Concerted Efforts?
Monetary Policy and Macro-Prudential Tools*

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Abstract

The inception of macro-prudential policy frameworks in the wake of the global financial crisis raises questions of how macro-prudential and monetary policies should be coordinated. We examine these questions through the lens of a macroeconomic model featuring nominal rigidities, housing, incomplete risk-sharing between borrower and saver households, and macro-prudential tools in the form of mortgage loan-to-value and bank capital requirements. We derive a welfare-based loss function which suggests a role for active macro-prudential policy to enhance risk-sharing. Macro-prudential policy faces tradeoffs, however, and complete macro-prudential stabilization is not generally possible in our model. Nonetheless, simulations of a housing boom and bust suggest that macro-prudential tools could alleviate debt-deleveraging and help avoid zero lower bound episodes, even when macro-prudential tools themselves impose only occasionally binding constraints on debt dynamics in the economy.

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1 Introduction

With the recovery in the UK economy broadening and gaining momentum in recent months, the Bank of England is now focussed on turning that recovery into a durable expansion. To do so, our policy tools must be used in concert. Carney (2014)

In the aftermath of the global financial crisis, macro-prudential policy frameworks have been established and developed across the world. The inception of these frameworks raises questions of how new policy tools should be operated and how macro-prudential and monetary policies should be coordinated.

These questions have particular resonance given the conditions currently facing policymakers in many advanced economies. Since the financial crisis, monetary policy has been set with a view to supporting economic activity and hence preventing inflation from falling below target. That has required a prolonged period of low, sometimes negative, short-term policy rates and a raft of so-called unconventional monetary policy measures. These policies have supported asset prices and kept borrowing costs low. However, these effects have also given rise to concerns that such monetary conditions may lead to levels of indebtedness that threaten financial stability. In some cases, macro-prudential policy instruments have been used to guard against these risks. So the policy mix in many economies has been 'loose' monetary policy and 'tight' macro-prudential policy.[1]

In this paper, we examine these questions through the lens of a simple and commonly used modeling framework. Our model is rich enough to generate meaningful policy tradeoffs, but sufficiently simple to deliver tractable expressions for welfare and analytical results under some parameterizations.

Our model incorporates borrowing constraints and nominal rigidities. These frictions give rise to meaningful roles for macro-prudential policy and monetary policy respectively. The financial friction takes the form of a collateral constraint, following Kiyotaki and Moore (1997) and Iacoviello (2005) among many others. The collateral constraint limits the amount that relatively impatient households can borrow. Specifically, their debt cannot exceed a particular fraction of the value of the housing stock that they own: there is a 'loan to value' (LTV) constraint which the macro-prudential authority can vary. In turn, borrowing by relatively impatient households is financed by saving by more patient households ('savers'). A perfectly competitive banking system intermediates the flow of saving from savers to borrowers. Banks are subject to a capital requirement and we assume that raising equity is costly (see also Justiniano et al., 2014). As such, variation in capital requirements generates fluctuations in the spread between borrowing and deposit rates, providing

[1]The quote at the start of the paper is from the opening statement at a press conference explaining the decision of Bank of England's Financial Policy Committee to limit the quantity of new lending at high loan-to-value ratios. That statement explains that: "The existence of macro-prudential tools allows monetary policy to focus on its primary responsibility of price stability. In other words, monetary policy does not need to be diverted to address a sector-specific risk in the housing market." Similarly, authorities in Canada have tightened macro-prudential policy several times since the financial crisis (see Krznar and Morsink, 2014) while the official policy rate has remained low. Conversely, the policy mix in Sweden has been a subject of much controversy and debate (see, for example, Jansson, 2014; Svensson, 2011).

the authorities with a second macro-prudential tool in addition to the LTV, albeit one that is costly to deploy. Finally, the nominal frictions are Calvo (1983) contracts, standard in the New Keynesian literature.

We derive a welfare-based loss function as a quadratic approximation to a weighted average of the utilities of borrowers and savers. The loss function has five (quadratic) terms. Two terms stem from the nominal rigidities in the model and are familiar from New Keynesian models: the policymaker seeks to stabilize the output gap and inflation. The remaining terms are generated by the financial friction: the policymaker seeks to stabilize the distribution of non-durable consumption and housing consumption between borrowers and savers—the 'consumption gap' and the 'housing gap' respectively. The presence of household heterogeneity therefore gives rise to objectives whose origin lies in the incompleteness of risk sharing between households in the economy. The final term in the loss function captures the costs of varying capital requirements, which themselves stem from the non-zero cost of equity we assume outside of steady state.

We use the model to study how monetary and macro-prudential policies should optimally respond to shocks. To build some intuition, we first focus on a linear approximation of the model around a steady state in which the borrowing constraint is always binding and the full value of housing can be used as collateral. We demonstrate that macro-prudential policy generally faces a tradeoff in stabilizing the distribution of consumption and the distribution of housing services even when prices are flexible and both macro-prudential tools are used. We also show that monetary policy alone has relatively little control over these distributions, particularly the distribution of housing between borrowers and savers. In other words, imperfect risk sharing is a real phenomenon whose consequences could be addressed by macro-prudential policies, but these policies also imply costs that must be accounted for when deploying them. This tradeoff prevents complete macro-prudential stabilization using the tools we study, even under flexible prices.

To examine the quantitative implications of the model for optimal monetary and macro-prudential policy, we explore numerical experiments designed to simulate a housing boom and subsequent house price correction, calibrated with reference to the experiences of the United Kingdom and the United States in the decades preceding and following 2008 (Figure 1). This allows us to examine the extent to which macro-prudential tools could have complemented monetary policy in achieving macroeconomic stabilization goals in the face of a house price fall large enough to force the nominal rate to the zero bound.

We find that macro-prudential policies, in the form of the LTV tool and bank capital requirements, allow for better stabilization of the consumption and housing gaps, but also allow monetary policy to fully stabilize the output gap and inflation because the short-term nominal interest rate does not hit the zero bound. In other words, during a house price correction the optimal conduct of macro-prudential policies mitigates the fall in the 'equilibrium real rate of interest' (that is, the real interest rate consistent with closing the output gap Ferguson, 2004).

When the LTV tool is not available to the policymaker, the behavior of policy during the recovery from a housing bust is broadly consistent with the 'loose' monetary policy and 'tight'

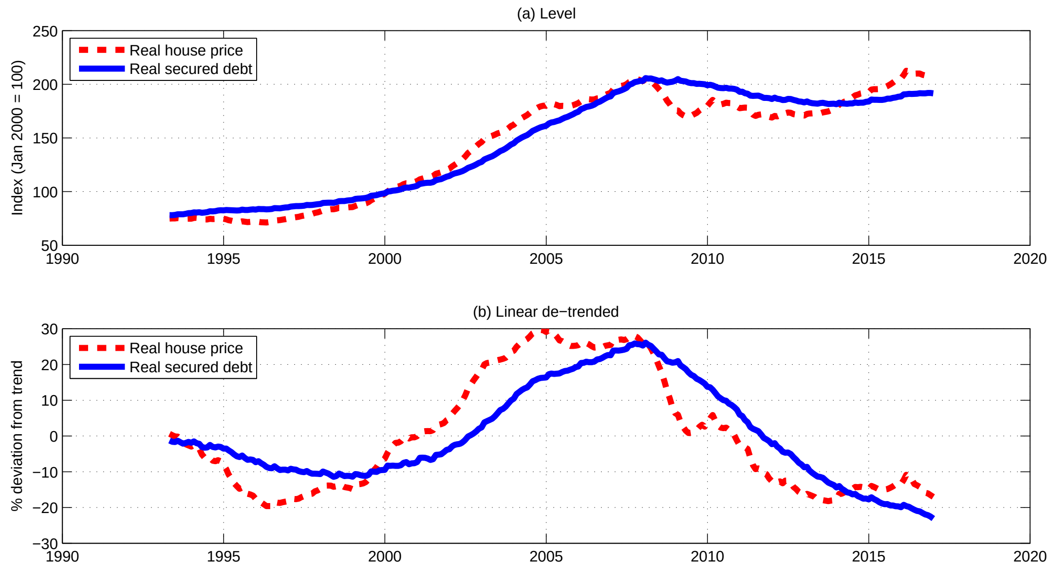

Figure 1: UK house prices and mortgage debt

macro-prudential policy mix observed in many economies in recent years. In that case, following an initial cut in response to the fall in house prices, capital requirements are progressively tightened. This macro-prudential policy behavior slows the speed of de-leveraging and ensures that the equilibrium real interest rate recovers more quickly than otherwise. The faster recovery of the equilibrium real interest rate allows monetary policy to liftoff from the zero bound earlier than would otherwise have been the case. However, after liftoff the optimal monetary policy stance is slightly stimulative, to cushion the aggregate demand effects from tightening capital requirements.

Our results contribute to a growing literature exploring the conduct and coordination of macro-prudential policy. In the context of optimal policy, Angelini et al. (2012), Bean et al. (2010) and de Paoli and Paustian (2017) consider the coordination between monetary and macro-prudential policies in models with similar frictions to ours. Those papers also find that there are cases in which it is optimal to adjust monetary and macro-prudential policy instruments in opposite directions. The welfare-based loss function in our model is similar to that derived by Andres et al. (2013) in a similar model, and bears similarity to Curdia and Woodford (2010). However, those authors focus on the analysis of optimal monetary policy and do not explore macro-prudential policy.

Other papers with a greater focus on macro-prudential policies include Angeloni and Faia (2013), Gertler et al. (2012), Clerc et al. (2015), Christiano and Ikeda (2016) and Aikman et al. (2018). The focus of each of these papers is on macro-prudential bank capital instruments, whereas we also consider a macro-prudential LTV tool. The potential of using LTV instruments is of particular relevance for countries like the UK, where mortgage lending is the single largest asset class on domestic banks' balance sheets, and is also the single largest liability class on households' balance sheets.

Finally, Eggertsson and Krugman (2012b) study the implications of households' debt-deleveraging for monetary policy in a model similar to ours, while Korinek and Simsek (2016) and Farhi and Werning (2016) study the theoretical implications of the zero bound constraint on monetary policy and distortions in financial markets for optimal macro-prudential policies.

The rest of the paper is organised as follows. Section 2 presents the model. Section 3 derives a linear-quadratic approximation of the equilibrium and discusses some analytical results. Section 4 illustrates the optimal joint conduct of monetary and macro-prudential policy via numerical simulations. Section 5 concludes.

2 Model

The economic agents in the model are households, banks, firms, and the government. Households are heterogeneous in their degree of patience. Banks transfer funds from savers to borrowers and fund their operations with a mix of deposits and equity. Firms produce goods for consumption. The government conducts monetary and macro-prudential policy.

2.1 Households

Patient households (i.e. savers, indexed by s) have a higher discount factor than impatient households (i.e. borrowers, indexed by b). We denote with $\xi \in (0,1)$ the mass of borrowers, and normalise the total size of the population to one. We also assume perfect risk sharing within each group.

2.1.1 Savers

A generic saver household $i \in [0, 1-\xi)$ decides how much to consume in goods $C_t^s(i)$ and housing services $H_t^s(i)$,[2] save in deposits $D_t^s(i)$ and equity $E_t^s(i)$ of financial intermediaries, and work $L_t^s(i)$, to maximise

$$\mathbb{W}_0^s(i) \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta_s^t \left[\left(1 - e^{-zC_t^s(i)}\right) + \frac{\chi_H^s e^{-u_t^h}}{1-\sigma_h} (H_t^s(i))^{1-\sigma_h} - \frac{\chi_L^s}{1+\varphi} (L_t^s(i))^{1+\varphi} \right] \right\}, \tag{1}$$

where $\beta_s \in (0,1)$ is the individual discount factor, $z > 0$ is the degree of absolute risk aversion, $\sigma_h \geq 0$ is the inverse elasticity of housing demand, $\varphi \geq 0$ is the inverse Frisch elasticity of labour supply, and χ_H^s, $\chi_L^s > 0$ are type-specific normalisation constants. Preferences include an aggregate housing preference shock, u_t^h, common to all households.

