise that mimics the current Japanese economy, the stochastic bubbles amount to around 40% of the real valuation of the public bonds, and the price level would jump by more than 200% immediately after the economy switched back to the *Ricardian* regime.

Key words: the fiscal theory of the price level, the quantity theory of money, stochastic bubbles, non-Ricardian fiscal policy, deflation, zero interest rates, the peso problem.
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1. Introduction

The coexistence of a mildly deflationary economy despite monetary expansion and near-zero rates of interest despite growing public debts—as experienced by the Japanese economy in the past two decades—is quite difficult to reconcile with the implications from standard monetary models. For example, the above combination of zero interest rates and mild deflations is quite different from the combination that emerges under Friedman's rule (Friedman 1969). In the latter economy, called *Ricardian* by Woodford (1995), where both *Ricardian* equivalence (Barro 1974) and the quantity theory of money (QTM) (Friedman 1956) hold tightly, the price level declines with monetary contraction, and the cost of retiring money stock is financed by tax revenues. Accordingly, neither the money stock nor the public bonds grow in nominal terms. As Buiter and Sibert (2007) prove, a deflationary economy with monetary expansion is indeed ruled out in standard monetary models.[3]

On the other hand, the price theory alternative to the QTM, the fiscal theory of the price level (FTPL) with a non-*Ricardian* fiscal rule, which is proposed by Woodford (1994) and others,[4] may not provide convincing explanations for the above deflationary phenomenon either. In Japan, the price level has been mildly deflationary since the primary budget balance started to decline in the early 1990s. Given the standard implications from the FTPL, however, deteriorating fiscal surpluses should yield not deflationary, but inflationary pressures. In addition, the FTPL usually works as an equilibrium selection device to restore uniqueness in a continuum of equilibria with speculative hyperinflations,[5] not in a deflationary economy.[6]

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[3] Benhabib et al. (2001) show that the adoption of Taylor's interest rate feedback rules may result in a steady state with near-zero interest rates and mild deflations in the presence of liquidity traps, but their steady state case with a constant real money balance involves a deflationary state with monetary contraction rather than monetary expansion.

[4] Others include Leeper (1991), Sims (1994), Cochrane (2001), and Bassetto (2002).

[5] Brock (1975), Obstfeld and Rogoff (1983, 1986, 2017), and others point out that speculative hyperinflations may not be ruled out in standard monetary models. Obstfeld and Rogoff propose a partial backing to the currency as a way to rule out speculative hyperinflations.

[6] McCallum (2001) opposes the FTPL partly because depending on the sequence of fiscal surpluses, the FTPL happens to pick up a particular initial price from the deflationary economy, which is not supported as a legitimate equilibrium. On the other hand, Buiter

In this paper, the non-*Ricardian* regime with the FTPL does not continue forever, but rather probabilistically switches back into the *Ricardian* regime with the QTM, which is regarded as an absorbing state. That is, the non-*Ricardian* regime is considered a temporary, or at most a persistent, deviation from the *Ricardian* regime. Given such an economic environment, the role that is played potentially by the FTPL is completely reversed. While the FTPL usually helps to restore uniqueness in a continuum of hyperinflationary equilibria, this setup with a switching possibility from the FTPL to the QTM instead helps to create a continuum of deflationary equilibria, which is accompanied by the stochastic bubbles [7] that work to relax the government's intertemporal budget constraint (GIBC).

As a consequence of the above setup, the economy experiences a large, one-off increase in the price level at the point of switching from the deflationary non-*Ricardian* economy to the *Ricardian* economy with the QTM, causing the above bubbles to burst because of a heavy devaluation triggered by such a price jump, and the remaining public bonds are repaid over time by tax revenues in a *Ricardian* manner. Thus, a government is able to operate a Ponzi scheme *only* in the presence of the stochastic bubbles during the deflationary non-*Ricardian* regime.

A major trick in this model is that in the deflationary non-*Ricardian* environment, *ex post* deflations (inflations) are always higher (lower) than expected deflations (inflations) when it is taken into account that there is a possibility of a large, one-off price rise at the point of switching. Accordingly, the *ex post* or actual nominal return, which is eventually negative, is always lower than the nominal rate of interest, which is determined by expected deflations (inflations) and at least zero, in the deflationary environment. Thus, the expected present value of the future public bonds converges to a positive constant in the GIBC as a result of discounting by lower actual nominal returns

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(2002) argues that the FTPL rests on a fundamental confusion between equilibrium conditions and budget constraints.

[7] As proposed by Blanchard and Watson (1982), Weil (1987), and others, stochastic bubbles are considered to be valuations above fundamentals, which correspond to the present value of fiscal surpluses in public bond pricing, but they burst with some probability. See LeRoy (2004a) and Martin and Ventura (2018) for a survey of rational bubbles.

in the GIBC, but it is discounted completely to zero by higher nominal rates of interest in the household's intertemporal budget constraint (HIBC). Given this contrast between the two budget constraints in the deflationary non-*Ricardian* regime, the terminal condition associated with the public bonds is satisfied in the HIBC, but the stochastic bubbles emerge in the GIBC, which in turn tentatively improve the fiscal surpluses despite the continuing primary deficits, and, indeed, create a deflationary pressure according to a conventional mechanism of the FTPL. In this case, the initial price level determined in the non-*Ricardian* regime becomes low relative to the level determined by the QTM to the extent that the stochastic bubbles are large in the GIBC.

Viewing the above feature from a different perspective, expected inflations register the information of an unprecedented possibility of large price jump at switching, which never appears in actual (*ex post*) inflations in the deflationary non-*Ricardian* regime. Consequently, the peso problem arises in the sense that the nominal rate of interest, which is equal to expected inflations plus the real rate of interest, is always upward biased relative to the realized (*ex post*) nominal return as long as the deflationary environment continues.

One of the most important policy implications from this model is that the real valuation of public bonds depends on how large the stochastic bubbles are in the GIBC, but it is completely independent of the present value of fiscal surpluses generated during the non-*Ricardian* regime. In this setup, the non-*Ricardian* economy is anchored eventually by the *Ricardian* economy, and the public bonds are ultimately financed by a heavy devaluation at the point of switching to the *Ricardian* regime, and by tax revenues after switching. Thus, *Ricardian* equivalence still holds in this FTPL setup. Sims (2016) and others claim that weaker fiscal discipline helps to create more inflationary pressures in the non-*Ricardian* regime.[8] In this environment, however, the price level is completely independent of how a non-*Ricardian* fiscal rule is implemented, and their claim is not relevant here.

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[8] As a historical perspective, Sims (2011) documents empirically a relationship between fiscal uncertainties and the US inflation of the 1970s, and Cochrane (1999) shows how the FTPL is consistent with a negative correlation between deficits and inflations in the US economy of the 1980s.

This paper is closely related to existing papers in monetary economics. As pointed out by LeRoy (2004) and Bloise and Reichlin (2008), among others, the GIBC is relaxed to the extent that rational bubbles are present in financial instruments, and accordingly the price level is still indeterminate even under the FTPL. In this paper, the government's constraint is relaxed directly by the emergence of the stochastic bubbles. It is seigniorage revenues in fiscal dominance in Sargent and Wallace (1981), and stochastic bubbles in the non-*Ricardian* regime in this model, that improve the fiscal surpluses in the GIBC, thereby creating deflationary pressures. While the condition under which the actual nominal return is always lower than the nominal rate of interest is responsible for the emergence of the stochastic and rational bubbles in this FTPL setup, Bassetto and Cui (2018) demonstrate that lower real returns, possibly driven by either dynamic inefficiency or the liquidity premium on government debts, may have negative implications for the FTPL. In their FTPL environment, the present value of fiscal surpluses is not well defined, primary deficits rather than surpluses are required, and the price level is still indeterminate with only its lower bound. Braun and Nakajima (2012) present a case where pessimistic views of a future debt crisis are not reflected in public bond pricing because of the presence of short sale constraints in the context of the FTPL.[9] While several papers, including Davig et al. (2010), and Bianchi and Ilut (2017), investigate possible macroeconomic impacts of switching among active/passive monetary/fiscal policies, this paper takes into consideration a switch of not only fiscal rules from non-Ricardian to Ricardian, but also pricing rules from the FTPL to the QTM.

One of the models closest to this paper is Davig et al. (2011), which share a similar structure of regime changes with this paper. Starting from active monetary policy and a stationary (passive) transfers process, their economy hits the fiscal limit as a consequence of a non-stationary (active) transfers process, and eventually enters the

[9] Several papers, not related to the FTPL, also provide potential reasons why public bonds are priced high despite a possible debt crisis. Sakuragawa and Sakuragawa (2016) demonstrate that public bonds are priced high, when public bonds serve as safe assets for those who strongly prefer domestic assets to foreign assets. Kobayashi and Ueda (2017) show that public bond yields are kept low despite a possible debt crisis, when a capital levy is imposed more mildly on public bonds than on private bonds at the time of a debt crisis.

absorbing state of active monetary/passive transfers policy either directly, or indirectly by way of passive monetary/active transfers policy. In the passive monetary/active transfers regime, their economy experiences sharp inflations, though with a small probability, which are registered in expected inflations. Thus, expected inflations are subject to the peso problem in the sense that they include the information concerning unprecedented inflationary states.

As mentioned above, this paper is motivated partly by several empirical facts concerning the Japanese economy. The deflationary economy accompanied by monetary expansion and zero rates of interest is a recent monetary phenomenon in Japan. As shown in Figure 1-1, the *Marshallian k*, which is defined as the ratio of outstanding Bank of Japan (BoJ) notes to nominal gross domestic product (GDP), was quite stable up to the early 1990s. That is, the price level was approximately proportional to the nominal macroeconomic scale, and it was broadly determined according to the QTM. However, when the call rates (the interbank money market rates) declined from just under 8% in 1990 to around 0.5% in 1995, the *Marshallian k* started to increase gradually, and the increase has accelerated since the mid-1990s. The nominal rate of interest has been fairly close to zero since 1995. In addition, the public bonds accumulated more quickly after the primary balance of the government's general account started to decline in the early 1990s, and became negative in the mid-1990s. As shown in Figure 1-2, the ratio of public bonds to nominal GDP has increased together with that of BoJ notes. As shown in Figure 1-3, on the other hand, a deflationary trend started from the early 1990s.

Another potentially important observation is that the consumers' expectations of future inflations tended to overestimate future inflations. According to Figure 1-4, the consumers' forecast of one-year ahead inflations has been always upward biased in the Consumer Confidence Survey (CCS) and the Opinion Survey (OS) since the mid-2000s, while the professional analysts' forecast tracked well actual inflations in the ESP Forecast. In the years 2010-2017, the consumers averagely overestimated one-year ahead inflations by 1.8% in the CCS, and by 3.5% in the OS. Such overestimation of future inflations is often interpreted as the consumers' inability to process information precisely, but it is here taken as their ability to consider the peso problem correctly.

In this paper, it is assumed that in the early 1990s, the Japanese fiscal policy switched from *Ricardian* to *non-Ricardian*, and the initial price level (for example, of year 1990) deviated slightly and downward from the level implied by the QTM. Consequently, a continuum of deflationary equilibria with the stochastic bubbles in the GIBC would have emerged since the early 1990s. If this were the case, then at a possible switching point in the future, the bubbles would burst because of a heavy devaluation caused by a large, one-off price increase. As implied by a calibration exercise that mimics the above-described Japanese economy, the stochastic bubbles amount to around 40% of the real valuation of the public bonds, and the price level would jump by more than 200% immediately after the economy switched to the *Ricardian* regime in the 2020s.

