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Dual Organ Markets: Coexistence of Living and Deceased Donors

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Dual Organ Markets: Coexistence of Living and Deceased Donors *

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Abstract

Lung transplantation is the only treatment for patients in the last stage of chronic lung diseases. Before April 2015, there were only two types of transplantation available: cadaveric-donor transplants and live-donor transplants. Ergin, Sönmez, and Ünver (2017) have proposed the idea of exchanging donors only for live-donor lung transplantation. The new technology, called hybrid transplantation, is now available as Dr. Oto and his team at Okayama University Hospital succeeded transplanting a cadaveric lung and a lobe of live lung to one patient at the same time. We point out that the new technology plays a key role in operating the cadaveric- and live-donor markets at the same time. In particular, the hybrid transplantation opens up a new type of exchanging donors. We investigate a mechanism of organizing transplants in terms of efficiency, fairness and incentive-compatibility. *Journal of Economic Literature* Classification Numbers : C78, D47, D71.

Keywords: Market Design; Multi-unit demand matching problem; Hybrid lung transplantation; Japanese mechanism; Priority mechanism.

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1 Introduction

Lung transplantation is the only treatment for patients with end-stage lung diseases. As of July 31, 2017, 339 patients are registered on the waiting list for deceased donor lung transplants in Japan, while 45 patients received transplants in 2015.[1] Deceased donor transplantation uses one or two lungs of a deceased donor to replace the diseased lungs of a patient.[2] Because the number of deceased donors is historically low for various reasons, the Japanese medical community has developed live donor lung transplantation which needs two living donors each of whom donates one lobe out of five to their intended patient. This transplants have been conducted to about 10 to 20 patients a year in recent years.[3] Although they are a substantial source for transplantation, living donors are conventionally constrained to be relatives of patients.[4] Moreover, they are medically constrained to be compatible with patients for blood types, tissue type, sizes, etc. These constraints are difficult to be met as a patient needs two compatible relatives. To overcome such difficulty, Ergin, Sönmez, and Ünver (2017) have recently proposed a novel transplantation modality of lung exchange – exchanges of donors between incompatible donors of patients – by applying the same idea for kidney exchange (Rapaport, 1986; Roth, Sönmez, and Ünver, 2004, 2005). Due to the multi-unit demand, the application is theoretically non-trivial, and is practically important as Ergin, Sönmez, and Ünver (2017) show the potential increase of transplants by about 80 % to 260 %, depending on sizes of exchanges, using the Japanese data.[5]

Another innovative transplant, called hybrid transplantation, has been successfully conducted by Prof. Oto and his team at Okayama University Hospital, Japan, on April 4, 2015.[6] The hybrid transplantation uses one lobe of living donor *and* one lung of a deceased donor to be transplanted to a patient.

We incorporate the possibility of hybrid transplantation for donor exchange in lung transplants, extending the Ergin, Sönmez, and Ünver (2017)'s model which exclusively takes up live donor transplantation for donor exchange. This extension does not only practically increase the number of saved patients but also opens up a new type of theoretical challenge. Let us discuss its practical importance and then the theoretical challenge.

An obvious benefit from hybrid transplantation is for a patient with only one compatible donor. There might also be a situation in which two patients each having two incompatible donors cannot

[1] The data is available on Japan Organ Transplant Network homepage, `http://www.jotnw.or.jp/datafile`. The cited numbers were retrieved on August 28, 2017.

[2] We refer to the donor by the male personal pronoun and to the patient by the female personal pronoun.

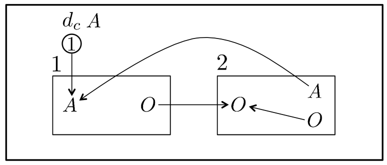
[3] See the Factbook by the Japan Society for Transplantation, `http://www.asas.or.jp/jst/pro/pro8.html`.

[4] "Conventionally" means that this practice is not illegal, but is followed by the medical community as the ethical guideline published in the Japan Society for Transplantation. A transplant from non-relatives of a patient needs special permission in hospitals conducting the transplant. The guideline is available on `http://www.asas.or.jp/jst/about/about12.html`, accessed on March 5, 2018.

[5] The numbers are for 50 patients. See Table II in Ergin, Sönmez, and Ünver (2017).

[6] See the official news at Okayama University, `https://www.okayama-u.ac.jp/eng/news/index_id4469.html`. The web was accessed on March 2, 1998.

Figure 1: Benefits from hybrid transplantation.



Note: In the figure, one lung from the decease donor, d_c, is available, patient 1 of blood type A has a donor of blood type O, and patient 2 of blood type O has two donors with blood types A and O. The allocation indicated by the arrows shows that with donor exchange, patient 1 receives the hybrid transplant, while patient 2 the living donor transplant.

exchange donors for living donor transplants but can do so under hybrid transplantation. Hybrid transplantation can enhance live-donor exchange in the sense that patients could not receive transplants even under donor exchange if only living donor transplantation were allowed (see Figure 1). Moreover, if we consider only two-way exchange under live donor transplantation, O-blood type patients do not benefit from donor exchange (Lemma 1 in Ergin, Sönmez, and Ünver, 2017). However, with the introduction of hybrid transplantation, such patients can receive transplants (see Figure 1).

Taking account into hybrid transplantation poses several new theoretical challenges. The model we introduce has one deceased donor and finitely many patients who bring a number of their relatives as (compatible or incompatible) living donors. A patient may have multi-unit demand as she is assigned two living donors for living donor transplantation, or is assigned one cadaveric lung and one living donor for hybrid transplantation, unlike the standard matching model with unit demand. Organs are indivisible goods, a cadaveric organ is taken to be a social endowment (common ownership), and living donors are those owned by patients (private ownership). Thus, a patient can simultaneously participate in the two "dual" markets for deceased donors and living donors.[7] Thus the treatment of hybrid transplantation naturally leads to the model with mixed ownership. Our model is the first real-life application of matching problems with multi-unit demand and mixed ownership.[8] In such a model, a desirable mechanism has not been discussed.

We search for a matching mechanism – a procedure of assigning donors to patients for transplants based on their medical types and preferences – which should have desirable properties of individual rationality, Pareto efficiency, fairness, and incentive compatibility of strategy-proofness. We focus on individual rational allocations in which each patient receives an assignment at least as good as the no transplant. The Pareto efficiency we use is the standard one. We propose a new fairness notion, called $\succeq$-fairness, based on the priority, $\succeq$, of patients. A patient regards an allocation $\succeq$-unfair if she can improve with the cadaveric lungs assigned to lower-priority patients whenever she can play the

[7]We use the term "dual organ markets" for the following two meanings. First, it expresses the multi-unit demand like Ergin, Sönmez, and Ünver (2017) do. Second, patients simultaneously participates in the two markets for deceased donors and living donors.

[8]An exception is Roth, Sönmez, and Ünver (2004). In their kidney exchange model, they have living donors and the waiting option of putting patients on higher priority in the waiting list of patients. The waiting option implicitly expresses the treatment of deceased donors. On the other hand, we explicitly model the situation in which a patient can access to and use resources in both markets for living donors and those for deceased donors.

role of assigned patients. Fair allocations are those not $\succeq$-unfair. Finally, the strategy-proofness we use is also the standard one where truth telling is a weak dominant strategy in the induced preference revelation game.

We examine the current Japanese mechanism that is only for assigning cadaveric lungs. We show in Proposition 3 that it is neither strategy-proof nor $\succeq$-fair, though it is individually rational and Pareto efficient. Instead, we propose a priority mechanism in which the highest-priority patient selects all of her favorite individual rational allocations, the second-highest patient selects all of her favorite among those selected by the highest-priority patient, and so on. Then we show in Theorem 1 that in regimes without donor exchange, including the current Japanese one, the priority mechanism recovers strategy-proofness and $\succeq$-fairness, still keeping individual rationality and Pareto efficiency. However, once we allow for donor exchange, the strategy-proofness will be violated (Proposition 4), because a lower-priority patient can, by misreporting, narrow down the higher-priority patient's allocations by providing her donors and then obtain more favorite cadaveric transplants. It turns out that such a negative result is from a general impossibility result (Theorem 2) under any regime with donor exchange: no mechanism is individually rational, Pareto efficient, $\succeq$-fair, and strategy-proof. Note that the priority mechanism satisfies all of the properties except for strategy-proofness. We observe that misreporting is risky in the priority mechanism in that if the medical types were different, a misreporting patient would get no-transplant instead of some transplant. With this observation, we introduce uncertainty about the medical type of the deceased donor as well as other patients' medical types and preferences. Then the priority mechanism is shown to be robust against manipulation, that is, the truth-telling profile is a Bayesian Nash equilibrium (Theorem 3).

We introduce the dual organ markets in Section 2. The model covers dual-donor organ exchange (Ergin, Sönmez, and Ünver, 2017) for dual-graft liver transplantation, bilateral living-donor lung transplantation, and simulataneous liver liver-kidney transplantation, with new strucgture of hybrid transplantation when both deceased and living donors coexist. Our description in the paper is for lung for simplicity, but the model is applicable to any other organ by selectively ignoring some parts of the model. In Section 3, several properties of mechanisms. All proofs are relegated to the Appendix. Section 4 discusses our main results regarding the current Japanese mechanism and the priority mechanism. Finally, Section 5 concludes. Omitted proofs are given in the Appendix.

1.1 Related literature

In the matching problem with unit demand, a model with social endowments is called a house allocation problem (Hylland and Zeckhauser, 1979); the one with private ownership is called a housing market (Shapley and Scarf, 1974); the one with mixed ownership is a house allocation problem with existing tenants (Abdulkadiroğlu and Sönmez, 1999) which Roth, Sönmez, and Ünver (2004) apply to donor exchange for kidney transplantation. In dichotomous preferences, Roth, Sönmez, and Ünver (2005) further investigates a priority mechanism under private ownership in which higher-priority patients narrow down their favorite allocations.

For multi-unit demand, a matching problem with social endowment is studied by, for example, Klaus and Miyagawa (2001); Budish and Cantillon (2012), while the one with private endowment is studied by Moulin (1995); Konishi, Quint, and Wako (2001). Its special case where objects are exogenously separated by types is a multiple-type market (e.g., Moulin 1995; Anno and Kurino 2016).

The most related paper is Ergin, Sönmez, and Ünver (2017) which is modeled as a matching problem with multi-unit demand and private endowment. Their model is a special case of our model when a deceased donor is not compatible with any patients. In this case, they focus on maximal matching instead of a mechanism. Our priority mechanism can achieve their maximal matching for a two-way donor exchange, too.

2 Model: Dual Organ Markets

2.1 Basics

We describe the model for dual organ markets. Although our theory covers both cases of lung and kidney, our description is for the most complex case of lung with multi-unit demand as the other organ of kidney with one unit demand can be easily understood with trivial modification.

There are one deceased donor who donates her (one or two) lungs and finitely many patients each of whom accompanies living donors. All of patients and donors have (medical) types for transplantation which determines which donor can donate to which patient. All patients have a priority for lungs from a deceased donor which is typically determined by their waiting time. Formally, a model is a list $(N, \{D_i^L\}_{i\in N}, D^C, (T, \trianglerighteq), \theta, \succeq)$ which satisfies the following conditions:

1. $N := \{1, \ldots, n\}$ is a finite set of patients. We suppose that each patient has authority to decide for transplantation with enough medical knowledge.[9] We assume that N contains at least two patients.

2. For each $i \in N$, D_i^L is the finite set of patient i's living donors. With this condition, patients have different numbers of their living donors.[10] We assume that $D_i^L \cap D_j^L = \emptyset$ for all $i, j \in N$ with $i \neq j$. That is, no living donor is shared by two patients. Since the main focus of this paper involves living donor exchange, we assume that at least two patients have multiple living donors. Let $D^L := \cup_{i\in N} D_i^L$ be the set of all living donors in the market.

3. $D^C := \{d_c\}$ where d_c is the cadaveric donor. We assume that the cadaveric donor is not one of the donors registering as a living donor of a patient, i.e., $D^C \cap D^L = \emptyset$. We denote $D := D^L \cup D^C$.

[9]This is because a patient can be considered to represent a team with her doctor. She does not necessarily know nor understand her own health status for transplantation, but she makes decision about whether to take a transplant. On the other hand, a medical doctor knows her health status, but cannot force her for a transplant.

[10]Note that this setting generalizes the assumption of Ergin, Sönmez, and Ünver (2017) in which each patient has exactly two living donors.

4. $(T, \trianglerighteq) = \times_{k=1}^{K}(T_k, \trianglerighteq_k)$ is a medical type space where there are K kinds of component medical type spaces, and for each $k \in \{1, \dots, K\}$, T_k is a finite set of k-th types equipped with a reflexive binary relation $\trianglerighteq_k$ where $t_k \trianglerighteq_k t'_k$ means that t_k is compatible with t'_k. For each $\{t, t'\} \in T = \times_{k=1}^{K} T_k$, a donor of type t is compatible with a patient of type t' if and only if for each $k \in \{1, \dots, K\}$, $t_k \trianglerighteq_k t'_k$. This is denoted as $t \trianglerighteq t'$.

   In case that a component medical type space is defined by blood types, written as $(T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$,

   $$T_{\mathcal{B}} := \{O, A, B, AB\}, \text{ and}$$

   $$\trianglerighteq_{\mathcal{B}} := \{(O, A), (O, B), (O, AB), (A, AB), (B, AB)\} \cup \{(X, X) \mid X \in \mathcal{B}\}.$$

   We call $(T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$ **the ABO type space**. We assume that the collection of component type spaces $\{(T_k, \trianglerighteq_k)\}_{k=1}^{K}$ contains the ABO type space.

5. $\theta = (\theta_{d_c}, \theta_1, \dots, \theta_n)$ represents the medical status of all agents. For brevity but slight confusion, we call it a type profile.

   - $\theta_{d_c} := (\theta_{d_c q}, \theta_{d_c T}) \in \{1, 2\} \times T$ is the medical status, or type, of the cadaveric donor where the cadaveric donor can supply $\theta_{d_c q}$ lungs and is of medical type $\theta_{d_c T}$. Let $\Theta_{d_c} := \{1, 2\} \times T$ denote the set of types of the cadaveric donor.
   - For each $i \in N$, $\theta_i = (\theta_i(i), (\theta_i(d))_{d \in D_i^L}) \in T^{\{i\} \times D_i^L}$ is the medical status, or type, of patient i which indicates her own medical type $\theta_i(i) \in T$ and her living donors' medical types $(\theta_i(d))_{d \in D_i^L} \in T^{D_i^L}$. We assume that the set of living donors of patient i contains at most one donor whose type is compatible with patient i. That is, for all $d, d' \in D_i^L, \theta_i(d) \trianglerighteq \theta_i(i)$ and $\theta_i(d') \trianglerighteq \theta_i(i)$ imply $d = d'$. This is because a patient would conduct a living-donor transplant with her own compatible donors if she has at least two compatible donors. In other words, our model captures the market for the patients who cannot have a transplant with only own donors. Note that this simplification is also employed in Ergin, Sönmez, and Ünver (2017).
   - Let Θ_i be the set of types of patient i. Let $\Theta := \Theta_{d_c} \times \Theta_1 \times \dots \times \Theta_n$ be the set of type profiles. For each $i \in N$, let $\Theta_{-i} := \Theta_{d_c} \times \prod_{j \neq i} \Theta_j$. For notational simplicity, given $\theta \in \Theta$, let $\theta(i)$ denote $\theta_i(i)$ for each $i \in N$, and $\theta(d)$ denote $\theta_i(d)$ for each $i \in N$ and each $d \in D_i^L$.

6. The symbol $\succeq$ represents a priority order for patients. Mathematically, it is a complete, transitive and anti-symmetric binary relation over N.

A dual organ market is the model above together with a preference profile of patients. We will introduce the preferences after defining allocations in the next subsection.

2.2 Allocation

To describe the notion of assignments, we first clarify what kind of transplants are potentially available for each patient $i \in N$ under a given type profile θ. A **transplant** for patient i is expressed by a pair $x_i = (x_i^C, x_i^L)$ where x_i^C is the number of cadaveric lungs and x_i^L is the set of living donors. Given $\theta \in \Theta$, we classify available transplants as follows.

1. **Double-lung transplantation** (also known as bilateral transplantation): Transplanting two lungs of a deceased donor to a patient. The set of such transplants for patient i is denoted by

$$X_i^{20}(\theta) := \begin{cases} \{(2, \emptyset)\} & \text{if } \theta_{d_{cq}} = 2 \text{ and } \theta_{d_cT} \trianglerighteq \theta(i), \\ \emptyset & \text{otherwise.} \end{cases}.$$

2. **Single-lung transplantation**: Transplanting a single lung of a deceased donor to a patient. The set of such transplants for patient i is denoted by

$$X_i^{10}(\theta) := \begin{cases} \{(1, \emptyset)\} & \text{if } \theta_{d_cT} \trianglerighteq \theta(i), \\ \emptyset & \text{otherwise.} \end{cases}$$

3. **Living-donor lung transplantation**: Transplanting two lobes from two living donors, one for each, to a patient. The set of such transplants for patient i is denoted by

$$X_i^{02}(\theta) := \left\{ (0, x^L) \;\middle|\; \begin{array}{l} i)\ x^L \in 2^{D^L} \text{ and } |x^L| = 2, \text{ and} \\ ii)\ \forall d \in x^L, \theta(d) \trianglerighteq \theta(i) \end{array} \right\}$$

4. **Hybrid lung transplantation**: Transplanting a single lung of a deceased donor and a lobe of a living donor to a patient. The set of such transplants for patient i is denoted by

$$X_i^{11}(\theta) := \begin{cases} \left\{ (1, x^L) \;\middle|\; \begin{array}{l} i)\ x^L \in 2^{D^L} \text{ and } |x^L| = 1, \text{ and} \\ ii)\ \forall d \in x^L, \theta(d) \trianglerighteq \theta(i) \end{array} \right\} & \text{if } \theta_{d_cT} \trianglerighteq \theta(i), \\ \emptyset & \text{otherwise.} \end{cases}$$

   This was newly conducted at Okayama University Hospital. In particular, let $\tilde{X}_i^{11}(\theta)$ be the set of hybrid transplants with i's own donor, i.e., $\tilde{X}_i^{11}(\theta) := \{(1, x^L) \in X_i^{11}(\theta) \mid x^L \subseteq D_i^L\}$.

5. **Null transplantation**: The transplant $(0, \emptyset)$, called the **null transplant**, means that patient i will not receive any transplant. The set $X_i^{00}(\theta) = \{(0, \emptyset)\}$ denotes the one that contains only the null transplant.[11]

[11] Actually, the set $X_i^{00}(\theta)$ does not depend on the type profile θ. However, for notational consistency, we do not use the notation without the reference for the type profile such as X_i^{00}.

Figure 2: Inflow and outflow are not balanced.

Then the set of potentially possible transplants for patient i under θ, $X_i(\theta)$, is defined as follows.

$$X_i(\theta) := X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup X_i^{11}(\theta) \cup X_i^{02}(\theta) \cup X_i^{00}(\theta).$$

We will later describe by preferences which transplant is sufficient or not for saving a patient.

Given a type profile $\theta \in \Theta$, an **allocation**, $a^\theta = (a_i^\theta)_{i \in N} = \left((a_i^{\theta C}, a_i^{\theta L})\right)_{i \in N} \in \prod_{i \in N} X_i(\theta)$, describes a distribution of transplants among patients. In particular, we say that $a_i^\theta \in X_i(\theta)$ is an **assignment** of patient i at a^θ, or a **transplant** of patient i at a^θ. We impose three conditions on allocations. The first is a physical constraint: the number of cadaveric lungs used at the allocation cannot exceed the number of cadaveric lungs supplied by the deceased donor. That is,

$$\sum_{i \in N} a_i^{\theta C} \leq \theta_{d_c q}. \tag{1}$$

The second is also a physical constraint: the number of lobes of lung that a living donor can donate is at most one. Note that for each patient i, $a_i^{\theta L} \in 2^{D^L}$ describes who donates a lobe of lung to i. Thus, to formalize the second condition, we just need to require that a living donor should not be included at two living-donor assignments at an allocation. That is,

$$\forall i, j \in N \text{ with } i \neq j, a_i^{\theta L} \cap a_j^{\theta L} = \emptyset. \tag{2}$$

The last condition is motivated by the allocations described in Figure 2. At the allocation in the left figure, patient i's donor d_i provides a lobe of lung to patient j, even though her intended patient i does not receive a transplant. This allocation would fail to achieve the goal of donor d_i who participates in the market to relieve patient i. At the allocation in the right figure, both patients i and j receive transplants. However, their treatment is very different. Patient i receives a donation from d_j in exchange for the donation by donors' d_{i2} and d_{i3}. This situation might arise when the type of patient i is so rare that she cannot find a compatible donor except for d_{i1}. In this case, if i finally finds d_j as her compatible donor, she and her own donors might be willing to accept d_j in exchange for donors' d_{i2} and d_{i3}. However, this type of exchange has a flavor of price mechanism which is strictly prohibited for the distribution of organs in most countries.

In this paper, we will not be involved in a radical interpretation of an allocation that needs a drastic change of Organ Transplant Law. To construct an allocation system without controversial concepts, we employ the cautious condition, requiring that each patient's benefit from the market as the number of lobes of lung from others' donors should not exceed the contribution of her own donors to other patients as the number of lobes of lung. That is,

$$\forall i \in N, \left|a_i^{\theta L} \backslash D_i^L\right| \leq \left|D_i^L \cap \left(\cup_{j \neq i} a_j^{\theta L}\right)\right|. \tag{3}$$

The lefthand side of the inequality is the number of lobes of lung from other patients' living donors, while the righthand side of the inequality is the number of i's own donors who donate to other patients. We call this condition the **Balanced Condition**, or BC for short. Note that the allocations in Figure 2 are excluded because patient j violates BC. Let us emphasize that this condition is implicitly employed in the Ergin, Sönmez, and Ünver's (2017) live-donor exchange model . Thus, our model with cadaveric and living donors under BC is a natural extension of their model. We denote the set of all allocations under θ by $\mathcal{A}(\theta)$. Moreover, let $\mathcal{A}$ be the set of all potentially possible allocations, i.e., $\mathcal{A} := \cup_{\theta \in \Theta} \mathcal{A}(\theta)$.

Our notion of allocations describes not only which patients are assigned lungs and lung lobes from donors, but also which transplants are conducted. This point is technically important. For example, if the assignment $(1, \{d_i\})$ for patient i just described the former, it would not be clear whether the actual transplant is single-lung or hybrid. For this reason, we interpret the assignment a_i^θ of agent i as the conducted transplant, and assume that the transplant using all of the lungs described in a_i^θ will be conducted.

Remark 1 (Lungs not described in an assignment). Based on the above interpretation, we explain how lungs that do not appear in an allocation are treated.

1. At an allocation $a^\theta \in \mathcal{A}(\theta)$, if $\sum_{i \in N} a_i^{\theta C} < \theta_{d_c q}$, then $\theta_{d_c q} - \sum_{i \in N} a_i^{\theta C}$ units of the cadaveric lungs are disposed at the allocation a^θ.

2. At an allocation $a^\theta \in \mathcal{A}(\theta)$, if a living donor $d_i \in D_i^L$ does not appear in any patient's assignment i.e., $d_i \notin a_1^{\theta L} \cup \ldots \cup a_n^{\theta L}$, then d_i does not receive any surgery operation at the allocation.

As a consequence of conditions imposed on allocations, we have the following simple pattern of allocations, the Balanced Condition holds with equality.