The budget constraint for patient household i is

$$\begin{aligned} P_t C_t^s(i) + D_t^s(i) + E_t^s(i) + (1+\tau^h) Q_t H_t^s(i) = \\ W_t^s L_t^s(i) + R_{t-1}^d D_{t-1}^s(i) + R_{t-1}^e E_{t-1}^s(i) + Q_t H_{t-1}^s(i) + \Omega_t^s(i) - T_t^s(i) - \Gamma_t(i), \end{aligned}$$

[2] We make the standard assumption that the flow of housing services is proportional to the stock of housing.

where P_t is the consumption price index, Q_t is the nominal house price, W_t^s is the nominal wage for savers, R_{t-1}^d is the nominal return on bank deposits, and R_{t-1}^e is the nominal return on bank equity.[3] The variable $T_t^s(i)$ captures lump-sum taxes while $\Omega_t^s(i)$ denotes the savers' share of remunerated profits from intermediate goods producers and from banks. The constant τ^h is a tax/subsidy on savers' housing that we assume is set to deliver an efficient steady state in the housing market. The final term in the budget constraint is a cost associated with deviations from some preferred portfolio level of bank equity $\tilde{\kappa} > 0$

$$\Gamma_t(i) \equiv \frac{\Psi}{2}\left[\frac{E_t^s(i)}{\tilde{\kappa}\xi D_t^b/(1-\xi)} - 1\right]^2 \frac{\tilde{\kappa}\xi D_t^b}{1-\xi},$$

with $\Psi > 0$. For analytical convenience, we express the adjustment cost relative to aggregate bank lending ξD_t^b, which savers take as given.[4]

2.1.2 Borrowers

A generic borrower household $i \in [1-\xi, 1]$ maximizes the same per-period utility as savers (1) but discounts the future at lower rate $\beta_b \in (0, \beta_s)$. The borrower's budget constraint is

$$P_t C_t^b(i) - D_t^b(i) + Q_t H_t^b(i) = W_t^b L_t^b(i) - R_{t-1}^b D_{t-1}^b(i) + Q_t H_{t-1}^b(i) + \Omega_t^b(i) - T_t^b(i),$$

where $D_t^b(i)$ is the amount of borrowing at time t, $T_t^b(i)$ are lump-sum taxes, including those used to obtain an efficient allocation of consumption in the model's steady state, and $\Omega_t^b(i)$ denotes profits from ownership of intermediate goods producing firms.

As common in the literature (e.g Kiyotaki and Moore, 1997), we assume that a collateral constraint limits impatient households' ability to borrow. In particular, their total liabilities cannot exceed a (potentially time-varying) fraction of their current housing wealth

$$D_t^b(i) \leq \Theta_t Q_t H_t^b(i),$$

where $\Theta_t \in [0, 1]$. The term Θ_t represents the maximum loan-to-value (LTV) ratio available to borrowers. The standard interpretation of such a constraint is that lenders (in this case, the financial intermediaries) require borrowers to have a stake in a leveraged investment to prevent moral hazard behavior. In our policy analysis, we will consider cases in which the macro-prudential authority sets the maximum LTV that banks can extend to borrowers. In this sense, the LTV ratio is part of the macro-prudential toolkit that we will study below.

[3]As in Benigno et al. (2014), the introduction of type-specific wages and exponential utility simplifies aggregation, and facilitates the derivation of a welfare criterion for the economy as a whole.

[4]The introduction of this adjustment cost function is a simple way to distinguish bank equity from bank debt (deposits), and captures the idea that deposits are generally more liquid, and thus easier for households to adjust. Little of substance would change in the first-order accurate solution to the model that we examine if we specified bank equity as a state-contingent claim.

2.2 Banks

A continuum of perfectly competitive banks, indexed by $k \in [0,1]$, raise funds from savers in the form of deposits and equity (their liabilities), and make loans (their assets) to borrowers. Bank k's balance sheet identity is

$$D_t^b(k) = D_t^s(k) + E_t^s(k). \tag{2}$$

In addition, we assume that equity must account for at least a fraction $\tilde{\kappa}_t$ of the total amount of loans banks extend to borrowers

$$E_t^s(k) \geq \tilde{\kappa}_t D_t^b(k). \tag{3}$$

The presence of equity adjustment costs breaks down the irrelevance of the capital structure (the Modigliani-Miller theorem). Savers demand a premium for holding equity, which banks pass on to borrowers in the form of a higher interest rate. From the perspective of the bank, equity is expensive, and thus deposits are the preferential source of funding. In the absence of any constraint, banks would choose to operate with zero equity and leverage would be unbounded. Equation (3) ensures finite leverage for financial intermediaries.

As in the case of the LTV parameter, we will consider two possible interpretations of $\tilde{\kappa}_t$. The first treats this variable as exogenous, relying on the notion that financial institutions target a certain leverage ratio due to market forces (Adrian and Shin, 2010). According to the second interpretation, while the constraint still plays the role of limiting banks' leverage, it is the macro-prudential authority that sets the capital requirement on financial institutions. In this sense, $\tilde{\kappa}_t$ becomes the second macro-prudential tool for the regulatory authority. Several recent contributions have discussed capital requirements as one of the key instruments to avoid financial crises in the future (e.g. Admati and Hellwig, 2014; Miles et al., 2013). In our analysis, we will focus on the interaction between capital requirements and interest rate setting.

Independently of its interpretation, the capital requirement constraint is always binding in equilibrium, exactly because financial intermediaries seek to minimize their equity requirement. If the capital constraint of all banks were slack, one bank could marginally increase its leverage, charge a lower loan rate, and take the whole market. Therefore, competition drives the banking sector against the constraint.

Banks' profits are

$$\mathcal{P}_t(k) \equiv R_t^b D_t^b(k) - R_t^d D_t^s(k) - R_t^e E_t^s(k) = [R_t^b - (1-\tilde{\kappa}_t)R_t^d - \tilde{\kappa}_t R_t^e] D_t^b(k),$$

where the second equality follows from substituting the balance sheet constraint (2) and the capital requirement (3) at equality. The zero-profit condition implies that the loan rate is a linear combination of the return on equity and the return on deposits

$$R_t^b = \tilde{\kappa}_t R_t^e + (1-\tilde{\kappa}_t) R_t^d,$$

where the time-varying capital requirement represents the weight on the return on equity. A surprise

rise in $\tilde{\kappa}_t$, whether due to an exogenous shock or a policy decision, forces banks to delever and raises credit spreads.

2.3 Production

A representative retailer combines intermediate goods according to a technology with constant elasticity of substitution $\varepsilon > 1$

$$Y_t = \left[\int_0^1 Y_t(f)^{\frac{\varepsilon-1}{\varepsilon}} df\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $Y_t(f)$ represents the intermediate good produced by firm $f \in [0,1]$. Expenditure minimisation implies that the demand for a generic intermediate good is

$$Y_t(f) = \left[\frac{P_t(f)}{P_t}\right]^{-\varepsilon} Y_t, \tag{4}$$

where $P_t(f)$ is the price of the variety produced by firm f and the aggregate price index is

$$P_t = \left[\int_0^1 P_t(f)^{1-\varepsilon} df\right]^{\frac{1}{1-\varepsilon}}.$$

Intermediate goods producers operate in monopolistic competition, are owned by savers and borrowers according to their shares in the population, and employ labour to produce variety f according to

$$Y_t(f) = A_t L_t(f). \tag{5}$$

Aggregate technology A_t follows a stationary autoregressive process in logs

$$\ln A_t \equiv a_t = \rho_a a_{t-1} + \epsilon_t^a,$$

with $\rho_a \in (0,1)$ and $\epsilon_t^a \sim \mathcal{N}(0, \sigma_a^2)$. To simplify aggregation, we assume $L_t(f)$ is a geometric average of borrower and saver labour, with weights reflecting the shares of the two types

$$L_t(f) \equiv [L_t^b(f)]^{\xi}[L_t^s(f)]^{1-\xi},$$

and the corresponding wage index is

$$W_t \equiv (W_t^b)^{\xi}(W_t^s)^{1-\xi}.$$

Intermediate goods producers set prices on a staggered basis. As customary, we solve their optimization problem in two steps. First, for given pricing decisions, firms minimize their costs, which implies that the nominal marginal cost M_t is independent of each firm's characteristics. The second step of the intermediate goods producers problem is to determine their pricing decision. As in Calvo (1983), we assume firms reset their price $\tilde{P}_t(f)$ in each period with a constant probability

$1-\lambda$, taking as given the demand for their variety, while the complementary measure of firms λ keeps their price unchanged. The optimal price setting decision for firms that do adjust at time t solves

$$\max_{\tilde{P}_t(f)} \mathbb{E}_t \left\{ \sum_{v=0}^{\infty} \lambda^i Q_{t,t+v} [(1+\tau^p)\tilde{P}_t(f) - M_{t+v}] Y_{t+v}(f) \right\},$$

subject to (4), where τ^p is a subsidy to make steady state production efficient. Intermediate goods producers are owned by households of both types in proportion to their shares in the population. Therefore, we assume that the discount rate for future profits is

$$Q_{t,t+v} \equiv (Q^b_{t,t+v})^{\xi} (Q^s_{t,t+v})^{1-\xi},$$

where $Q^j_{t,t+v} = \beta_j z e^{-z(C^j_{t+v}(i) - C^j_t(i))}$ is the stochastic discount factor between period t and $t+v$ of type $j = \{b, s\}$.

2.4 Equilibrium

Because of the assumption of risk-sharing within each group, all households of a given type consume the same amount of goods and housing services and work the same number of hours. Therefore, in what follows, we drop the index i and characterize the equilibrium in terms of type-aggregates. Similarly, because all financial intermediaries make identical decisions in terms of interest rate setting, we drop also the index k and simply refer to the aggregate balance sheet of the banking sector.

For a given specification of monetary and macro-prudential policy, an imperfectly competitive equilibrium for this economy is a sequence of quantities and prices such that households and intermediate goods producers optimise subject to the relevant constraints, final good producers and banks make zero profits, and all markets clear.[5] In particular, for the goods market, total production must equal the sum of consumption of the two types plus the resources spent for portfolio adjustment costs[6]

$$Y_t = \xi C^b_t + (1-\xi) C^s_t + \Gamma_t, \tag{6}$$

where

$$\Gamma_t \equiv \int_0^{1-\xi} \Gamma_t(i) di = \frac{\Psi}{2} \left(\frac{\tilde{\kappa}_t}{\tilde{\kappa}} - 1 \right)^2 \tilde{\kappa} \xi D^b_t.$$

We assume housing is in fixed supply (i.e., land). The housing market equilibrium then requires

$$H = \xi H^b_t + (1-\xi) H^s_t, \tag{7}$$

[5] Appendix A reports the equilibrium conditions for the private sector and the details of aggregation.