This paper is organized as follows. In Section 2, a monetary version of an exchange economy is employed to analyze a *Ricardian* fiscal rule with the QTM and a non-*Ricardian* fiscal rule with the FTPL separately. Section 3 presents a simple model in which the non-*Ricardian* regime with the FTPL probabilistically switches back to the *Ricardian* regime with the QTM. In Section 4, several numerical examples shed light on some interpretations of the current Japanese economy. Section 5 concludes.

2. Basic framework

2.1. *Ricardian* economy

In this section, a simple monetary model of exchange economy, proposed by Brock (1975), Obstfeld and Rogoff (1983), and Kocherlakota and Phelan (1999), is presented as a basic framework. The representative household has the following preference over streams of consumption (c_t) and the real money balance ($\frac{M_t}{P_t}$):

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + v\left(\frac{M_t}{P_t}\right)\right], \tag{1}$$

where $0 < \beta < 1$, and u and v are twice differentiable, strictly increasing, and strictly concave.

The maximization of the objective function characterized by equation (1) is subject to

$$B_{t+1} = P_{t+1}\left(y - \tau_{t+1}\right) - P_{t+1}c_{t+1} - \left(M_{t+1} - M_t\right) + R_t B_t, \tag{2}$$

where y is a constant endowment stream of consumption goods, c_t is the real amount of consumption goods, τ_t is a real lump-sum tax, P_t is the price of consumption, M_t is the nominal money balance, B_t is the nominal amount of public bonds, and R_t is the nominal gross rate of interest. From the assumption of an exchange economy, consumption is invariant at endowment y net of constant government expenditure g.

$$c_t = y - g \tag{3}$$

Suppose that M_t grows at the rate of μ. Then, equation (2) is solved in a recursive manner as follows:

$$\lim_{T\to\infty}\frac{B_T + M_T}{\prod_{s=-1}^{T-1} R_s} + \sum_{\tau=-1}^{\infty}\frac{P_{\tau+1}\left(y-g\right)}{\prod_{s=-1}^{\tau} R_s} = \left(R_{-1}B_{-1} + M_{-1}\right) + \sum_{\tau=-1}^{\infty}\left[\frac{P_{\tau+1}\left(y-\tau_{\tau+1}\right)}{\prod_{s=-1}^{\tau} R_s} - \frac{1}{\prod_{s=-1}^{\tau-1} R_s}\left(1-\frac{1}{R_\tau}\right)M_\tau\right], \tag{4}$$

where $R_{-1} = 1$, $B_0 = R_{-1}B_{-1}$, $M_0 = M_{-1}$, and $\prod_{s=-1}^{-2} R_s = 1$.[10]

Thus, the limiting condition dictates that

$$\lim_{T\to\infty}\frac{B_T + M_T}{\prod_{s=-1}^{T-1} R_s} = 0. \tag{5}$$

As long as equation (5) holds, neither the money stock nor the public bonds serve as net wealth.

Focusing on time *t* and time *t*+1 consumption, the above maximization problem is reformulated as follows:

$$\max_{c_t, c_{t+1}, M_t, \eta_{t+1}} \left[\begin{array}{l} \beta\left[u\left(c_{t+1}\right) + v\left(\dfrac{M_{t+1}}{P_{t+1}}\right)\right] + \left[u\left(c_t\right) + v\left(\dfrac{M_t}{P_t}\right)\right] \\ +\eta_{t+1}\left\{\begin{array}{l}\left[P_{t+1}\left(y-\tau_{t+1}\right) - P_{t+1}c_{t+1} - \left(M_{t+1} - M_t\right)\right] \\ +R_t\left[P_t\left(y-\tau_t\right) - P_t c_t - \left(M_t - M_{t-1}\right) + R_{t-1}B_{t-1}\right] - B_{t+1}\end{array}\right\}\end{array}\right] \tag{6}$$

where η_{t+1} is a Lagrange multiplier.

Given equation (3), the two first-order conditions are obtained.

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[10] There is no trade at time –1. As Niepelt (2004) shows, if there is an intertemporal trade between time –1 and time 0, then only a *Ricardian* fiscal policy is admissible.

$$\beta R_t \frac{P_t}{P_{t+1}} = 1 \tag{7}$$

$$v'\left(\frac{M_t}{P_t}\right) = u'(y-g)\left(1-\frac{1}{R_t}\right) \tag{8}$$

Here, the following functional forms are assumed:

$$u(c) = \frac{1}{1-\frac{1}{\sigma}} c^{1-\frac{1}{\sigma}} \text{ if } 0<\sigma \text{ and } \sigma \neq 1, \text{ and } u(c) = \ln(c) \text{ if } \sigma = 1, \tag{9}$$

for $u(c)$, and

$$v\left(\frac{M}{P}\right) = \frac{\lambda}{1-\frac{1}{\sigma}}\left(\overline{c}+\frac{M}{P}\right)^{1-\frac{1}{\sigma}} \text{ if } 0<\sigma \text{ and } \sigma \neq 1, \text{ and}$$

$$v\left(\frac{M}{P}\right) = \lambda \ln\left(\overline{c}+\frac{M}{P}\right) \text{ if } \sigma = 1, \tag{10}$$

for $v\left(\frac{M}{P}\right)$ where $0<\lambda<1$, and $\overline{c}$ denotes per capita consumption, which is taken as given by the representative consumer.[11]

In equation (10), a part of consumption is assumed to serve as commodity currencies. One important consequence of the presence of $\overline{c}$ in equation (10) is that the nominal (net) rate of interest ($R_t - 1$) is bounded from the above at $\frac{M_t}{P_t} = 0$, and asymptotically approaches zero as $\frac{M_t}{P_t}$ goes to infinity (see Figure 2-1). While σ in equation (9) represents an elasticity of intertemporal substitution, σ in equation (10) determines the interest elasticity of money demand. In an environment with constant endowment y and government expenditure g, the latter interpretation of σ is relevant.

Substituting equations (9) and (10) into equations (7) and (8) leads to:

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[11] This specification of $v\left(\frac{M}{P}\right)$ follows Kocherlakota and Phelan (1999).

$$\frac{P_t}{P_{t+1}} = \frac{1}{\beta R_t} = \frac{1}{\beta}\left[1-\lambda\left(1+\frac{M_t}{P_t(y-g)}\right)^{-\sigma}\right]. \tag{11}$$

Here, $0<\lambda<1$ guarantees a positive $\frac{P_t}{P_{t+1}}$ and a finite P_{t+1}, even if $\frac{M_t}{P_t(y-g)}$ converges to zero.

Let us begin with a *Ricardian* case where the QTM and *Ricardian* equivalence hold jointly. Under the QTM, the price level and the money balance grow at the same rate μ. Substituting $\frac{P_t^R}{P_{t+1}^R} = \frac{1}{1+\mu}$ into equation (11) leads to

$$R_t = \frac{1+\mu}{\beta}, \tag{12}$$

$$P_t^R = \frac{1}{\left[\frac{\lambda(1+\mu)}{1+\mu-\beta}\right]^{\sigma}-1}\frac{M_t}{y-g}, \tag{13}$$

where P_t^R denotes the price of consumption goods in the *Ricardian* economy, and is indeed proportional to the nominal amount of money balances M_t.

Here, κ represents a constant *Marshallian k*, and is defined as $\frac{M_t}{P_t^R(y-g)}$. Given κ, λ is set as follows.

$$\lambda = (1+\kappa)^{\frac{1}{\sigma}}\frac{1+\mu-\beta}{1+\mu} \tag{14}$$

The nominal primary budget balance is assumed to be proportional to the nominal public bonds, net of seigniorage revenues $M_t - M_{t-1}$ or μM_{t-1}:

$$P_t^R(\tau_t - g) = (R_{t-1}-\gamma)B_{t-1} - \mu M_{t-1}, \tag{15}$$

where $0<\gamma<1$. That is, seigniorage revenues are reimbursed as a lump-sum subsidy to households. Hence, the nominal public bonds evolve according to

$$\begin{aligned} B_t &= R_{t-1}B_{t-1} - P_t^R(\tau_t - g) - (M_t - M_{t-1}) \\ &= \gamma B_{t-1} = \gamma^t B_0 \end{aligned} \tag{16}$$

That is, the public bonds are repaid over time by tax revenues. Thus, in the sense of Woodford (1995), a fiscal rule specified by equation (15) is *Ricardian*.

Given the above fiscal rule, equation (5) holds as long as $\frac{\beta\gamma}{1+\mu}<1$.

$$\lim_{T\to\infty}\frac{\gamma^T B_0+(1+\mu)^T M_0}{\left(\frac{1+\mu}{\beta}\right)^T}=\lim_{T\to\infty}\beta^T\left[\left(\frac{\gamma}{1+\mu}\right)^T B_0+M_0\right]=0 \tag{17}$$

Note that Friedman's rule with zero interest rates ($R_t=1$) cannot hold in this setup. Substituting $\mu=\beta-1<0$ into equation (14) leads to $\lambda=0$, which is inconsistent with the parameter restriction $0<\lambda<1$. Nevertheless, if μ is close to $\beta-1$, but still larger than $\beta-1$, then the economy approximately follows Friedman's rule.[12]

This setup may yield a continuum of hyperinflationary or deflationary equilibria, if the economy starts with an initial price other than P_0^R given by equation (13). The economy with $P_0>P_0^R$ is hyperinflationary; $\frac{P_{t+1}}{P_t}\to\frac{\beta}{1-\lambda}>1+\mu$, $\frac{M_t}{P_t}\to 0$, and $R_t\to\frac{1}{1-\lambda}>\frac{1+\mu}{\beta}$. Then, the terminal condition is derived as

$$\lim_{T\to\infty}\frac{\gamma^T B_0+(1+\mu)^T M_0}{\prod_{s=0}^{T-1}R_s}\le\lim_{T\to\infty}\frac{\gamma^T B_0+(1+\mu)^T M_0}{\left(\frac{1+\mu}{\beta}\right)^T}=\lim_{T\to\infty}\beta^T\left[\left(\frac{\gamma}{1+\mu}\right)^T B_0+M_0\right]=0.$$

Thus, as long as $\frac{\beta\gamma}{1+\mu}<1$, equation (5) holds, and the hyperinflationary economy is supported as a continuum of equilibria.

On the other hand, the economy with $P_0<P_0^R$ is deflationary; $\frac{P_{t+1}}{P_t}\to\beta<1$, $\frac{M_t}{P_t}\to\infty$, and $R_t\to 1$. Then, the terminal condition associated with the money stock is derived as follows:

[12] Buiter and Sibert (2007) rigorously prove that Friedman's rule with $\mu=\beta-1<0$ is inconsistent with the transversality condition.

$$\lim_{T\to\infty}\frac{(1+\mu)^T M_0}{\prod_{s=0}^{T-1} R_s}=\frac{1}{\Lambda(R_0)}\lim_{T\to\infty}(1+\mu)^T M_0,$$

where $\Lambda(R_0)=\prod_{s=0}^{\infty} R_s>1$ Hence, $\lim_{T\to\infty}\frac{(1+\mu)^T M_0}{\prod_{s=0}^{T-1} R_s}$ diverges to infinity if $\mu>0$, it converges to a positivev constant or $\frac{M_0}{\Lambda(R_0)}$ (>0) if $\mu=0$, and it converges to zero if $\beta-1<\mu<0$. As Buiter and Sibert (2007) show, a deflationary economy with monetary expansion or constant money stock is not supported as a continuum of equilibria.

2.2. Non-*Ricardian* economy

As discussed in the introduction, the FTPL can be considered an equilibrium selection device in the context of a continuum of hyperinflationary equilibria. Consider a case where the initial price level P_0^{NR}, which is determined by the FTPL, is larger than P_0^R given by equation (13). As equation (11) implies, $\frac{P_{t+1}^{NR}}{P_t^{NR}}>1+\mu$ with $P_0^{NR}>P_0^R$.