Proposition 1. *Under any type profile $\theta \in \Theta$, every allocation $a^\theta \in \mathcal{A}(\theta)$ is balanced in the following sense.*

$$\forall i \in N, \left|a_i^{\theta L} \backslash D_i^L\right| = \left|D_i^L \cap \left(\cup_{j \neq i} a_j^{\theta L}\right)\right|.$$

Namely, for each $i \in N$, the number of lobes of a lung donated to i from other patients' donors is balanced with the number of i's donors who donate to other patients.

		Donation of living donors to non-relatives	
		unacceptable	acceptable
Hybrid transpl.	unacceptable	Regime O	Regime E
	acceptable	Regime H	Regime EH

Table 1: Four regimes

2.3 Preference

We formulate the preferences of patients. To this end, it is useful to have the notations $\mathcal{R}(Z)$ and $\mathcal{P}(Z)$ for any finite set Z: $\mathcal{R}(Z)$ is the set of complete and transitive binary relations on Z, while $\mathcal{P}(Z)$ is the set of complete, transitive, and anti-symmetric binary relations on Z.

We assume that each patient has a preference on the set of transplantation types $\{20, 10, 11, 02, 00\}$, where $20, 10, 11, 02$, and 00 stand for double-lung, single-lung, hybrid, living-donor, and null transplantation, respectively. That is, each patient $i \in N$ has a strict preference $R_i \in \mathcal{P}(\{20, 10, 11, 02, 00\})$ on the set of transplantation types. Note that the formulation of a preference is free from a given type profile θ. Let $\mathcal{R}$ be the set of preferences, i.e., $\mathcal{R} := \mathcal{P}(\{20, 10, 11, 02, 00\})$. For each $R_i \in \mathcal{R}$, the anti-symmetric part and symmetric part of R_i are denoted by P_i and I_i, respectively. For each $R_i \in \mathcal{R}$, a transplantation type α is **acceptable** at R_i if she prefers α to the null transplantation 00. Let $Ac_i(R_i)$ be the set of acceptable transplantation types at R_i, i.e., $Ac_i(R_i) := \{\alpha \mid \alpha\ P_i\ 00\}$.

Based on a preference $R_i \in \mathcal{R}$ over transplantation types, we induce a preference over available transplants under a type profile θ. Namely, given $\theta \in \Theta$, we assume that a patient $i \in N$ with her preference $R_i \in \mathcal{R}$ has a preference $R_i(\theta) \in \mathcal{R}\left(X_i(\theta)\right)$ defined as follows;

$$\forall \alpha, \beta \in \{20, 10, 11, 02, 00\}, \forall x_i \in X_i^{\alpha}(\theta), \forall y_i \in X_i^{\beta}(\theta), x_i\ R_i(\theta)\ y_i \Leftrightarrow \alpha\ R_i\ \beta.$$

Note that, in the induced preference $R_i(\theta)$, patients are indifferent between two transplants in the same type transplantation. Without any confusion, we abuse the notation R_i to represent $R_i(\theta)$. That is, we write $x_i\ R_i\ y_i$ instead of $x_i\ R_i(\theta)\ y_i$ for two transplants $x_i, y_i \in X_i(\theta)$ under a type profile θ.

Given a type profile θ, a transplant $x_i \in X_i(\theta)$ is called **acceptable** to agent i with her preference R_i if she prefers s transplant x_i to nothing, i.e., $x_i\ P_i\ (0, \emptyset)$. Let $Ac_i(R_i; \theta)$ be the set of acceptable transplants for agent i with R_i under θ.

A **preference profile** is a list $R = (R_i)_{i \in N} \in \mathcal{R}^N$ consisting of preferences of all patients. The set of all preference profiles $\mathcal{R}^N$ is called the **preference domain**.

2.4 Regimes: Legal Constraints

The feasibility of an allocation is determined not only by medical technologies but also by social environments. By social environment we mean legal and ethical one that stipulates whether a medically possible transplant is socially acceptable and implementable without much administrative and monetary burden. In our paper we examine two types of constraints for social environments: (i) hybrid transplantation and (ii) donation of living donors to non-relative patients. According to how these constraints are treated, we consider the following (2×2) kinds of regimes (See Table 1).

2.4.1 Regime *O*

Regime *O* is the environment before the introduction of live-donor exchange and hybrid transplantation technology.[12] Since each patient cannot have a live-donor transplant with her own donors, only cadaveric (single-lung or double-lung) transplantation is possible. Thus, given $\theta \in \Theta$, an allocation $a^\theta \in \mathcal{A}(\theta)$ is **feasible** under regime O if for each $i \in N$, $a_i^\theta \in X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup X_i^{00}(\theta)$. Let $\mathcal{A}^O(\theta)$ be the set of feasible allocations at θ under regime O.

2.4.2 Regime *E*

Regime *E* is the environment with the introduction of live-donor exchange to the original market. Donor exchanges (Ergin, Sönmez, and Ünver, 2017) are allowed, in addition to cadaveric (single-lung or double-lung) and living donor transplantation. Thus, given $\theta \in \Theta$, an allocation $a^\theta \in \mathcal{A}(\theta)$ is **feasible** under regime E if for each $i \in N$, $a_i^\theta \in X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup X_i^{02}(\theta) \cup X_i^{00}(\theta)$. Let $\mathcal{A}^E(\theta)$ be the set of feasible allocations at θ under regime E.

2.4.3 Regime *H*

Regime *H* is the environment with the introduction of hybrid transplantation technology to the original market. The hybrid transplant between a patient and one of her own donors is allowed, in addition to cadaveric lung transplantation. Thus, given $\theta \in \Theta$, an allocation $a^\theta \in \mathcal{A}(\theta)$ is **feasible** under regime H if for each $i \in N$, $a_i^\theta \in X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$. Let $\mathcal{A}^H(\theta)$ be the set of feasible allocations at θ under regime H.

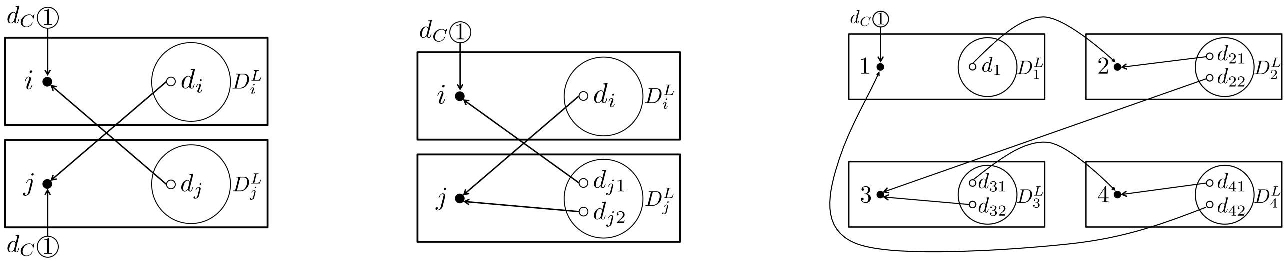
2.4.4 Regime *HE*

Regime *HE* is the environment with the introduction of both live-donor exchange and hybrid transplantation technology to the original market. All types of (cadaveric lung, living donor, hybrid) transplantation are possible. Thus, given $\theta \in \Theta$, every allocation $a^\theta \in \mathcal{A}(\theta)$ is **feasible** under regime HE. Let $\mathcal{A}^{HE}(\theta)$ be the set of feasible allocations at θ under regime HE. The new patterns of exchange that our paper advocates is an exchange of donors for hybrid transplantation. At the left allocation in Figure 3, both patients i and j receive a hybrid transplant with the other's donor. At the middle in Figure 3, patient i receives a hybrid transplant while j receives a living-donor transplant. This allocation suggests that the hybrid transplant makes the living-donor transplant possible.[13] Theoretically,

[12] Because this is the "original" state of the market, we call the regime "O".

[13] The literature on kidney exchange considers the system that allows patients to exchange their donors with the right to receive a cadaveric kidney (Roth, Sönmez, and Ünver, 2004). Note that it contains a exchange of living-donor kidney

Figure 3: Hybrid exchange opens up new patterns of allocation.



the number of living-donor transplants caused by a hybrid transplant can be any large number. At the right in Figure 3, three living-donor transplants are implemented.

Remark 2. By definition, we have the following relations among the sets of feasible allocations under various regimes.

$$\begin{array}{ccc} \mathcal{A}^O(\theta) & \subseteq & \mathcal{A}^E(\theta) \\ \cap\!\!| & & \cap\!\!| \\ \mathcal{A}^H(\theta) & \subseteq & \mathcal{A}^{HE}(\theta) \end{array}$$

2.5 Dual organ markets under various regimes

Now we summarize our model. A **dual organ market under regime Y** $\in \{O, E, H, HE\}$ consists of the following components:

1. $(N, \{D_i^L\}_{i \in N}, D^C, (T, \trianglerighteq), \theta, \succeq)$ as described in Section 2.1.

2. $R = (R_i)_{i \in N} \in \mathcal{R}^N$, a preference profile as described in Section 2.3;

3. $\mathcal{A}^Y(\theta)$, the set of feasible allocations under regime Y as described in Section 2.4.

We assume that there is a clearinghouse whose goal is to distribute transplants among agents in a "desirable" way.[14] To do so, it needs to collect the decentralized information about

- types of market participants $\theta = (\theta_i)_{i \in N}$ and

- preferences of patients R.

Together with the type of the cadaveric donor, the clearinghouse processes the information in determining a feasible allocation.[15] The procedure is called a mechanism. Formally, a **mechanism under regime** Y is a function φ from $\mathcal{R}^N \times \Theta$ to $\cup_{\theta \in \Theta} \mathcal{A}^Y(\theta)$ such that for each $(R; \theta) \in \mathcal{R}^N \times \Theta$,

and deceased-donor kidney. What is new in the middle allocation in Figure 3 is that patient i who receives a hybrid transplant exchanges her living-donor with patient j's living-donor.

[14]In the next section, we will discuss what allocations are desirable.

[15]We assume that the type information of the cadaveric donor is known by the clearinghouse whenever a cadaveric donor appears. This is natural because it is reported by a medical doctor who is in charge of the cadaveric donor. The clearinghouse shares the information with medical doctors in charge who are out of our model.

$\varphi(R;\theta) \in \mathcal{A}^Y(\theta)$. That is, the clearinghouse has patients report their types and preferences. Expressing the information by type and preference profile $(R;\theta) \in \mathcal{R}^N \times \Theta$, the clearinghouse uses a mechanism φ to determine an allocation $\varphi(R;\theta) \in \mathcal{A}^Y(\theta)$ feasible under regime Y.

3 Properties of Mechanisms

In a general indivisible goods resource allocation problem, the desirable properties for a mechanism to satisfy are individual rationality, efficiency, fairness, and incentive compatibility. We introduce these properties for our dual organ market. We fix a regime $Y \in \{O, E, H, HE\}$ throughout this section.

3.1 Individual rationality

We introduce individual rationality for our model. It is a condition on allocations in which any patient does not hurt from participating in a market. In other words, it is not the case that any patient receives her transplant worse than the null transplant $(0, \emptyset)$. Thus, the individual rationality is the minimum natural requirement for any allocation to be implemented. Formally we define:

Definition 1. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **individually rational** at $(R;\theta) \in \mathcal{R}^N \times \Theta$ if for each patient $i \in N$, a_i^θ is acceptable at R_i. We denote by $\mathcal{I}^Y(R;\theta)$ the set of all individually rational allocations at $(R;\theta)$. Moreover, a mechanism under regime Y, φ, is **individually rational** if for each $(R;\theta) \in \Theta \times \mathcal{R}^N$, the selected allocation $\varphi(R;\theta) \in \mathcal{A}^Y(\theta)$ is individually rational at $(R;\theta)$.

3.2 Efficiency

We define three notions of efficiency. The first is the standard notion of Pareto efficiency. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **Pareto efficient** at $(R;\theta) \in \mathcal{R}^N \times \Theta$ if there is no allocation $b^\theta \in \mathcal{A}^Y(\theta)$ such that for each $i \in N$, $b_i^\theta \ R_i \ a_i^\theta$, and for some $i \in N$, $b_i^\theta \ P_i \ a_i^\theta$. A mechanism under regime Y, φ, is **Pareto efficient** if for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, the selected allocation $\varphi(R;\theta)$ is Pareto efficient at $(R;\theta)$.

We next introduce the notions of non-wastefulness for our dual organ market. The standard notion of non-wastefulness is defined for indivisible goods resource allocation problems under common ownership and unit demand where no agent initially owns an object and an agent consumes one object. The non-wastefulness means that "unused" objects cannot be assigned to benefit some agent without affecting anybody else's assignment (Balinski and Sönmez, 1999). On the other hand, our dual organ market has mixed ownership and multi-unit demand. Thus the unused objects available for a patient are disposed cadaveric lungs and a lobe of her own donors who do not donate at the original allocation. Thus, each patient could potentially access to the unused objects in addition to the original assignment without affecting anybody else's assignment. Although the set of better opportunities can be well captured by the set of unused objects under unit demand, the appropriate extension to our model with multi-unit demand and mixed ownership should include both unused objects and the original individual assignment. If a patient finds a better transplant within her potentially accessible objects, she might feel that the original assignment is wasteful in terms of opportunity. The following notion of induced allocation explicitly describes each patient's potentially accessible objects.

Definition 2 (Induced allocation, or shadow allocation)**.** Given a type profile $\theta \in \Theta$ and an allocation $a^\theta = \big((a_i^{\theta C}, a_i^{\theta L})\big)_{i \in N} \in \mathcal{A}(\theta)$, we define the **induced allocation** $\overline{a^\theta} = \big((\overline{a}_i^{\theta C}, \overline{a}_i^{\theta L})\big)_{i \in N}$ as follows: For each $i \in N$,

$$(i) \quad \overline{a}_i^{\theta C} := a_i^{\theta C} + \Big(\theta_{d_c q} - \textstyle\sum_{j \in N} a_j^{\theta C}\Big), \text{ and}$$
$$(ii) \quad \overline{a}_i^{\theta L} := a_i^{\theta L} \cup \{d \in D_i^L \big| d \notin \cup_{j \neq i} a_j^{\theta L}\}.$$

In words, patient i's induced cadaveric lung assignment $\overline{a}_i^{\theta C}$ is the sum of numbers of cadaveric lung she receives at a^θ and the disposed cadaveric lungs at a^θ. Patient i's induced live donor assignment $\overline{a}_i^{\theta L}$ denotes the union of her assignment at a^θ and her own donors who do not donate to other patient at a^θ. Thus, $\overline{a}_i^\theta$ formalizes the potentially accessible resources of patient i at a^θ without changing other patients' assignment. Now, we are ready to introduce non-wastefulness.

Definition 3. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **non-wasteful** at $(R; \theta) \in \Theta \times \mathcal{R}^N$ if it is not the case that there exist $i \in N$ and $b_i \in X_i(\theta)$ such that (i) $b_i \; P_i \; a_i^\theta$, (ii) $(b_i; a_{-i}^\theta) \in \mathcal{A}^Y(\theta)$, and (iii) $b_i^C \leq \overline{a}_i^{\theta C}$ and $b_i^L \subseteq \overline{a}_i^{\theta L}$.

The condition says that patient i cannot find (i) a better transplant (ii) which is allowed under regime Y without affecting others' assignments, and (iii) which can be constructed within patient i's accessible resources at a^θ. For example, if $a_i^\theta = (1, \emptyset)$ is a single-lung transplant, then $b_i = (1, \{d_i\})$ can be a hybrid transplant if patient i's compatible own donor d_i does not donate at a^θ, i.e., $d_i \in \overline{a}_i^L$.

In addition to being non-wasteful, an allocation is strongly non-wasteful if it maximizes the number of cadaveric lungs used among individually rational allocations. That is,

Definition 4. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **strongly non-wasteful** at $(R; \theta) \in \mathcal{R}^N \times \Theta$ if a^θ is non-wasteful, and

$$\sum_{i \in N} a_i^{\theta C} = \max_{b^\theta \in \mathcal{I}^Y(R;\theta)} \sum_{j \in N} b_j^{\theta C}.$$

We say that a mechanism under regime Y, φ, is **(strongly) non-wasteful** if for each $(R; \theta) \in \mathcal{R}^N \times \Theta$, the selected allocation $\varphi(R; \theta) \in \mathcal{A}^Y(\theta)$ is (strongly) non-wasteful at $(R; \theta)$.

Example 1 (Non-wasteful but strongly wasteful allocation)**.** Let $(T, \trianglerighteq) = (T_\mathcal{B}, \trianglerighteq_\mathcal{B})$.[16] Let $\theta \in \Theta$ be such that $\theta_{d_c} = (2, A)$ and $\theta(1) = \theta(2) = A$. Suppose also that for each $i \in N \backslash \{1, 2\}$ and each $d \in D^L$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be a preference profile described by the following table.

[16]This assumption enable us to simplify the description without any loss of generality as it corresponds with the assumption that all patients and donors have the identical medical type except for the ABO blood type.

Figure 4: Logical relationship among efficiency concepts

Under regime O or H. Under regime E or HE.

Note: SNW stands for strong non-wastefulness, NW for non-wastefulness, and PE for Pareto efficiency.

R_1	10	00	$\cdots$
R_2	20	00	$\cdots$

Thus patients 1 and 2 can receive donation only from the cadaveric donor. Moreover, patient 1 finds the only single-lung transplant acceptable, and patient 2 the only double-lung transplant acceptable. Consider an allocation $a^\theta := ((1, \emptyset), (0, \emptyset))$ where one cadaveric lung is transplanted to patient 1, but the other is disposed so that patient 2 receives the null transplant.[17] Obviously a^θ is non-wasteful. However, one lung is disposed and the two lungs can be transplanted to patient 2 in another allocation $b^\theta := ((0, \emptyset), (2, \emptyset)) \in \mathcal{I}^Y(R; \theta)$, and thus allocation a^θ is strongly wasteful. Note that $\sum_{i \in N} a_i^{\theta C} = 1 < 2 = \sum_{i \in N} b_i^{\theta C}$. $\diamondsuit$

The following remark clarifies the logical relationship among our efficiency notions (See Figure 4).

Remark 3. We have the following three statements. Sentences without a reference for regime hold under any regime. The proofs are in the Appendix. Let $(R; \theta) \in \mathcal{R}^N \times \Theta$.

1. If an allocation is strongly non-wasteful at $(R; \theta)$, then it is non-wasteful at $(R; \theta)$. The converse is not true.

2. If an allocation is Pareto efficient at $(R; \theta)$, then it is non-wasteful at $(R; \theta)$. The converse is also true only if $Y \in \{O, H\}$.

3. There is no logical relationship between Pareto efficiency and strong non-wastefulness under any regime $Y \in \{E, HE\}$.

3.3 Fairness

In our dual organ market, we have a priority $\succeq$ given as one component of the market. The priority expresses the right of patients receiving cadaveric lungs. In this subsection we introduce the notion of fairness with respect to the priority.

In our notion, each patient i has the the following minimum ethical view that if some patient, j, of lower priority than i uses cadaveric lungs, then patient i can take them from j for her transplant, except when the role of j is critical for a transplant for a patient, k, of higher priority than i.

[17] The assignments for $i \in N \backslash \{1, 2\}$ are necessarily the null transplant $(0, \emptyset)$, and thus they are omitted in the description of the allocations.

Definition 5. Under regime Y, patient $i \in N$ regards an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ as **$\succeq$-unfair** at $(R;\theta) \in \mathcal{R}^N \times \Theta$ *if* there is $b^\theta \in \mathcal{I}^Y(R;\theta)$ such that (i) $b_i^\theta \; P_i \; a_i^\theta$, (ii) $b_i^{\theta C} > a_i^{\theta C}$ and for each $j \in N$ with $j \succ i$, $b_j^{\theta C} = a_j^{\theta C}$ and (iii) for each $j \in N$, $a_j^\theta \; P_j \; b_j^\theta$ implies $a_j^{\theta C} > b_j^{\theta C}$. Moreover, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **$\succeq$-fair** at $(R;\theta) \in \mathcal{R}^N \times \Theta$ if no agent regards a^θ as $\succeq$-unfair at $(R;\theta)$.

In words, a^θ is $\succeq$-unfair at $(R;\theta)$ if a patient i can find an individually rational allocation b^θ in which (i) i is better off, (ii) with cadaveric lungs taken up from a lower-priority patient (iii) without making any other patient worse off except for the one who is taken away a cadaveric lung. We say that a mechanism under regime Y, φ, is **$\succeq$-fair** if for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, the selected allocation $\varphi(R;\theta) \in \mathcal{A}^Y(\theta)$ is $\succeq$-fair at $(R;\theta)$.

The following remark clarifies the logical relationship between $\succeq$-fairness and axioms previously introduced.

Remark 4. We have the following two statements. Sentences without a reference for regime hold under any regime.

1. $\succeq$-fairness does not imply individual rationality. Under any regime $Y \in \{E, HE\}$, even the combination of Pareto efficiency and $\succeq$-fairness does not imply individual rationality.

2. There is no logical relationship between $\succeq$-fairness and any one of the three efficiency notions of non-wastefulness, strong non-wastefulness, and Pareto efficiency.

The following impossibility asserts that strong non-wastefulness is too demanding as an efficiency notion as long as we employ individual rationality and $\succeq$-fairness as basic axioms for a mechanism.

Proposition 2. *Under any regime $Y \in \{O, E, H, HE\}$, there exists $(R;\theta) \in \mathcal{R}^N \times \Theta$ such that no allocation is individually rational, strongly non-wasteful, and $\succeq$-fair at $(R;\theta)$.*

3.4 Incentive compatibility

We employ strategy-proofness as our incentive compatibility condition. In our model, each patient has two pieces of information about herself: type θ_i and preference R_i. We do not treat these information symmetrically, since the former is verifiable by the other medical doctors, while the latter is her private information. Thus, we assume that her reporting type θ_i is sincere, while her reporting preference R_i may not. In other words, preference reporting is the only source of strategic manipulation.

Assumption 1 (Sincere reporting about types)**.** *Each patient reports her type sincerely.*

Under the above assumption, the following is a standard definition of *strategy-proofness.*

Definition 6. A mechanism under regime Y, φ, is **strategy-proof** if for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, each $i \in N$, and each $R_i' \in \mathcal{R}$, $\varphi_i(R;\theta) \; R_i \; \varphi_i(R_i', R_{-i};\theta)$.

4 Main Results

In this section, we first investigate the mechanism currently used in Japan under regime O.[18] We show that the Japanese mechanism is individually rational and Pareto efficient, but is neither $\succeq$-fair nor strategy-proof. Then, we propose priority mechanisms that can be used for any regime, and show its prominence for our dual organ market. To define these mechanisms, we use the notation $Top(\succeq, M)$ which is the highest-priority patient among those in the non-empty set M of patients. That is, for each $i \in M$, $Top(\succeq, M) \succeq i$.

4.1 Case study: the Japanese mechanism

We describe the Japanese mechanism that works only under regime O. To this end, we introduce the following notations. For each $(R;\theta) \in \mathcal{R}^N \times \Theta$,

- $N(R;\theta) := \{i \in N \mid (2,\emptyset) \in Ac_i(R_i;\theta) \text{ or } (1,\emptyset) \in Ac_i(R_i;\theta)\}$ is the set of all candidates of the deceased donor transplantation;
- $N^{20}(R;\theta) = \{i \in N(R;\theta) \mid 20\ P_i\ 00\ P_i\ 10\}$ is the set of all patients who think the only the double-lung transplants as acceptable;
- $N^{21}(R;\theta) = \{i \in N(R;\theta) \mid 20\ P_i\ 10\ P_i\ 00\}$ is the set of all patients who prefer the double-lung transplants to single-lung ones;
- $N^{10}(R;\theta) = \{i \in N(R;\theta) \mid 10\ P_i\ 00\ P_i\ 20\}$ is the set of all patients who think the only single-lung transplants as acceptable;
- $N^{12}(R;\theta) = \{i \in N(R;\theta) \mid 10\ P_i\ 20\ P_i\ 00\}$ is the set of all patients who prefer the single-lung transplants to double-lung ones.