[6] The resource constraint follows from combining the budget constraints of the two types (aggregated over their respective measures) with the financial intermediaries balance sheet, under the assumption that the government adjusts residually the lump-sum transfers to savers.

where H is the total available stock of housing.[7] Finally, in the credit market, total bank loans must equal total household borrowing. Thus, the aggregate balance sheet for the financial sector respects

$$\xi D_t^b = (1-\xi)(D_t^s + E_t^s),$$

where per-capita real private debt (derived from the borrowers' budget constraint) evolves according to

$$\frac{D_t^b}{P_t} = \frac{R_{t-1}^b}{\Pi_t}\frac{D_{t-1}^b}{P_{t-1}} + C_t^b - Y_t + \frac{Q_t}{P_t}(H_t^b - H_{t-1}^b) + \mathcal{T}^b,$$

and $\mathcal{T}^b$ is a subsidy that ensures the steady state allocation is efficient.

3 Linear-Quadratic Framework

Our ultimate objective is to study the optimal joint conduct of monetary and macro-prudential policy. In this section, we aim to obtain some analytical results following the approach of the optimal monetary policy literature (e.g. Clarida et al., 1999; Woodford, 2003), and derive a linear-quadratic approximation to our model with nominal rigidities and financial frictions. We approximate the model around a zero-inflation steady state in which the collateral constraint binds. An appropriate choice of taxes and subsidies ensures that the steady state allocation is efficient. Appendix C reports the details of the derivations.

3.1 Quadratic Loss Function

To derive the welfare-based loss function, we take the average of the per-period utility functions of borrowers and savers, weighting each type according to their share in the population. We assume that policymakers discount the future at rate β_s.[8] A second-order approximation of the resulting objective gives

$$\mathcal{L}_0 \propto \frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta_s^t \left(x_t^2 + \lambda_\pi \pi_t^2 + \lambda_\kappa \kappa_t^2 + \lambda_c \tilde{c}_t^2 + \lambda_h \tilde{h}_t^2 \right), \tag{8}$$

where lower-case variables denote log-deviations from the efficient steady state, $x_t \equiv y_t - y_t^*$ is the efficient output gap (with y_t^* representing the efficient level of output), $\tilde{c}_t \equiv c_t^b - c_t^s$ is the consumption gap between borrowers and savers, and $\tilde{h}_t \equiv h_t^b - h_t^s$ is the housing gap between borrowers and savers. The weights on deviations of inflation and capital requirements from target are

$$\lambda_\pi \equiv \frac{\varepsilon}{\gamma} \qquad \text{and} \qquad \lambda_\kappa \equiv \frac{\psi\eta}{\sigma+\varphi},$$

[7] The absolute level of this variable plays no role in the analysis.

[8] Benigno et al. (2014) assume that the discount factor of the two types is the same in the limit ($\beta_b \to \beta_s$), and that borrowing/lending position are determined by the initial distribution of wealth. We retain the heterogeneity in the discount factors but effectively assume that the policymaker is chosen among the population of savers. This choice is obviously arbitrary but we can solve the optimal policy problem for any value of $\beta \in (0,1)$.

where $\gamma \equiv (1-\beta\lambda)(1-\lambda)(\sigma+\varphi)/\lambda$, while the weights on the consumption and housing gaps are

$$\lambda_c \equiv \frac{\xi(1-\xi)\sigma(1+\sigma+\varphi)}{(1+\varphi)(\sigma+\varphi)} \qquad \text{and} \qquad \lambda_h \equiv \frac{\sigma_h \xi(1-\xi)}{\sigma+\varphi}.$$

The loss function (8) features two sets of terms. The first includes the efficient output gap and inflation—the standard variables that appear in the welfare-based loss function of a large class of New Keynesian models. Their presence in the loss function reflects the two distortions associated with price rigidities. First, such rigidities open up a “labour wedge”, causing the level of output to deviate from its efficient level. Second, staggered price setting implies an inefficient dispersion in prices, which is proportional to the rate of inflation.

The second set of terms in (8), comprising the consumption gap and the housing gap, arise from the heterogeneity between household types and, in particular, from the fact that one group of households are credit-constrained while the others are not. The collateral constraint generates an inefficiency because of incomplete insurance. In the absence of the collateral constraint, households could insure each other against variation in their housing and consumption bundles. The collateral constraint limits the amount of borrowing that can take place to carry out this insurance in full. As a result, risk sharing is imperfect. An analogous argument applies to housing. Imperfect risk sharing therefore becomes a source of welfare losses the policymaker accounts for when setting policy optimally. Finally, the term $\lambda_\kappa \kappa_t^2$ accounts for the costs associated with the use of capital requirements as a policy tool.[9]

3.2 Linearised Constraints

In this section, we combine the linearised equilibrium relations to obtain a parsimonious set of constraints for the optimal policy problem in our linear-quadratic setting.[10] To simplify the derivations, we assume $\Theta = 1$ (a 100% LTV ratio). We return to the case $\Theta < 1$ in the quantitative analysis. Appendix D provides additional details on the derivations.

On the supply side, as common in the literature, we can rewrite the Phillips curve in terms of the efficient output gap

$$\pi_t = \gamma x_t + \beta \mathbb{E}_t \pi_{t+1} + u_t^\pi, \tag{9}$$

where u_t^π is an exogenous cost-push shock.

On the demand side, we write an aggregate demand curve in terms of the output gap and the consumption gap

$$x_t - \xi \tilde{c}_t = -\sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + \mathbb{E}_t(x_{t+1} - \xi \tilde{c}_{t+1}) + \nu_t^c, \tag{10}$$

where ν_t^c is a combination of exogenous shocks defined in the appendix. In a standard representative

[9] If we treat the leverage ratio as an exogenous—albeit time-varying—constraint, the costs of its fluctuations become independent of policy, and thus irrelevant for ranking alternative policies in terms of welfare. In this case, the welfare objective only has four terms but the policymaker has one fewer tool for stabilization purposes.

[10] Unless otherwise stated, lower-case variables denote log-deviations from steady state. For a generic variable Z_t, with steady state value Z, $z_t \equiv \ln(Z_t/Z)$.

agent model, the consumption gap is zero, and all agents behave like the savers in our economy. The consumption gap in (10) summarises the impact of debt obligations, house prices and LTV ratios on aggregate demand due to the lack of risk sharing.

We derive an equation for the housing gap by taking the difference of the housing demand equations between borrowers and savers. The resulting expression is

$$\tilde{h}_t = -\frac{\omega - \xi(\beta_s - \beta_b)}{\sigma_h \xi \omega}(i_t - \mathbb{E}_t \pi_{t+1}) + \frac{\beta_s - \beta_b}{\sigma_h \omega}(q_t - \mathbb{E}_t q_{t+1}) - \frac{\sigma}{\sigma_h \xi}(x_t - \mathbb{E}_t x_{t+1}) + \frac{\sigma}{\sigma_h}\tilde{c}_t + \frac{\tilde{\mu}}{\sigma_h \omega}\theta_t - \frac{1-\tilde{\mu}}{\sigma_h \omega}\psi \kappa_t + \nu_t^h, \quad (11)$$

where ω and ν_t^h are combinations of fundamental parameters and shocks, respectively, defined in the appendix.

To complete the description of the housing block, we take a population-weighted average of the housing demand equation to obtain an aggregate house price equation that reads as

$$q_t = -(i_t - \mathbb{E}_t \pi_{t+1}) + \frac{\sigma \omega}{\omega + \beta}\mathbb{E}_t x_{t+1} + \frac{\xi \tilde{\mu}}{\omega + \beta}\theta_t - \frac{\xi(1-\tilde{\mu})}{\omega + \beta}\psi \kappa_t + \frac{\beta}{\omega + \beta}\mathbb{E}_t q_{t+1} + \nu_t^q, \quad (12)$$

where ν_t^q is a combination of fundamental shocks defined in the appendix.

In the neighborhood of a steady state in which the borrowing constraint binds, debt is a function of the LTV constraint, house prices and the housing gap

$$d_t^b = \theta_t + q_t + (1-\xi)\tilde{h}_t. \quad (13)$$

We can keep track of its dynamics via the borrowers' budget constraint

$$d_t^b = \frac{1}{\beta_s}(i_{t-1} + \psi \kappa_{t-1} + d_{t-1}^b - \pi_t) + (1-\xi)(\tilde{h}_t - \tilde{h}_{t-1}) + \frac{1-\xi}{\eta}\tilde{c}_t. \quad (14)$$

The set of endogenous states in the model consists of private debt, the nominal interest rate, and the leverage ratio. Since we solve the optimal policy problem under discretion, with endogenous state variables, we need to keep track of the effects of current actions on future losses through the evolution of the states. In this respect, we can simplify the optimal policy problem by defining a single composite state variable

$$S_t \equiv d_t^b + i_t + \psi \kappa_t - \beta_s(1-\xi)\tilde{h}_t, \quad (15)$$

which captures the burden of debt at maturity for borrowers relative to the discounted quantity of housing owned. Using this composite state variable, we can rewrite the borrowing constraint at equality as

$$S_t = \theta_t + q_t + i_t + \psi \kappa_t + (1-\beta_s)(1-\xi)\tilde{h}_t. \quad (16)$$

Similarly, the law of motion of debt can be rewritten as the law of motion of the single state variable as

$$S_t = \frac{1}{\beta_s}(S_{t-1} - \pi_t) + i_t + \psi\kappa_t + (1-\beta_s)(1-\xi)\tilde{h}_t + \frac{1-\xi}{\eta}\tilde{c}_t. \tag{17}$$

In this way, we have reduced the endogenous states from three to one, and we can characterise the effects of current decisions on future losses via the variable S_t only.

The joint optimal monetary and macro-prudential problem under discretion consists of minimizing (8) subject to the constraints (9)-(14), or, equivalently, (9)-(12), (16), and (17). In general, the policymaker can choose three instruments (the nominal interest rate, i_t, the LTV ratio, θ_t, and the capital requirement, κ_t) to implement the optimal plan. To fix ideas before deriving optimal policy plan in the most general case, however, we start from a simple benchmark without nominal rigidities in which debt is issued in real terms. This special case allows us to focus on the characteristics of macro-prudential policy, abstracting from its interactions with monetary policy.

3.3 Optimal Macro-Prudential Policy under Flexible Prices

To highlight the effects of macro-prudential policy, we focus on the efficient equilibrium of the model. With flexible prices ($\lambda \to 0$) and no markup shocks ($u_t^m = 0,\ \forall t$), productivity fully determines output, which becomes exogenous ($y_t = y_t^*$), so that the output gap is always zero ($x_t = 0,\ \forall t$). In addition, the weight on inflation in (8) is zero ($\lambda_\pi = 0$). Hence, the first two terms in the loss function disappear. The job of the policy authority then becomes to stabilize the consumption and housing gap, and—if used as an instrument—minimize the volatility of capital requirements.

The Phillips curve is no longer a constraint for the optimal policy plan. Since the output gap is always zero, the aggregate demand curve determines the consumption gap as a function of the real interest rate r_t

$$\xi\tilde{c}_t = \sigma^{-1} r_t + \xi\mathbb{E}_t\tilde{c}_{t+1} - \nu_t^c.$$

The other constraints correspond to equation (11), (12), (16), and (17), after imposing a zero output gap in all periods. In addition, because of the assumption that debt is in real terms, inflation disappears from the model and the nominal interest rate is replaced by the real interest rate.

Appendix E.1 sets up the Lagrangian that describes the optimal policy problem under discretion and derives the first order conditions. After solving out for the Lagrange multipliers, we can express the optimal policy plan in terms of two targeting rules. The first instructs the policymaker how to set capital requirements in response to an opening up of the consumption and housing gap

$$\kappa_t = \Phi_c\tilde{c}_t + \Phi_h\tilde{h}_t, \tag{18}$$

where Φ_c and Φ_h are coefficients defined in the appendix, where we also demonstrate that both coefficients are positive for empirically plausible parameterizations of the model.