Suppose that the primary balance with seigniorage revenues never responds to the nominal amount of public bonds:

$$P_t^{NR}(\tau_t-g)=P_t^{NR}\varepsilon-\mu M_t, \tag{18}$$

where ε and μ are positive. Again, seigniorage revenues are reimbursed as a lump-sum subsidy to households. Thus, the real balance of the public bonds evolves according to

$$\begin{aligned}B_{t+1}&=R_tB_t-P_{t+1}^{NR}(\tau_{t+1}-g)-\mu M_t\\&=R_tB_t-P_{t+1}^{NR}\varepsilon.\end{aligned} \tag{19}$$

In the sense that the public bonds may not be repaid completely by tax revenues, a fiscal rule specified by equation (18) is non-*Ricardian*.

Together with equation (7), equation (19) is rewritten as follows:

$$\frac{B_t}{P_t^{NR}}=\beta\frac{B_{t+1}}{P_{t+1}^{NR}}+\beta\varepsilon. \tag{20}$$

Equation (20) is solved in a recursive manner.

$$\begin{aligned}\frac{B_0}{P_0^{NR}} &= \sum_{\tau=0}^{\infty}\beta^{\tau+1}\varepsilon + \lim_{T\to\infty}\beta^T \frac{B_T}{P_T^{NR}} \\ &= \frac{\beta\varepsilon}{1-\beta} + \lim_{T\to\infty}\prod_{s=0}^{T-1}\left(\beta\frac{P_s^{NR}}{P_{s+1}^{NR}}\right)\frac{B_T}{P_0^{NR}} \\ &= \frac{\beta\varepsilon}{1-\beta} + \lim_{T\to\infty}\prod_{s=0}^{T-1}\left(\frac{1}{R_s}\right)\frac{B_T}{P_0^{NR}}\end{aligned} \tag{21}$$

As equation (19) implies, the nominal public bonds B_t grow at a rate less than R_t, as long as $\varepsilon > 0$, $R_t > 1$, and $P_t^{NR} > 0$. Thus, the second term of the third line in equation (20) converges to zero, or

$$\lim_{T\to\infty}\frac{B_T}{\prod_{t=0}^{T-1} R_t} = 0 . \tag{22}$$

An essential aspect of equation (21) is that it cannot hold for any initial price other than a particular price P_0^{NR}. According to the FTPL, the initial price P_0^{NR} is chosen such that equation (21) may hold with equation (22), or

$$\frac{B_0}{P_0^{NR}} = \frac{\beta\varepsilon}{1-\beta}. \tag{23}$$

From (23) with $\kappa = \dfrac{M_0}{P_0^R\left(y-g\right)}$, the following condition is obtained.

$$\frac{P_0^R}{P_0^{NR}} < 1 \rightarrow \frac{M_0}{B_0} < \frac{1-\beta}{\beta}\frac{\kappa}{\dfrac{\varepsilon}{y-g}} \tag{24}$$

In this way, with inequality (24) satisfied, the FTPL as a selection device allows us to choose a particular value for the initial price $P_0^{NR} > P_0^R$ such that equation (23) holds in a continuum of hyperinflationary equilibria.

3. Non-*Ricardian* economy as a deviation from the *Ricardian* economy

3.1. Three features of the model

In the previous section, the *Ricardian* economy with the QTM and the non-*Ricardian* economy with the FTPL are explored separately. In this section, however, the latter is considered a temporary, or at most a persistent, deviation from the former; with a small probability, the non-*Ricardian* regime with the FTPL switches back into the *Ricardian* regime with the QTM. As demonstrated in this section, the setup with such a switching possibility allows space for the stochastic bubbles that work to relax the GIBC (government's intertemporal budget constraint) during the deflationary non-*Ricardian* regime. Thus, the public bonds are now backed not only by tax revenues in the current non-*Ricardian* regime before switching and the future *Ricardian* regime after switching, but also by the stochastic bubbles during the former regime. In this way, the fiscal surpluses tentatively improve with the stochastic bubbles despite the continuing budget deficits and, indeed, create a deflationary pressure through a conventional mechanism of the FTPL. Immediately after the economy switches to *Ricardian*, however, the bubbles burst because of a heavy devaluation caused by a large, one-off increase in the price level, and the remaining public bonds are repaid by tax revenues over time from then onward.

In this section, the following three points are analyzed with adequate care. First, there are potentially two cases in which the non-*Ricardian* economy switches to the *Ricardian* economy. In one case, the non-*Ricardian* economy is hyperinflationary, and the price level jumps *down* to the *Ricardian* level at switching. In the other case, the non-*Ricardian* economy is deflationary, and the price level jumps *up* to the *Ricardian* level at switching. During the non-*Ricardian* regime, the expected inflation is formed with consideration for such a downward or upward price jump. As demonstrated in this section, however, the formation of price expectations is consistent with deflationary paths, but not with hyperinflationary paths during the non-*Ricardian* regime. In addition, the terminal condition associated with the money stock is satisfied in the deflationary non-*Ricardian* economy even with monetary expansion, which contrasts sharply with a deflationary case presented in Section 2.1.

Second, as long as the economy remains non-*Ricardian*, *ex post* deflations always exceed expected deflations, thereby balancing a possible large price rise at switching. Consequently, the nominal rate of interest, determined by real returns minus expected deflations, is at least zero, whereas the actual nominal return, determined by real returns minus *ex post* deflations, turns out to be negative. Such eventually negative returns are responsible for the emergence of the stochastic bubbles that work to relax the GIBC during the non-*Ricardian* regime. Accordingly, the initial price level decreases to the extent that the stochastic bubbles are large. On the other hand, the representative consumer applies at least zero nominal interest rates, thereby discounting to zero the expected present value of the future public bonds. That is, the stochastic bubbles appears in the GIBC, but not in the household's intertemporal budget constraint. Accordingly, the public bonds with the stochastic bubbles never serve as net wealth for the household in the deflationary non-*Ricardian* regime.

Third, the real valuation of public bonds depends on how large the stochastic bubbles are in the GIBC, but it is completely independent of the present value of fiscal surpluses that are generated during the non-*Ricardian* regime. In this FTPL environment, the non-*Ricardian* economy is anchored eventually by the *Ricardian* economy, and, sooner or later, the public bonds are financed ultimately by a heavy devaluation when switching to the *Ricardian* regime, and tax revenues from then onward. Thus, *Ricardian* equivalence still holds in this setup. Sims (2016) and others claim that weaker fiscal discipline (lower ε in this setup) helps to create more inflationary pressures in the non-*Ricardian* regime. In this environment, however, the price level is completely independent of how a non-*Ricardian* fiscal rule is implemented, and their claim is not relevant in this model.

3.2. Dynamics in the price level

Let us start with the non-*Ricardian* regime where a fiscal rule follows equation (18) with possibly zero or even negative ε. With a probability π, the non-*Ricardian* economy switches back into the *Ricardian* economy with a fiscal rule specified by equation (15), whereas the economy remains non-*Ricardian* with a probability $1-\pi$. Note that every varibale evolves deterministically in each regime.

Because uncertainty arises due to the above probabilistic switching, the objective function is reformulated as

$$\sum_{t=0}^{\infty} \beta^t E_0 \left[u(c_t) + v\left(\frac{M_t}{P_t}\right) \right], \tag{1'}$$

where E_0 is the conditional expectation operator as of time 0. Thus, the first-order condition (7) is redefined as

$$\beta R_t E_t\left(\frac{P_t^{NR}}{P_{t+1}}\right)=1. \tag{7'}$$

Equation (11) is rewritten as

$$E_t\left(\frac{P_t^{NR}}{P_{t+1}}\right)=\frac{1}{\beta R_t}=\frac{1}{\beta}\left[1-\lambda\left(1+\frac{M_t}{P_t\left(y-g\right)}\right)^{-\sigma}\right]. \tag{11'}$$

If the non-*Ricardian* economy switches to the *Ricardian* economy, then P_t^{NR} jumps up or down to P_{t+1}^R, which is determined by equation (13). Otherwise, P_t^{NR} changes to P_{t+1}^{NR}. Thus, the expected deflation is defined as

$$E_t\left(\frac{P_t^{NR}}{P_{t+1}}\right)=\left(1-\pi\right)\frac{P_t^{NR}}{P_{t+1}^{NR}}+\pi\frac{P_t^{NR}}{P_{t+1}^{R}}. \tag{25}$$

Together with equation (11'), equation (25) is solved for $\frac{P_t^{NR}}{P_{t+1}^{NR}}$.

$$\frac{P_t^{NR}}{P_{t+1}^{NR}}=\frac{1}{1-\pi}\left[\frac{1}{\beta}\left[1-\lambda\left(1+\frac{M_t}{P_t\left(y-g\right)}\right)^{-\sigma}\right]-\pi\frac{P_t^{NR}}{P_{t+1}^{R}}\right] \tag{26}$$

As discussed in Section 2, if $P_0^{NR} > P_0^R$, the inflation rate is higher than μ. Then, when the economy stays in the hyperinflationary non-*Ricardian* regime for a long time, $\frac{P_t^{NR}}{P_{t+1}^{R}}$ becomes large at switching. Consequently, the right-hand side of equation (26) eventually becomes negative, and positive prices can no longer be supported during the non-*Ricardian* regime. In other words, inflations need to be high to balance a big price slump at switching, but too high inflations are inconsistent with the upper bound of the nominal rate of interest in the neighborhood of $\frac{M_t}{P_t}=0$. Hence, a hyperinflationary case with $P_0^{NR} > P_0^R$ is ruled out when the economy has a chance to switch from the non-*Ricardian* economy to the *Ricardian* economy with a probability $\pi > 0$. On the other hand, if the economy is deflationary with $P_0^{NR} < P_0^R$, the deflationary process is accelerated in the non-*Ricardian* regime. Throughout the deflationary non-*Ricardian* regime, positive prices are still supported in equation (26) because $\frac{P_t^{NR}}{P_{t+1}^{R}}$ converges to zero.

As demonstrated below, the terminal condition associated with the money stock holds in the above deflationary non-*Ricardian* regime in spite of monetary expansion. The lifetime budget constraint is expressed as follows in the deflationary non-*Ricardian* regime.

$$\begin{aligned}
B_0 + M_0 &= E_0\left[\sum_{\tau=-1}^{\infty}\frac{P_{\tau+1}(\tau_{t+1}-g)}{\prod_{s=-1}^{\tau}R_s} + \sum_{\tau=-1}^{\infty}\frac{1}{\prod_{s=-1}^{\tau-1}R_s}\left(1-\frac{1}{R_\tau}\right)M_\tau + \lim_{T\to\infty}\frac{B_T+M_T}{\prod_{s=-1}^{T-1}R_s}\right] \\
&= \sum_{\tau=0}^{\infty}\frac{(1-\pi)^\tau \pi P_{\tau+1}^{R}(\tau_{t+1}-g)}{\prod_{s=0}^{\tau}R_s} + \sum_{\tau=0}^{\infty}\frac{(1-\pi)^\tau \pi (B_{\tau+1}+M_{\tau+1})}{\prod_{s=0}^{\tau}R_s} \qquad (27) \\
&+ \sum_{\tau=-1}^{\infty}\frac{(1-\pi)^{\tau+1} P_{\tau+1}^{NR}(\tau_{t+1}-g)}{\prod_{s=-1}^{\tau}R_s} + \sum_{\tau=-1}^{\infty}\frac{(1-\pi)^\tau}{\prod_{s=-1}^{\tau-1}R_s}\left(1-\frac{1}{R_\tau}\right)M_\tau + \lim_{T\to\infty}\frac{(1-\pi)^T(B_T+M_T)}{\prod_{s=-1}^{T-1}R_s}
\end{aligned}$$

In the *Ricardian* regime, the terminal conditions associated with the money stock and the public bonds are satisfied by construction.