Note that the sets $N^{20}(R;\theta)$, $N^{21}(R;\theta)$, $N^{10}(R;\theta)$, and $N^{12}(R;\theta)$ are disjoint.

Definition 7 (Japanese mechanism)**.** Under the regime O, the **Japanese mechanism**, φ^J, selects an allocation for each $(R;\theta) \in \mathcal{R}^N \times \Theta$ as follows.

First, we consider the case with $\theta_{d_c q} = 2$. If $N(R;\theta) = \emptyset$, then let $\varphi^J(R;\theta) = \big((0,\emptyset), \cdots, (0,\emptyset)\big)$. Otherwise, let $i_1 := Top(\succeq; N(R;\theta))$.

<u>Case 1:</u> $i_1 \in N^{20}(R;\theta) \cup N^{21}(R;\theta)$.

For each $i \in N$,

$$\varphi^J_i(R;\theta) = \begin{cases} (2,\emptyset) \text{ if } i = i_1, \\ (0,\emptyset) \text{ if } i \neq i_1. \end{cases}$$

<u>Case 2:</u> $i_1 \in N^{10}(R;\theta) \cup N^{12}(R;\theta)$ and $\big(N^{10}(R;\theta) \cup N^{12}(R;\theta)\big) \setminus \{i_1\} \neq \emptyset$.

[18] We formalize the mechanism based on the recipient selection rule described in `http://www.jotnw.or.jp/jotnw/law_manual/pdf/rec-lungs.pdf`.

Let $i_2 := Top(\succeq; \left(N^{10}(R;\theta) \cup N^{12}(R;\theta)\right) \backslash \{i_1\})$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1, \emptyset) \text{ if } i = i_1 \text{ or } i = i_2, \\ (0, \emptyset) \text{ if } i \in N \backslash \{i_1, i_2\}. \end{cases}$$

Case 3: $N^{10}(R;\theta) \cup N^{12}(R;\theta) = \{i_1\}$.

Case 3.1: $N^{21}(R;\theta) \neq \emptyset$.

Let $i_3 := Top(\succeq; N^{21}(R;\theta))$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1, \emptyset) \text{ if } i = i_1 \text{ or } i = i_3, \\ (0, \emptyset) \text{ if } i \in N \backslash \{i_1, i_3\}. \end{cases}$$

Case 3.2: $N^{21}(R;\theta) = \emptyset$.

Case 3.2.1: $i_1 \in N^{10}(R;\theta)$ and $N^{20}(R;\theta) \neq \emptyset$.

Let $i_4 := Top(\succeq; N^{20}(R;\theta))$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (2, \emptyset) \text{ if } i = i_4, \\ (0, \emptyset) \text{ if } i \neq i_4. \end{cases}$$

Case 3.2.2: $i_1 \in N^{12}(R;\theta)$ or $N^{20}(R;\theta) = \emptyset$.

For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1, \emptyset) \text{ if } i = i_1, \\ (0, \emptyset) \text{ if } i \neq i_1. \end{cases}$$

Next, we consider the case with $\theta_{d_cq} = 1$. If $N^{21}(R;\theta) \cup N^{12}(R;\theta) \cup N^{10}(R;\theta) = \emptyset$, then let $\varphi^J(R;\theta) = \left((0,\emptyset), \cdots, (0,\emptyset)\right)$. Otherwise,

Case 4: $N^{12}(R;\theta) \cup N^{10}(R;\theta) \neq \emptyset$.

Let $i_5 := Top(\succeq; N^{12}(R;\theta) \cup N^{10}(R;\theta))$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1, \emptyset) \text{ if } i = i_5, \\ (0, \emptyset) \text{ if } i \neq i_5. \end{cases}$$

Case 5: $N^{12}(R;\theta) \cup N^{10}(R;\theta) = \emptyset$.

Let $i_6 := Top(\succeq; \left(N^{21}(R;\theta)\right)$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1, \emptyset) \text{ if } i = i_6, \\ (0, \emptyset) \text{ if } i \neq i_6. \end{cases}$$

The Japanese mechanism has aspiration for a strongly non-wasteful allocation in a situation where the compatible highest-priority patient has the unit demand with rejecting two units. In that situation, if (1) there is no lower-priority patient with unit demand and (2) at least one lower-priority patient demands two units, then the mechanism skips the compatible highest-priority patient, and assigns two lungs to the compatible highest-priority patient who demands two units to reduce the number of disposed cadaveric lungs (See Case 3.2.1 in the definition of the Japanese mechanism).[19]

[19]More precisely, this type of skip is not applied if the cadaveric donor is not one of the relatives of the compatible highest-priority patient at Case 3.2.1. Since we assume that $D^C \cap D^L = \emptyset$, this case is omitted from the description of the Japanese mechanism.

Table 2: Preferences for Example 2

R_1	10	20	00	$\cdots$
R_2	20	00	$\cdots$	
R'_1	10	00	$\cdots$	

However, the attempt to reduce the number of disposed cadaveric lungs is not fully achieved by the current mechanism, i.e., there is another situation where it fails to assign cadaveric lungs in a strongly non-wasteful manner. Moreover, the anomalistic allocation by the mechanism can be a source of unfairness and strategic manipulation. These points are captured by the following example.

Example 2 (Flaws of the Japanese mechanism)**.** Suppose that only patients 1 and 2 have the identical type with the two units of cadaveric lungs. That is, let $\theta \in \Theta$ be such that $\theta_{d_c q} = 2$, $\theta(1) = \theta(2) = \theta_{d_c T}$, and for each $i \in N \backslash \{1, 2\}$, $\theta_{d_c T} \not\trianglerighteq \theta(i)$. Suppose also that patient 1 has higher priority than patient 2 does, i.e., $1 \succ 2$.

Flaw 1 (φ^J is strongly wasteful). Let $R \in \mathcal{R}^N$ be a preference profile in Table 2. Then $N(R;\theta) = \{1, 2\}$, $N^{12}(R;\theta) = \{1\}$, $N^{20}(R;\theta) = \{2\}$, $N^{21}(R;\theta) = N^{10}(R;\theta) = \emptyset$ and $i_1 = 1$. Thus, Case 3.2.2 in the definition of φ^J applies and thus $\varphi^J(R;\theta) = \big((1,\emptyset),(0,\emptyset)\big)$. However, since $\big((0,\emptyset),(2,\emptyset)\big)$ is individually rational at $(R;\theta)$, $\varphi^J(R;\theta)$ is strongly wasteful at $(R;\theta)$.

Flaw 2 (φ^J is not $\succeq$-fair nor strategy-proof). Let $R'_1 \in \mathcal{R}$ be a preference in Table 2. Let $R' := (R'_1, R_2)$. Then $N(R';\theta) = \{1, 2\}$, $N^{10}(R';\theta) = \{1\}$, $N^{20}(R';\theta) = \{2\}$, and $i_1 = 1$. Thus Case 3.2.1 in the definition of φ^J applies and then $i_4 = 2$ so that $\varphi^J(R';\theta) = ((0,\emptyset),(2,\emptyset))$. Higher-priority patient 1 can be better off from a lung assigned to lower-priority patient 2 at $\varphi^J(R';\theta)$. Thus, $\varphi^J(R';\theta)$ is not $\succeq$-fair at $(R';\theta)$.

Now consider the strategic deviation of patient 1 from R'_1 to R_1. Then, the allocation selected at $(R_1, R_2;\theta) = (R;\theta)$ is $\big((1,\emptyset),(0,\emptyset)\big)$ (See the calculation at Flaw 1). Thus, $\varphi^J_1(R_1, R_2;\theta) = (1,\emptyset)\ P'_1\ (0,\emptyset) = \varphi^J_1(R'_1, R_2;\theta)$, which violates strategy-proofness. $\Diamond$

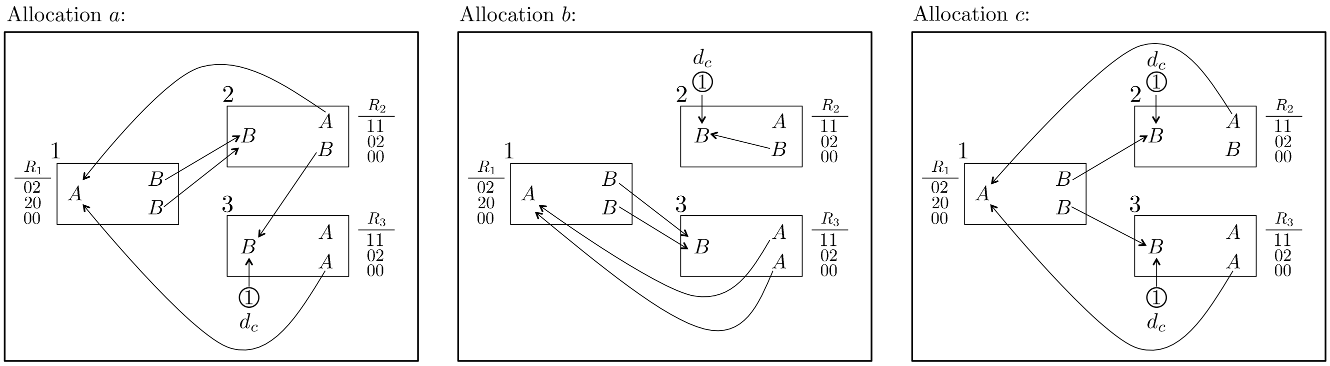
The following summarizes the properties of the Japanese mechanism. The proof is in the Appendix.

Proposition 3. *Under regime O, the Japanese mechanism φ^J is (i) individually rational, (ii) Pareto efficient, (iii) not strongly non-wasteful, (iv) not $\succeq$-fair, and (v) not strategy-proof.*

4.2 Priority mechanism

To introduce the priority mechanism, we need the following notation. Let $\sigma : \{1, \ldots, n\} \to N$ be the bijection that represents the priority order. That is, for each $i \in \{1, \ldots, n\}$, σ selects the i-th highest-priority patient, i.e., $\sigma(1) \succ \sigma(2) \succ \cdots \succ \sigma(n)$.

Figure 5: Allocations a, b and c in Example



Definition 8 (Priority mechanism)**.** The **priority correspondence under regime** Y, Φ^Y, is the nonempty-valued correspondence from $\mathcal{R}^N \times \Theta$ to $\cup_{\theta \in \Theta} \mathcal{A}^Y(\theta)$ that selects feasible allocations in $\mathcal{A}^Y(\theta)$ for each $(R;\theta) \in \mathcal{R}^N \times \Theta$ as follows.

Round 0: $\Phi_0^Y(R;\theta)$ selects all individually rational allocations at $(R;\theta)$. That is, $\Phi_0^Y(R;\theta) := \mathcal{I}^Y(R;\theta)$.

Round $k \in \{1, \ldots, n\}$: $\Phi_k^Y(R;\theta)$ selects all allocations that k^{th}-priority patient, $\sigma(k)$, prefers the most among those in $\Phi_{k-1}^Y(R;\theta)$. That is, $\Phi_k^Y(R;\theta) := \left\{ a^\theta \in \Phi_{k-1}^Y(R;\theta) \mid \forall b^\theta \in \Phi_{k-1}^Y(R;\theta), a_{\sigma(k)}^\theta \; R_{\sigma(k)} \; b_{\sigma(k)}^\theta \right\}$.

Let $\Phi^Y(R;\theta) := \Phi_n^Y(R;\theta)$.

A **priority mechanism** under regime Y, φ^P, is a selection from the priority correspondence, i.e., for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, $\varphi^P(R;\theta) \in \Phi^Y(R;\theta)$.

Example 3 (Priority mechanism)**.** This example describes the procedure of the priority mechanism (Figure 5). There are three patients each of whom has exactly two living donors. For each $i \in \{1,2,3\}$, the types of patient i and her donors are shown in box i in the figures. For example, in box 1, patient 1's type is shown in the left-hand side of the box, i.e., $\theta_1(1) = A$, while her two donors have the identical type B as shown in the right-hand side of the same box. Suppose that patient 1 has the highest priority, patient 2 the second, and patient 3 the third. The preferences of patients are given as follows. $R_1 : 02, 20, 00$, $R_2 : 11, 02, 00$, $R_3 : 11, 02, 00$.

In the first round, patient 1 chooses her assignment. Since her best is living-donor transplantation, all feasible allocations in which patient 1 receives a living donor transplant are selected by the priority correspondence. Such allocations are abundant, and some of them are given in Figure 5: allocations a, b, and c. Note that all three allocations are indifferent for patient 1, while patient 2 and 3 are not.

In the second round, patient 2 chooses her assignment. Since her best is hybrid transplantation, some allocations selected in the first round are rejected. Among allocations a, b, and c, allocation a is rejected, since it assigns a living-donor transplant to patient 2. On the other hand, allocations b and c remain selected at the second round, since it assigns a living-donor transplant to patient 1 and a hybrid transplant to patient 2. Note that patient 3 is not indifferent between b and c.

In the third round, patient 3 chooses her assignment. Since her best is hybrid transplantation, some allocations selected in the second round are rejected. Among allocations b and c, allocation b is rejected, since it assigns a living-donor transplant to patient 3.

Consequently, the priority correspondence contains allocation c. As shown in this example, in each round, the priority mechanism selects the best feasible allocations for the chooser at the round not to harm the higher-priority patients.[20] $\Diamond$

Remark 5. The difference in regimes makes some differences on the nature of priority mechanisms. The following items 2 and 3 are basic. The proofs are in the Appendix.

1. Under any regime $Y \in \{O, E, H, HE\}$, if both φ and $\tilde{\varphi}$ are priority mechanisms, then they are welfare equivalent. That is, for each $(R; \theta) \in \mathcal{R}^N \times \Theta$ and each $i \in N$, $\varphi_i(R; \theta) \; I_i \; \tilde{\varphi}_i(R; \theta)$.

2. Under any regime $Y \in \{O, H\}$, Φ^Y is single-valued. Thus a priority mechanism is unique.

3. Under any regime $Y \in \{E, HE\}$, Φ^Y may not be single-valued. Thus a priority mechanism may not be unique.

4.2.1 Performance of priority mechanism under regimes O and H

Under regimes without donor exchange (regime O or H), the performance of the priority mechanism is quite good:

Theorem 1. *Under any regime $Y \in \{O, H\}$, the priority mechanism is individually rational, Pareto efficient, $\succeq$-fair, and strategy-proof.*

The priority mechanism overcomes the two major flaws of the Japanese mechanism – the violation of $\succeq$-fairness and strategy-proofness. Moreover, the prominence is kept even under the introduction of hybrid transplantation. However, by Proposition 2, strong non-wastefulness is not overcome. This is viewed as an inevitable cost for a mechanism to be individually rational and $\succeq$-fair.

4.2.2 Performance of priority mechanism under regimes E and HE

Under regimes with donor exchange (regime E or HE), each priority mechanism keeps its good performance for the normative side.

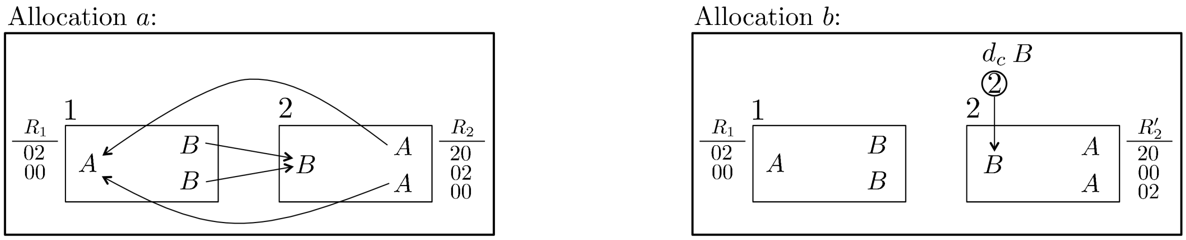
Proposition 4. *Under any regime $Y \in \{E, HE\}$, each priority mechanism is individually rational, Pareto efficient, and $\succeq$-fair. However, it is not strategy-proof.*

One of the critical differences between regimes with and without donor exchange lies on the degree of manipulability of the mechanisms. In particular, under regimes with donor exchange, a patient can hide her own donors by rejecting hybrid and living-donor transplants in her preference.[21] Recall that

[20] Rigorously speaking, a variant of c is also selected by the priority correspondence. It is the allocation in which the donation from the patient 3's second donor is replaced by the one from patient 3's first donor. Note that c and its variant are indifferent for all patients.

[21] Note that hiding donors has no positive effect for that patient under the regimes without donor exchange.

Figure 6: Allocations a and b in Example 4



we focus on balanced allocations (Proposition 1). Thus, if a patient pretends that she cannot accept any hybrid and living-donor transplants, then no other patients can use her living-donors. This type of strategic behavior is a source for the manipulation of a mechanism. Consequently, every priority mechanism is manipulable. To see this, consider the following example.

Example 4 (Manipulation of the priority mechanism)**.** There are two patients, patients 1 and 2, each of whom has exactly two living donors. The priority is $1 \succ 2$. Their medical types and preferences are illustrated in Figure 6.

Under true preferences (R_1, R_2), patient 1 can accept only living-donor transplant, while patient 2 prefers a deceased-donor transplant to a living-donor transplant. Then the priority mechanism selects allocation a in the left figure as follows. In the first round, patient 1 chooses allocation a, since it is her most preferred transplant. In the second round, allocation a is the only allocation that patient 2 can choose. Thus allocation a is selected. Note that this allocation cannot be implemented without patient 2's living donors.

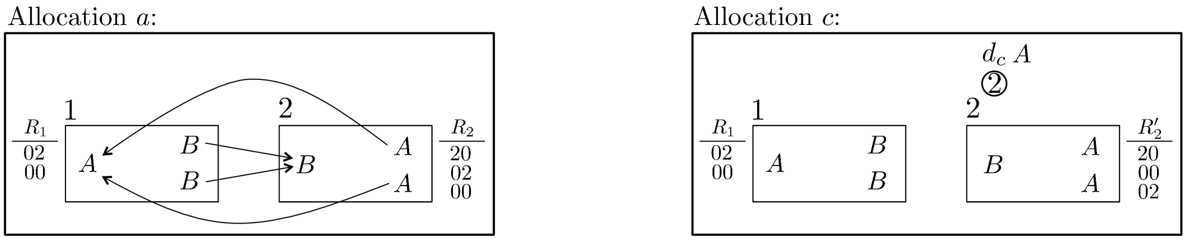
Now suppose that patient 2 deviates from R_2 to R_2'. At R_2', patient 2 hides the fact that she can accept a living-donor transplant. This is a kind of "truncated strategy" well-known in two-sided matching models (Roth and Sotomayor, 1990). Note that, for patient 1, to get a living donor transplant, the living donors of patient 2 are critical. Consequently, since the hiding strategy of patient 2 narrows down the opportunity for patient 1, she cannot help but choose the null transplant in the first round of the priority mechanism. This enables patient 2 to get a more preferred transplant, i.e., the deceased-donor transplant. $\Diamond$

A natural question arising from Proposition 4 is: Is there a mechanism satisfying all axioms listed in the proposition? The answer is negative.

Theorem 2. *Under any regime $Y \in \{E, HE\}$, no mechanism is individually rational, Pareto efficient, $\succeq$-fair, and strategy-proof.*

Since no mechanism satisfies all axioms listed in Theorem 2, we have to give up at least one of the axioms to design a plausible mechanism under a regime with donor exchange. Since Proposition 4 says that under regimes with donor exchange, each priority mechanism satisfies three of our four basic axioms, priority mechanisms attain one of the best we can choose. Moreover, we will see a posi-

Figure 7: Allocations a and c in Example



tive aspect of priority mechanisms which are robust against strategic manipulation under incomplete information, although they are not strategy-proof under complete information.

4.3 Preference revelation game with incomplete information under priority mechanism

The successful manipulation of the priority mechanism in Example 4 heavily depends on the complete information setup. In practice, the type of deceased donors is uncertain, and the types of other patients and living donors are usually not open to the public. Thus the manipulation of the mechanism pertains to the risk of missing an opportunity for a transplant. To see this point, let us consider the previous example with one modification.

Example 5 (Manipulation of the priority mechanism is risky)**.** Consider Example 4 where patient 2 of blood type B gets the deceased donor of the same blood type by manipulating the priority mechanism with the truncation strategy. However, when the blood type of the deceased donor is type A instead of type B, her truncation strategy ends up with no transplants for both patients (Figure 7). In other words, she cannot receive a living-donor transplant as well as a deceased-donor transplant because the truncation strategy not only narrows down the opportunity for the higher-priority patient but also the one for herself. In this sense, the manipulation strategy is risky.

Summing up, without accurate information about resources, i.e., types of deceased-donor and other patients' living donors, the manipulation behavior may be harmful to the manipulator, too. In that sense, the truncation strategy is "a double-edge sword" for the manipulator.[22] $\lozenge$

Motivated by the above example, we introduce incomplete information into our model. We assume that each patient can observe only her own type and preference, not the others'. That is, she knows her own preference and type (R_i, θ_i), but does not know other patients' $(R_{-i}; \theta_{-i})$, including the deceased donor's.[23] Formally, we consider a preference revelation game $G = (N, D^C, \{D_i^L\}_{i\in N}, (T, \trianglerighteq), \Theta_{d_c}, \{\mathcal{R} \times \Theta_i\}_{i\in N}, \{u_i^*\}_{i\in N}, Y, \varphi^P, \{p_i\}_{i\in N})$, where

[22]In Lemma 2 of the Appendix, we show that all successful manipulation strategies of the priority mechanism are necessarily "double-edge" in the sense that they always narrow down the possible assignment for the manipulator.

[23]This setup is suitable for Japan, because patients simply register for the Japan Organ Transplant Network without any communication with other patients.

1. The symbols $N, D^C, \{D_i^L\}_{i\in N}, (T, \trianglerighteq), \Theta_{d_c}$ are the same as in the complete information model: Each of them represents the set of patients, the set of cadaveric donors, the collection of the set of living donors, the type space for the medical status of patients and donors and the type space of the cadaveric donor, respectively.

2. For each $i \in N$, $\mathcal{R} \times \Theta_i$ denotes patient i's action set. It also represents patient i's "type space" in the standard Bayesian game terminology.

3. For each $i \in N$, $u_i^* : \{20, 10, 11, 02, 00\} \times \big(\mathcal{R} \times \Theta_i\big) \to \mathbb{R}$ is a state-dependent utility function. For each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, $u_i^*(\cdot \mid R_i, \theta_i)$ represents R_i. Without any confusion, given $\theta \in \Theta$, for each $x_i^\theta \in X_i(\theta)$, $u_i^*(x_i^\theta \mid R_i, \theta_i)$ denotes the value of $u_i^*(\cdot \mid R_i, \theta_i)$ at the transplantation type of x_i^θ.