Rule (18) is static and requires the policymaker to increase capital requirements whenever either a consumption gap or a housing gap (or both) opens up. Positive consumption or housing gaps signal excess demand by borrowers, and hence require a tightening of financial conditions. The policymaker achieves this goal by raising capital requirements, thus reducing credit and making it more expensive for borrowers.

The second rule is dynamic and forward looking

$$\tilde{c}_t + \Omega_h \tilde{h}_t = \Omega_c \mathbb{E}_t \tilde{c}_{t+1}, \tag{19}$$

where Ω_h and Ω_c are coefficients defined in the appendix. Given that the static targeting rule sets capital requirement, we can think of rule (19) as implicitly determining the optimal LTV ratio. In particular, the combination of current consumption and housing gaps on the left-hand side must be proportional to the expected future consumption gap.

Through the effects on the composite state variable (and in particular debt, but also current interest rates and capital requirements) optimal policy affects future losses. Under discretion, the policymaker cannot commit to any policy in the future. Therefore, current decisions need to take into account the effects on future outcomes. For example, if a negative shock hits the economy in the current period and opens up the consumption and housing gaps, the policymaker should relax capital requirements to minimize its macroeconomic impact. However, once the shock dies out, such an expansionary policy will contribute to overheat the economy. The policymaker should use the LTV ratio to ensure that current policy is not excessively costly in the future by allowing for excessive leverage. The example is purely illustrative and does not necessarily mean that the policymaker should always adjust the macro-prudential instruments in opposite directions.

The targeting rules derived above illustrate that it is not generally possible for macro-prudential policy to achieve full stabilization: that is, a stable equilibrium with $\tilde{c}_t = \tilde{h}_t = \kappa_t, \forall t$. We can demonstrate this using an informal proof by contradiction. If it was possible to deliver such an allocation using these targeting rules, then those rules also imply that $\kappa_t = 0$ (from (18)) and $\mathbb{E}_t \tilde{c}_{t+1} = 0$ (from (19)). These conditions (together with the conjectured full stabilization allocations) can be substituted into the Euler equation and the real version of (17) to give:

$$S_t = \beta_s^{-1} S_{t-1} + \sigma \nu_t^c$$

which implies an explosive trajectory for the composite state variable, since $\beta_s < 1$.

In the next section we shall see how the features of optimal macro-prudential policy under discretion in the efficient equilibrium extend to the case with nominal rigidities and interact with the optimal conduct of conventional monetary policy.

3.4 Monetary and Macro-Prudential Policies with Sticky Prices

As in the previous section, we report here the optimal targeting rules for monetary and macro-prudential policy under discretion, and refer to Appendix E.2 for the details of the derivation.

The first result is that the introduction of sticky prices does not change the nature of the static tradeoffs for the macro-prudential authority. Equation (18) continues to describe the optimal setting of capital requirements.[11]

A second static targeting rule characterizes optimal monetary policy

$$x_t + \gamma\lambda_\pi \pi_t + \Lambda_c \tilde{c}_t = 0, \tag{20}$$

where Λ_c is a coefficient defined in the appendix. Equation (20) resembles the standard New Keynesian monetary targeting rule under discretion, but also includes an adjustment for the consumption gap. Since the coefficient on the consumption gap Λ_c is positive for empirically relevant calibrations, monetary policy will typically "lean against the wind". Everything else equal, a shock that opens a positive consumption gap requires a negative combination of the output gap and inflation (appropriately weighted), contrary to the standard case (Clarida et al., 1999) in which the same combination should be set equal to zero.

The last targeting rule extends the optimal setting of the LTV ratio to the case of sticky prices

$$\tilde{c}_t + \Upsilon_x x_t + \Upsilon_\pi \pi_t + \Upsilon_h \tilde{h}_t = \Upsilon_c \mathbb{E}_t \tilde{c}_{t+1}, \tag{21}$$

where Υ_x, Υ_π, Υ_h, and Υ_c are coefficients defined in the appendix. With sticky prices, the optimal setting of LTV ratios needs to take into account also current inflation and output gap.

As in the efficient equilibrium, the current policy decisions affect future outcomes and losses through the composite state variable. A negative shock that hits the economy in the current period not only creates distributional effects, opening up a consumption and housing gap, but also negatively affects inflation and the output gap. The policymaker should respond to the shock by relaxing monetary and financial conditions, without losing sight of effects of the current policy response on future outcomes and losses. A simple extension of the argument in Section 3.3 shows that full stabilization is generally not possible, even in the absence of cost-push shocks.

In sum, the rules (18), (20), and (21) reveal a rich interaction between monetary and macro-prudential policy that operates via the output gap and inflation on the monetary side, and the consumption and housing gap on the macro-prudential side. The next section makes the targeting criteria above operational in the context of a house price boom scenario that mimics some aspects of the recent crisis.

4 Quantitative Experiments

In this section, we use our model to study the interaction of monetary and macro-prudential policies in a stylized simulation of a house price boom. Our objective is to explore the interaction of monetary and macro-prudential policies under alternative 'policy configurations', when accounting for the presence of occasionally binding constraints. We consider several policy configurations, specifying which policy instruments may be used, which policymakers (monetary and/or macro-prudential)

[11] Formally, in the appendix we show that a subset of the first order conditions for the problem with sticky prices are identical to the static first order conditions of the case with flexible prices and real debt. Therefore, we can derive the same static macro-prudential rule.

may use them and the objectives of the policymaker(s). Occasionally binding constraints are of interest because policy instruments (the short-term nominal interest rate and capital requirements) may be constrained by a lower bound and because the borrowing constraint may become slack. Both of these cases will change the ability of policy to stabilize the economy.

To provide a somewhat more realistic dynamic structure of the model, we incorporate a slow-moving borrowing limit in the same way as Guerrieri and Iacoviello (2017) and Justiniano et al. (2015). This modification is intended to generate more persistent movements in debt and its marginal value (that is, the multiplier μ). In principle, it is possible to incorporate a wide range of additional frictions to enhance the dynamic properties of the model (as Guerrieri and Iacoviello, 2017, do, for example). Here, we focus on the slow-moving borrowing limit largely because it does not affect the derivation of the welfare-based loss function while introducing some quantitative relevance.

Specifically, we assume that borrowers face the following borrowing constraint:

$$D_t^b(i) \leq \gamma_d D_{t-1}^b(i) + (1-\gamma_d)\,\Theta_t Q_t H_t^b(i) \tag{22}$$

where $\gamma_d \in [0,1)$ is a parameter controlling the extent to which the debt limit depends on the household's debt in the previous period. As argued by Guerrieri and Iacoviello (2017), this formulation can be interpreted as capturing the idea that only a fraction of borrowers experience a change to their borrowing limit each period (which may be associated with moving or re-mortgaging). One implication of this formulation is that movements in debt adjust only gradually to changes in the value of the housing stock, which is consistent with the data in Figure 1. The modification to the specification of the borrowing limit affects the Euler equation and housing demand equation of borrowers as shown in Appendix F.1. When $\gamma_d = 0$ and $\Theta = 1$ the model collapses to the version discussed in Section 3.4.

4.1 Parameter Values

The parameter values used for the simulation exercises are shown in Table 1. Most of the parameter values are taken from the careful estimation of a similar (though richer) model on US data by Guerrieri and Iacoviello (2017). We focus discussion on the remaining parameters.

We set $\beta_b = 0.99$, implying that borrowers are less patient than implied by the estimate of 0.992 by Guerrieri and Iacoviello (2017). The relative discount factors of borrowers and lenders are crucial for the extent to which changes in house price expectations cause the borrowing constraint to become slack. Our lower value of β_b increases the steady state value of the borrowing constraint multiplier so that larger shocks are required for the constraint to become slack. We set the slope of the Phillips curve in line with the assumption in Eggertsson and Woodford (2003).[12]

Two parameters that are important in determining the response to housing demand shocks are

[12]This requires a Calvo price adjustment parameter of 0.8875, a little lower than the estimate of Guerrieri and Iacoviello (2017).

	Description	Value	Comments/issues/questions
β_s	Saver discount factor	0.995	Guerrieri and Iacoviello (2017)
σ	Inverse elasticity of substitution (consumption)	1	Guerrieri and Iacoviello (2017)
φ	Inverse Frisch elasticity	1	Guerrieri and Iacoviello (2017)
γ_d	Debt limit inertia	0.7	Guerrieri and Iacoviello (2017)
Θ	Debt limit (fraction of house value)	0.9	Guerrieri and Iacoviello (2017)
γ	Slope of Phillips curve	0.024	Eggertsson and Woodford (2003)
β_b	Borrower discount factor	0.99	See text.
ξ	Fraction of borrowers in economy	0.57	Cloyne et al. (2016)
η	Debt: GDP ratio	1.8	BIS data (1990–2000)
ψ	Elasticity of funding cost to capital ratio	0.05	See text.
σ_h	Inverse elasticity of substitution (housing)	25	See text.
ρ_h	Housing demand shock persistence	0.85	See text.

Table 1: Parameter Values

the persistence of the shock and the intertemporal substitution elasticity. We assume a moderate level of persistence, setting $\rho_h = 0.85$, which is somewhat smaller than the modal estimate of 0.98 from Guerrieri and Iacoviello (2017). We set $\sigma_h = 25$ which implies that housing demand is rather insensitive to movements in the real house price. Guerrieri and Iacoviello (2017) assume $\sigma_h = 1$, but also incorporate habit formation in the sub-utility function for housing. The high degree of habit formation that they estimate implies that the short-run elasticity of housing demand to changes in the house price is much lower than unity. By setting $\sigma_h = 25$, we aim to replicate this qualitative behavior without complicating the model and particularly the derivation of the welfare-based loss function.[13]

The remaining parameters are set with reference to UK data. To set ξ we refer to the analysis in Cloyne et al. (2016), who study the behavior of households by tenure type. Their data imply that UK household shares are roughly: 30% homeowners; 40% mortgagors and 30% renters. Since our model does not include renters, we set $\xi = \frac{0.4}{0.3+0.4} \approx 0.57$ so that it represents the relative population shares of mortgagors and homeowners in the data.[14]

We set η with reference to UK household debt to GDP ratios. According to BIS data, this ratio averaged around 60% between 1990 and 2000.[15] Around three quarters of household debt is mortgage debt, which suggests setting $\eta = 0.6 \times 0.75 \times 4 \approx 1.8$ (since η is the ratio of debt to *quarterly* GDP).

We assume the steady state capital ratio $\widetilde{\kappa}$ is 10%, close to the average reported in Meeks (2017) for UK banks over the period 1990-2008. Given that, the key determinant of the transmission of changes in κ_t through to credit spreads is the parameter $\psi \equiv \Psi\widetilde{\kappa}$. In its final report to the

[13] Of course, our approach also reduces the long-run elasticity of housing demand to house prices, so that it is less flexible than the introduction of habit formation.

[14] Guerrieri and Iacoviello (2017) estimate that the fraction of labor income accruing to borrowers is around 0.5. Since labor income is allocated in proportion to population share in our model, this suggests a similar value for ξ.

[15] This period precedes the run up in house prices before the financial crisis. We use this period for our calibration as we aim to mimic the pre-crisis house price rise in our simulation.