When the economy remains non-*Ricardian*, R_t converges to one. Then, the last term of the third line in equation (27), $\lim_{T\to\infty}\frac{(1-\pi)^T M_T}{\prod_{s=-1}^{T-1}R_s}$ converges to zero if $(1-\pi)(1+\mu)<1$.

$$\lim_{T\to\infty}\frac{(1-\pi)^T M_T}{\prod_{s=-1}^{T-1}R_s} \le \lim_{T\to\infty}\left[(1-\pi)(1+\mu)\right]^T M_0 = 0 \qquad (28)$$

Accordingly, as long as μ is positive but less than $\frac{\pi}{1-\pi}$, the terminal condition associated with the money stock holds even in the deflationary non-Ricardian economy with monetary expansion. That is, a continuum of deflationary equilibria with monetary expansion is impossible in standard monetary models, but it may be possible in this setup with probabilistic switching.

As equations (25) and (26) imply, as long as the deflationary non-*Ricardian* regime continues, the *ex post* deflation $\frac{P_t^{NR}}{P_{t+1}^{NR}}$ always exceeds the expected deflation $E_t\left(\frac{P_t^{NR}}{P_{t+1}}\right)$, given that the price level jumps up with probability π. Then, the actual (*ex post*) nominal return, defined as $\frac{1}{\beta}\frac{P_{t+1}^{NR}}{P_t^{NR}}$, is always lower than the nominal rate of interest $R_t = \frac{1}{\beta}\frac{1}{E_t\left(\frac{P_t^{NR}}{P_{t+1}}\right)}$. That is, the nominal net rate of interest is at least zero, whereas the actual nominal net return turns out to be negative.

$$\frac{1}{R_t} = \beta E_t\left(\frac{P_t^{NR}}{P_{t+1}}\right) < \beta\frac{P_t^{NR}}{P_{t+1}^{NR}} \tag{29}$$

Inequality (29) is recognized as the peso problem in the sense that expected deflations are always upward biased relative to actual (*ex post*) deflations. Then, it is later employed in proving that the terminal condition associated with the public bonds still holds for the household's intertemporal budget constraint (27) in spite of the presence of the stochastic bubbles in the GIBC.

3.3. A continuum of deflationary equilibria with the stochastic bubbles in the GIBC

The primary balance of the non-*Ricardian* economy is determined by equation (18), and the public bonds evolve according to

$$B_{t+1} = R_t B_t - P_{t+1}\left(\tau_t - g\right) - \mu M_t = R_t B_t - P_{t+1}\varepsilon\ . \tag{30}$$

Here, a non-*Ricardian* fiscal rule (18) is assumed to continue immediately after switching to the *Ricardian* regime, and a fiscal rule shifts to equation (15) one period after.

Together with equation (7'), equation (30) is rewritten as follows:

$$\beta E_t\left(\frac{P_t^{NR}}{P_{t+1}}\right)\left(\frac{B_{t+1}}{P_{t+1}} + \varepsilon\right) = \frac{P_t^{NR}}{P_{t+1}}\frac{B_t}{P_t^{NR}}\ .$$

Taking the conditional expectation operator E_t of both sides of the above equation, the following is obtained.

$$\beta\left[E_t\left(\frac{B_{t+1}}{P_{t+1}}\right)+\varepsilon\right]=\frac{B_t}{P_t^{NR}} \tag{31}$$

The expected real public bond is determined by

$$E_t\left(\frac{B_{t+1}}{P_{t+1}}\right)=(1-\pi)\frac{B_{t+1}}{P_{t+1}^{NR}}+\pi\frac{R_tB_t-P_{t+1}^{R}\varepsilon}{P_{t+1}^{R}}. \tag{32}$$

Together with equation (32), equation (31) is rewritten as

$$\frac{B_t}{P_t^{NR}}=\beta(1-\pi)\frac{B_{t+1}}{P_{t+1}^{NR}}+\beta(1-\pi)\varepsilon+\beta\pi\frac{R_tB_t}{P_{t+1}^{R}}.$$

The above equation is solved in a recursive manner as follows:

$$\begin{aligned}\frac{B_0}{P_0^{NR}}&=\sum_{\tau=0}^{\infty}\beta^{\tau}(1-\pi)^{\tau}\left[\beta(1-\pi)\varepsilon+\beta\pi\frac{R_\tau B_\tau}{P_{\tau+1}^{R}}\right]+\lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{B_T}{P_T^{NR}}\\&=\frac{\beta(1-\pi)\varepsilon}{1-\beta(1-\pi)}+\sum_{\tau=0}^{\infty}\beta^{\tau}(1-\pi)^{\tau}\beta\pi\frac{R_\tau B_\tau}{P_{\tau+1}^{R}}+\lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{B_T}{P_T^{NR}}.\end{aligned} \tag{33}$$

Equation (33) represents the GIBC, and serves as a vital part in determining the initial price P_0^{NR} given B_0 in the context of the FTPL. The public bonds are now financed by the primary balance during the current non-*Ricardian* regime ($\frac{\beta(1-\pi)\varepsilon}{1-\beta(1-\pi)}$), a heavy devaluation caused by a one-off price rise at switching together with tax revenues during the future *Ricardian* regime ($\sum_{\tau=0}^{\infty}\beta^{\tau}(1-\pi)^{\tau}\beta\pi\frac{R_\tau B_\tau}{P_{\tau+1}^{R}}$), and the bubbles appearing tentatively in the GIBC if any ($\lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{B_T}{P_T^{NR}}\geq 0$).

Let us rewrite equation (33) to illuminate an interesting feature of the GIBC. If $\varepsilon=0$, then the nominal public bonds grow at exactly the rate of R_t, or $B_t=\prod_{s=0}^{t-1}R_sB_0$. Compared with the case of $\varepsilon=0$, the real valuation of public bonds decreases with the repayment through tax revenues, or the present value of the primary balance $\frac{\beta(1-\pi)\varepsilon}{1-\beta(1-\pi)}$, and equation (33) is rewritten as follows:

$$\begin{aligned}\frac{B_0}{P_0^{NR}} &= \frac{\beta(1-\pi)\varepsilon}{1-\beta(1-\pi)} + \sum_{\tau=0}^{\infty}\beta^{\tau}(1-\pi)^{\tau}\beta\pi\left[\frac{R_\tau B_\tau}{P_{\tau+1}^{R}}\right] + \lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{B_T}{P_T^{NR}} \\ &= \frac{\beta(1-\pi)\varepsilon}{1-\beta(1-\pi)} + \left\{\left\{\sum_{\tau=0}^{\infty}\beta^{\tau}(1-\pi)^{\tau}\beta\pi\left[\frac{R_\tau \prod_{s=0}^{\tau-1} R_s B_0}{P_{\tau+1}^{R}}\right] + \lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{\prod_{s=0}^{T-1} R_s B_0}{P_T^{NR}}\right\} - \frac{\beta(1-\pi)\varepsilon}{1-\beta(1-\pi)}\right\} \\ &= \sum_{\tau=0}^{\infty}\beta^{\tau}(1-\pi)^{\tau}\beta\pi\left[\frac{R_\tau \prod_{s=0}^{\tau-1} R_s B_0}{P_{\tau+1}^{R}}\right] + \lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{\prod_{s=0}^{T-1} R_s B_0}{P_T^{NR}}.\end{aligned} \tag{34}$$

Accordingly, $\frac{B_0}{P_0^{NR}}$ is independent of ε.

In this setup, the primary balance ε has impacts not on the real valuation of public bonds in the non-*Ricardian* regime, but on the distribution of the repayment of the public bonds by tax revenues between the non-*Ricardian* regime and the *Ricardian* regime. That is, the repayment by tax revenues obviously increases with ε during the non-*Ricardian* regime, but it decreases with ε during the *Ricardian* regime. In this way, *Ricardian* equivalence still holds in this FTPL setup.

Let us below demonstrate that there emerges a continuum of deflationary equilibria. Firstly, the initial price equal to the *Ricardian* level ($P_0^{NR} = P_0^{R}$) is consistent with equation (34) without any stochastic bubble ($\lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{\prod_{s=0}^{T-1} R_s B_0}{P_T^{NR}} = 0$). Suppose that $R_t = \frac{1+\mu}{\beta}$ and $P_t^{R} = (1+\mu)^{t} P_0^{R}$ as in the *Ricardian* economy. Then, equation (34) is rewritten as

$$\begin{aligned}\frac{B_0}{P_0^{NR}} &= \sum_{\tau=0}^{\infty}\beta^{\tau}(1-\pi)^{\tau}\beta\pi\frac{\left(\frac{1+\mu}{\beta}\right)^{\tau+1}}{(1+\mu)^{\tau+1}}\frac{B_0}{P_0^{R}} + \lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{\left(\frac{1+\mu}{\beta}\right)^{T}}{(1+\mu)^{T}}\frac{B_0}{P_0^{R}} \\ &= \sum_{\tau=0}^{\infty}(1-\pi)^{\tau}\pi\frac{B_0}{P_0^{R}} + \lim_{T\to\infty}(1-\pi)^{T}\frac{B_0}{P_0^{R}} = \frac{B_0}{P_0^{R}}.\end{aligned} \tag{34'}$$

Hence, $\lim_{T\to\infty}\beta^{T}(1-\pi)^{T}\frac{\prod_{s=0}^{T-1} R_s B_0}{P_T^{NR}} = 0$, and $P_0^{NR} = P_0^{R}$.

Secondly, the initial price less than the *Ricardian* level ($P_0^{NR} < P_0^{R}$) is also consistent with equation (34) with the stochastic bubbles ($0 < \lim_{T\to\infty} \beta^T (1-\pi)^T \frac{\prod_{s=0}^{T-1} R_s B_0}{P_T^{NR}} < \infty$).

Given $P_0^{NR} < P_0^R$, $\frac{M_t}{P_t^{NR}} \to \infty$, and $R_t \to 1$, $\prod_{s=0}^{T-1} R_s B_0$ reaches a certain constant $\hat{B}$ in the deflationary non-*Ricardian* regime. From equation (26), $\frac{P_{t+1}^{NR}}{P_t^{NR}} \to \beta(1-\pi)$. Thus, the last term of the second line in equation (34) is rewritten as

$$\begin{aligned} &\lim_{T\to\infty} \beta^T (1-\pi)^T \frac{\prod_{s=0}^{T-1} R_s B_0}{P_T^{NR}} = \lim_{T\to\infty} \prod_{s=0}^{T-1} \left(\frac{1}{\frac{1}{\beta(1-\pi)} \frac{P_{s+1}^{NR}}{P_s^{NR}}} \right) \frac{\prod_{s=0}^{T-1} R_s B_0}{P_0^{NR}} \\ &= \frac{1}{\Phi\left(P_0^{NR}\right)} \lim_{T\to\infty} \left(\frac{1}{\frac{\beta(1-\pi)}{\beta(1-\pi)}} \right)^T \frac{\hat{B}}{P_0^{NR}} = \frac{1}{\Phi\left(P_0^{NR}\right)} \frac{\hat{B}}{P_0^{NR}} > 0, \end{aligned} \tag{35}$$

where $\Phi\left(P_0^{NR}\right) = \prod_{s=0}^{\infty} \frac{1}{\beta(1-\pi)} \frac{P_{s+1}^{NR}}{P_s^{NR}} > 1$.