4. A priority mechanism φ^P under regime Y is fixed.

5. For each $i \in N$, $p_i : \mathcal{R}^N \times \Theta \to [0, 1]$ is a probability distribution that represents patient i's prior belief. We assume that p_i has full support, i.e., for each $(R; \theta) \in \mathcal{R}^N \times \Theta$, $p_i(R; \theta) > 0$. Note that we do not place the common prior assumption. For each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, the posterior belief is denoted by $p_i(\cdot \mid R_i, \theta_i)$, i.e., it is the function from $\mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ to $[0, 1]$ defined as $p_i(R_{-i}; \theta_{-i} \mid R_i, \theta_i) := \frac{p_i(R_i, R_{-i}; \theta_i, \theta_{-i})}{\sum_{(R'_{-i}; \theta'_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}} p_i(R_i, R'_{-i}; \theta_i, \theta'_{-i})}$.

Now we make an assumption on the players' utility functions that reflects the huge gap in utilities between acceptable transplants and unacceptable ones. To describe it, for each $i \in N$ and each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, we introduce the following two notations:

- $\overline{UD}(u_i^*; R_i, \theta_i) := \begin{cases} \max\limits_{\alpha \in Ac_i(R_i)} u_i^*(\alpha \mid R_i, \theta_i) - \min\limits_{\alpha \in Ac_i(R_i)} u_i^*(\alpha \mid R_i, \theta_i) & \text{if } Ac_i(R_i) \neq \emptyset, \\ 0 \quad \text{if } Ac_i(R_i) = \emptyset. \end{cases}$

- $\underline{p_i}(R_i, \theta_i) := \min\limits_{(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}} p_i(R_{-i}; \theta_{-i} \mid R_i, \theta_i)$.

In words, $\overline{UD}(u_i^*; R_i, \theta_i)$ denotes the utility difference between the best acceptable transplantation and the worst acceptable one which in turn shows the maximal gain from the quality improvement when a patient gets an acceptable transplant instead of another. Given patient i's own type $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, the most unlikely event occurs with probability $\underline{p_i}(R_i, \theta_i)$ in patient i's perspective. Note that $\underline{p_i}(R_i, \theta_i) > 0$, since we assume that p_i has a full support.

Assumption 2 (Huge utility gap between acceptable and unacceptable transplants)**.** *Even at the most unlikely event, the expected utility loss from the worst acceptable transplant to the best unacceptable one is so huge that a patient cannot recover it even if she gets an additional utility $\overline{UD}(u_i^*; R_i, \theta_i)$ in every other event. Formally, for each $i \in N$ and each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$ with at least one acceptable* *transplantation type at R_i, we have the following inequality:*

$$\underline{p}_i(R_i,\theta_i)\Big(\min_{\alpha\in Ac_i(R_i)} u_i^*(\alpha\mid R_i,\theta_i)-u_i^*(00\mid R_i,\theta_i)\Big) > (1-\underline{p}_i(R_i,\theta_i))\Big(\left|\mathcal{R}^{N\setminus\{i\}}\times\Theta_{-i}\right|-1\Big)\overline{UD}(u_i^*;R_i,\theta_i).$$

Since an assignment in our model represents a transplant, an unacceptable transplant can be interpreted as the death of the patient. Thus, it is natural that there is a huge utility gap between acceptable and unacceptable transplants.

For each player $i \in N$, a strategy is a function $s_i : \mathcal{R}\times\Theta_i \to \mathcal{R}\times\Theta_i$ such that for each $(R_i,\theta_i)\in\mathcal{R}\times\Theta_i$, the submitted type is sincere one, i.e., the second coordinate of $s_i(R_i,\theta_i)$ is θ_i. Recall that the medical condition of a patient and her donors θ_i is verifiable by medical doctors (See Assumption 1).[24] Let $\mathcal{S}_i$ be the set of patient i's strategies. The identity mapping $s_i^*\in\mathcal{S}_i$ is called **the truth-telling strategy**.

Before introducing the equilibrium concept of the game G, we use the following simplifying notation. For each $(R;\theta)=(R_1,\ldots,R_n;\theta_{d_c},\theta_1,\ldots,\theta_n)\in\mathcal{R}^N\times\Theta$, we sometimes denote it as $(\theta_{d_c};(R_1,\theta_1),\ldots,(R_n,\theta_n))$. Moreover, when we focus on a patient i, we denote it as $(\theta_{d_c};(R_i,\theta_i);(R_j,\theta_j)_{j\neq i})$. A strategy profile $s=(s_1,\ldots,s_n)\in\prod_{i\in N}\mathcal{S}_i$ is a **Bayesian Nash equilibrium** in G if for each $i\in N$, each $(R_i,\theta_i)\in\mathcal{R}\times\Theta_i$, and each $R_i'\in\mathcal{R}$,

$$\sum_{(R_{-i};\theta_{-i})\in\mathcal{R}^{N\setminus\{i\}}\times\Theta_{-i}} p_i(R_{-i};\theta_{-i}\mid R_i,\theta_i)u_i^*\left(\varphi_i^P(\theta_{d_c};s_i(R_i,\theta_i);\big(s_j(R_j,\theta_j)\big)_{j\neq i})\;\Big|\;R_i,\theta_i\right)$$

$$\geq \sum_{(R_{-i};\theta_{-i})\in\mathcal{R}^{N\setminus\{i\}}\times\Theta_{-i}} p_i(R_{-i};\theta_{-i}\mid R_i,\theta_i)u_i^*\left(\varphi_i^P(\theta_{d_c};(R_i',\theta_i);\big(s_j(R_j,\theta_j)\big)_{j\neq i})\;\Big|\;R_i,\theta_i\right).$$

The main result of this subsection shows that each priority mechanism is robust against strategic manipulation even under regimes with living-donor exchange under incomplete information. To show that positive result, we need some specification and simplification that decently approximate the real world. We make the following three assumptions.

Assumption 3 (The number of component type spaces are two)**.** *The collection of component type spaces $\{(T_k,\trianglerighteq_k)\}_{k=1}^K$ is simplified to the one with the length two, i.e., $K=2$, such that $(T_1,\trianglerighteq_1)$ is the blood type space $(T_{\mathcal{B}},\trianglerighteq_{\mathcal{B}})$; $(T_2,\trianglerighteq_2)$ is the other factor space that needs coincidence, i.e., for all $t_2,t_2'\in T_2$, $t_2\trianglerighteq_2 t_2'\Leftrightarrow t_2=t_2'$; T_2 contains at least four elements. We denote it as $T_2=\{I,II,III,IV,\ldots\}$.*

This assumption seems too specific, but can reasonably accommodate the current practice. For example, consider the simplest space $T_{\mathcal{B}}\times T_2=T_{\mathcal{B}}\times\{l,s\}\times\{c_1,c_2\}$ where the component space $\{l,s\}$ of T_2 expresses the sizes of lungs, large (l) or small (s); moreover, the component space $\{c_1,c_2\}$ of T_2 does the types of leucocyte. There are many types of leucocyte which are an important compatibility

[24]Each patient knows that other patients submit own medical type honestly. However, she does not know which types are realized.

condition. However, these types cannot be classifed with clear formula.[25] For this reason, the cross-match test is used and there would be at least two types which are incompatible with each other.[26] What is common within T_2 space is that it needs coincidence of types for a donor and a recipient. With these two component spaces of T_2, it is reasonable to assume at least four elements in T_2. Hence our example space satisfies Assumption 3. Note that T_2 can be any larger cardinality as long as it contains four elements.

Assumption 4 (On the number of living donors of each patient). *The following three conditions hold.*[27]

1. *Each patient has at most two living donors, i.e., for each $i \in N$, $|D_i^L| \leq 2$.*
2. *At least four patients have multiple living donors, i.e., there exist distinct $i, j, k, \ell \in N$ such that for each $m \in \{i, j, k, \ell\}$, $|D_m^L| = 2$.*
3. *The highest-priority patient $\sigma(1)$ has multiple living donors, i.e., $|D_{\sigma(1)}^L| \geq 2$.*

The first two conditions in Assumption 4 are quite weak, because hundreds of patients are in line for deceased donors in Japan. The last condition seems strong, but actually weak, because our notion of donors include potential donors who are incompatible donors and usually do not show up in hospitals. Theoretically speaking, this assumption is for simplification. That is, to maintain our main result of Theorem 3, we can drop Assumption 4 when there are at least four patients and each of them faces further uncertainty about the number of the other patients' living donors.

Assumption 5 (Type space restriction). *For each $i \in N$, we redefine the type space Θ_i to slightly restrict a feasible type profile.*[28]

$$\Theta_i := \left\{ \theta_i \in T^{\{i\} \cup D_i^L} \,\middle|\, \begin{array}{l} i)\ \forall d, d' \in D_i^L, \theta_i(d) \trianglerighteq \theta_i(i) \text{ and } \theta_i(d) \trianglerighteq \theta_i(i) \Rightarrow d = d', \\ ii)\ \forall d \in D_i^L, \theta_i(d) \neq \theta_i(i) \end{array} \right\}.$$

The new definition of the type space excludes that a patient has a living donor who is not only compatible with the intended patient but also has the identical type with the patient. Let us emphasize

[25] We would like to thank Prof. Takahiro Oto for numerous useful comments from his expertise.

[26] Although red blood cell has four types of A, B, O, AB, leucocyte also has many types. The human leucocyte is first classified into three types of A, B, DR, and then each of A, B, DR is classified into dozens of antigen types. Moreover, there are unknown types, i.e., the antigens have not yet been exhausted. It is known that every human being has two of the HLA (human leucocyte antigen)s. Thus, for example, when each of A, B, DR types are assumed to contain 20 antigens, human HLA types are $\binom{60}{2} = 1770$. The extreme diversity and the existence of unknown types make it hard to specify which type is compatible with the given type. For this reason, in practice, the cross-match test is carried out to experiment whether the patient's and donor's blood have the immunological rejection in HLA type. In our example, the HLA type is described by the two types for simplicity. Note that our general setup, especially the fact that T_2 can be large as long as it contains four types, allows more complex type spaces.

[27] A special case of the model satisfying this assumption is a market formed by four or more patients with exactly two living donors for each.

[28] The role of Assumption 5 is critical only when $Y = HE$. Theorem 3 can be proved without it if $Y \in \{O, E, H\}$.

that the new definition does not necessarily exclude a compatible living donor. It only excludes the complete coincidence between the type of a patient and her own donor. Since T_2 can be any large set, the restriction of the type space makes almost no loss of generality.[29]

Now we are ready to state our main result of this subsection that asserts each priority mechanism is robust against strategic behavior under incomplete information even when the living donor exchange is allowed.

Theorem 3. *The truth-telling strategy profile* $s^* = (s_1^*, \ldots, s_n^*) \in \prod_{i \in N} \mathcal{S}_i$ *is a Bayesian Nash equilibrium in* G*.*

5 Conclusion

We introduce a dual organ markets where patients are in the two markets for deceased donors and living donors. We investigated the properties of the priority mechanism. Without donor exchange, the priority mechanism is shown to be individually rational, Pareto efficient, fair, and strategy-proof. However, once we allow for donor exchange, we lose its strategy-proofness. Because patients' manipulation is risky, we show that the priority mechanism is robust against any manipulation by showing that the truth-telling profile is a Bayesian Nash equilibrium.

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[29] For example, a cross-match type space based on individual tissue-type can be large.

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Appendix: Proofs

Proof of Proposition 1. Let $a^\theta \in \mathcal{A}(\theta)$ be arbitrary. First, we show two claims.

Claim 1. $\bigcup_{i\in N} \left(a_i^{\theta L}\backslash D_i^L\right) = \bigcup_{i\in N} \left[D_i^L \cap \left(\bigcup_{j\neq i} a_j^{\theta L}\right)\right]$.

To show that the LHS of the equality is a subset of the RHS, let $d \in \cup_{i\in N}\left(a_i^{\theta L}\backslash D_i^L\right)$. Then there is $i \in N$ such that $d \in a_i^{\theta L}\backslash D_i^L$. Since $d \in D^L = \cup_{j\in N} D_j^L$ and $d \notin D_i^L$, there is $j_0 \in N\backslash\{i\}$ such that $d \in D_{j_0}^L$. Moreover, $d \in a_i^{\theta L} \subseteq \cup_{j\neq j_0} a_j^{\theta L}$. Thus $d \in D_{j_0}^L \cap \left(\cup_{j\neq j_0} a_j^{\theta L}\right)$.

To show the converse, let $d \in \cup_{i\in N}\left[D_i^L \cap \left(\cup_{j\neq i} a_j^{\theta L}\right)\right]$. Then there is $i \in N$ such that $d \in D_i^L \cap \left(\cup_{j\neq i} a_j^{\theta L}\right)$. Thus, there is $j_0 \in N\backslash\{i\}$ such that $d \in a_{j_0}^{\theta L}$. Since $D_i^L \cap D_{j_0}^L = \emptyset$, $d \in D_i^L$ implies $d \notin D_{j_0}^L$. Thus $d \in a_{j_0}^{\theta L}\backslash D_{j_0}^L$. □

Claim 2. Both $\bigcup_{i\in N}\left(a_i^{\theta L}\backslash D_i^L\right)$ and $\bigcup_{i\in N}\left[D_i^L \cap \left(\bigcup_{j\neq i} a_j^{\theta L}\right)\right]$ are a direct union.

Let $i, i' \in N$ be distinct. If $d \in \left(a_i^{\theta L}\backslash D_i^L\right) \cap \left(a_{i'}^{\theta L}\backslash D_{i'}^L\right)$, then $d \in a_i^{\theta L} \cap a_{i'}^{\theta L}$. This violates the second condition of an allocation (2). Thus $\cup_{i\in N}\left(a_i^{\theta L}\backslash D_i^L\right)$ is direct a union.

If $d \in \left[D_i^L \cap \left(\bigcup_{j\neq i} a_j^{\theta L}\right)\right] \cap \left[D_{i'}^L \cap \left(\bigcup_{j\neq i'} a_j^{\theta L}\right)\right]$, then $d \in D_i^L \cap D_{i'}^L$, a contradiction. Thus, $\cup_{i\in N}\left[D_i^L \cap \left(\cup_{j\neq i} a_j^{\theta L}\right)\right]$ is also a direct union. □

Now we turn back to the proof of Proposition 1. We have

$$\sum_{i\in N}\left|a_i^{\theta L}\backslash D_i^L\right| = \left|\bigcup_{i\in N}\left(a_i^{\theta L}\backslash D_i^L\right)\right| = \left|\bigcup_{i\in N}\left[D_i^L \cap \left(\bigcup_{j\neq i} a_j^{\theta L}\right)\right]\right| = \sum_{i\in N}\left|D_i^L \cap \left(\bigcup_{j\neq i} a_j^{\theta L}\right)\right|. \qquad (*)$$

Note that the first and the third equalities follow from Claim 2, while the second one follows from Claim 1. On the other hand, BC requires

$$\forall i \in N, \left|a_i^{\theta L}\backslash D_i^L\right| \leq \left|D_i^L \cap \left(\cup_{j\neq i} a_j^{\theta L}\right)\right|.$$

For the equality $(*)$ to be true, BC must hold with equality for each $i \in N$. □

To show Remark 3, we need the following lemma.

Lemma 1. *Under any regime $Y \in \{O, H\}$, non-wastefulness implies individual rationality.*

Proof. Let $(R;\theta) \in \mathcal{R}^N \times \Theta$. We show the contrapositive. Suppose that $a^\theta \in \mathcal{A}^Y(\theta)$ is not individually rational at $(R;\theta)$. Then we have a patient $i \in N$ such that $(0,\emptyset)\ P_i\ a_i^\theta$. Since $Y \in \{O, H\}$, no patient $j \in N\backslash\{i\}$ uses a lobe of a donor in D_i^L at a^θ. That is, for each $j \in N\backslash\{i\}$, $a_j^{\theta L} \cap D_i^L = \emptyset$. Thus, letting $b_i^\theta := (0,\emptyset)$, we have (i) $b_i^\theta\ P_i\ a_i^\theta$, (ii) $(b_i^\theta; a_{-i}^\theta) \in \mathcal{A}^Y(\theta)$ and (iii) $b_i^{\theta C} = 0 < \overline{a}_i^{\theta C}$ and $b_i^{\theta L} = \emptyset \subseteq \overline{a}_i^{\theta L}$. This means that a^θ is wasteful at $(R;\theta)$. □

Proof of Remark 3. (Item 1) The first part is trivial. The second one is already shown in Example 1.

(Item 2) Since the first part is trivial, we only show the second part. Let $Y \in \{O, H\}$ and $(R;\theta) \in \mathcal{R}^N \times \Theta$. Suppose that $a^\theta \in \mathcal{A}^Y(\theta)$ is non-wasteful at $(R;\theta)$. Let $b^\theta \in \mathcal{A}^Y(\theta)$ be such that for each $i \in N$, $b_i^\theta \ R_i \ a_i^\theta$. It is sufficient to show that there is no $i \in N$ such that $b_i^\theta \ P_i \ a_i^\theta$. We consider four cases separately according to the distribution of cadaveric lungs. Note that a^θ is individually rational at $(R;\theta)$ by Lemma 1. Consequently, b^θ is also individually rational at $(R;\theta)$.

Case 1: $\exists i \in N$ s.t. $a_i^\theta \in X_i^{20}(\theta)$. Note that no patient, except for patient i, receives a non-null transplant at a^θ, since all cadaveric lungs are used by i and the regime under consideration is $Y \in \{O, H\}$. Thus, at the induced allocation, $\overline{a}_i^\theta = (2, D_i^L)$. If patient i's assignment b_i^θ is not $(2, \emptyset)$, i.e., $b_i^\theta \in X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$, then $b_i^\theta \ P_i \ a_i^\theta$, $b_i^{\theta C} \leq \overline{a}_i^{\theta C}$, and $b_i^{\theta L} \subseteq \overline{a}_i^{\theta L}$. This violates the non-wastefulness of a^θ at $(R;\theta)$. Thus $b_i^\theta = (2, \emptyset)$. Note that as $b_i^{\theta C} = 2$ and $Y \in \{O, H\}$, no patient, except for patient i, receives a non-null transplant at b^θ. Thus $b^\theta = a^\theta$.

Case 2: $\sum_{i \in N} a_i^{\theta C} = 2$ and $\nexists i \in N$ s.t. $a_i^\theta \in X_i^{20}(\theta)$. Then there exist $i, j \in N$ such that $i \neq j$ and $a_i^{\theta C} = a_j^{\theta C} = 1$. Note that no patient, except for patients i and j, receives a non-null transplant at a^θ, since all cadaveric lungs are used by i and j, and the regime under consideration is $Y \in \{O, H\}$. Thus, at the induced allocation, $\overline{a}_i^\theta = (1, D_i^L)$ and $\overline{a}_j^\theta = (1, D_j^L)$.

If $b_i^\theta = (2, \emptyset)$, then $b_j^{\theta C} = 0$. Since $Y \in \{O, H\}$, $b_j^\theta = (0, \emptyset)$. However, this violates the fact that $b_j^\theta \ R_j \ a_j^\theta \ P_j \ (0, \emptyset)$, as $b_j^\theta \ R_j \ a_j^\theta$, a_j^θ $(\neq (0, \emptyset))$ is individually rational at $(R;\theta)$. Thus $b_i^\theta \neq (2, \emptyset)$. Thus, as $Y \in \{O, H\}$, $b_i^\theta \in X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$. Note that $b_i^{\theta C} \leq \overline{a}_i^{\theta C}$ and $b_i^{\theta L} \subseteq \overline{a}_i^{\theta L}$. Therefore, $b_i^\theta \ P_i \ a_i^\theta$ is impossible as it violates the non-wastefulness of a^θ at $(R;\theta)$. Thus $b_i^\theta \ I_i \ a_i^\theta$. Since $Y \in \{O, H\}$, $b_i^\theta = a_i^\theta$. By the identical argument, we have $b_j^\theta = a_j^\theta$. Note that, as $Y \in \{O, H\}$, no patient, except for patients i and j, receives a non-null transplant at b^θ. Thus $b^\theta = a^\theta$.

Case 3: $\sum_{i \in N} a_i^{\theta C} = 1$. We treat two cases separately according to the number of available cadaveric lungs.

Case 3.1: $\theta_{d_c q} = 1$. We can show that $b^\theta = a^\theta$ by the same argument as the proof for Case 1.

Case 3.2: $\theta_{d_c q} = 2$. Let $i \in N$ be the patient such that $a_i^{\theta C} = 1$. Note that, since $Y \in \{O, H\}$, no patient, except for patient i, receives a non-null transplant at a^θ. Note also that one cadaveric lung is disposed at a^θ. Thus, the induced allocation is as follows: For each $j \in N$,

$$\overline{a}_j^\theta = \begin{cases} (2, D_j^L) \text{ if } j = i, \\ (1, D_j^L) \text{ if } j \neq i. \end{cases}$$

Note that, since $Y \in \{O, H\}$, b_i^θ satisfies that $b_i^{\theta C} \leq \overline{a}_i^{\theta C}$ and $b_i^{\theta L} \subseteq \overline{a}_i^{\theta L}$. Therefore, $b_i^\theta \ P_i \ a_i^\theta$ is impossible as it violates the non-wastefulness of a^θ at $(R;\theta)$. Thus $b_i^\theta \ I_i \ a_i^\theta$. Since $Y \in \{O, H\}$, $b_i^\theta = a_i^\theta$. Thus $b_i^{\theta C} = 1$.

Since patient i uses one unit of cadaveric lung at b^θ, for each $j \in N \backslash \{i\}$, $b_j^{\theta C} \leq 1$. Thus $b_j^\theta \in X_j^{10}(\theta) \cup \tilde{X}_j^{11}(\theta) \cup X_j^{00}(\theta)$. Note that b_j^θ satisfies that $b_j^{\theta C} \leq \overline{a}_j^{\theta C}$ and $b_j^{\theta L} \subseteq \overline{a}_j^{\theta L}$, since $Y \in \{O, H\}$. Therefore, $b_j^\theta \ P_j \ a_j^\theta$ is impossible as it violates the non-wastefulness of a^θ at $(R;\theta)$. Thus $b_j^\theta \ I_j \ a_j^\theta$. Since $Y \in \{O, H\}$, $b_j^\theta = a_j^\theta = (0, \emptyset)$. Summing up with $a_i^\theta = b_i^\theta$, we obtain $b^\theta = a^\theta$.

Case 4: $\sum_{i\in N} a_i^{\theta C} = 0$. Note that no patient receives a non-null transplant at a^θ, since $Y \in \{O, H\}$. Note also that all cadaveric lungs are disposed at a^θ. Thus the induced allocation is: for each $i \in N$, $\overline{a}_i^\theta = (\theta_{d_cq}, D_i^L)$. If patient i's assignment b_i^θ is in $X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$ and $b_i^\theta \ P_i \ a_i^\theta$, then a^θ is wasteful at $(R;\theta)$. Thus, $b_i^\theta \notin X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$ or $a_i^\theta \ R_i \ b_i^\theta$. Note that since $Y \in \{O, H\}$, $b_i^\theta \notin X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$ is impossible. Thus $a_i^\theta \ R_i \ b_i^\theta$. Because we assume that $b_i^\theta \ R_i \ a_i^\theta$, this implies $a_i^\theta \ I_i \ b_i^\theta$. Since $Y \in \{O, H\}$, $b_i^\theta = a_i^\theta$. Thus $b^\theta = a^\theta$.

As $b^\theta = a^\theta$ for all of the four cases, no patient prefers b^θ to a^θ. This completes the proof of Item 2.