BIS, Macroeconomic Assessment Group (2010) estimate that a 1 percentage point rise in capital requirements would have a peak effect on GDP of between -0.05% and -0.35%. A partial equilibrium thought experiment implies that this effect would be generated by an increase in (annualized) short-term nominal interest rates of between 0.2pp and 1.4pp when $\sigma = 1$, as in our calibration. However, in our model, credit spreads are only faced by about 60% of households, so the spread would need to increase by around 0.3–2.3pp on an annualized basis. We assume that the change in spreads is 2 percentage points, towards the top of this range. Taken together, these assumptions imply a value of Ψ solving $(0.02/4) = \Psi \times 0.10 \times 0.1$, or equivalently, a value of ψ given by $(0.02/4)/0.1 = 0.05$.[16]

4.2 Simulation Methodology

Our simulation is designed to generate a prolonged rise in the real price of housing followed by a sharp fall. The simulation is calibrated to deliver a similar increase in real house prices that was observed in pre-crisis UK data (Figure 1). The fall in prices is much more extreme than observed in the United Kingdom because our aim is to generate a sufficiently large downturn that monetary policy may be constrained by the zero bound purely as a consequence of the house price fall. In this respect, perhaps, the simulation more closely resembles the US experience.

To generate a steady increase in house prices, we apply a sequence of unanticipated shocks to housing preferences $u_t^h, t = 1, \ldots, K$. In each period, we solve the model (applying the occasionally binding constraints when necessary) conditional on agents' information up to that date. For the boom phase, the sequence of shocks is increasing, so that $u_t^h > u_{t-1}^h$ for $t = 1, \ldots, K-1$. The 'bust' occurs in period K: a large negative housing preference shock is realised, which acts to reverse most of the previous increase in house prices. To match the slow pre-crisis increase in house prices we set assume that the boom lasts for 30 quarters ($K = 31$).

We use a piecewise linear solution approach to account for the possibility that (a) the short-term nominal interest rate is constrained by the zero lower bound and/or (b) borrowers' debt limit (22) does not bind. This approach takes account of the possibility that the occasionally binding constraints may apply in future periods, but does not account for the *risk* that future shocks may cause the constraints to bind. This means that our solution approach does not account for the skewness that may be generated in the expected distribution of future outcomes (e.g. of output and inflation) by the possibility of being constrained in future. Our solution methodology is therefore similar to the OccBin toolkit developed by Guerrieri and Iacoviello (2015). Appendix F.4 contains a detailed description of our method.

4.3 Alternative Policy Configurations

In our simulations, we study the macroeconomic effects of a housing boom under several alternative assumptions about the conduct of monetary and macro-prudential policies. These assumptions are motivated by the nature of central bank remits in the past and the way that the introduction of new

[16] Meeks (2017) estimates that a 50 basis point rise in capital requirements raises mortgages spreads by 20-25 annualized basis points at its peak. This implies a somewhat lower value of $\psi = 0.0125$.

Policy variant	Policy assumptions	
	Monetary	Macro-prudential
Flexible inflation targeting	minimize $\mathcal{L}_0^{FIT}$	Inactive ($\theta_t = \kappa_t = 0$)
Leaning against the wind	minimize $\mathcal{L}_0$	Inactive ($\theta_t = \kappa_t = 0$)
Macro-prudential leadership	minimize $\mathcal{L}_0^{FIT}$	minimize $\mathcal{L}_0^{MaP}$
Full co-ordination	minimize $\mathcal{L}_{\prime}$	

Table 2: Alternative policy assumptions used in simulations

macro-prudential policy instruments may affect those remits in the future. However, a common assumption that policy is set to minimize a loss function subject to constraints on their actions, as advocated in the context of monetary policy by Svensson (1999).

Our alternative assumptions are based around the following decomposition of the welfare-based loss function given in equation (8):

$$\mathcal{L}_0 \propto \underbrace{\mathbb{E}_0 \sum_{t=0}^{\infty} \beta_s^t \left(x_t^2 + \lambda_\pi \pi_t^2 \right)}_{\equiv \mathcal{L}_0^{FIT}} + \underbrace{\mathbb{E}_0 \sum_{t=0}^{\infty} \beta_s^t \left(\lambda_\kappa \kappa_t^2 + \lambda_c \tilde{c}_t^2 + \lambda_h \tilde{h}_t^2 \right)}_{\equiv \mathcal{L}_0^{MaP}} \tag{23}$$

The first component of the loss function, $\mathcal{L}_0^{FIT}$, is intended to capture the objectives encoded in the 'flexible inflation targeting' mandates of many central banks in the pre-crisis period. The second component, $\mathcal{L}_0^{MaP}$, captures 'macro-prudential' considerations. There are many potential ways to allocate objectives to different policymakers. For example, de Paoli and Paustian (2017) examine a case in which the concern for output stabilization is divided between the monetary policymaker and macro-prudential policymaker. Our decomposition is based on the observation that if we take the limiting case in which the share of borrowers collapses to zero ($\xi \to 0$) and (hence) the banking sector disappears ($\Psi \to 0$), the model collapses to a standard New Keynesian model in which the only friction arises from price stickiness and the only policy instrument is the short-term nominal interest rate. So the existence of the financial frictions in our model generates both additional terms in the loss function and additional instruments with which to address them.

On the basis of the decomposition of the welfare-based loss function in (23), Table 2 details the set of policy assumptions (or delegation schemes) that we consider in the next subsections. We explain the motivation and present the results for each variant in turn.

We assume that the policymakers set policy in a time-consistent manner and are therefore unable to use promises of future policy actions to improve stabilization outcomes today.[17] One motivation for studying time consistent policies is to limit the power of monetary policy at the zero bound. It is well known that optimal commitment policies can be very effective at mitigating the effects of the zero bound in standard New Keynesian models (see, for example, Eggertsson

[17] Formally, we solve for a Markov-perfect policy. In each period, policymakers acts as a Stackelberg leader with respect to private agents and future policymakers. Current policymakers takes the decision rules of future policymakers as given. In equilibrium, the decisions of policymakers in the current period satisfy the decision rule followed by future policymakers. See Appendix F for technical details.

and Woodford, 2003). Our setting, therefore, maximizes the potential scope for macro-prudential policies to improve outcomes when used alongside monetary policy.

Our analysis therefore contributes to an emerging literature studying monetary and macro-prudential policies under discretion. Bianchi and Mendoza (2013) argue that then nature of financial frictions generates an inherent time inconsistency problem for macro-prudential policymakers. Using a model similar to ours, Laureys and Meeks (2017) demonstrate the striking result that discretionary policies can generate better outcomes than a class of simple macro-prudential policy rules (to which policymakers commit) that have been widely studied in the existing literature.

4.4 The Baseline: Flexible Inflation Targeting

Our first experiment considers the case in which a monetary policymaker pursues a 'flexible inflation targeting' mandate, with no macro-prudential policy in place. That is, we assume that the monetary policymaker is tasked with using the short-term nominal interest rate to minimize the loss function $\mathcal{L}_0^{FIT}$ and macro-prudential policy is not used. We use $\mathcal{L}_0^{FIT}$ as a simple characterization of the pre-crisis monetary policy arrangements in which central banks used the short-term nominal interest rate to pursue stabilization objectives defined in terms of inflation and aggregate real activity. As such, it represents a natural benchmark policy assumption against which to compare the alternatives considered in subsequent sub-sections.

These assumptions imply that, when unconstrained by the zero bound, the monetary policymaker implements a standard flexible inflation-targeting criterion

$$x_t + \gamma\lambda_\pi\pi_t = 0, \tag{24}$$

which is identical to the case of the baseline New Keynesian models (Clarida et al., 1999; Woodford, 2003). Despite the additional richness of our model relative to the canonical New Keynesian model, this 'static' optimality condition is preserved because the policymaker's current decisions have no effect on the ability of future policymakers to set policy optimally. That is, in the absence of the zero bound, even though the model contains two endogenous state variables neither of them constrain the ability of future policymakers to stabilize the output gap and inflation by an appropriate choice of the nominal interest rate. However, the monetary policymaker may be constrained by the zero lower bound on the policy rate.

We use the flexible inflation targeting case as the baseline against which alternative policy configurations will be compared in the following subsections. As we will see, the broad contours of the housing boom in the baseline simulation match some of the qualitative features of the Great Moderation period: house prices rise strongly (and debt increases substantially, albeit by less than house valuations), but the output gap and inflation were well stabilized with relatively low nominal interest rates.

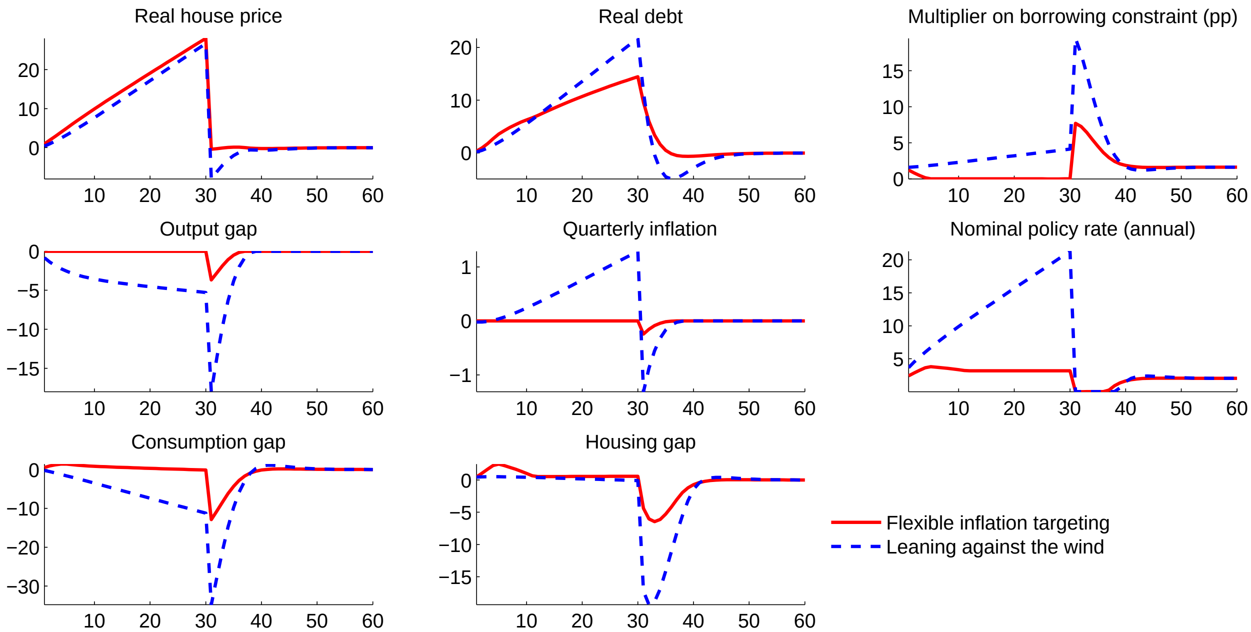

Figure 2: A housing boom/bust: 'flexible inflation targeting' and 'leaning against the wind'

4.5 Leaning Against the Wind

Our second policy configuration assumes that the monetary policymaker sets the short-term nominal interest rate to minimize the welfare-based loss function, $\mathcal{L}_0$. Macro-prudential policy is again assumed to be inactive (so that $\kappa_t = \theta_t = 0, \forall t$). This setting is therefore one in which the monetary policymaker can be said to 'lean against the wind' by setting the short-term nominal interest rate to address both the traditional focus of monetary policy (that is $\mathcal{L}_0^{FIT}$) and macro-prudential considerations (as captured by $\mathcal{L}_0^{MaP}$).