That is, $\lim_{T\to\infty} \beta^T (1-\pi)^T \frac{\prod_{s=0}^{T-1} R_s B_0}{P_T^{NR}}$ converges to neither zero nor infinity, but to a positive constant. A reason for this is that in the limit, the real value of the public bonds $\frac{B_t}{P_t^{NR}}$ grows at the rate of deflation $\frac{1}{\beta(1-\pi)}$, which is exactly equal to an inverse of a discount factor $\beta(1-\pi)$. This part corresponds to the stochastic bubbles that work to relax the GIBC during the deflationary non-*Ricardian* regime. The presence of the stochastic bubbles tentatively improves the fiscal surpluses, and it creates a deflationary pressure according to a conventional mechanism of the FTPL. Accordingly, P_0^{NR} becomes lower than P_0^R to the extent that the stochastic bubbles are larger. In this way, a government is able to operate a Ponzi scheme as long as the deflationary non-*Ricardian* economy continues.

Let us finally prove that the terminal condition associated with the public bonds still holds in spite of the presence of the stochastic bubbles during the deflationary non-*Ricardian* regime. With $P_0^{NR} < P_0^R$, $\frac{M_t}{P_t^{NR}} \to \infty$, $R_t \to 1$, $P_t^{NR} \to 0$, and equation (29), the nominal amount of public bonds reaches a certain constant $\tilde{B}$ in the deflationary non-*Ricardian* regime. When the economy remains non-*Ricardian*, $R_t \to 1$ and $B_t \to \tilde{B}$. Then, the last term of the third line in equation (27), converges to zero.

$$\lim_{T\to\infty} \frac{(1-\pi)^T B_T}{\prod_{s=-1}^{T-1} R_s} \le \lim_{T\to\infty} (1-\pi)^T \tilde{B} = 0 . \tag{36}$$

In the deflationary non-*Ricardian* regime, the terminal condition associated with the public bonds holds in the household's intertemporal budget constraint, but the stochastic bubbles are still present in the GIBC.[13] A major reason for this seemingly puzzling phenomenon is that the stochastic bubbles emerge under the negative *ex post* nominal return (defined as real returns minus *ex post* deflations), but the representative consumer applies the nominal rate of interest, which is determined not by the *ex post*, but by the expected deflation, to evaluate the terminal condition. Let us compare equation (35) with equation (36).

$$\begin{aligned} &\lim_{T\to\infty} \beta^T (1-\pi)^T \frac{B_T}{P_0^{NR}} = \lim_{T\to\infty} (1-\pi)^T \prod_{s=0}^{T-1} \left(\beta \frac{P_s^{NR}}{P_{s+1}^{NR}} \right) \frac{B_T}{P_0^{NR}} \\ &> \lim_{T\to\infty} (1-\pi)^T \prod_{s=0}^{T-1} \left[\beta E_t \left(\frac{P_s^{NR}}{P_{s+1}} \right) \right] \frac{B_T}{P_0^{NR}} = \lim_{T\to\infty} (1-\pi)^T \prod_{s=-1}^{T-1} \left(\frac{1}{R_s} \right) \frac{\tilde{B}}{P_0^{NR}} = 0 \end{aligned} \tag{37}$$

Given a possible price rise at switching to the *Ricardian* economy, the above inequality is established by the fact that actual deflations always exceed expected deflations in the non-*Ricardian* regime, as implied by inequality (29).

In this setup with probabilistic switching, the FTPL in which the initial price is determined by the GIBC, serves not as an equilibrium selection device among a continuum of hyperinflationary equilibria as in standard monetary models, but as an instrument to generate a continuum of deflationary equilibria with the stochastic bubbles in the GIBC. The GIBC is satisfied not only for $P_0^{NR} = P_0^R$, but also for $P_0^{NR} < P_0^R$

[13] Bloise and Reichlin (2008) show that the bubbles in infinite-maturity public debts relax the GIBC, and that their presence is consistent with the terminal condition as long as the supply of such public debts is declining over time in real terms. In the current setup, on the other hand, the nominal amount of public bonds converges to a positive constant.

with the stochastic bubbles. If the initial price P_0^{NR} is chosen as $P_0^{NR} < P_0^R$, then a deflationary pressure is generated, and the bubbles are created; in fact, a deflationary pressure and the stochastic bubbles interact with each other thanks to the tentative relaxation of the GIBC.

In this way, a deflationary economy emerges despite monetary expansion, and the price of the public bonds is high despite the continuing primary deficits and growing public debts. With the terminal condition satisfied, the deflationary non-*Ricardian* regime is supported as a legitimate continuum of equilibria. Such a seemingly paradoxical phenomenon is sustained only by the possibility that the non-*Ricardian* economy sooner or later reverts to the *Ricardian* world. Once it switches to the *Ricardian*, the economy experiences a sudden and difficult turnaround phase before everything returns to normal. That is, the bubbles burst because of a significant devaluation caused by a one-off price rise at the switching point, before the remaining public bonds are repaid over time by tax revenues, and the prices gradually increase with monetary expansion according to the QTM.

4. Some numerical examples and calibration exercises

In this section, several numerical examples are presented first to illuminate the general properties of the deflationary non-*Ricardian* regime, and then to demonstrate how this model mimics the current Japanese economy.

4.1. Numerical examples

Let us begin with the case where $\beta = 0.96$, $\mu = 0.01$, $\kappa = 0.1$, $\pi = 0.05$, σ is either 1 or 0.1, and λ is determined by equation (14). Here, $(1+\mu)(1-\pi)<1$ is satisfied. Note that σ is interpreted as the interest elasticity of money demand in this context. In terms of the initial conditions, $y-g=100$, ε is set at zero, $M_0 = 100$, and $B_0 = 100$. Given this set of parameters, $P_0^R = 10$. Then, any initial price P_0^{NR} less than 10 is consistent with the deflationary non-*Ricardian* economy. Thus, P_0^{NR} is set at 8.

As shown in Figure 3-1, the nominal net rate of interest ($R_t - 1$) decreases mildly from 5.1% to 4.6% over 30 years if σ is 1, but it declines substantially from 4.1% to almost zero if $\sigma = 0.1$. The relative size of the money stock or the *Marshallian k* ($\frac{M_t}{P_t^{NR}(y-g)}$) increases from 12.5% to 24.5 % if $\sigma = 1$, and from 12.5% to 75.8% if $\sigma = 0.1$. Figure 3-2 shows that the price level P_t^{NR} declines despite monetary expansion in the non-*Ricardian* regime, whereas P_t^R grows at the rate of 1% in the *Ricardian* regime. With $\sigma = 0.1$, for example, $P_{19}^{NR} = 4.1$ in year 19, but $P_{20}^R = 12.2$ in year 20. Thus, if the economy switches back into the *Ricardian* regime in year 20, then the price level jumps by around 200%. As a consequence of this one-off large price increase, the public bonds are heavily devalued when switching to the *Ricardian* economy.

As Figure 3-3 demonstrates, the expected present value of the future public bonds in the GIBC, which is discounted according to the actual nominal return ($\beta^T(1-\pi)^T \frac{\prod_{s=0}^{T-1} R_s B_0}{P_T^{NR}}$ in equation (34)), converges to a positive constant, whereas the value that is discounted according to the nominal rate of interest ($(1-\pi)^T \prod_{s=-1}^{T-1}\left(\frac{1}{R_s}\right)\frac{B_T}{P_0^{NR}}$ in equation (35)) converges to zero. The share of the stochastic bubbles amounts to 26% of the real valuation of public bonds ($\frac{B_0}{P_0^{NR}}$) under the above assumption. This numerical example demonstrates that the stochastic bubbles are present in the GIBC, but that they never constitute any net wealth for the household in the deflationary non-*Ricardian* regime.

4.2. Calibration exercises

Next, let us present calibration exercises that mimic the recent Japanese economy. It is assumed that in 1990, the fiscal policy switched from *Ricardian* to non-*Ricardian*,[14] and the price level of 1990 (the initial price level) deviated slightly and downward from

[14] According to Ito et al. (2011), the estimation result based on the net public bonds indicates that Japanese fiscal policy switched from a stationary *Ricardian* rule to a non-stationary non-*Ricardian* rule in the early 1990s.

the level implied by the QTM as a consequence of some deflationary shocks.[15] The exercises are constructed such that the relative amounts of Bank of Japan notes (BoJ notes) and the public bonds during the 1990-2016 period can be matched approximately with those predicted by the model.

A set of parameters is chosen as follows. β is set at close to 1 or 0.99 to yield low nominal rates of interest, σ is either 0.1, 0.05, or 0.01 to reproduce a substantial decrease in interest rates over time, κ is 0.078, which is equal to the 1980–1995 average *Marshallian* k ($E\left[\frac{M_t}{P_t(y-g)}\right]$), μ is 0.033, which corresponds to the 2000–2016 average growth of BoJ notes, and λ is determined by equation (14). In this setup with constant consumption, σ is interpreted as the interest elasticity of money demand, and extremely low σ is consistent with the fact that the *Marshallian* k was relatively stable with positive nominal interest rates. Note that even with low σ, the interest elasticity is infinite at zero nominal rates of interest.

Given $y-g=100$, the primary balance ε is set at –2.9, which is obtained from the 2000–2016 average ratio of the primary balance of the government's general account to nominal GDP. Setting 1990 as the starting year, M_{1990} is standardized as 100, and B_{1990} is set at 493, because the outstanding BoJ notes and public bonds amounted to 40 trillion yen and 196 trillion yen, respectively, in 1990. Given the above set of parameters, P_{1990}^{R} is computed as 12.8; then, the initial price (P_{1990}^{NR}) needs to be less than 12.8 to present a deflationary case.

Being consistent with $(1+0.033)(1-\pi)<1$, π is set at 0.04. P_{1990}^{NR} must be less than 12.8, and it is set at 10.7 for $\sigma=0.1$, 11.6 for $\sigma=0.05$, and 12.7 for $\sigma=0.01$. As shown in Figure 3-4, the predictions and observations are matched approximately in the years of 1990–2016. In a case of $\sigma=0.01$, a slightly downward deviation of the 1990 price from the level determined by the QTM gurantees reasonable predictions.

[15] Such deflationary shocks may include the tremendous decline in asset pricing which was triggered by consecutive sharp interest hikes by the BoJ. The BoJ raised the call rates from 6.4% in December 1989 to 8.5% in March 1991. Upon these hikes, the Nikkei index declined from close to 40,000 yen in December 1989 to less than 20,000 in October 1990, and land pricing also started to decline from early 1991.

As Figure 3-5 shows, the predicted inflation rate captures the actual trend except in year 2014 when the consumption tax rate was raised from 5% to 8%. If σ is 0.05, the predicted nominal rate of interest is close to zero in the 2010s, but it still declines a little more slowly than the observations did. If σ is set at 0.01 together with $P_{1990}^{NR} = 12.7$, then the predicted rate of interest approaches zero in the early 2000s. As shown in Figure 3-6, the stochastic bubbles amount to about 40% of the real valuation of public bonds in these calibration exercises. In the deflationary environment of the years 2000-2017, one-year ahead inflations are overestimated by around 2.4% in either σ, which degree of over-forecasting is quite comparable with those reported in Figure 1-4 (1.8% in the CCS and 3.5% in the OS).