(Item 3) We show the statement by two examples. First, we show that Pareto efficiency does not imply strong non-wastefulness. Obviously, allocation a^θ described in Example 1 is Pareto efficient, while it is strongly wasteful. Next, we show that strong non-wastefulness does not imply Pareto efficiency under any regime $Y \in \{E, HE\}$. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that patients 1 and 2 have multiple living donors. Let $d_{11}, d_{12} \in D_1^L$ and $d_{21}, d_{22} \in D_2^L$. Let $\theta \in \Theta$ be such that $\theta_{d_cT} = AB$, $\theta(1) = \theta(d_{21}) = \theta(d_{22}) = A$ and $\theta(2) = \theta(d_{11}) = \theta(d_{12}) = B$. Suppose that for each $i \in N\backslash\{1,2\}$ and each $d \in D^L\backslash\{d_{11}, d_{12}, d_{21}, d_{22}\}$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be such that

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_2 & 02 & 00 & \cdots \end{array}$$

Consider the allocation $a^\theta := ((0,\emptyset),(0,\emptyset))$. The allocation is trivially non-wasteful at $(R;\theta)$. Note that no feasible allocation uses a cadaveric lung, since the deceased donor is not compatible with any patients. Thus a^θ is strongly non-wasteful at $(R;\theta)$. On the other hand, since it is Pareto dominated by allocation $((0,\{d_{21}, d_{22}\}),(0,\{d_{11}, d_{12}\}))$, a^θ is not Pareto efficient at $(R;\theta)$. □

Proof of Remark 4. (First half of Item 1: $\succeq$-fairness $\not\Rightarrow$ individual rationality) We prove it by an example. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Let i be the highest-priority patient. Let $\theta \in \Theta$ be such that $\theta_{d_c} = (2, A)$ and $\theta(i) = A$. Let $R \in \mathcal{R}^N$ be such that $\begin{array}{c|cccc} R_i & \cdots & 00 & 20 & \cdots \end{array}$. An allocation where i receives $(2,\emptyset)$ is trivially $\succeq$-fair because the highest-priority patient receives all cadaveric lungs. However, $(2,\emptyset)$ is not acceptable for patient i, i.e., the allocation is not in $\mathcal{I}^Y(R;\theta)$.

(The latter half of Item 1: Pareto efficiency and $\succeq$-fairness $\not\Rightarrow$ individual rationality) We prove it by an example. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that patients 1 and 2 have multiple living donors. Let $d_{11}, d_{12} \in D_1^L$ and $d_{21}, d_{22} \in D_2^L$. Let $\theta \in \Theta$ be such that $\theta_{d_cT} = AB$, $\theta(1) = \theta(d_{21}) = \theta(d_{22}) = A$ and $\theta(2) = \theta(d_{11}) = \theta(d_{12}) = B$. Suppose that for each $i \in N\backslash\{1,2\}$ and each $d \in D^L\backslash\{d_{11}, d_{12}, d_{21}, d_{22}\}$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be such that

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_2 & 00 & 02 & \cdots \end{array}$$

Consider allocation $a^\theta := ((0, \{d_{21}, d_{22}\}), (0, \{d_{11}, d_{12}\})) \in \mathcal{A}^Y(\theta)$. Note that a^θ is not individually rational at $(R;\theta)$ because a_2^θ is not acceptable for patient 2.

We claim that a^θ is Pareto efficient and $\succeq$-fair at $(R;\theta)$. Since no patient is compatible with the cadaveric lung, a^θ is trivially $\succeq$-fair at $(R;\theta)$. Next we show the Pareto efficiency. Suppose that $b^\theta \in \mathcal{A}^Y(\theta)$ is such that $b_1^\theta \; R_1 \; a_1^\theta$ and $b_2^\theta \; R_2 \; a_2^\theta$. Note that $X_1^{02} = \{(0, \{d_{21}, d_{22}\})\}$. Thus, since the living-donor transplantation, 02, is the top choice at R_1, $b_1^\theta = a_1^\theta$. By Proposition 1, patient 2 receives two lobes of other patient's donors because her own donors (d_{21} and d_{22}) donate two lobes to other patient in total. Thus $b_2^\theta = (0, \{d_{11}, d_{12}\})$. Thus $b^\theta = a^\theta$. This completes the proof of Pareto efficiency.

(Item 2) We show it by an example. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that $1 \succ 2$. Let $\theta \in \Theta$ be such that $\theta_{d_c} = (2, A)$ and $\theta(1) = \theta(2) = A$. Suppose that for each $i \in N\backslash\{1,2\}$ and each $d \in D^L$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be such that

$$\begin{array}{c|cccc} R_1 & 10 & 20 & 00 & \cdots \\ R_2 & 20 & 00 & \cdots & \end{array}$$

First, consider allocation $a^\theta := ((0, \emptyset), (2, \emptyset)) \in \mathcal{A}^Y(\theta)$. This allocation is non-wasteful, strongly non-wasteful, and Pareto efficient at $(R;\theta)$, because the two cadaveric lungs go to patient 2 whose top choice is $(2, \emptyset)$ and the market has no living donor compatible with a patient. However, a^θ is not $\succeq$-fair at $(R;\theta)$, because patient 1 can be better off by using cadaveric lungs assigned to lower-priority patient (patient 2). Thus, neither non-wastefulness, strong non-wastefulness, nor Pareto efficiency implies $\succeq$-fairness.

Next, consider allocation $b^\theta := ((2, \emptyset), (0, \emptyset))$. Since the highest-priority patient 1 uses all cadaveric lungs, b^θ is trivially $\succeq$-fair at $(R;\theta)$. However, b^θ is wasteful at $(R;\theta)$, because patient 1 can be better off by disposing of one cadaveric lung without affecting patient 2's assignment. Formally, letting $c_1^\theta := (1, \emptyset)$, (i) $c_1^\theta \; P_1 \; b_1^\theta$, (ii) $(c_1^\theta, b_2^\theta) \in \mathcal{A}^Y(\theta)$ and (iii) $c_1^{\theta C} = 1 \leq \overline{b}_1^{\theta C}$ and $c_1^{\theta L} = \emptyset \subseteq \overline{b}_1^{\theta L}$. This means that b^θ is wasteful at $(R;\theta)$. Thanks to Remark 3, b^θ is strongly wasteful at $(R;\theta)$, and not Pareto efficient at $(R;\theta)$. Thus $\succ$-fairness does not imply any one of non-wastefulness, strong non-wastefulness and Pareto efficiency. □

Proof of Proposition 2. We prove it by an example. We use the same example as in Example 1. Assume, without loss of generality, that the priority is given as $1 \succ 2$. Obviously, $\mathcal{I}^Y(R;\theta)$ consists of just three allocations: $a^\theta := \big((1, \emptyset), (0, \emptyset)\big)$, $b^\theta := \big((0, \emptyset), (2, \emptyset)\big)$, and $c^\theta := \big((0, \emptyset), (0, \emptyset)\big)$. Allocation a^θ is $\succeq$-fair, but is strongly wasteful, since the two cadaveric lungs can be used at allocation b^θ. Next, allocation b^θ is strongly non-wasteful, but is not $\succeq$-fair, since agent 1 can be better off by using one cadaveric lung assigned for agent 2 who is of lower priority than agent 1. Finally, allocation c^θ is wasteful, since one of the agents 1 and 2 can be better off by using the disposed lungs. Thus, no individually rational allocation satisfies $\succeq$-fairness and strong non-wastefulness at $(R;\theta)$. □

Proof of Proposition 3. Item (i) is trivial. Items (iii) to (v) are shown in Example 2. So it remains to show (ii). Since Pareto efficiency is equivalent to non-wastefulness under regime O by Remark 3, we show that φ^J is non-wasteful. Note that under regime O, non-wastefulness only requires that no patient be better off by using a disposed cadaveric lung. This is straightforward. □

Proof of Remark 5. (Item 1) Trivial.

(Item 2) Let $Y \in \{O, H\}$. We show that for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, $\Phi^Y(R;\theta)$ is a singleton. Let $a^\theta, b^\theta \in \Phi^Y(R;\theta)$. By Item 1, for each $i \in N, a_i^\theta \; I_i \; b_i^\theta$. Note that regimes O and H allow only transplants in $X_i^{20}(\theta), X_i^{10}(\theta), \tilde{X}_i^{11}(\theta)$, and $X_i^{00}(\theta)$. Thus, $a_i^\theta \; I_i \; b_i^\theta$ implies $a_i^\theta = b_i^\theta$. Thus $a^\theta = b^\theta$.

(Item 3) Let $Y \in \{E, HE\}$. We show that for some $(R;\theta) \in \mathcal{R}^N \times \Theta$, $\Phi^Y(R;\theta)$ contains at least two allocations. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that patients 1 and 2 have multiple living donors. Let $d_{11}, d_{12} \in D_1^L$ and $d_{21}, d_{22} \in D_2^L$. Let $\theta \in \Theta$ be such that $\theta_{d_cT} = AB$, $\theta(1) = \theta(d_{21}) = A$, $\theta(2) = \theta(d_{11}) = B$ and $\theta(d_{12}) = \theta(d_{22}) = O$. Suppose that for each $i \in N \backslash \{1,2\}$ and each $d \in D^L \backslash \{d_{11}, d_{12}, d_{21}, d_{22}\}$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be such that

R_1	02	00	$\cdots$
R_2	02	00	$\cdots$

Note that the only compatible donors for patients in $N \backslash \{1,2\}$ are d_{12} and d_{22}. Thus, if a patient $i \in N \backslash \{1,2\}$ receives a non-null transplant at an allocation, it must be $(0, \{d_{12}, d_{22}\})$. However, in that case, patients 1 and 2 also receive a non-null transplant with receiving at least one other patient's living donor (Proposition 1). Because there are not enough compatible donors for patients 1 and 2 to receive a non-null transplant without d_{12} and d_{22}, every feasible allocation assigns the null transplant $(0, \emptyset)$ to patients in $N \backslash \{1,2\}$.

For any priority order $\succeq$, $\Phi^Y(R;\theta)$ contains all allocations where patients 1 and 2 receive living-donor transplants. Note that both $((0, \{d_{12}, d_{21}\}), (0, \{d_{11}, d_{22}\}))$ and $\big((0, D_2^L), (0, D_1^L)\big)$ are in $\Phi^Y(R;\theta)$. □

Note that in the following proof, there is no specification of regime except for strategy-proofness.

Proof of Theorem 1. We assume, without loss of generality, that $1 \succ 2 \succ \ldots \succ n$.

(Individual rationality) Trivial.

(Pareto efficiency) Let $(R;\theta) \in \mathcal{R}^N \times \Theta$. Suppose to the contrary that there is an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ such that for each $i \in N$, $a_i^\theta \; R_i \; \varphi_i^P(R;\theta)$ and for some $i \in N$, $a_i^\theta \; P_i \; \varphi_i^P(R;\theta)$. Note that $a^\theta \in \mathcal{I}^Y(R;\theta) = \Phi_0^Y(R;\theta)$, since a^θ Pareto-dominates an individually rational allocation $\varphi^P(R;\theta)$. Let $i \in N$ be the highest-priority patient among those who prefer a^θ to $\varphi^P(R;\theta)$. Note that this implies that for each $j \in N$ with $j \succ i$, $a_j^\theta \; I_j \; \varphi_j^P(R;\theta)$. Thus $a^\theta \in \Phi_{i-1}^Y(R;\theta)$. Since $a_i^\theta \; P_i \; \varphi_i^P(R;\theta)$, $\varphi^P(R;\theta) \notin \Phi_i^Y(R;\theta) \supseteq \Phi^Y(R;\theta)$, a contradiction.

($\succeq$-fairness) Let $(R;\theta) \in \mathcal{R}^N \times \Theta$. Let $a^\theta := \varphi^P(R;\theta)$. Suppose to the contrary that there is an allocation $b^\theta \in \mathcal{I}^Y(R;\theta)$ such that for some $i \in N$, (i) $b_i^\theta \; P_i \; a_i^\theta$, (ii) $b_i^{\theta C} > a_i^{\theta C}$ and for each $j \in N$ with $j \succ i$, $b_j^{\theta C} = a_j^{\theta C}$ and (iii) for each $j \in N$, $a_j^\theta \; P_j \; b_j^\theta$ implies $a_j^{\theta C} > b_j^{\theta C}$.

If for each $j \in N$ with $j \succ i$, $b_j^\theta \; R_j \; a_j^\theta$, then allocation a^θ is excluded from the priority correspondence at some step, the latest step i, of the algorithm. That is, there is $j \in \{1, \ldots, i\}$ such that $a^\theta \notin \Phi_j^Y(R;\theta) \supseteq \Phi^Y(R;\theta)$, a contradiction. Thus there is $j \in N$ with $j \succ i$ such that $a_j^\theta \; P_j \; b_j^\theta$. By Item (iii), $a_j^{\theta C} > b_j^{\theta C}$. This contradicts the second part of Item (ii).

(Strategy-proofness) Let $Y \in \{O, H\}$. Suppose to the contrary that there are $(R;\theta) \in \mathcal{R}^N \times \Theta, i \in N$ and $R'_i \in \mathcal{R}$ such that $\varphi_i^P(R'_i, R_{-i};\theta) \; P_i \; \varphi_i^P(R;\theta)$. For notational simplicity, let $R' := (R'_i, R_{-i})$.

Since regime Y does not allow for donor exchange, any preference misreporting cannot affect the individual assignment of higher-priority patients in the priority mechanism. That is, for each $j \in N$ with $j \succ i$, $\varphi_j^P(R;\theta) \; I_j \; \varphi_j^P(R';\theta)$. Thus, since indifferent individual assignments are identical in regime Y, for each $j \in N$ with $j \succ i$, $\varphi_j^P(R;\theta) = \varphi_j^P(R';\theta)$. Note that $\varphi^P(R';\theta)$ is individually rational at $(R;\theta)$ because $\varphi_i^P(R';\theta) \; P_i \; \varphi_i^P(R;\theta) \; R_i \; (0,\emptyset)$ and $R'_{-i} = R_{-i}$. Thus $\varphi^P(R';\theta) \in \mathcal{I}^Y(R;\theta) = \Phi_0^Y(R;\theta)$. Thus we obtain $\varphi^P(R';\theta) \in \Phi_{i-1}^Y(R;\theta)$. Hence $\varphi^P(R;\theta) \notin \Phi_i^Y(R;\theta) \supseteq \Phi^Y(R;\theta)$, a contradiction. □

Proof of Proposition 4. The proof of individual rationality, Pareto efficiency, and $\succeq$-fairness is identical with that of Theorem 1. Non-strategy-proofness is by Example 4. □

Proof of Theorem 2. Suppose to the contrary that a mechanism under $Y \in \{E, HE\}$, φ, satisfies all axioms stated in Theorem 2. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that patients 1 and 2 have multiple living donors and that $1 \succ 2$. Let $d_{11}, d_{12} \in D_1^L$ and $d_{21}, d_{22} \in D_2^L$. Let $\theta \in \Theta$ be such that $\theta_{d_c} = (2, O)$, $\theta(1) = \theta(d_{21}) = \theta(d_{22}) = A$ and $\theta(2) = \theta(d_{11}) = \theta(d_{12}) = B$. Let $R_1, R'_1, R_2, R'_2 \in \mathcal{R}$ be preferences of patients 1 and 2 described by the following table.

R_1	02	10	00	$\cdots$	
R'_1	02	20	10	00	$\cdots$
R_2	10	02	00	$\cdots$	
R'_2	10	00	$\cdots$		

Let $R_{-\{1,2\}} \in \mathcal{R}^{N \setminus \{1,2\}}$ be a preference profile such that each patient $j \in N \setminus \{1,2\}$ has $(0,\emptyset)$ as the most preferred in R_j. Note that each patient in $N \setminus \{1,2\}$ receives $(0,\emptyset)$, no matter when patients 1 and 2 submit any preference because φ is individually rational. In the subsequent part of the proof, we omit their assignments in the description of an allocation. First, we show the following claim.

Claim. $\varphi(R_1, R_2, R_{-\{1,2\}};\theta) = ((1,\emptyset),(1,\emptyset))$.

Suppose to the contrary that $\varphi(R_1, R_2, R_{-\{1,2\}};\theta) \neq ((1,\emptyset),(1,\emptyset))$. Note that the only individually rational and Pareto efficient allocations at $(R_1, R_2, R_{-\{1,2\}};\theta)$ are $a^\theta := ((0,\{d_{21}, d_{22}\}),(0,\{d_{11}, d_{12}\}))$ and $b^\theta := ((1,\emptyset),(1,\emptyset))$. Thus, by the contradiction hypothesis, $\varphi(R_1, R_2, R_{-\{1,2\}};\theta) = a^\theta$. On the other hand, b^θ is the only individually rational and Pareto efficient allocation at $(R_1, R'_2, R_{-\{1,2\}};\theta)$.

Thus $\varphi(R_1, R_2', R_{-\{1,2\}}; \theta) = b^\theta$. Therefore, $\varphi_2(R_1, R_2', R_{-\{1,2\}}; \theta)\ P_2\ \varphi_2(R_1, R_2, R_{-\{1,2\}}; \theta)$, a violation of strategy-proofness of φ. This completes the proof of Claim.

Next, we consider patient 1's assignment at $\varphi(R_1', R_2, R_{-\{1,2\}}; \theta)$. Note that since φ is individually rational, $\varphi_1(R_1', R_2, R_{-\{1,2\}}; \theta)$ is one of the following assignments: $(0, \emptyset), (1, \emptyset), (2, \emptyset), (0, \{d_{21}, d_{22}\})$. We separately derive a contradiction for each case.

Case 1: $\varphi_1(R_1', R_2, R_{-\{1,2\}}; \theta) = (0, \emptyset)$. Then $\varphi_1(R_1, R_2, R_{-\{1,2\}}; \theta) = (1, \emptyset)\ P_1'\ (0, \emptyset) = \varphi_1(R_1', R_2, R_{-\{1,2\}}; \theta)$, a violation of strategy-proofness of φ.

Case 2: $\varphi_1(R_1', R_2, R_{-\{1,2\}}; \theta) = (1, \emptyset)$. Since $\varphi(R_1', R_2, R_{-\{1,2\}}; \theta)$ is Pareto efficient at $(R_1', R_2, R_{-\{1,2\}}; \theta)$, the assignment of patient 2 is $\varphi_2(R_1', R_2, R_{-\{1,2\}}; \theta) = (1, \emptyset)$, i.e., $\varphi(R_1', R_2, R_{-\{1,2\}}; \theta) = b^\theta$. However, b^θ is not $\succeq$-fair at $(R_1', R_2, R_{-\{1,2\}}; \theta)$, because $c^\theta := ((2, \emptyset), (0, \emptyset)) \in \mathcal{I}^Y(R_1', R_2, R_{-\{1,2\}}; \theta)$. This violates the $\succeq$-fairness of φ.

Case 3: $\varphi_1(R_1', R_2, R_{-\{1,2\}}; \theta) = (2, \emptyset)$. Note that no patient in $N \backslash \{2\}$ uses patient 2's living donor. By Proposition 1, patient 2's assignment does not other's living donor. Moreover, patient 2 cannot use cadaveric lung, since all two units of cadaveric lungs are assigned to patient 1. Thus the assignment of patient 2 is $\varphi_2(R_1', R_2, R_{-\{1,2\}}; \theta) = (0, \emptyset)$, i.e., $\varphi(R_1', R_2, R_{-\{1,2\}}; \theta) = c^\theta$. However, c^θ is Pareto dominated by a^θ at $(R_1', R_2, R_{-\{1,2\}}; \theta)$. This violates Pareto efficiency of φ.

Case 4: $\varphi_1(R_1', R_2, R_{-\{1,2\}}; \theta) = (0, \{d_{21}, d_{22}\})$. We have $\varphi_1(R_1', R_2, R_{-\{1,2\}}; \theta) = (0, \{d_{21}, d_{22}\})\ P_1\ (1, \emptyset) = \varphi_1(R_1, R_2, R_{-\{1,2\}}; \theta)$, a violation of strategy-proofness of φ.

Since the above four cases exhaust all possibilities of $\varphi_1(R_1', R_2, R_{-\{1,2\}}; \theta)$, we conclude that $\varphi(R_1', R_2, R_{-\{1,2\}}; \theta)$ is not well-defined, a contradiction. □

To prove Theorem 3, we need two lemmas. Lemma 2 says that if a patient can successfully manipulate a priority mechanism, then the assignment under the true preference necessarily contains other patient's intended donor. Consequently, the transplant is an acceptable one at the true preference. Moreover, at the false preference, the assignment under the true preference is evaluated as unacceptable. That is, successful manipulation forces the patient to pretend that she cannot accept a transplantation type which is actually an acceptable one.

Lemma 2. *Under any regime $Y \in \{E, HE\}$, for each $(R; \theta) \in \mathcal{R}^N \times \Theta$, each $i \in N$, and each $R_i' \in \mathcal{R}$, if $\varphi_i^P(R_i', R_{-i}; \theta)\ P_i\ \varphi_i^P(R_i, R_{-i}; \theta)$, then*

(i) $\varphi_i^P(R_i, R_{-i}; \theta) \in \left(X_i^{11}(\theta) \backslash \tilde{X}_i^{11}(\theta)\right) \cup X_i^{02}(\theta)$. Consequently, $\varphi_i^P(R_i, R_{-i}; \theta) \neq (0, \emptyset)$.

(ii) $\varphi_i^P(R_i', R_{-i}; \theta)\ P_i'\ (0, \emptyset)\ P_i'\ \varphi_i^P(R_i, R_{-i}; \theta)$.

Proof. Let $(R; \theta) \in \mathcal{R}^N \times \Theta$, $i \in N$ and $R_i' \in \mathcal{R}$. Suppose $\varphi_i^P(R_i', R_{-i}; \theta)\ P_i\ \varphi_i^P(R_i, R_{-i}; \theta)$. For notational simplicity, let $b^\theta := \varphi^P(R_i', R_{-i}; \theta)$ and $a^\theta := \varphi^P(R_i, R_{-i}; \theta)$. Assume, without loss of generality, that $1 \succ 2 \succ \ldots \succ n$.

Proof of Item *(i)*. Suppose to the contrary that $a_i^\theta \notin \left(X_i^{11}(\theta) \backslash \tilde{X}_i^{11}(\theta)\right) \cup X_i^{02}(\theta)$. Namely, a_i^θ is a double-lung, single-lung, hybrid with own donor, or null transplant. We claim

$$\forall j \in N \text{ with } j \succ i, b_j^\theta \; I_j \; a_j^\theta. \tag{4}$$

To show (4), suppose to the contrary that at least one patient $j \in N$ with $j \succ i$ is not indifferent between b_j^θ and a_j^θ, i.e., $b_j^\theta \; P_j \; a_j^\theta$ or $a_j^\theta \; P_j \; b_j^\theta$. We derive a contradiction for each case separately. Without loss of generality, suppose that j is the highest-priority patient who has such a preference.

Case 1: $a_j^\theta \; P_j \; b_j^\theta$. Let c^θ be such that

$$c_k^\theta = \begin{cases} (0, \emptyset) \text{ if } k = i, \\ a_k^\theta \text{ if } k \neq i. \end{cases}$$

Note that $c^\theta \in \mathcal{A}^Y(\theta)$, since a_i^θ does not use other's donor. Moreover, $c^\theta \in \mathcal{I}^Y(R_i', R_{-i}; \theta)$, since the only difference between $(R_i, R_{-i}; \theta)$ and $(R_i', R_{-i}; \theta)$ is patient i's preference. Thus $c^\theta \in \Phi_0^Y(R_i', R_{-i}; \theta)$. By the definition of j, for each $k < j$, $c_k^\theta = a_k^\theta \; I_k \; b_k^\theta$. Thus $c^\theta \in \Phi_{j-1}^Y(R_i', R_{-i}; \theta)$. Since $a_j^\theta \; P_j \; b_j^\theta$, $b^\theta \notin \Phi_j^Y(R_i', R_{-i}; \theta) \supseteq \Phi^Y(R_i', R_{-i}; \theta)$. However, $b^\theta = \varphi^P(R_i', R_{-i}; \theta) \in \Phi^Y(R_i', R_{-i}; \theta)$, a contradiction.