Figure 2 compares outcomes generated by 'leaning against the wind' with those from the 'flexible inflation targeting' benchmark. Several important results stand out.

First, leaning against the wind requires an extreme increase in the short-term nominal interest rate during the boom phase. Relative to flexible inflation targeting, the main success of leaning against the wind during the boom phase is a stable housing gap. The effects on the other variables in the loss function are large. Tighter monetary policy pushes output below potential for the duration of the boom, by a sizable amount. However, relative to the flexible inflation targeting benchmark, tighter monetary policy has very little effect on house prices and debt is actually *higher*.[18] Higher interest rates increase debt serving costs and reduce aggregate activity, thus raising the debt to GDP ratio as argued by Svensson (2013). These results chime with the commonly held view among monetary policymakers that a policy of leaning against the wind in the pre-crisis period would have required substantial increases in the short-term policy rate, with substantial effects on the variables

[18]The negative output gap is accompanied by a negative consumption gap so that borrowers' consumption is particularly weak. As a result, the borrowing constraint always binds (see top right panel) and the house price boom generates a pronounced increase in debt.

that appear explicitly in central bank mandates (output and inflation) but relatively little effect on debt and house prices (see, for example, Bean et al., 2010; Jansson, 2017).

One reason why our simulation exhibits these properties is that the preference shocks used to generate the simulation have a very strong and direct effect on house prices. The shocks generate an increase in the housing demand of both savers and borrowers. Given the fixed aggregate stock of housing, this results in an increase in house prices in equilibrium. This means that the ability of monetary policy actions to influence the housing market is limited. Some proponents of leaning against the wind may argue that higher interest rates should moderate expectations of future house prices and reduce housing demand in the near term. While present in our model, this mechanism is relatively weak compared with the direct effects of the preference shocks.

Second, while the increase in the policy rate during the housing boom generates a substantial fall in output, inflation is persistently *above* target. This reflects the recursive nature of our simulation: the boom is generated by a sequence of unanticipated increases in housing demand. At each point in time, house prices are expected to gradually fall back to steady state, a process that will be accompanied by a loosening of monetary policy and a subsequent boom in output. Because the Phillips curve in the model is relatively flat, inflation responds more to expected inflation (which is above target because of the anticipated boom) than to the current output gap.[19] A general implication of this observation is that swings in the short-term interest rate are much more pronounced under a leaning against the wind policy.

Finally, and relatedly, the bust phase creates a more severe recession under the leaning against the wind policy. Just as during the boom phase the policy prescribes a tighter course for monetary policy, the bust implies that, absent the zero bound constraint, it would be optimal to set the policy rate substantially below zero. This exacerbates the effect of the zero bound, generating substantial shortfalls in the output gap, inflation and the consumption and housing gaps. The deep recession is also associated with a collapse in house prices and a much sharper deleveraging process, compared with the outcomes under flexible inflation targeting.

The effect of alternative monetary/macro-prudential policy configurations during a recovery from a recession is of topical relevance, since many macro-prudential policy regimes have been put in place during a period when monetary policy has been constrained by the zero bound. The policy normalization phase (as economies recover) is therefore likely to be the first time we see active macro-prudential policies in many economies. The appropriate policy mix during this period is an open question, which our simulations shed some light on.

We can explain the effect of alternative policy configurations through the lens of a simple approach favored by some monetary policymakers. This approach uses the concept of the 'equilibrium real rate of interest' as a metric by which to assess the appropriate stance of monetary policy.[20] Structural models have an internally consistent analogue of the equilibrium rate (in our model it

[19]Figure G.1 in Appendix G demonstrates this by plotting the recursive paths underlying the simulation.

[20]The equilibrium real rate is sometimes called the 'neutral rate' or 'natural rate' but we avoid this terminology to avoid confusion with model-based concepts.

would be the real interest rate that prevails in the efficient equilibrium).[21] However, many policymakers favor a more general concept, namely the real rate of interest consistent with a zero output gap. For example, Ferguson (2004) defines it as "the level of the real federal funds rate that, if allowed to prevail for a couple of years, would place economic activity at its potential".

We can uncover the equilibrium real rate in our model by writing equation (10) as

$$x_t = \mathbb{E}_t x_{t+1} - \sigma^{-1} \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^{eq} \right)$$

where

$$r_t^{eq} \equiv \sigma \left[\nu_t^c + \xi \left(\tilde{c}_t - \mathbb{E}_t \tilde{c}_{t+1} \right) \right]$$

corresponds to the 'general' definition of the equilibrium real interest rate: if the short term real interest rate were to track the path of r^{eq}, then the output gap would be closed.

In our simulations, there are no consumption preference shocks, so that $\nu_t^c = 0, \forall t$ and hence:

$$r_t^{eq} = -\sigma \xi \mathbb{E}_t \Delta \tilde{c}_{t+1}$$

These observations reveal that the equilibrium real interest rate is endogenous in our model: it is a function of the consumption gap which will be affected by debt, among other factors. This property is shared with the models of Eggertsson and Krugman (2012a) and Benigno et al. (2014), who also assume that the household sector includes distinct borrowers and savers.

Figure 3 zooms in on the 'bust' phase of Figure 2 and explores the relative performance of 'flexible inflation targeting' (FIT) and 'leaning against the wind' (LATW), through the lens of the equilibrium real interest rate. We observe that *all* goal variables in the welfare-based loss function are considerably further from target in the LATW case: social welfare is unambiguously lower when the policymaker is tasked with using the short-term policy rate to minimize social welfare losses rather than focusing on a flexible inflation targeting mandate.

This result follows from the fact that LATW policy prescribes very active use of the nominal interest rate. In the context of a large negative shock, the zero bound is therefore a more severe constraint on the policymaker. The recovery of a very large initial consumption gap implies a very negative neutral rate, that unwinds very slowly. The presence of the zero bound on the policy rate prevents the policy maker from closing the output gap and the anticipation of a longer period of constrained monetary policy depresses aggregate demand in the near term, validating the expectation of a prolonged liquidity trap.

4.6 Macro-prudential Leadership

We now investigate whether outcomes can be materially improved by using macro-prudential policies. We first assume that monetary and macro-prudential policymakers pursue independent ob-

[21] The existence of internally consistent neutral rates in DSGE models has allowed researchers to examine the extent to which policymakers have set policy close to a model-based estimate of the neutral rate. See, for example, Cúrdia et al. (2015).

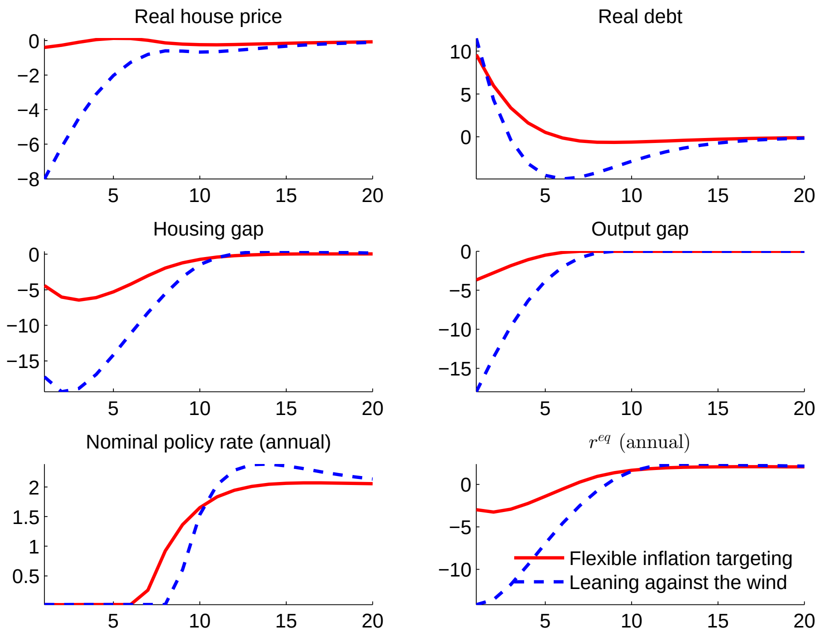

Figure 3: Housing bust and recovery: 'flexible inflation targeting' and 'leaning against the wind'

jectives. The macro-prudential policymaker is assumed to minimize $\mathcal{L}_0^{MaP}$ using the instruments θ_t and κ_t. The monetary policymaker sets the policy rate i_t to minimize $\mathcal{L}_0^{FIT}$. Both policymakers act under discretion.

We assume that the macro-prudential policymaker sets its instruments first within each period: it is the (within period) 'leader'.[22] The assumption of macro-prudential leadership means that the argument regarding the irrelevance of state variables for the monetary policymaker in Section 4.4 also applies in this case. So the monetary policymaker will implement the targeting criterion (24) unless prevented from doing so by the zero lower bound. The macro-prudential policymaker internalizes this behavior when setting policy at the beginning of the period.[23]

Figure 4 compares outcomes under macro-prudential leadership with those under flexible inflation targeting. We observe that an independently operated macro-prudential policy enables the monetary policymaker to completely stabilize the output gap and inflation (dashed blue lines). There is no recession when the real house price falls because macro-prudential policy is aggressively loosened. The tightening of borrowing conditions implies that the multiplier μ varies much less than in the case in which only monetary policy is used (solid red lines). The loosening of macro-prudential policy when house prices fall results in a much slower fall in aggregate debt (less deleveraging), which helps to avoid large movements in the consumption and housing gaps. Since borrowers are not required to reduce their debt levels as rapidly as would be the case with a fixed LTV, their consumption does not fall as drastically. Accordingly, a much smaller cut in the nominal policy rate is required to stabilize the output gap and inflation, so that monetary policy is not

[22] de Paoli and Paustian (2017) also study this timing protocol, among others.

[23] To implement this variant, we add (24) as a constraint (alongside the private sector optimality conditions) to an optimal policy problem that minimizes $\mathcal{L}_0^{MaP}$ using θ_t and κ_t.