What would happen to the Japanese economy if the regime switched from non-*Ricardian* to *Ricardian*? Figure 3-7 demonstrates that a difference between P_t^{NR} and P_t^{R} becomes larger and larger as time goes on. For example, consider $\sigma = 0.1$, $P_{2019}^{NR} = 10.8$, and $P_{2020}^{R} = 34.0$. Thus, if the economy switched back to *Ricardian* in 2020, then the price level would jump by more than 200%, the accumulated nominal bonds would be devalued heavily from 230% to 75% in terms of the ratio to nominal GDP, the *Marshallian k* would fall from 25% to 8% ($= \kappa$), and the nominal rate of interest would rise suddenly from near-zero rates to more than 4% ($= \frac{1+\mu}{\beta}$). If the switching occurred in 2026, the price level would jump by even 364%, and the relative amount of public bonds would reduce more dramatically from 311% to 68%.

5. Conclusion

A deflationary economy with monetary expansion and growing public debts under near-zero rates of interest is hard to reconcile with the implications from standard monetary models with the QTM, and it is also difficult to justify using alternative monetary models with the FTPL. However, if the latter models are viewed as a temporary, or at most a persistent, deviation from the former, then a deflationary economy with monetary expansion can be characterized as a legitimate continuum of equilibria. Given such probabilistic switching from the FTPL to the QTM, the stochastic bubbles may emerge in a deflationary environment, thereby relaxing the GIBC (government's intertemporal budget constraint), but they burst because of a heavy devaluation caused by a large, one-off increase in the price level at switching. In this way, a government is able to operate a Ponzi scheme *only* in the deflationary non-*Ricardian* economy.

In terms of policy implications, *Ricardian* equivalence still holds in this FTPL setup. The price level decreases with the size of the stochastic bubbles, but it is completely independent of how a non-*Ricardian* fiscal policy is implemented. A major reason for this is that the non-*Ricardian* economy is eventually anchored by the *Ricardian* economy, and the public bonds accumulated under the continuing primary deficits are sooner or later repaid by a heavy devaluation at switching, and by a *Ricardian* fiscal policy after switching. Given this implication from this FTPL setup, weaker (stronger) fiscal discipline never generates more inflationary (deflationary) pressures on the price level; thus, a fiscal policy may not be employed as an instrument to control the price level.

There are also positive implications from this model. As the calibration exercises demonstrate, once a fiscal policy shifts to non-*Ricardian*, a slightly downward deviation of the initial price level from the level determined by the QTM helps to generate reasonable predictions. In addition, a transition from the non-*Ricardian* regime to the *Ricardian* regime is never smooth, but rather discontinuous in terms of the price level, the relative amounts of the public bonds and the money stock, and the nominal rate of interest, in the presence of the stochastic bubbles that eventually burst at switching. Under this scenario, it is predicted that the deflationary Japanese economy accompanied by monetary expansion and growing public debts–which has been often viewed as the new normal in practical policy debates–will experience a sudden and difficult reversal before everything returns to the old normal.

One important issue to be resolved is whether the implications from this model with money supply as a policy instrument can survive with a policy environment with interest rate feedback rules, which is more realistic as monetary policy. In particular, the public bonds are priced high because of the presence of the stochastic bubbles that work to relax the GIBC in this model. But, there may be alternative factors that help to generate liquidity premiums in public bond pricing under interest rate feedback rules. More realistic policy environments may need to be considered to identify possible sources of high public bond pricing.

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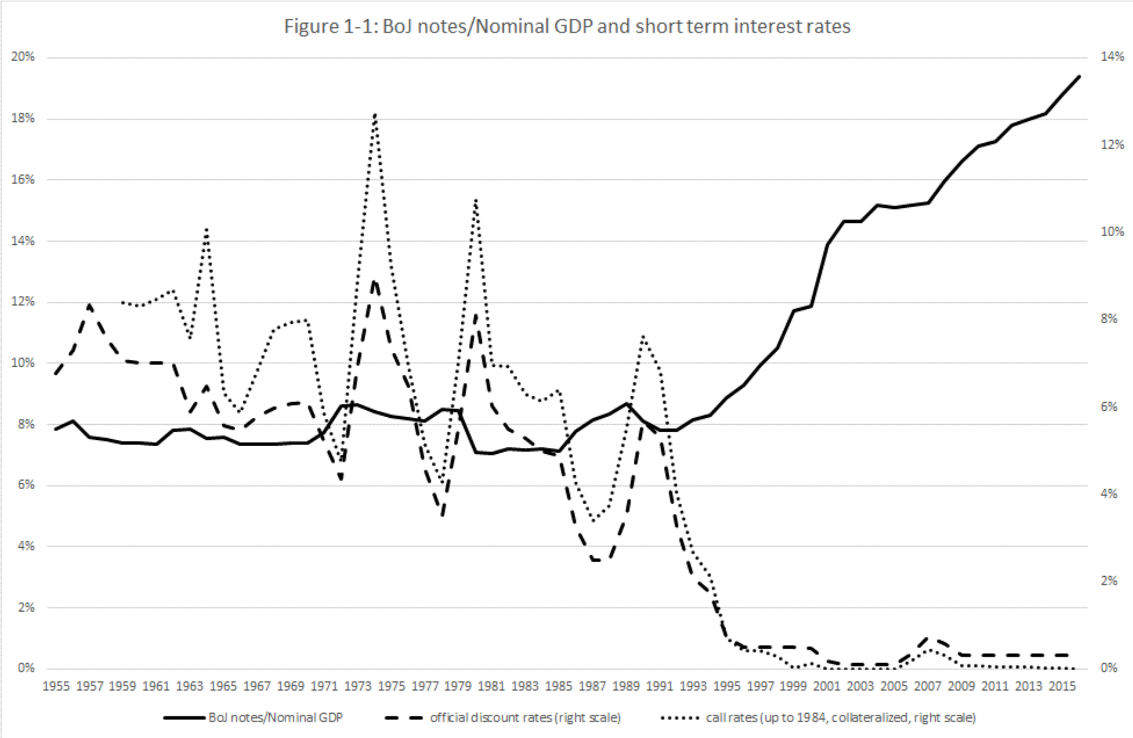
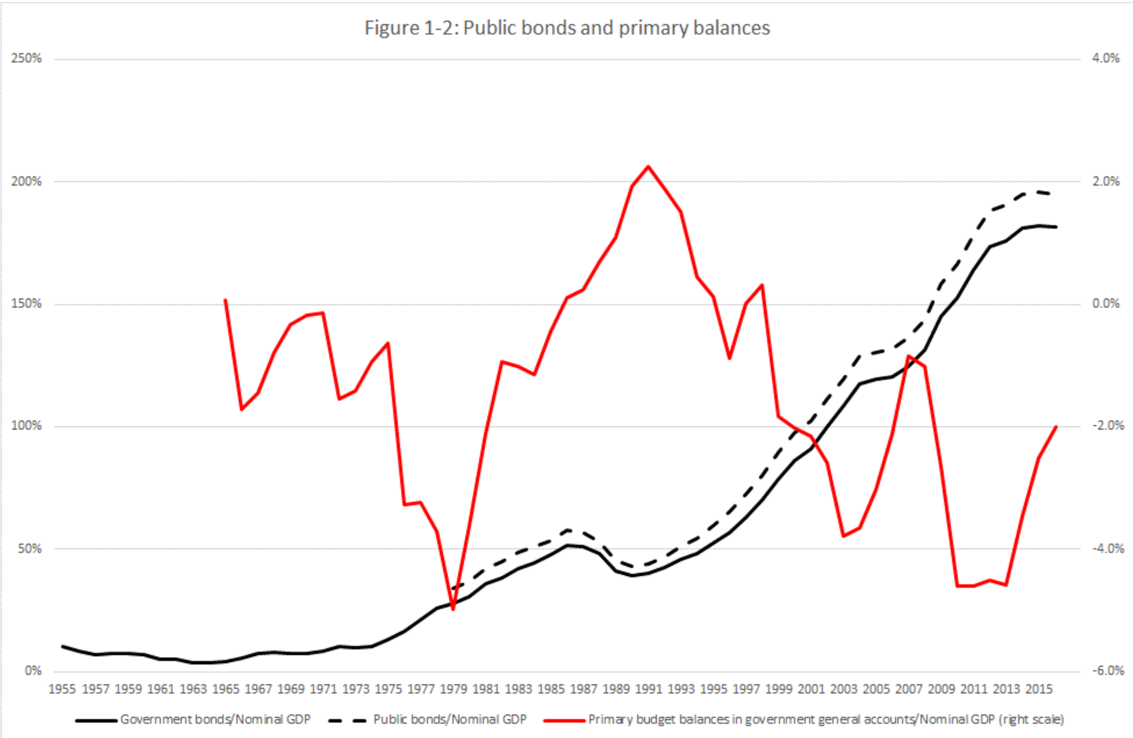


Sources: Bank of Japan, and Cabinet Office.



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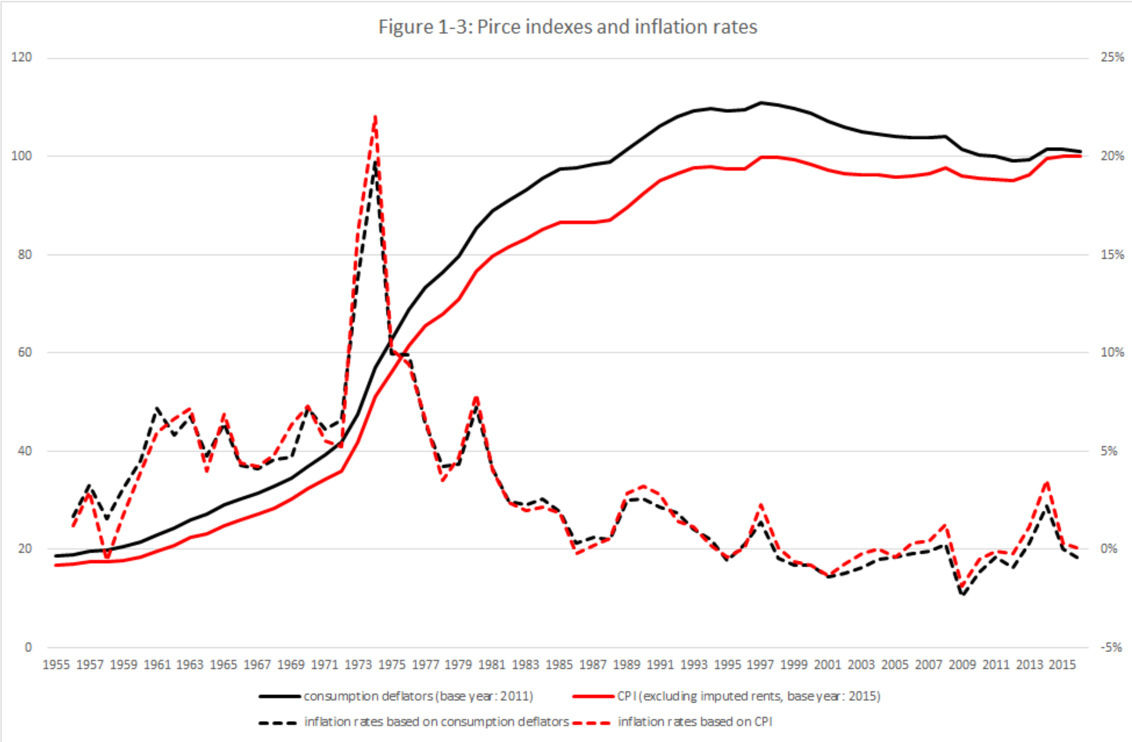