Case 2: $b_j^\theta \; P_j \; a_j^\theta$. Since $b^\theta \in \mathcal{I}^Y(R_i', R_{-i}; \theta)$ and $b_i^\theta \; P_i \; a_i^\theta \; R_i \; (0, \emptyset)$, $b^\theta \in \mathcal{I}^Y(R_i, R_{-i}; \theta)$. Thus $b^\theta \in \Phi_0^Y(R_i, R_{-i}; \theta)$. By the definition of j, for each $k < j$, $a_k^\theta \; I_k \; b_k^\theta$. Thus $b^\theta \in \Phi_{j-1}^Y(R_i, R_{-i}; \theta)$. Thus, $a^\theta \notin \Phi_j^Y(R_i, R_{-i}; \theta) \supseteq \Phi^Y(R_i, R_{-i}; \theta)$. However, $a^\theta = \varphi^P(R_i, R_{-i}; \theta) \in \Phi^Y(R_i, R_{-i}; \theta)$, a contradiction.

Summing up Cases 1 and 2, we get (4).

Now we complete the proof of Item *(i)*. Note that as we have seen in Case 2, $b^\theta \in \mathcal{I}^Y(R_i, R_{-i}; \theta)$. Thus, by (4), $b^\theta \in \Phi_{i-1}^Y(R_i, R_{-i}; \theta)$. Since $b_i^\theta \; P_i \; a_i^\theta$, $a^\theta \notin \Phi_i^Y(R_i, R_{-i}; \theta) \supseteq \Phi^Y(R_i, R_{-i}; \theta)$. However, $a^\theta = \varphi^P(R_i, R_{-i}; \theta) \in \Phi^Y(R_i, R_{-i}; \theta)$, a contradiction. □

Proof of Item *(ii)*: We show the first part of Item *(ii)*. Since φ^P is individually rational, it is obvious that $b_i^\theta = \varphi_i^P(R_i', R_{-i}; \theta) \; R_i' \; (0, \emptyset)$. If $b_i^\theta = (0, \emptyset)$, then $(0, \emptyset) \; P_i \; a_i^\theta = \varphi_i^P(R_i, R_{-i}; \theta)$, a violation to individual rationality of φ^P. Thus we obtain $b_i^\theta \; P_i' \; (0, \emptyset)$.

Next, we show the second part of Item *(ii)*, i.e., $(0, \emptyset) \; P_i' \; a_i^\theta$. Suppose to the contrary that $a_i^\theta \; R_i' \; (0, \emptyset)$. Then, both a^θ and b^θ are individually rational at both $(R_i, R_{-i}; \theta)$ and $(R_i', R_{-i}; \theta)$, since the only difference between $(R_i, R_{-i}; \theta)$ and $(R_i', R_{-i}; \theta)$ is patient i's preference. We consider the following two cases separately, and derive a contradiction for each.

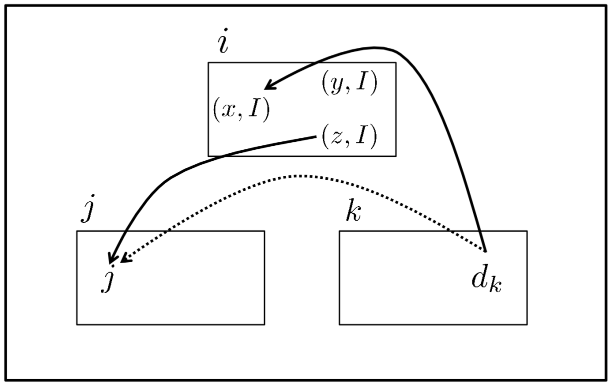
Case 1: $\forall j \in N$ with $j \succ i$, $a_j^\theta \; I_j \; b_j^\theta$. By $b^\theta \in \mathcal{I}^Y(R_i, R_{-i}; \theta)$ and the assumption for Case 1, $b^\theta \in \Phi_{i-1}^Y(R_i, R_{-i}; \theta)$. Since $b_i^\theta \; P_i \; a_i^\theta$, $a^\theta \notin \Phi_i^Y(R_i, R_{-i}; \theta) \supseteq \Phi^Y(R_i, R_{-i}; \theta)$. Thus, $a^\theta = \varphi^P(R_i, R_{-i}; \theta) \notin \Phi^Y(R_i, R_{-i}; \theta)$, a contradiction.

Case 2: Let $j \in N$ be the highest-priority patient with $j \succ i$ and not $a_j^\theta \; I_j \; b_j^\theta$.

Case 2.1: $a_j^\theta \; P_j \; b_j^\theta$. By $a^\theta \in \mathcal{I}^Y(R_i', R_{-i}; \theta)$ and the assumption for Case 2.1, $a^\theta \in \Phi_{j-1}^Y(R_i', R_{-i}; \theta)$. Since $a_j^\theta \; P_j \; b_j^\theta$, $b^\theta \notin \Phi_j^Y(R_i', R_{-i}; \theta) \supseteq \Phi^Y(R_i', R_{-i}; \theta)$. Thus, $b^\theta = \varphi^P(R_i', R_{-i}; \theta) \notin \Phi^Y(R_i', R_{-i}; \theta)$, a contradiction.

Case 2.2: $b_j^\theta \; P_j \; a_j^\theta$. By $b^\theta \in \mathcal{I}^Y(R_i, R_{-i}; \theta)$ and the assumption for Case 2.2, $b^\theta \in \Phi_{j-1}^Y(R_i, R_{-i}; \theta)$.

Figure 8: Patient with a profitable deviation necessarily plays a critical role in the allocation.



Since $b_j^\theta \; P_j \; a_j^\theta$, $a^\theta \notin \Phi_j^Y(R_i, R_{-i};\theta) \supseteq \Phi^Y(R_i, R_{-i};\theta)$. Thus, $a^\theta = \varphi^P(R_i, R_{-i};\theta) \notin \Phi^Y(R_i, R_{-i};\theta)$, a contradiction. □

Lemma 3 says that a patient i has a profitable deviation only if she has a donor d_i such that a donor with i's type cannot donate to a patient with d_i's type. To see why this is true, let us take a look at Figure 8. Let i be a patient who has donors with identical type except for the blood type. In that figure, x, y, and z represent the blood types of patient i, 1st donor of i, and 2nd donor of i, respectively. Suppose that patient i has a profitable deviation. That is, she has a false preference with which she can get a more preferable transplant to the one with the true preference. Suppose also that the figure describes the allocation when patient i submits her true preference. Let us call the allocation a. By Lemma 2, patient i gets a hybrid transplant with other's donor or living donor transplant at a. In either case, she uses a living donor from other patient, say d_k. By Proposition 1, one of the patient i's donors donates to a patient, say j.

We claim that group i's contribution at a must be critical to maintain the welfare level of other patients. If $x \trianglerighteq_1 z$ holds, then the donor d_k can donate to patient j directly, because the blood type compatibility relation $\trianglerighteq_1 = \trianglerighteq_{\mathcal{B}}$ is transitive. This means that the patients in $N \backslash \{i\}$ can attain the welfare level of a without the contribution of group i. Consequently, even if patient i reports a preference which states that a_i is unacceptable, other patients can keep consuming transplants indifferent with a_{-i}. Thus patient i's deviation has no effect on the priority mechanism. This contradicts that patient i has a profitable deviation. Thus, x cannot be blood-type-compatible with z, i.e., $x \not\trianglerighteq_1 z$.

Lemma 3. *Let $Y \in \{E, HE\}$, $(R;\theta) \in \mathcal{R}^N \times \Theta$ and $i \in N$. Suppose that for each $d \in D_i^L$, $\theta(d)$ is identical with $\theta(i)$ except for the blood type, i.e., $\theta_2(d) = \theta_2(i)$. Then,*

$$\left[\exists R_i' \in \mathcal{R} \text{ s.t. } \varphi_i^P(R_i', R_{-i};\theta) \; P_i \; \varphi_i^P(R_i, R_{-i};\theta)\right] \Rightarrow \left[\exists d \in D_i^L \text{ s.t. } \theta_1(i) \not\trianglerighteq_1 \theta_1(d)\right].$$

Proof. Assume, without loss of generality, $1 \succ 2 \succ \ldots \succ n$. Suppose to the contrary that

$$\forall d \in D_i^L, \theta_1(i) \trianglerighteq_1 \theta_1(d). \tag{5}$$

Fix patient i's profitable deviation $R'_i \in \mathcal{R}$. For notational simplicity, let $b^\theta := \varphi^P(R'_i, R_{-i}; \theta)$ and $a^\theta := \varphi^P(R; \theta)$. The proof consists of three steps.

<u>Step 1:</u> Defining new allocations b'^θ and a'^θ. We define b'^θ and a'^θ in $\mathcal{A}^Y(\theta)$ based on b^θ and a^θ. Since the way to generate a'^θ from a^θ is the same as the one for b'^θ from b^θ, we only describe the construction of b'^θ in detail. The definition of b'^θ varies according to the number of other patient's living donors in $b_i^{\theta L}$. Each case below corresponds to the case where the number is 0, 1 and 2, respectively.

<u>Case 1:</u> $b_i^\theta \in X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$. For each $m \in N$,

$$b'^\theta_m := \begin{cases} (0, \emptyset) \text{ if } m = i, \\ b_m^\theta \text{ if } m \neq i. \end{cases}$$

Obviously, $b'^\theta \in \mathcal{A}^Y(\theta)$.

<u>Case 2:</u> $b_i^\theta \in X_i^{11}(\theta) \backslash \tilde{X}_i^{11}(\theta)$ or $\left[b_i^\theta \in X_i^{02}(\theta) \text{ and } b_i^{\theta L} \cap D_i^L \neq \emptyset\right]$. Let $d_i \in D_i^L$ be i's donor who donates to other patient, say $k \in N\backslash\{i\}$, at b^θ, i.e., $d_i \in b_k^{\theta L}$. Let $d_\ell \in D_\ell^L$ be other patient's donor who donates to i at b^θ, i.e., $\ell \neq i$ and $d_\ell \in b_i^{\theta L}$. For each $m \in N$,

$$b'^\theta_m := \begin{cases} (0, \emptyset) \text{ if } m = i, \\ (b_k^{\theta C}, (b_k^{\theta L} \backslash \{d_i\}) \cup \{d_\ell\}) \text{ if } m = k, \\ b_m^\theta \text{ if } m \notin \{i, k\}. \end{cases}$$

We claim that $b'^\theta \in \mathcal{A}^Y(\theta)$. To show this, it is sufficient to prove that d_ℓ is compatible with k, i.e., $\theta(d_\ell) \trianglerighteq \theta(k)$. First, we show

$$\begin{aligned} \theta_2(d_\ell) &\trianglerighteq_2 \theta_2(i) && (\because d_\ell \in b_i^{\theta L}) \\ &\trianglerighteq_2 \theta_2(d_i) && (\because \text{Assumption of Lemma 3}) \\ &\trianglerighteq_2 \theta_2(k). && (\because d_i \in b_k^{\theta L}) \end{aligned}$$

Since the binary relation $\trianglerighteq_2$ is the equality "=", it is transitive. Thus $\theta_2(d_\ell) \trianglerighteq_2 \theta_2(k)$. Similarly,

$$\begin{aligned} \theta_1(d_\ell) &\trianglerighteq_1 \theta_1(i) && (\because d_\ell \in b_i^{\theta L}) \\ &\trianglerighteq_1 \theta_1(d_i) && (\because \text{The contradiction hypothesis (5)}) \\ &\trianglerighteq_1 \theta_1(k). && (\because d_i \in b_k^{\theta L}) \end{aligned}$$

Since the binary relation $\trianglerighteq_1$ is transitive, $\theta_1(d_\ell) \trianglerighteq_1 \theta_1(k)$. In sum, $\theta(d_\ell) \trianglerighteq \theta(k)$. Thus $b'^\theta \in \mathcal{A}^Y(\theta)$.

<u>Case 3:</u> $b_i^\theta \in X_i^{02}(\theta)$ and $b_i^{\theta L} \cap D_i^L = \emptyset$. Let $d_{i1}, d_{i2} \in D_i^L$ be i's donor who donate to other patient(s), say $k, \ell \in N\backslash\{i\}$, at b^θ, i.e., $d_{i1} \in b_k^{\theta L}$ and $d_{i2} \in b_\ell^{\theta L}$. Let $d_p, d_q \in D^L \backslash D_i^L$ be other patient's donors

who donate to i at b^θ, i.e., $b_i^{\theta L} = \{d_p, d_q\}$. If $k = \ell$, then let b'^θ be such that for each $m \in N$,

$$b'^\theta_m := \begin{cases} (0, \emptyset) \text{ if } m = i, \\ (0, \{d_p, d_q\}) \text{ if } m = k, \\ b^\theta_m \text{ if } m \notin \{i, k\}. \end{cases}$$

If $k \neq \ell$, then let b'^θ be such that for each $m \in N$,

$$b'^\theta_m := \begin{cases} (0, \emptyset) \text{ if } m = i, \\ (b_k^{\theta C}, (b_k^{\theta L} \backslash \{d_{i1}\}) \cup \{d_p\}) \text{ if } m = k, \\ (b_\ell^{\theta C}, (b_\ell^{\theta L} \backslash \{d_{i2}\}) \cup \{d_q\}) \text{ if } m = \ell, \\ b^\theta_m \text{ if } m \notin \{i, k, \ell\}. \end{cases}$$

In either case, the proof for $b'^\theta \in \mathcal{A}^Y(\theta)$ is the same as the one given in Case 2. Thus we omit it.

In the same manner, we define a'^θ based on a^θ.[30] Note that every patient, except for i, receives the same transplantation type at b'^θ and b^θ in each case. This is true at a'^θ and a^θ. Thus we have

$$\forall k \in N \backslash \{i\}, b'^\theta_k \; I_k \; b^\theta_k \text{ and } a'^\theta_k \; I_k \; a^\theta_k. \tag{6}$$

Step 2: We show $\forall j \in N$ with $j \succ i$, $b^\theta_j \; I_j \; a^\theta_j$. Suppose to the contrary that for some $j \in N$ with $j \succ i$, $b^\theta_j \; P_j \; a^\theta_j$ or $a^\theta_j \; P_j \; b^\theta_j$. Let j be the highest-priority patient among the patients who are not indifferent between b^θ and a^θ.

First, suppose $b^\theta_j \; P_j \; a^\theta_j$. Note that $b'^\theta \in \mathcal{I}^Y(R;\theta)$ by $b'^\theta_i = (0, \emptyset)$ and (6). Thus $b'^\theta \in \Phi^Y_0(R;\theta)$. By the definition of j and (6), for each $k \in N$ with $k \succ j$, $a^\theta_k \; I_k \; b^\theta_k \; I_k \; b'^\theta_k$. Thus $b'^\theta \in \Phi^Y_{j-1}(R;\theta)$. By (6), $b'^\theta_j \; I_j \; b^\theta_j \; P_j \; a^\theta_j$. Thus we conclude $a^\theta \notin \Phi^Y_j(R;\theta) \supseteq \Phi^Y(R;\theta)$, contradicting $a^\theta = \varphi^P(R;\theta) \in \Phi^Y(R;\theta)$.

Next, suppose that $a^\theta_j \; P_j \; b^\theta_j$. Note that $a'^\theta \in \mathcal{I}^Y(R'_i, R_{-i};\theta)$ by $a'^\theta_i = (0, \emptyset)$ and (6). Thus $a'^\theta \in \Phi^Y_0(R'_i, R_{-i};\theta)$. By the definition of j and (6), for each $k \in N$ with $k \succ j$, $b^\theta_k \; I_k \; a^\theta_k \; I_k \; a'^\theta_k$. Thus $a'^\theta \in \Phi^Y_{j-1}(R'_i, R_{-i};\theta)$. By (6), $a'^\theta_j \; I_j \; a^\theta_j \; P_j \; b^\theta_j$. Thus we conclude $b^\theta \notin \Phi^Y_j(R'_i, R_{-i};\theta) \supseteq \Phi^Y(R'_i, R_{-i};\theta)$. This contradicts $b^\theta = \varphi^P(R'_i, R_{-i};\theta) \in \Phi^Y(R'_i, R_{-i};\theta)$.

In either case, we obtain a contradiction. This completes the proof of Step 2.

Step 3: We complete the proof. Since b^θ satisfies that $b^\theta_i \; P_i \; a^\theta_i \; R_i \; (0, \emptyset)$ and $\forall k \in N \backslash \{i\}, b^\theta_k = \varphi^P_k(R'_i, R_{-i};\theta) \; R_k \; (0, \emptyset)$, we have $b^\theta \in \mathcal{I}^Y(R;\theta) = \Phi^Y_0(R;\theta)$. Thus, by Step 2, $b^\theta \in \Phi^Y_{i-1}(R;\theta)$. Since $b^\theta_i \; P_i \; a^\theta_i$, we conclude $a^\theta \notin \Phi^Y_i(R;\theta) \supseteq \Phi^Y(R;\theta)$. This contradicts $a^\theta = \varphi^P(R;\theta) \in \Phi^Y(R;\theta)$. □

Proof of Theorem 3. Without loss of generality, assume that $1 \succ 2 \succ \ldots \succ n$. Suppose to the

[30] By Lemma 2, a^θ_i is a hybrid transplant with other's donor or a living donor transplant. Thus Case 1 is redundant to define a'^θ.

contrary that for some $i \in N$, $(R_i^*, \theta_i^*) \in \mathcal{R} \times \Theta_i$, and $R_i \in \mathcal{R}$,

$$\begin{aligned} &\sum_{(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}} p_i(R_{-i};\theta_{-i} \mid R_i^*, \theta_i^*) u_i^* \left(\varphi_i^P(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i}) \,\middle|\, R_i^*, \theta_i^* \right) \\ < &\sum_{(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}} p_i(R_{-i};\theta_{-i} \mid R_i^*, \theta_i^*) u_i^* \left(\varphi_i^P(\theta_{d_c}; (R_i, \theta_i^*); (R_j, \theta_j)_{j \neq i}) \,\middle|\, R_i^*, \theta_i^* \right). \end{aligned} \tag{7}$$

A direct consequence of the hypothesis (7) is that at least one $(R'_{-i};\theta'_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, we have

$$u_i^* \left(\varphi_i^P(\theta'_{d_c}; (R_i^*, \theta_i^*); (R'_j, \theta'_j)_{j \neq i}) \,\middle|\, R_i^*, \theta_i^* \right) < u_i^* \left(\varphi_i^P(\theta'_{d_c}; (R_i, \theta_i^*); (R'_j, \theta'_j)_{j \neq i}) \,\middle|\, R_i^*, \theta_i^* \right) \tag{8}$$

Thus, patient i is not the highest-priority patient, i.e., $i \neq 1$. Since $\varphi_i^P(\theta'_{d_c}; (R_i^*, \theta_i^*); (R'_j, \theta'_j)_{j \neq i})$ is at least as good as $(0, \emptyset)$ at R_i^*, patient i's true preference R_i^* has at least one acceptable transplantation type, i.e., $Ac_i(R_i^*) \neq \emptyset$. Thus we can apply Assumption 2 to patient i. Moreover, by Lemma 2, $\varphi_i^P(\theta'_{d_c}; (R_i^*, \theta_i^*); (R'_j, \theta'_j)_{j \neq i}) \in \left(X_i^{11}(\theta_i^*, \theta'_{-i}) \setminus \tilde{X}_i^{11}(\theta_i^*, \theta'_{-i}) \right) \cup X_i^{02}(\theta_i^*, \theta'_{-i})$ and $(0, \emptyset)$ P_i $\varphi_i^P(\theta'_{d_c}; (R_i^*, \theta_i^*); (R'_j, \theta'_j)_{j \neq i})$. Consequently, at least one of the following two statements holds:

$$02 \text{ is acceptable at } R_i^*, \text{ but not acceptable at } R_i. \tag{9}$$

$$11 \text{ is acceptable at } R_i^*, \text{ but not acceptable at } R_i. \tag{10}$$

In the subsequent part of the proof, we show

$$\exists (R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i} \text{ s.t. } \left[\begin{array}{c} \varphi_i^P(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i}) \in Ac_i(R_i^*) \\ \text{and} \\ \varphi_i^P(\theta_{d_c}; (R_i, \theta_i^*); (R_j, \theta_j)_{j \neq i}) = (0, \emptyset) \end{array} \right]. \tag{11}$$

Note that (11) contradicts inequality (7) because Assumption 2 states that even if misreporting R_i is successful for every $(\tilde{R}_{-i}; \tilde{\theta}_{-i}) \in \left(\mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i} \right) \setminus \{(R_{-i};\theta_{-i})\}$, i.e., $\varphi_i^P(R_i, \tilde{R}_{-i}; \theta_i^*, \tilde{\theta}_{-i})$ P_i^* $\varphi_i^P(R_i^*, \tilde{R}_{-i}; \theta_i^*, \tilde{\theta}_{-i})$, the expected utility gain from misreporting is canceled out by the failure of misreporting at $(R_{-i};\theta_{-i})$ (See statement (11)).

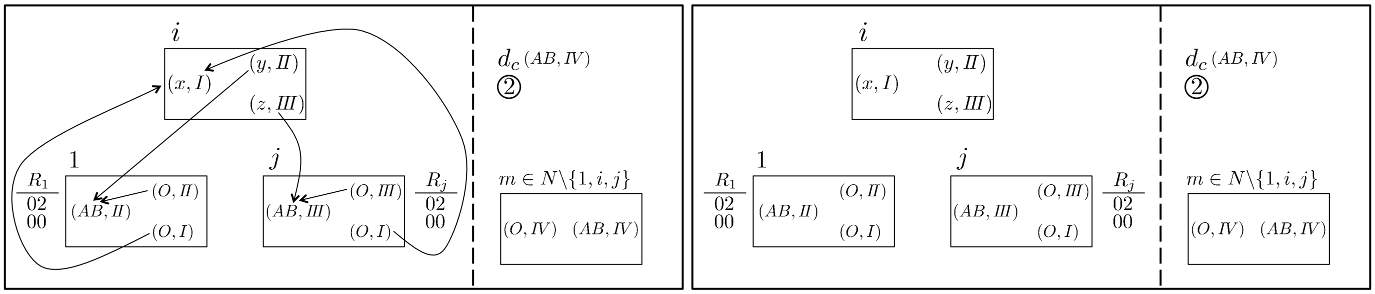
In the following, we construct $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ to show (11) for each case separately. For notational simplicity, we will use the following notation.

$$a := \varphi^P(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i}) \text{ and } b := \varphi^P(\theta_{d_c}; (R_i, \theta_i^*); (R_j, \theta_j)_{j \neq i}).$$

That is, a denotes the allocation under truth-telling at $(R_{-i};\theta_{-i})$ constructed in the case under consideration, and b does the allocation under misreporting at $(R_{-i};\theta_{-i})$ constructed in the case under consideration.

By Assumption 4, let $j, k \in N \setminus \{1, i\}$ be distinct patients who have two living donors. Let $D_1^L = \{d_{11}, d_{12}\}$, $D_j^L = \{d_{j1}, d_{j2}\}$, and $D_k^L = \{d_{k1}, d_{k2}\}$.