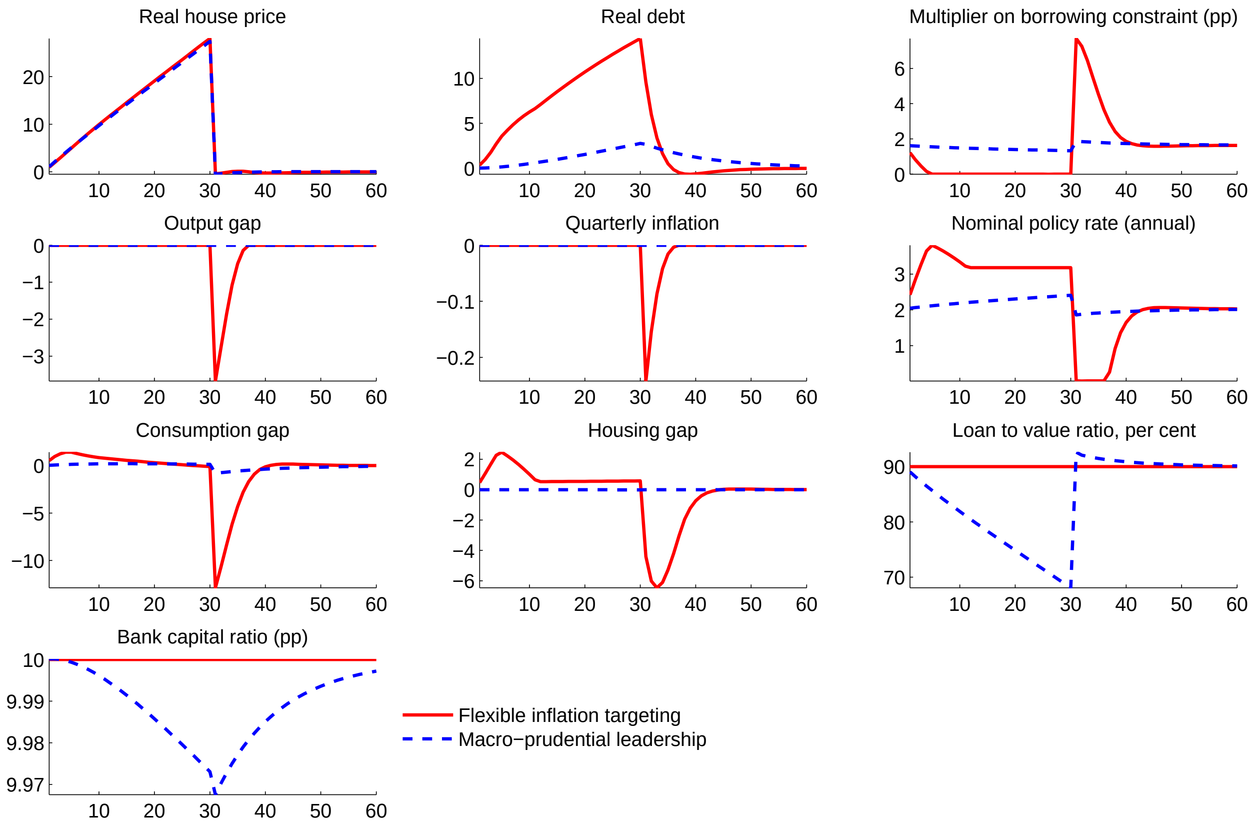

Figure 4: A housing boom/bust: 'flexible inflation targeting' and 'macro-prudential leadership'

constrained by the zero lower bound.

The use of the LTV as a macro-prudential policy instrument also has implications for macroeconomic dynamics during the housing boom. In the initial phase, a tightening of macro-prudential policy limits the increase in debt and succeeds in stabilizing the housing gap. In the boom phase, debt hardly increases because macro-prudential policy tightens steadily, reducing the maximum LTV by more than 20 percentage points — essentially offsetting the direct effect of the house price increase on borrowing limits.

The power of the LTV instrument is illustrated by the fact that the optimal trajectory for the bank capital ratio (κ) is virtually unchanged. The main reason for this result is that the effect of changes in the capital ratio on spreads (ψ) is very small. This means that very large changes in capital requirements are required to have a substantial influence on spreads. But large changes in κ also incur a direct cost in the 'macro-prudential' loss function $\mathcal{L}_0^{MaP}$. The net effect is that the capital ratio is adjusted relatively little, with a correspondingly small marginal effect on outcomes.[24]

[24]This effect is explained further in Appendix F.3.

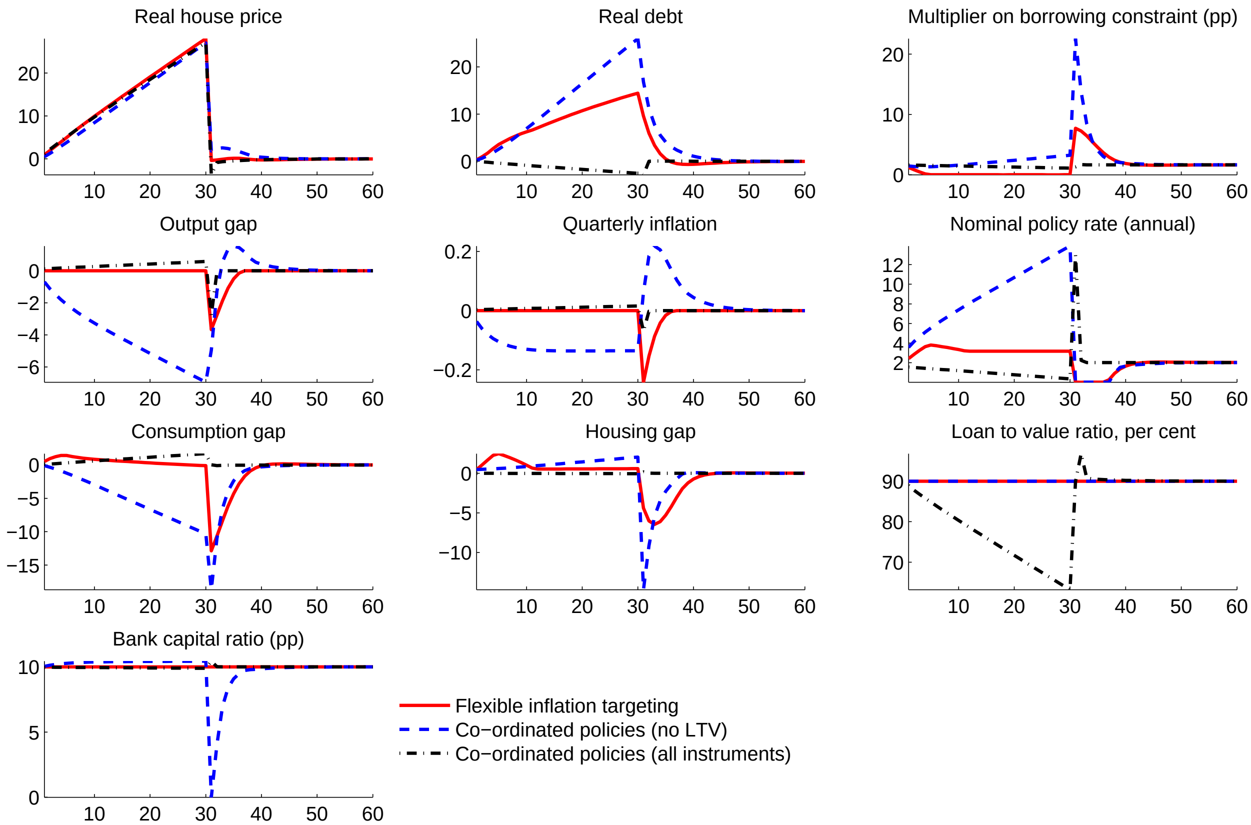

Figure 5: A housing boom/bust: 'flexible inflation targeting' and 'full co-ordination'

4.7 Full Co-ordination

Our final experiment considers the case in which the monetary and macro-prudential policy instruments are set jointly to minimize the welfare-based loss function $\mathcal{L}_0$. Because the LTV limit (θ) is so powerful, as demonstrated by the results in Section 4.6, we consider the cases in which all instruments may be used and also the case in which the policy instruments are restricted to the short-term nominal interest rate (i) and the bank capital ratio (κ).

Figure 5 compares outcomes under full co-ordination with the benchmark flexible inflation targeting policy. When the LTV limit is not used (blue dashed lines), outcomes under coordination are qualitatively similar to those under the leaning against the wind policy. In this case, because we assume that the LTV instrument is not used, the only difference with leaning against the wind is that the bank capital ratio is available as an additional policy instrument. The welfare-based loss function implies that changes in the bank capital ratio are costly. As explained above, when combined with the fact that the effects of bank capital ratios on spreads and (hence) macroeconomic outcomes are relatively small, in equilibrium the bank capital ratio is adjusted relatively little during the boom phase. As with leaning against the wind, co-ordinated policy generates a substantial increase in the nominal policy rate during the boom phase, resulting in a negative output gap and consumption gap. The bust is also associated with a prolonged period at the zero lower bound.

However, the ability to use the bank capital ratio as an instrument generates some differences with the leaning against the wind policy. Macro-prudential policy is tightened during the boom and loosened dramatically in the bust phase. Relative to leaning against the wind, these actions help to mitigate the effects on inflation and the output and consumption gaps during both the boom and the bust.[25]

During the bust phase, the bank capital ratio is cut aggressively such that the lower bound (assumed to be zero) binds for one period.[26] Loosening macro-prudential policy so aggressively at the beginning of the bust phase imposes direct welfare costs. However, by doing so the coordinated policymaker is able to shorten the duration of the liquidity trap and stimulate spending. Indeed, although the bust phase is associated with (temporarily) large consumption and housing gaps, the recession is limited: both the output gap and inflation move quickly into positive territory.

Turning to the case in which all instruments are used (black dot-dashed lines), we observe that the ability to set the LTV limit has a material effect on the results. The LTV limit is reduced substantially during the boom, offsetting the effects of house prices on debt, which actually falls. The tightening in LTV policy during the boom is accompanied by a loosening of monetary policy and bank capital ratios. Indeed, the nominal policy rate drops very close to the zero bound towards the end of the boom phase, supporting a slightly positive output gap. During the bust, the stance implied by the individual instruments is reversed: LTV policy is loosened substantially, while monetary policy and capital ratios *tighten*.[27] This configuration of policy instruments during the bust phase is the opposite of the 'loose monetary, tight macro-prudential' adopted in some countries since the post-crisis period. However, Figure 5 makes clear that the nature of the optimal policy mix depends critically on the set of instruments that are available to the policymaker. Importantly, instruments that are broadly equivalent to the LTV ratio in our model have not been widely available or operational in many economies.

With this in mind, we again focus in on the bust phase of the simulation. Figure 6 compares outcomes under the 'leaning against the wind' policy with co-ordinated interest rate and bank capital policy. Recall that these policy configurations both assume that the policymaker is attempting to minimize the (discounted) social welfare loss, $\mathcal{L}_0$. The only difference lies in whether the policymaker has access to the capital requirement alongside the short-term nominal interest rate.

Figure 6 reveals that allowing the policymaker to use capital requirements alongside the policy rate improves outcomes substantially. In the period of the housing bust, capital ratios are set at

[25]Inflation undershoots during the boom because the contraction in output, while smaller than under leaning against the wind, is also more persistent. See Figure G.2 in Appendix G.

[26]Since the zero bound on the short-term nominal rate is also binding in this period, the nominal rate of return to saving is zero and the return to borrowing is negative. Imposing a zero bound on saving rates can be justified by the assumption that there exists a zero interest alternative saving instrument (eg cash). Allowing the borrowing rate to become negative creates an arbitrage opportunity for borrowers by borrowing from banks and investing the loans in higher return savings instruments. However, they cannot exploit it as borrowing is limited by the collateral constraint.

[27]Indeed, the small recession in response to the drop in housing demand is an optimal response rather than a result of the zero bound binding.