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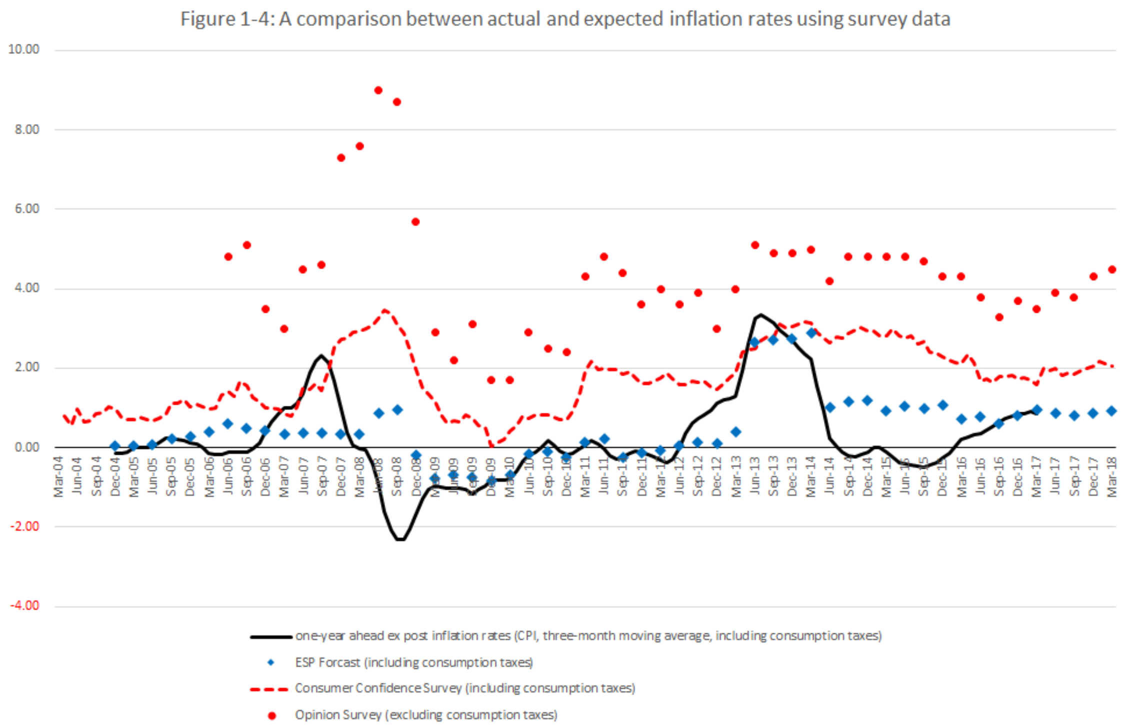


Sources: Ministry of Internal Affairs and Communications for CPI, Japan Center for Economic Research for the ESP Forecast which asks professional analysts about future prices, Bank of Japan for the Opinion Survey, and Cabinet Office for the Consumer Confidence Survey, the two surveys of which ask consumers about future prices.

Figure 2-1: The shape of a money demand function

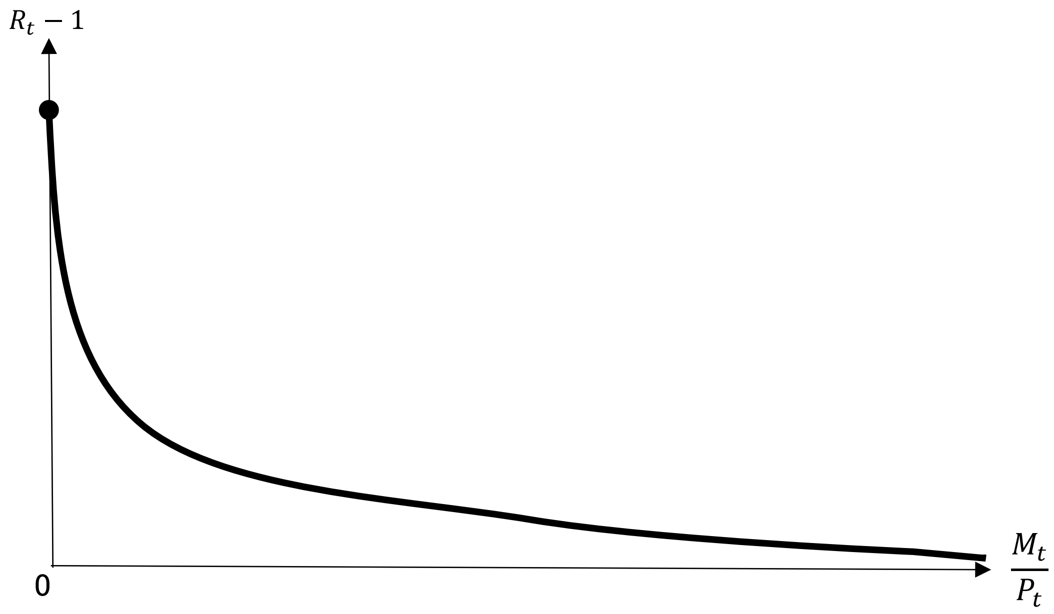


Figure 3-1: Central bank notes/Nominal GDP and nominal rates of interest

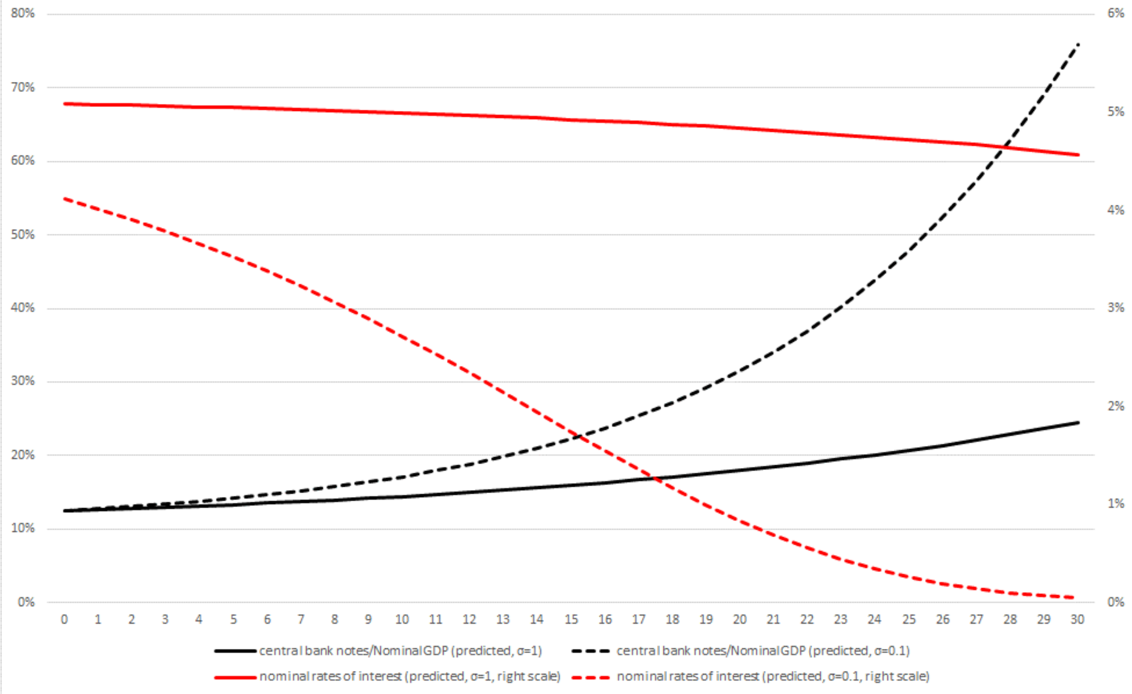
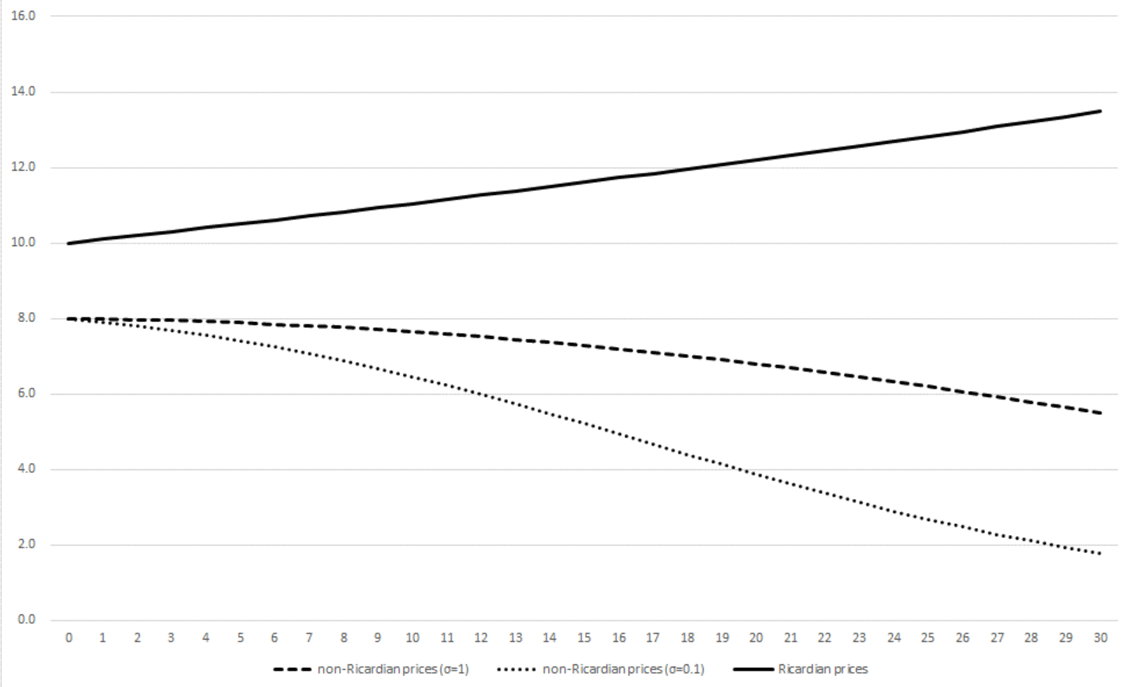
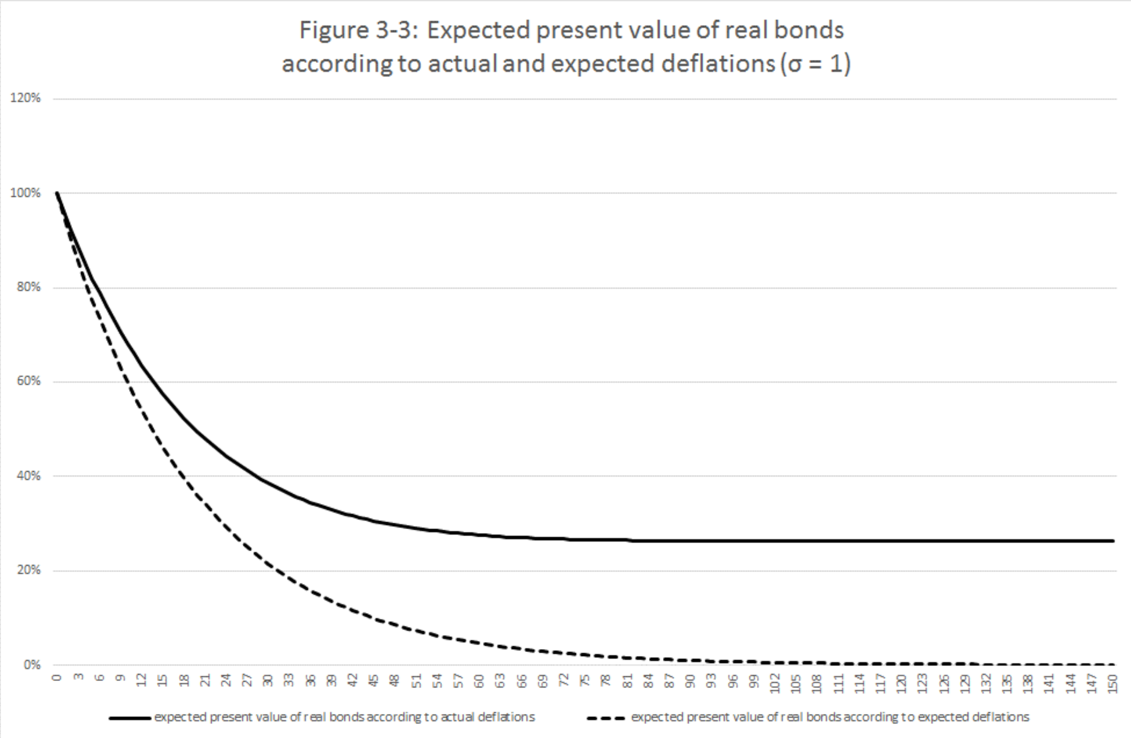
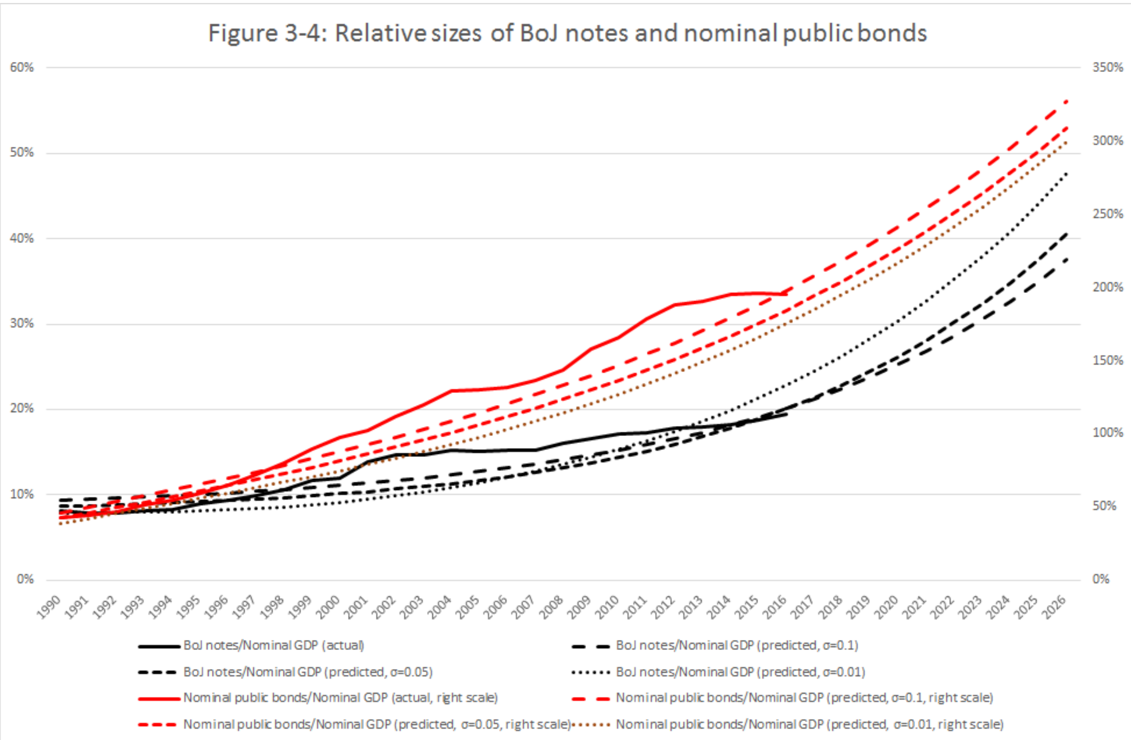