Figure 9: Allocations a and b in Case 1.1.1.



Case 1: Patient i has two living donors, i.e., $|D_i^L| = 2$. Let $D_i^L = \{d_{i1}, d_{i2}\}$.
Case 1.1: (9) holds.
Case 1.1.1: $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_{i1}), \theta^*_{i2}(d_{i2})\}| = 3$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, II), \theta^*_i(d_{i2}) = (z, III)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D_m^L$.

$$\Big\{\theta_{d_c} = \Big(2, (AB, IV)\Big) \quad \begin{cases} \theta_1(1) = (AB, II) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, I) \end{cases} \begin{cases} \theta_j(j) = (AB, III) \\ \theta_j(d_{j1}) = (O, III) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.[31]

$$\begin{array}{r|lll} R_1 & 02 & 00 & \cdots \\ R_j & 02 & 00 & \cdots \end{array}$$

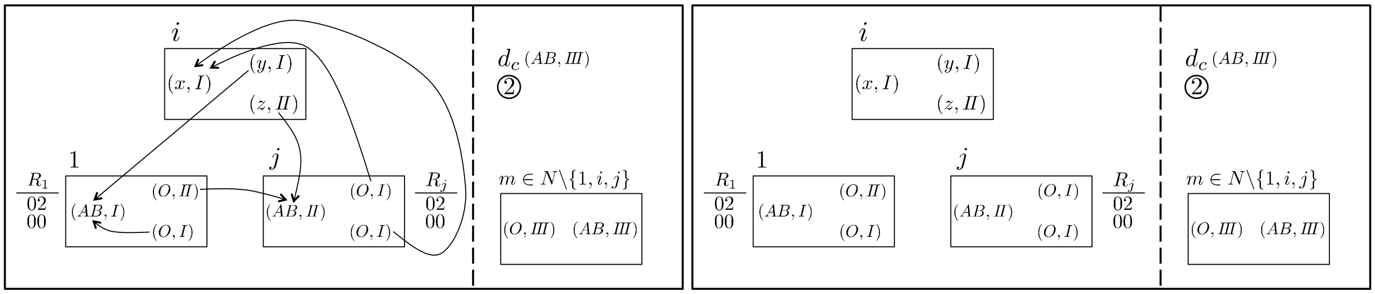
Claim 1.1.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, \{d_{i1}, d_{11}\}) \text{ if } m = 1, \\ (0, \{d_{12}, d_{j2}\}) \text{ if } m = i, \\ (0, \{d_{i2}, d_{j1}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

Proof. Since the above allocation is the only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, φ^P selects it. □
Claim 1.1.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$, $b_m = (0, \emptyset)$.
Proof. Since 02 is unacceptable at R_i, patients 1 and j cannot receive living donor transplants. Moreover the cadaveric lung is not compatible with any patient. □
Case 1.1.2: $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_{i1}), \theta^*_{i2}(d_{i2})\}| = 2$.

[31] The preferences of patients in $N\setminus\{1, i, j\}$ are omitted because they are free. In the later cases, the omitted preferences are free, too.

Figure 10: Allocations a and b in Case 1.1.2.1.



Case 1.1.2.1: $\theta^*_{i2}(d_{i1}) \neq \theta^*_{i2}(d_{i2})$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, I), \theta^*_i(d_{i2}) = (z, II)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (AB, III)\Big) \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, I) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_j & 02 & 00 & \cdots \end{array}$$

Claim 1.1.2.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, i, j\}, \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

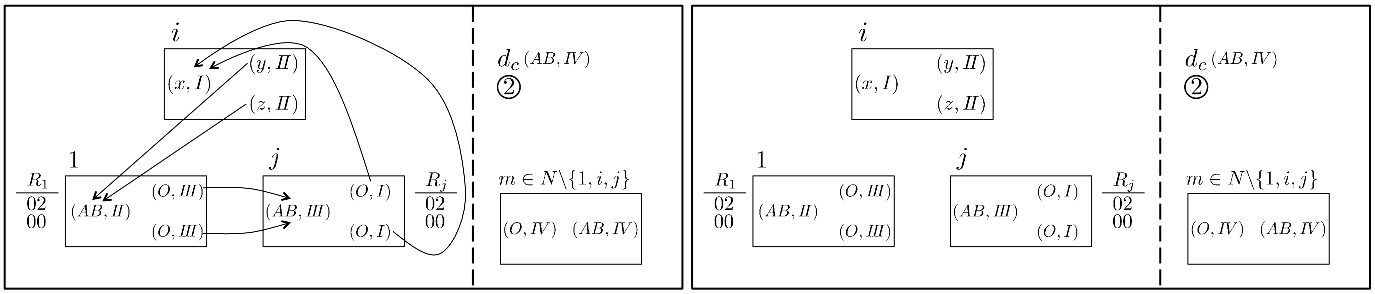
Proof. Since the above allocation is the only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, φ^P selects it. □

Claim 1.1.2.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$, $b_m = (0, \emptyset)$.

Proof. Since 02 is unacceptable at R_i, b_i cannot be 02. Moreover the cadaveric lung is not compatible with patient i. Thus $b_i = (0, \emptyset)$. This implies that d_{i2} does not donate to any patient (Proposition 1). Thus, $b_j = (0, \emptyset)$, since 02 is the only acceptable transplantation type for patient j. This implies that d_{j1} and d_{j2} do not donate to any patient. Thus $b_1 = (0, \emptyset)$. □

Case 1.1.2.2: $\theta^*_{i2}(d_{i1}) = \theta^*_{i2}(d_{i2})$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, II), \theta^*_i(d_{i2}) = (z, II)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

Figure 11: Allocations a and b in Case 1.1.2.2.



- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (AB, IIV)\Big) \quad \begin{cases} \theta_1(1) = (AB, II) \\ \theta_1(d_{11}) = (O, III) \\ \theta_1(d_{12}) = (O, III) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, III) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_j & 02 & 00 & \cdots \end{array}$$

<u>Claim 1.1.2.2a:</u> For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, D^L_i) \text{ if } m = 1, \\ (0, D^L_j) \text{ if } m = i, \\ (0, D^L_1) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1, i, j\}. \end{cases}$$

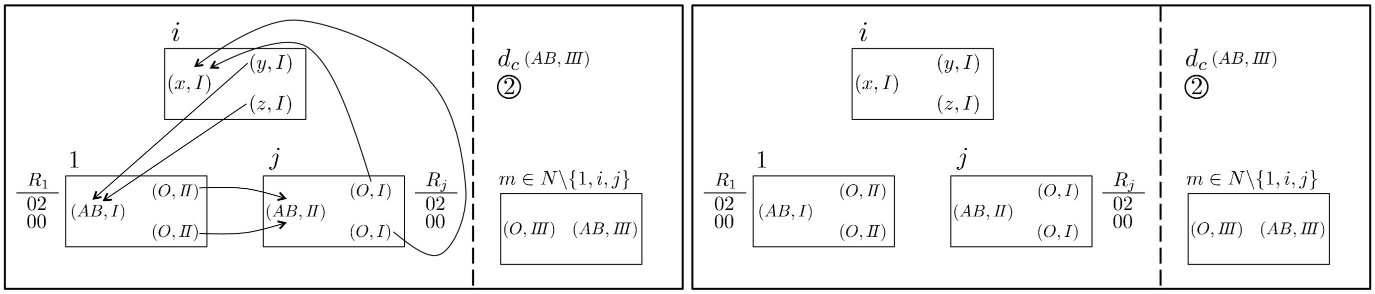
Proof. Since the above allocation is the only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j \neq i})$, φ^P selects it. □

<u>Claim 1.1.2.2b:</u> For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$, $b_m = (0, \emptyset)$.

Proof. Since 02 is unacceptable at R_i, b_i cannot be 02. Moreover the cadaveric lung is not compatible with patient i. Thus, $b_i = (0, \emptyset)$. This implies that d_{i1} and d_{i2} do not donate to any patient (Proposition 1). Thus $b_1 = (0, \emptyset)$ since 02 is the only acceptable transplantation type for patient 1. This implies that d_{11} and d_{12} do not donate to any patient. Thus $b_j = (0, \emptyset)$. □

<u>Case 1.1.3:</u> $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_{i1}), \theta^*_{i2}(d_{i2})\}| = 1$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, I), \theta^*_i(d_{i2}) = (z, I)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

Figure 12: Allocations a and b in Case 1.1.3.



- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (AB, III)\Big) \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, II) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, III) \\ \theta_m(d) = (AB, III) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_j & 02 & 00 & \cdots \end{array}$$

Claim 1.1.3a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, i, j\}, \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

Proof. Since the above allocation is the only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, φ^P selects it. □

Claim 1.1.3b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (0, D^L_j) \text{ if } m = 1, \\ (0, D^L_1) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, j\}. \end{cases}$$

Proof. Since 02 is unacceptable at R_i, b_i cannot be 02. Moreover the cadaveric lung is not compatible with patient i. Thus $b_i = (0, \emptyset)$. The only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, φ^P selects it. □

Case 1.2: (10) holds.

Case 1.2.1: $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_{i1}), \theta^*_{i2}(d_{i2})\}| = 3$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, II), \theta^*_i(d_{i2}) = (z, III)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: For $m \in N \setminus \{1,i,j\}$ and $d \in D^L_m$,

$$\left\{\theta_{d_c} = \left(2,(O,I)\right)\right. \quad \begin{cases}\theta_1(1) = (AB, I)\\ \theta_1(d_{11}) = (O,I)\\ \theta_1(d_{12}) = (O,II)\end{cases} \quad \begin{cases}\theta_j(j) = (AB, II)\\ \theta_j(d_{j1}) = (O,I)\\ \theta_j(d_{j2}) = (O,I)\end{cases} \quad \begin{cases}\theta_m(m) = (O, IV)\\ \theta_m(d) = (AB, IV)\end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|cccc} R_1 & 02 & 20 & 00 & \cdots \\ R_j & 02 & 00 & \cdots & \end{array}$$

<u>Claim 1.2.1a:</u> For the above $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1,j\},\\ X^{11}_m(\theta^*_i, \theta_{-i}) \text{ if } m = i,\\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1,i,j\}.\end{cases}$$

Proof. Since the allocation described in the left hand side of Figure 13 belongs to $\mathcal{I}^Y\left(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}\right) = \Phi^Y_0\left(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}\right)$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since a_1 is 02, d_{12} donates to a patient (Proposition 1). Since only patient j can receive d_{12}'s donation, $d_{12} \in a^L_j$. Since the only acceptable transplantation type for j is 02, $a_j \in X^{02}_j(\theta^*_i, \theta_{-i})$. Since a_1 and a_j is 02, d_{11}, d_{j1} and d_{j2} donate to a patient respectively (Proposition 1). Since two of them donate to patient 1, the remaining one donates to patient i. Since no other living donor is compatible with patient i, $a_i \in X^{11}_i(\theta^*_i, \theta_{-i})$. □

<u>Claim 1.2.1b:</u> For the above $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases}(2,\emptyset) \text{ if } m = 1,\\ (0,\emptyset) \text{ if } m \in N\setminus\{1\}.\end{cases}$$

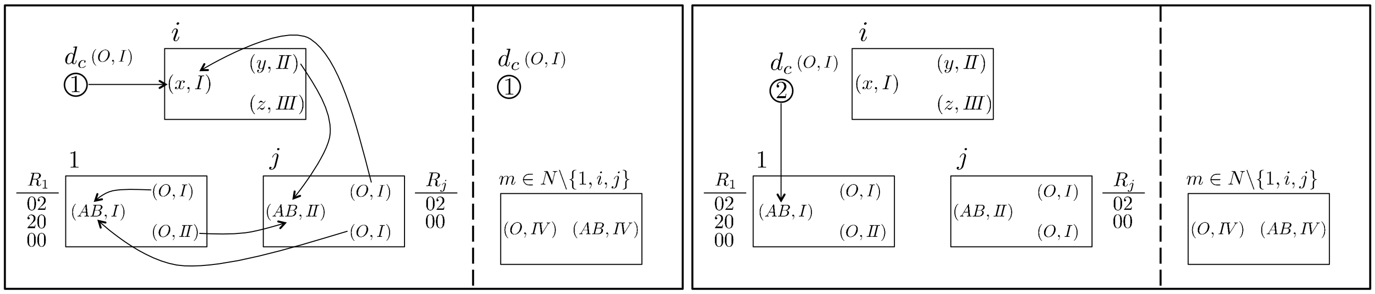
Proof. In a similar manner to Claim 1.2.1a, we can show that b_1 is 02 only if b_i is 11. However, since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2,\emptyset)$. Thus each patient in $N\setminus\{1\}$ cannot use a cadaveric lung. Thus b_m is not $20, 10$ or 11 for each $m \in N\setminus\{1\}$. Since patient j cannot find two compatible living donors in $D^L\setminus D^L_1$, $b_j \notin X^{02}_j(\theta^*_i, \theta_{-i})$. Thus $b_j = (0,\emptyset)$. Since patient $m \in N\setminus\{1,j\}$ cannot find two compatible living donors in $D^L\setminus(D^L_1 \cup D^L_j)$, $b_m \notin X^{02}_m(\theta^*_i, \theta_{-i})$. Thus $b_m = (0,\emptyset)$. □

<u>**Case 1.2.2:**</u> $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_{i1}), \theta^*_{i2}(d_{i2})\}| = 2$.

<u>**Case 1.2.2.1:**</u> $\theta^*_{i2}(d_{i1}) \neq \theta^*_{i2}(d_{i2})$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, I), \theta^*_i(d_{i2}) = (z, II)$, where $x, y, z \in \mathcal{B}$. We consider the following three cases separately.

<u>**Case 1.2.2.1.1:**</u> $y \not\trianglerighteq_1 x$. Define $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

Figure 13: Allocations a and b in Case 1.2.1.



- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (x, I) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, III) \end{cases} \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, III) \end{cases} \begin{cases} \theta_k(k) = (AB, III) \\ \theta_k(d_{k1}) = (O, I) \\ \theta_k(d_{k2}) = (O, I) \end{cases}$$

$$\begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

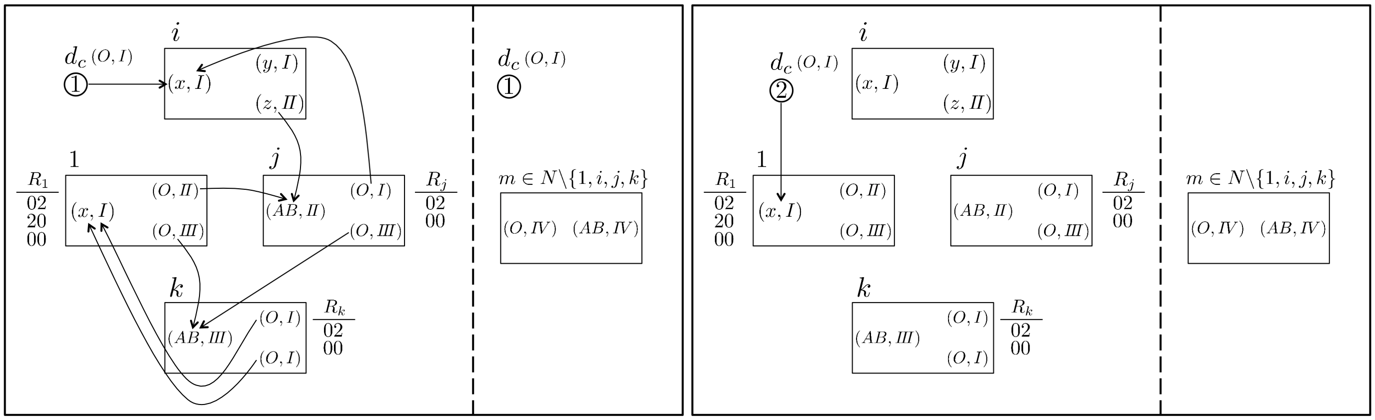
R_1	02	20	00	$\cdots$
R_j	02	00	$\cdots$	
R_k	02	00	$\cdots$	

Claim 1.2.2.1.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, j, k\}, \\ X^{11}_m(\theta^*_i, \theta_{-i}) \text{ if } m = i, \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j, k\}. \end{cases}$$

Proof. Since the allocation described in the left hand side of Figure 14 belongs to $\mathcal{I}^Y(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}) = \Phi^Y_0(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since a_1 is 02, d_{11} and d_{12} donate to a patient respectively (Proposition 1). Since only patients j and k can receive d_{11}'s and d_{12}'s donation respectively, $d_{11} \in a^L_j$ and $d_{12} \in a^L_k$. Since the only acceptable transplantation type for j and k is 02, $a_j \in X^{02}_j(\theta^*_i, \theta_{-i})$ and $a_k \in X^{02}_k(\theta^*_i, \theta_{-i})$. Thus $a^L_j = \{d_{11}, d_{i2}\}$ and $a^L_k = \{d_{12}, d_{j2}\}$. Since a_j and a_k are 02, d_{j1}, d_{k1} and d_{k2} donate to a patient respectively (Proposition 1). Since two of them donate to patient 1, the remaining one donates to patient i (Recall that $y \not\trianglerighteq_1 x$). Since no other living donor is compatible with patient i, $a_i \in X^{11}_i(\theta^*_i, \theta_{-i})$. □

Figure 14: Allocations a and b in Case 1.2.2.1.1.



Claim 1.2.2.1.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.2.1.1a, we can show that b_1 is 02 only if b_i is 11. However, since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Thus each patient in $N\backslash\{1\}$ cannot use a cadaveric lung. Thus b_m is not $20, 10$ or 11 for each $m \in N\backslash\{1\}$. Since patient k cannot find two compatible living donors in $D^L\backslash D^L_1$, $b_k \notin X^{02}_k(\theta^*_i, \theta_{-i})$. Thus $b_k = (0, \emptyset)$. Since patient $m \in N\backslash\{1, k\}$ cannot find two compatible living donors in $D^L\backslash(D^L_1 \cup D^L_k)$, $b_m \notin X^{02}_m(\theta^*_i, \theta_{-i})$. Thus $b_m = (0, \emptyset)$. □

Case 1.2.2.1.2: $y = x$. Note that $y = x$ implies that $\theta^*_i(i) = (x, I) = (y, I) = \theta^*_i(d_{i1})$. Thus, by Assumption 5, this case is excluded.

Case 1.2.2.1.3: $y \trianglerighteq_1 x$ and $y \neq x$. Note that $y \trianglerighteq_1 x$ and $y \neq x$ imply that the combination of x and y is one of the following five: $(O, A), (O, B), (O, AB), (A, AB), (B, AB)$. Note also that each of them satisfies $x \not\trianglerighteq_1 y$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ as follows:

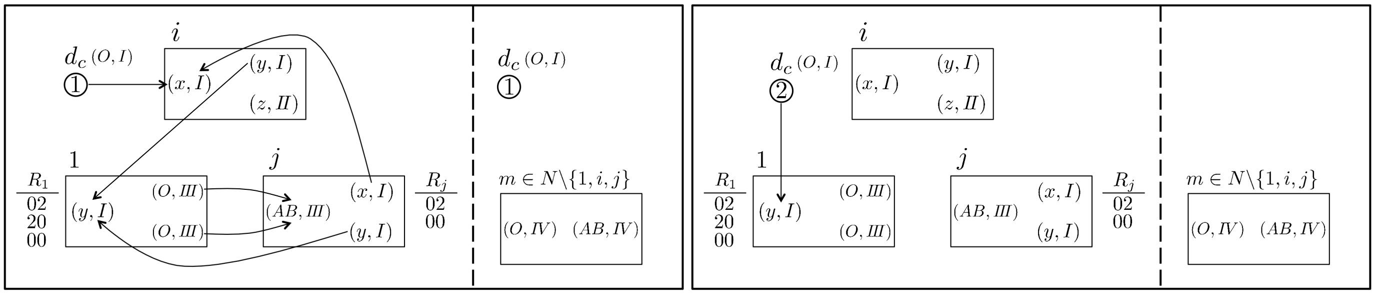
- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (y, I) \\ \theta_1(d_{11}) = (O, III) \\ \theta_1(d_{12}) = (O, III) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, III) \\ \theta_j(d_{j1}) = (x, I) \\ \theta_j(d_{j2}) = (y, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

R_1	02	20	00	$\cdots$
R_j	02	00	$\cdots$	

Figure 15: Allocations a and b in Case 1.2.2.1.3.



Claim 1.2.2.1.3a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, \{d_{i1}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{j1}\}) \text{ if } m = i, \\ (0, \{d_{11}, d_{12}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1, i, j\}. \end{cases}$$

Proof. Since the above allocation belongs to $\mathcal{I}^Y(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i}) = \Phi_0^Y(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i})$, $a_1 \in X_1^{02}(\theta_i^*, \theta_{-i})$. Since only d_{i1} and d_{j2} are compatible with patient 1, $a_1 = (0, \{d_{i1}, d_{j2}\})$ (Recall that $x \not\trianglerighteq_1 y$). Since a_1 is 02, d_{11} and d_{12} donate to a patient respectively (Proposition 1). Since only patients j can receive d_{11}'s and d_{12}'s donation, $a_j = (0, \{d_{11}, d_{12}\})$. Since a_j is 02, d_{j1} donates to a patient (Proposition 1). Since patient 1 cannot receive d_{j1}'s donation, it goes to i, i.e., $d_{j1} \in a_i^L$. Since no other living donor is compatible with patient i, $a_i = (1, \{d_{j1}\})$. □

Claim 1.2.2.1.3b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.2.1.3a, we can show that b_1 is 02 only if b_i is 11. However, since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Thus each patient in $N\backslash\{1\}$ cannot use a cadaveric lung. Thus b_m is not $20, 10$ or 11 for each $m \in N\backslash\{1\}$. Since patient j cannot find two compatible living donors in $D^L\backslash D_1^L$, $b_j \notin X_j^{02}(\theta_i^*, \theta_{-i})$. Thus $b_j = (0, \emptyset)$. Since patient $m \in N\backslash\{1, j\}$ cannot find two compatible living donors in $D^L\backslash(D_1^L \cup D_j^L)$, $b_m \notin X_m^{02}(\theta_i^*, \theta_{-i})$. Thus $b_m = (0, \emptyset)$. □

Case 1.2.2.2: $\theta_{i2}^*(d_{i1}) = \theta_{i2}^*(d_{i2})$. Without loss of generality, let $\theta_i^*(i) = (x, I), \theta_i^*(d_{i1}) = (y, II), \theta_i^*(d_{i2}) = (z, II)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, I) \\ \theta_1(d_{12}) = (O, III) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, III) \end{cases} \quad \begin{cases} \theta_k(k) = (AB, III) \\ \theta_k(d_{k1}) = (O, II) \\ \theta_k(d_{k2}) = (O, I) \end{cases}$$

$$\begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

R_1	02	20	00	$\cdots$
R_j	02	00	$\cdots$	
R_k	02	00	$\cdots$	

Claim 1.2.2.2a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, j, k\} \\ X^{11}_m(\theta^*_i, \theta_{-i}) \text{ if } m = i \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j, k\}. \end{cases}$$

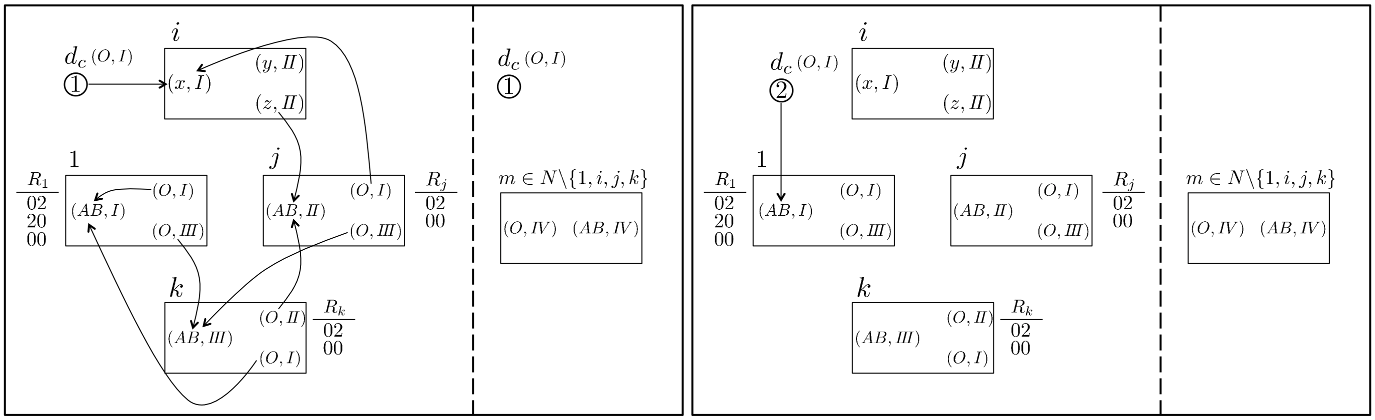
Proof. Since the allocation described in the left hand side of Figure 16 belongs to $\mathcal{I}^Y(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}) = \Phi^Y_0(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since a_1 is 02, d_{12} donates to a patient (Proposition 1). Since only patients k can receive d_{12}'s donation, $d_{12} \in a^L_k$. Since the only acceptable transplantation type for patient k is 02, $a_k \in X^{02}_k(\theta^*_i, \theta_{-i})$. Thus $a^L_k = \{d_{12}, d_{j2}\}$. Since d_{j2} donates to patient k, patient j receives an acceptable transplant, i.e. $a_j \in X^{02}_j(\theta^*_i, \theta_{-i})$. Thus a^L_j consists of d_{k1} and one of d_{i1} and d_{i2} (Note that if $a^L_j = D^L_i$, then d_{k1} cannot donate to any patient). Since a_1, a_j and a_k are 02, d_{11}, d_{j1} and d_{k2} donate to a patient respectively (Proposition 1). Since two of them donate to patient 1, the remaining one donates to patient i, i.e., a_i uses a living donor. Since no other living donor is compatible with patient i, $a_i \in X^{11}_i(\theta^*_i, \theta_{-i})$. □

Claim 1.2.2.2b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.2.2a, we can show that b_1 is 02 only if b_i is 11. However, since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Thus each patient in $N\setminus\{1\}$ cannot use a cadaveric lung. Thus b_m is not $20, 10$ or 11 for each $m \in N\setminus\{1\}$. Since patient k cannot find two compatible living donors in $D^L\setminus D^L_1$, $b_k \notin X^{02}_k(\theta^*_i, \theta_{-i})$. Thus $b_k = (0, \emptyset)$. Since patient j cannot find two compatible living donors in $D^L\setminus(D^L_1 \cup D^L_k)$, $b_j \notin X^{02}_j(\theta^*_i, \theta_{-i})$. Thus $b_j = (0, \emptyset)$. Since patient

Figure 16: Allocations a and b in Case 1.2.2.2.



$m \in N\backslash\{1,j,k\}$ cannot find two compatible living donors in $D^L\backslash(D_1^L \cup D_j^L \cup D_k^L)$, $b_m \notin X_m^{02}(\theta_i^*, \theta_{-i})$. Thus $b_m = (0, \emptyset)$. □

Case 1.2.3: $|\{\theta_{i2}^*(i), \theta_{i2}^*(d_{i1}), \theta_{i2}^*(d_{i2})\}| = 1$. Without loss of generality, let $\theta_i^*(i) = (x, I), \theta_i^*(d_{i1}) = (y, I), \theta_i^*(d_{i2}) = (z, I)$, where $x, y, z \in \mathcal{B}$. By the definition of Θ_i, at least one of d_{i1} and d_{i2} is not compatible with patient i, i.e., $y \not\trianglerighteq_1 x$ or $z \not\trianglerighteq_1 x$. Without loss of generality, assume that $y \not\trianglerighteq_1 x$. Thus $x \neq AB$. By Lemma 3, $x \not\trianglerighteq_1 y$ or $x \not\trianglerighteq_1 z$. Thus $x \neq O$. Summing up, we have $x \in \{A, B\}$. Without loss of generality, we assume $x = A$ till the end of Case 1.2.3.[32]

Note that since $y \not\trianglerighteq_1 x$, $y \in \{B, AB\}$. Moreover, we have the following two claims that narrow down the combination of x, y and z.

Claim 1.2.3: The combination of x, y and z, written as (x, y, z), is one of the following five: (i) (A, B, O), (ii) (A, B, B), (iii) (A, B, AB), (iv) (A, AB, O), and (v) (A, AB, B).

Proof. First, we show that $y = B$ or $z \in \{O, B\}$ by contradiction. Suppose to the contrary that $y \neq B$ and $z \notin \{O, B\}$. Since $y \in \{B, AB\}$, $y = AB$. Since $z \in \mathcal{B}\backslash\{O, B\} = \{A, AB\}$, we have $x = A \trianglerighteq_1 AB = y$ and $x = A \trianglerighteq_1 z$, contradicting Lemma 3.

Now we complete the proof of Claim 1.2.3. Note that $x = A$ and $y \in \{B, AB\}$. First consider the case with $y = B$. Since $z = A$ is impossible by Assumption 5, we have (i), (ii), and (iii). Next consider the case with $y = AB$. By the fact shown in the previous paragraph, we have $z \in \{O, B\}$. Thus we have (iv) and (v). □

We omit the proof for the case (v) because it is same as the one for case (iii). Let us consider the following two cases of 1.2.3.1 and 1.2.3.2 separately.

Case 1.2.3.1: (x, y, z) is $(i)(A, B, O)$ or $(iv)(A, AB, O)$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ as follows:

[32]The same argument works for the case with $x = B$ by replacing A with B and B with A in the proof given here.

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D_m^L$.

$$\left\{\theta_{d_c} = \left(1, (O, I)\right)\right. \quad \begin{cases} \theta_1(1) = (AB, II) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, I) \end{cases} \quad \begin{cases} \theta_j(j) = (O, I) \\ \theta_j(d_{j1}) = (A, I) \\ \theta_j(d_{j2}) = (O, II) \end{cases} \quad \begin{cases} \theta_m(m) = (O, III) \\ \theta_m(d) = (AB, III) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|cccc} R_1 & 02 & 00 & \cdots & \\ R_j & 02 & 11 & 00 & \cdots \end{array}$$

Claim 1.2.3.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N \setminus \{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, \{d_{11}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{j1}\}) \text{ if } m = i, \\ (0, \{d_{12}, d_{i2}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N \setminus \{1, i, j\}. \end{cases}$$

Proof. Since the allocation described in the LHS of Figure 17 belongs to $\mathcal{I}^Y\left(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i}\right) = \Phi_0^Y\left(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i}\right)$, $a_1 \in X_1^{02}(\theta_i^*, \theta_{-i})$. Since only d_{11} and d_{j2} are compatible with patient 1, $a_1 = (0, \{d_{11}, d_{j2}\})$. Note that this implies that a_j^L contains at least one living donor, i.e., a_j is 11 or 02.

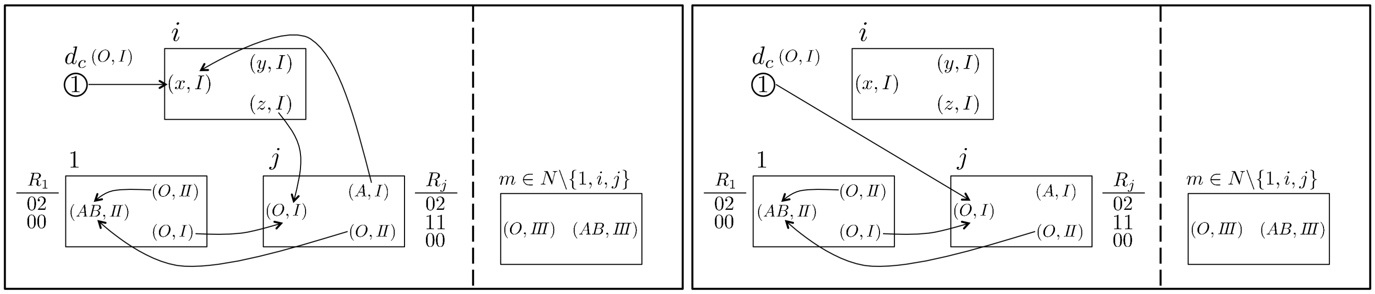
Since $\theta_{d_c q} = 1$, a_i cannot be 20. Moreover, a_i cannot be 10 since it implies that a_j is not 11 ($\because$ patient j cannot use the cadaveric lung) and not 02 ($\because$ patient j cannot receive a donation from d_{i2} by Proposition 1). Moreover, a_i cannot be 02 since it implies that d_{i1} who has no compatible patient donates to a patient. Moreover, a_i cannot be 00 ($\because$ Since 11 is acceptable at R_i^*, the allocation described in the left hand side of Figure 17 excludes the allocations that assign $(0, \emptyset)$ to patient i). Summing up, $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. The allocation described in the LHS of Figure 17 enable patient j to receive 02 under the condition that $a_1 \in X_1^{02}(\theta_i^*, \theta_{-i})$ and $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. Thus a_j is 02. Thus $a_j = (0, \{d_{12}, d_{i2}\})$ ($\because$ Only d_{12} and d_{i2} are compatible living donors with patient j). Thus $a_i = (1, \{d_{j1}\})$. □

Claim 1.2.3.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N \setminus \{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (0, \{d_{11}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{12}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N \setminus \{1, j\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.3.1a, we can show that $b_1 = (0, \{d_{11}, d_{j2}\})$. Note that this implies that b_j^L contains at least one living donor, i.e., b_j is 11 or 02.

Figure 17: Allocations a and b in Case 1.2.3.1.



In a similar manner to Claim 1.2.3.1a, we can show that b_i is not $20, 10$ or 02. Moreover, since 11 is unacceptable at R_i, b_i is not 11. Thus $b_i = (0, \emptyset)$. Since patient j cannot find compatible living donors in $D^L \backslash D^L_1$, b_j is not 02. Thus b_j is 11 since the allocation described in the right hand side of Figure 17 is available. Since it is the only allocation that assigns 02 to patient 1 and 11 to patient j, we are done. □

Case 1.2.3.2: (x, y, z) is $(ii)(A, B, B)$ or $(iii)(A, B, AB)$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\left\{\theta_{d_c} = \Big(1, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (AB, II) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (B, I) \end{cases} \quad \begin{cases} \theta_j(j) = (B, I) \\ \theta_j(d_{j1}) = (A, I) \\ \theta_j(d_{j2}) = (O, II) \end{cases} \quad \begin{cases} \theta_m(m) = (O, III) \\ \theta_m(d) = (AB, III) \end{cases}\right.$$

- The definition of R_{-i}.

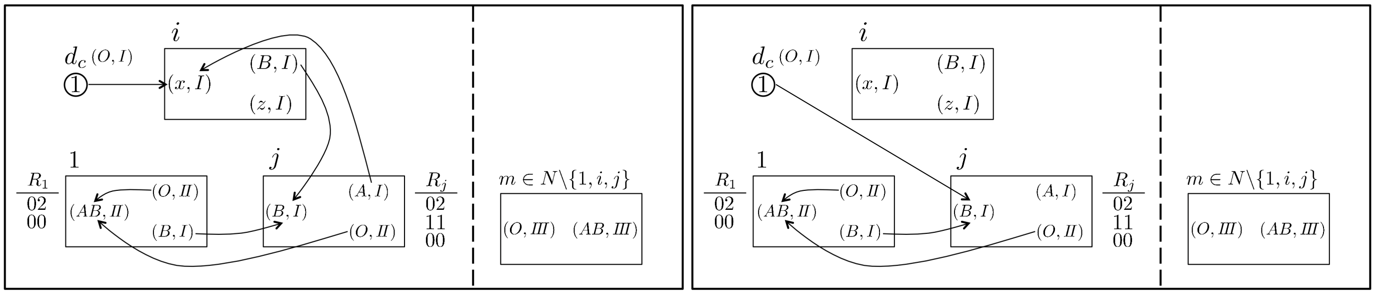
R_1	02	00	$\cdots$	
R_j	02	11	00	$\cdots$

Claim 1.2.3.2a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, \{d_{11}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{j1}\}) \text{ if } m = i, \\ (0, \{d_{12}, d_{i1}\}) \text{ if } m = j \text{ and } (iii) \text{ holds}, \\ (0, \{d_{12}, d_{i1}\}) \text{ or } (0, \{d_{12}, d_{i2}\}) \text{ if } m = j \text{ and } (ii) \text{ holds}, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1, i, j\}. \end{cases}$$

Proof. Since the allocation described in the LHS of Figure 18 belongs to $\mathcal{I}^Y\left(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}\right) = \Phi^Y_0\left(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}\right)$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since only d_{11} and d_{j2} are compatible with patient 1, $a_1 = (0, \{d_{11}, d_{j2}\})$. Note that this implies that a^L_j contains at least one living donor, i.e., a_j is 11 or 02.

Figure 18: Allocations a and b in Case 1.2.3.2.



Since $\theta_{d_cq} = 1$, a_i cannot be 20. Moreover, a_i cannot be 10 since it implies that a_j is not 11 ($\because$ patient j cannot use the cadaveric lung) and not 02 ($\because$ patient j cannot receive a donation from a patient in D_i^L by Proposition 1). Moreover, a_i cannot be 02 since it implies that both d_{i1} and d_{i2} donate to a patient respectively. Note that d_{12} also donates to a patient since a_1 is 02. However, since the economy can receive donation from at most two of d_{i1}, d_{i2} and d_{12}, one of d_{i1} and d_{i2} cannot donate any patient. Thus a_i is not 02. Moreover, a_i cannot be 00 ($\because$ Since 11 is acceptable at R_i^*, the allocation described in the left hand side of Figure 18 excludes the allocations that assign $(0, \emptyset)$ to patient i). In sum, $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. The allocation described in the left hand side of Figure 18 enable patient j to receive 02 under the condition that $a_1 \in X_1^{02}(\theta_i^*, \theta_{-i})$ and $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. Thus a_j is 02. Thus a_j^L consists of d_{12} and a donor in D_i^L ($\because$ Donors in D_j^L are not compatible with patient j). Thus $a_i = (1, \{d_{j1}\})$. $\square$

Claim 1.2.3.2b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (0, \{d_{11}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{12}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1, j\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.3.2a, we can show that $b_1 = (0, \{d_{11}, d_{j2}\})$. Note that this implies that b_j^L contains at least one living donor, i.e., b_j is 11 or 02.

In a similar manner to Claim 1.2.3.2a, we can show that b_i is not $20, 10$ or 02. Moreover, since 11 is unacceptable at R_i, b_i is not 11. Thus $b_i = (0, \emptyset)$. Since patient j cannot find compatible living donors in $D^L \backslash D_1^L$, b_j is not 02. Thus b_j is 11 since the allocation described in the right hand side of Figure 18 is available. Since it is the only allocation that assigns 02 to patient 1 and 11 to patient j, we are done. $\square$

Case 2: Patient i has one living donors, i.e., $|D_i^L| = 1$. Let $D_i^L = \{d_i\}$. Note that, by Proposition 1, patient i never receives a living donor transplant at any profile in $\mathcal{R}^N \times \Theta$. Thus, (10) holds.

Case 2.1: $|\{\theta_{i2}^*(i), \theta_{i2}^*(d_i)\}| = 2$. Without loss of generality, let $\theta_i^*(i) = (x, I)$ and $\theta_i^*(d_i) = (y, II)$, where $x, y \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\left\{\theta_{d_c} = \Big(2, (O, I)\Big)\right. \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, I) \\ \theta_1(d_{12}) = (O, III) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, III) \end{cases} \quad \begin{cases} \theta_k(k) = (AB, III) \\ \theta_k(d_{k1}) = (O, II) \\ \theta_k(d_{k2}) = (O, I) \end{cases}$$

$$\begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

R_1	02	20	00	$\cdots$
R_j	02	00	$\cdots$	
R_k	02	00	$\cdots$	

Claim 2.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, j, k\}, \\ X^{11}_m(\theta^*_i, \theta_{-i}) \text{ if } m = i, \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j, k\}. \end{cases}$$

Proof. Since the allocation described in the left hand side of Figure 19 belongs to $\mathcal{I}^Y(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}) = \Phi^Y_0(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since a_1 is 02, d_{12} donates to a patient (Proposition 1). Since only patients k can receive d_{12}'s donation, $d_{12} \in a^L_k$. Since the only acceptable transplantation type for patient k is 02, $a_k \in X^{02}_k(\theta^*_i, \theta_{-i})$. Thus $a^L_k = \{d_{12}, d_{j2}\}$. Since d_{j2} donates to patient k, patient j receives an acceptable transplant, i.e. $a_j \in X^{02}_j(\theta^*_i, \theta_{-i})$. Thus $a^L_j = \{d_i, d_{k1}\}$ ($\because$ Only d_i and d_{k1} are compatible with patient j). Thus patient i receives a donation from a living donor, i.e., $a_i \in X^{11}_i(\theta^*_i, \theta_{-i})$. □

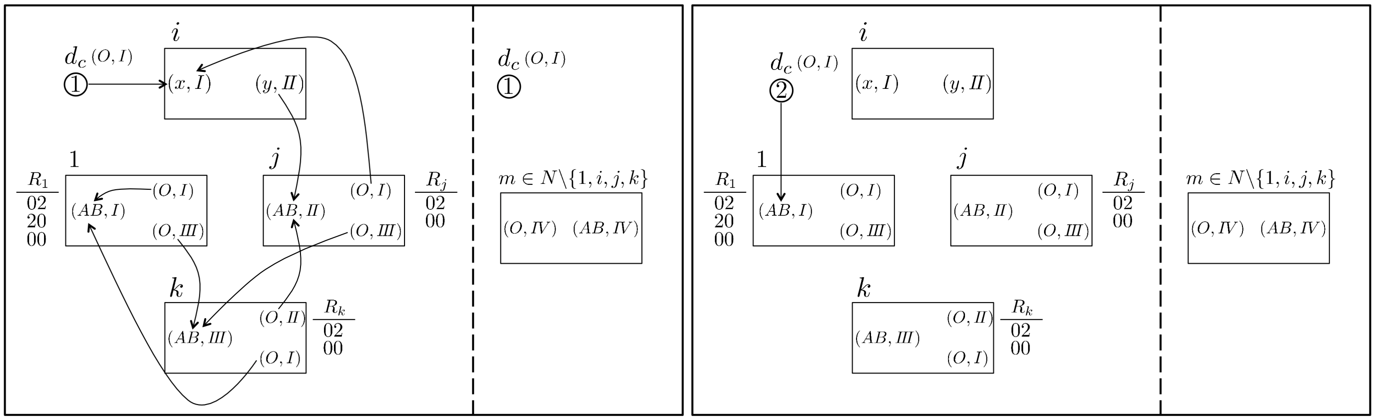
Claim 2.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1\}. \end{cases}$$

Proof. In a similar manner to Case 2.1a, we can show that b_1 is 02 only if b_i is 11. Since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Since b_1 does not use a living donor, d_{11} and d_{12} do not donate to any patient. Since patient k cannot find two compatible living donors in $D^L\setminus D^L_1$, b_k is not 02. Thus $b_k = (0, \emptyset)$. Since b_k does not use a living donor, d_{k1} and d_{k2} do not donate to any patient. Since patient j cannot find two compatible living donors in $D^L\setminus(D^L_1 \cup D^L_k)$, b_j is not 02. Thus $b_j = (0, \emptyset)$. Since patient i cannot use a cadaveric lung, b_i is not $20, 10$ or 11. □

Case 2.2: $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_i)\}| = 1$. Without loss of generality, let $\theta^*_i(i) = (x, I)$ and $\theta^*_i(d_i) = (y, I)$, where $x, y \in \mathcal{B}$. Note that $x \not\trianglerighteq_1 y$ by Lemma 3. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

Figure 19: Allocations a and b in Case 2.1.



- The definition of θ_{-i}: For $m \in N \setminus \{1, i, j\}$ and $d \in D_m^L$,

$$\left\{\theta_{d_c} = \Big(2, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (y, I) \\ \theta_1(d_{11}) = (x, I) \\ \theta_1(d_{12}) = (O, II) \end{cases} \quad \begin{cases} \theta_m(m) = (O, III) \\ \theta_m(d) = (AB, III) \end{cases}\right.$$

- The definition of R_{-i}.

$$R_1 \,\Big|\, 11 \quad 20 \quad 00 \quad \cdots$$

Claim 2.2a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (1, \{d_i\}) \text{ if } m = 1, \\ (1, \{d_{11}\}) \text{ if } m = i, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1, i\}. \end{cases}$$

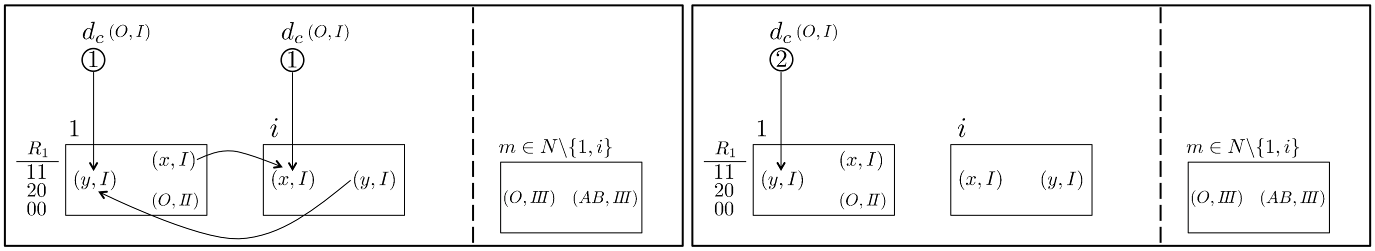
Proof. Since the above allocation belongs to $\mathcal{I}^Y(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i}) = \Phi_0^Y(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X_1^{11}(\theta_i^*, \theta_{-i})$. Since d_i is the only living donor compatible with patient 1, $a_1 = (1, \{d_i\})$. By Proposition 1, one of d_{11} and d_{12} donates to a patient. Since donor d_{12} has no compatible patient, d_{11} donates to patient i. Thus $a_i = (1, \{d_{11}\})$. $\square$

Claim 2.2b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1\}. \end{cases}$$

Proof. Note that patient 1 cannot receive a hybrid transplant with own donor since $x \not\trianglerighteq_1 y$. Since 11 is unacceptable at R_i, b_1 is not 11. Thus $b_1 = (2, \emptyset)$. Since patient 1 uses two units of cadaveric lungs, patient i cannot use a cadaveric lung. Thus b_i is not $20, 10$ or 11. Thus $b_i = (0, \emptyset)$. $\square$

Figure 20: Allocations a and b in Case 2.2.



Case 3: D_i^L contains no living donor. Note that, by Proposition 1, patient i never receives a living donor transplant or a hybrid transplant at any profile in $\mathcal{R}^N \times \Theta$. Thus, by Lemma 2, patient i cannot manipulate φ^P. $\square$