Figure 3-2: Non-Ricardian and Ricardian price levels







Sources: The author's calculation, Bank of Japan, Cabinet Office, and Ministry of Finance.

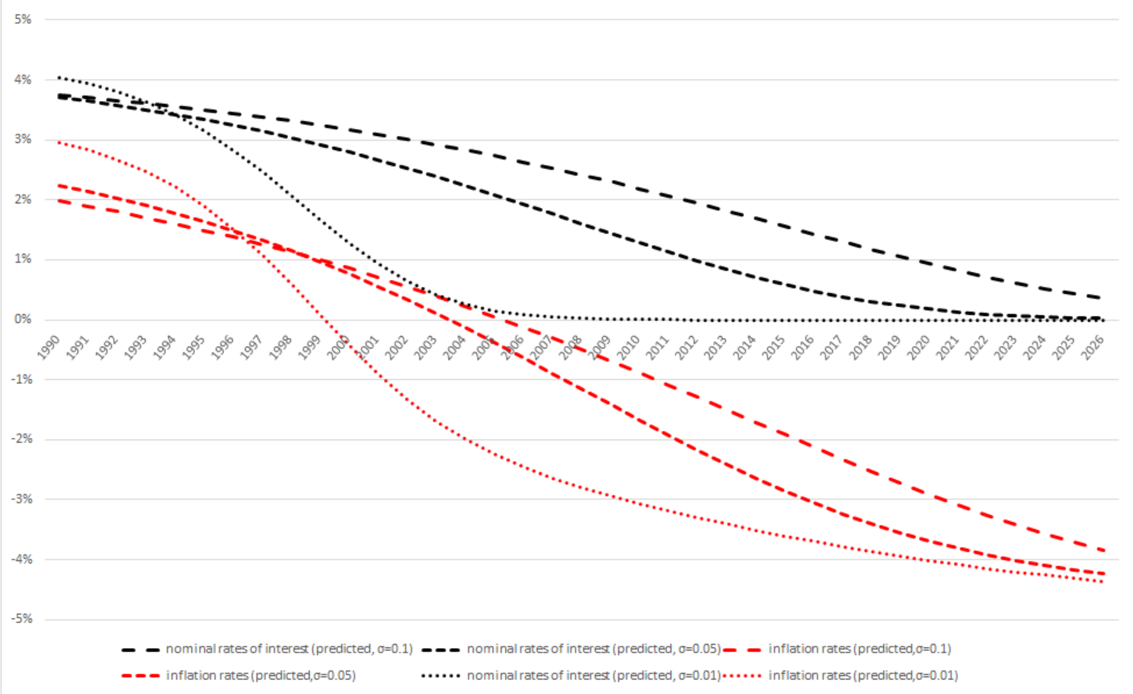


Figure 3-5: Nominal rates of interest and inflation rates

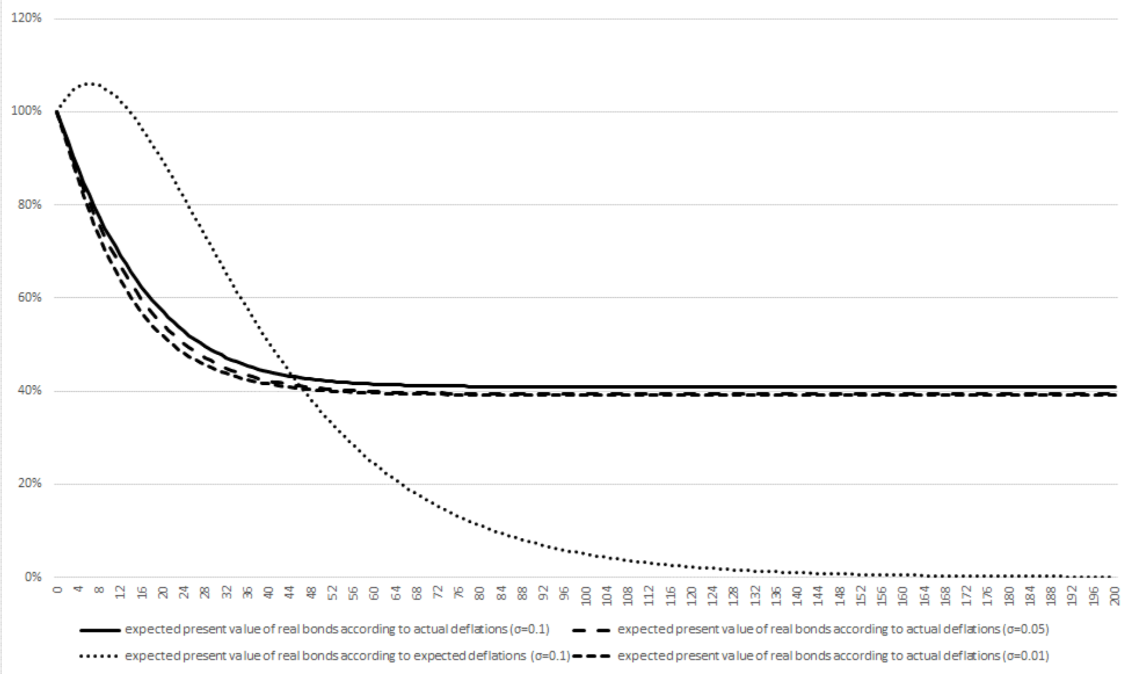


Figure 3-6: Expected present value of real bonds according to actual and expected deflations

Figure 3-7: Non-Ricardian and Ricardian price levels

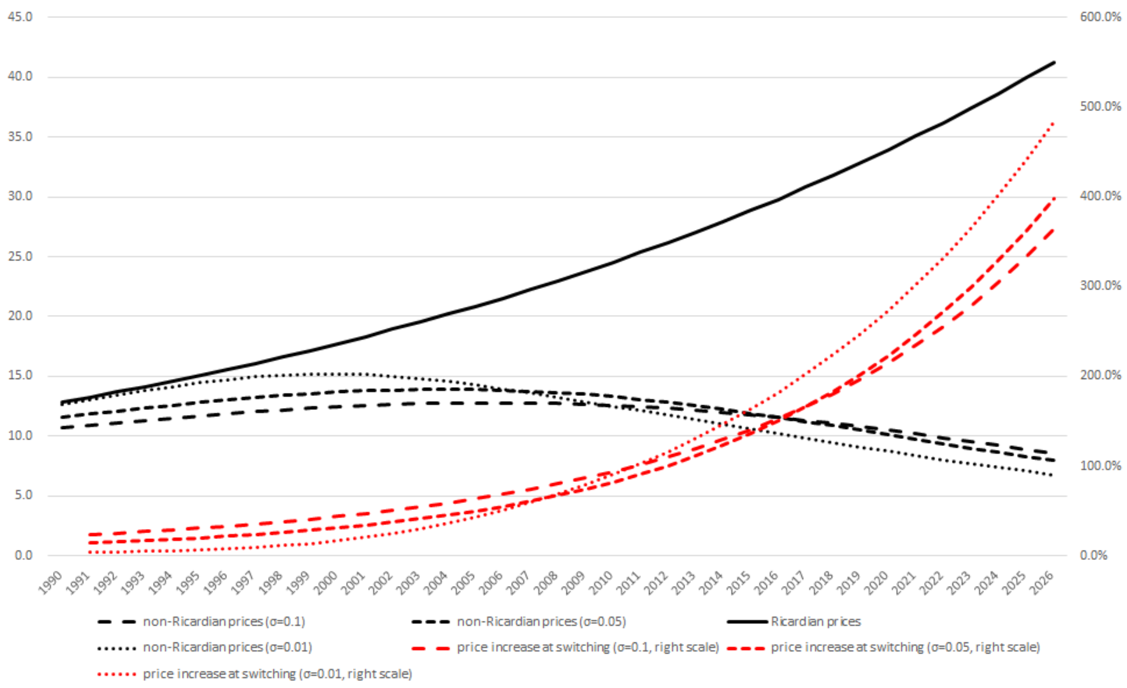


Figure 3-8: Relative sizes of nominal public bonds in non-Ricardian regime and at switching