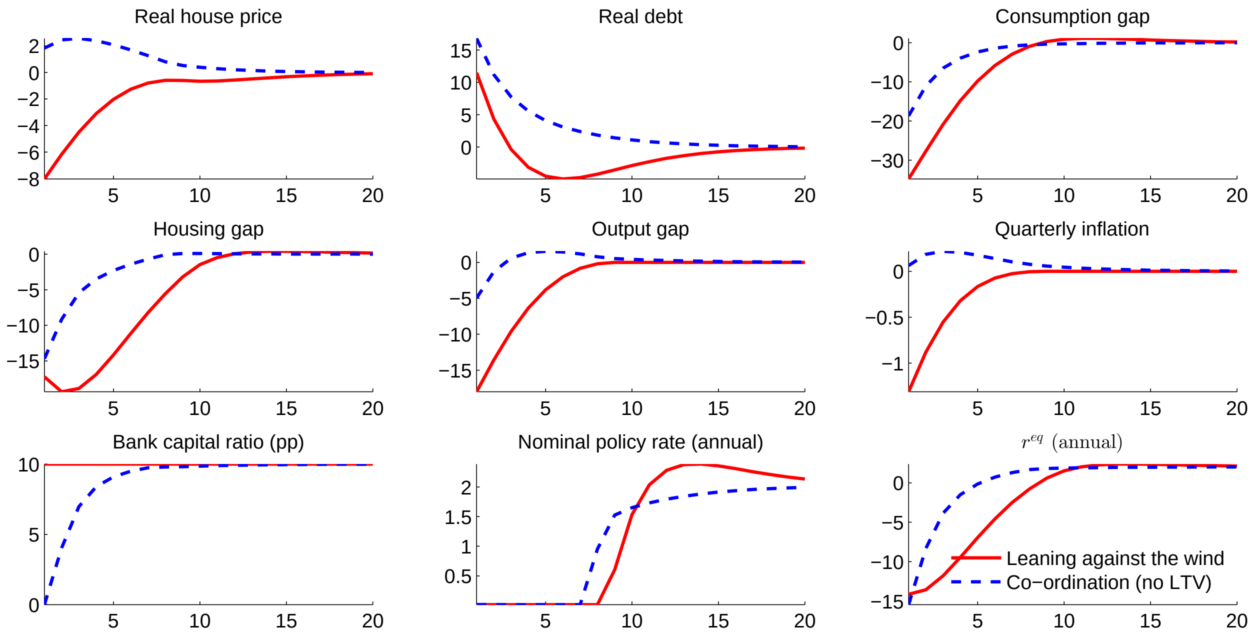

Figure 6: Housing bust and recovery: 'leaning against the wind' and co-ordinated bank capital and interest rate policies

their lower bound. Even though the policy rate is initially constrained, low capital ratios reduce borrowing costs and help to support the consumption gap. The consumption gap converges back to zero relatively slowly, so that the equilibrium real interest rate recovers back to normal more quickly than under the LATW policy. This permits an earlier liftoff from the zero bound, though the policy stance is expansionary even after liftoff. The expectation that policy will be conducted in this way means that, relative to the LATW case, long-term real rates are low. Relatively low real interest rates moderate the initial fall in the output gap during the house price correction. One interpretation of Figure 6 is that the coordinated policy stance during the recovery phase is consistent with the typical monetary/macro-prudential policy mix favored by policymakers in recent years. During the recovery phase, macro-prudential policy is tightening (as bank capital ratios rise) alongside a prolonged period of relatively loose monetary policy.

5 Conclusion

We develop a simple model to examine the interaction of monetary and macro-prudential policies. Our model is rich enough to generate meaningful policy tradeoffs, but sufficiently simple to deliver tractable expressions for welfare and analytical results under some parameterizations. We derive a welfare-based loss function as a quadratic approximation to a weighted average of the utilities of borrowers and savers.

We use the model to study how monetary and macro-prudential policies (LTV ratios and capital requirements) should optimally respond to shocks. To build intuition, we derive some analytical results under restrictive assumptions on parameters and nature of the constraints. In this simplified setting, we demonstrate that macro-prudential policy generally faces a trade-off in stabilizing the distribution of consumption and the distribution of housing services, even when prices are flexible and both macro-prudential tools are used. We also show that monetary policy alone has relatively little control over these distributions, particularly the distribution of housing between borrowers and savers. In other words, imperfect risk sharing is a real phenomenon whose consequences could be addressed by macro-prudential policies. Nevertheless, these policies also imply costs that must be accounted for in deploying them. This tradeoff prevents complete macro-prudential stabilization given the tools we study even under flexible prices.

We use the model to explore a simulation of a prolonged boom followed by a sharp fall in house prices. When there is a single monetary policymaker pursuing a 'flexible inflation targeting' mandate (minimizing a loss function that includes only the output gap and inflation), the house price fall causes a recession because monetary policy is constrained by the zero bound. During the housing boom, increases in the policy rate required to stabilize the output gap and inflation are moderated because the borrowing constraint becomes slack. Although the output gap and inflation are fully stabilized during the boom, welfare losses are incurred because monetary policy is unable to stabilize the consumption and housing gaps. Allowing macro-prudential policies to focus on stabilizing the consumption and housing gaps improves welfare substantially. Indeed, the existence of macro-prudential policy implies that there is no recession after the house price fall: if the LTV tool is used optimally, monetary policy is able to stabilize the output gap and inflation without hitting the zero bound.

Our results have important implications for the current economic environment. As economic conditions improve in many countries, and central banks move away from the effective lower bound, macro-prudential decisions may have non-negligible effects on the nature of the recovery. Indeed, it would seem prudent, as our initial quote of Carney (2014) suggests, to use all policy tools in concert.

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Appendix

A Private Sector Optimality Conditions and Aggregation

This appendix reports the optimality conditions of the private sector (savers, borrowers, and intermediate goods producers) and the details of the aggregation.

A.1 Savers

Starting with savers, the first order condition for deposits is

$$\mathbb{E}_t\left[\beta_s e^{-z(C^s_{t+1}(i)-C^s_t(i))}\frac{R^d_t}{\Pi_{t+1}}\right]=1,$$

where $\Pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate. The corresponding condition for bank equity is

$$\mathbb{E}_t\left[\beta_s e^{-z(C^s_{t+1}(i)-C^s_t(i))}\frac{R^e_t}{\Pi_{t+1}}\right]=1+\Psi\left[\frac{E^s_t(i)}{\tilde{\kappa}\xi D^b_t/(1-\xi)}-1\right].$$

Combining the two Euler equation, we can obtain the no-arbitrage condition between equity and deposits

$$\mathbb{E}_t\left[\beta_s e^{-z(C^s_{t+1}(i)-C^s_t(i))}\frac{R^e_t-R^d_t}{\Pi_{t+1}}\right]=\Psi\left[\frac{E^s_t(i)}{\tilde{\kappa}\xi D^b_t/(1-\xi)}-1\right].$$

After rearranging, the first order condition for housing services can be written as

$$(1+\tau^h)\frac{Q_t}{P_t}=\frac{\chi^s_H e^{-u^h_t}H^s_t(i)^{-\sigma_h}}{e^{-zC^s_t(i)}}+\mathbb{E}_t\left[\beta_s e^{-z(C^s_{t+1}(i)-C^s_t(i))}\frac{Q_{t+1}}{P_{t+1}}\right].$$

The labour supply condition is

$$W^s_t=\frac{\chi^s_L L^s_t(i)^{\varphi}}{e^{-zC^s_t(i)}}.$$

The budget constraint at equality completes the list of first order conditions for savers.

A.2 Borrowers

Moving on to borrowers, we attach a Lagrange multiplier normalised by the real marginal utility of consumption ($\tilde{\mu}_t(i)ze^{-zC^b_t(i)}/P_t$) to the collateral constraint. The first order condition for borrowed funds is

$$\mathbb{E}_t\left[\beta_b e^{-z(C^b_{t+1}(i)-C^b_t(i))}\frac{R^b_t}{\Pi_{t+1}}\right]=1-\tilde{\mu}_t(i).$$

The first order condition for housing demand is

$$(1-\Theta\tilde{\mu}_t(i))\frac{Q_t}{P_t}=\frac{\chi^b_H e^{-u^h_t}H^b_t(i)^{-\sigma_h}}{e^{-zC^b_t(i)}}+\mathbb{E}_t\left[\beta_b e^{-z(C^b_{t+1}(i)-C^b_t(i))}\frac{Q_{t+1}}{P_{t+1}}\right].$$

The labour supply condition is

$$W_t^b = \frac{\chi_L^b L_t^b(i)^\varphi}{e^{-zC_t^b(i)}}.$$

The equilibrium conditions for borrower households include the complementary slackness condition

$$\tilde{\mu}_t(i)[D_t^b - \Theta_t Q_t H_t^b(i)] = 0.$$

The budget constraint at equality completes the list of first order conditions for borrowers.

A.3 Firms

The text reports the optimality condition for banks, which is the result of perfect competition in the financial sector.

To derive the expression for the marginal cost, we solve the dual problem

$$\min_{L_t^b, L_t^s} \frac{W_t^b}{P_t} L_t^b(f) + \frac{W_t^s}{P_t} L_t^s(f),$$

subject to the technological constraint given by the production function. Let $M_t(f)$ be the multiplier on the constraint (the real marginal cost). The first order conditions for the two types of labour are

$$\begin{aligned} \frac{W_t^b}{P_t} &= \xi M_t(f) A_t L_t^b(f))^{\xi-1} L_t^s(f)^{1-\xi} &&= \xi M_t(f) \frac{Y_t(f)}{L_t^b(f)} \\ \frac{W_t^s}{P_t} &= (1-\xi) M_t(f) A_t L_t^b(f))^{\xi} L_t^s(f)^{-\xi} &&= (1-\xi) M_t(f) \frac{Y_t(f)}{L_t^s(f)}. \end{aligned}$$

Taking the ratio between the two first order conditions above shows that at the optimum all firms choose the same proportion of labor of the two types. As a consequence, the marginal cost is independent of firm-specific characteristics ($M_t(f) = M_t$). Furthermore, if we take a geometric average of the two first order conditions above, with weights ξ and $1-\xi$, respectively, we obtain the expression for the marginal cost

$$M_t = \frac{W_t/P_t}{\xi^\xi (1-\xi)^{1-\xi} A_t},$$

where the expression for the aggregate wage index is reported in the text.

Intermediate goods producers set prices on a staggered basis. Their optimality condition can be summarised by a non-linear Phillips curve

$$\frac{X_{1t}}{X_{2t}} = \left(\frac{1-\lambda \Pi_t^{\varepsilon-1}}{1-\lambda} \right)^{\frac{1}{-\varepsilon}},$$

where X_{1t} represents the present discounted value of real costs

$$X_{1t} = \frac{\varepsilon}{\varepsilon - 1} z e^{-zC_{t+1}} Y_t M_t + \beta\lambda \mathbb{E}_t(\Pi_t^{\varepsilon} X_{1t+1}),$$

and X_{2t} represents the present discounted value of real revenues

$$X_{2t} = (1 + \tau^p) z e^{-zC_{t+1}} Y_t + \beta\lambda \mathbb{E}_t(\Pi_t^{\varepsilon-1} X_{2t+1}).$$

A.4 Aggregation

To aggregate within types, we simply integrate over the measure of households in each group. Consumption of savers and borrowers is

$$\int_0^{1-\xi} C_t^s(i)di = (1-\xi)C_t^s \qquad \text{and} \qquad \int_{1-\xi}^1 C_t^b(i)di = \xi C_t^b,$$

while housing demand is

$$\int_0^{1-\xi} H_t^s(i)di = (1-\xi)H_t^s \qquad \text{and} \qquad \int_{1-\xi}^1 H_t^b(i)di = \xi H_t^b.$$

In the credit market, total bank loans must equal total household borrowing

$$\int_0^1 D_t^b(k)dk = \int_{1-\xi}^{\xi} D_t^b(i)di = \xi D_t^b.$$

Similarly, for deposits and equity holdings we have

$$\int_0^1 D_t^s(k)dk = \int_0^{1-\xi} D_t^s(i)di = (1-\xi)D_t^s \qquad \text{and} \qquad \int_0^1 E_t^s(k)dk = \int_0^{1-\xi} E_t^s(i)di = (1-\xi)E_t^s.$$

Using these expressions, we obtain the aggregate balance sheet for the financial sector and the economy-wide capital constraint reported in the text.

Labour market clearing requires

$$\int_0^1 L_t^s(f)df = \int_0^{1-\xi} L_t^s(i)di = (1-\xi)L_t^s \qquad \text{and} \qquad \int_0^1 L_t^b(f)df = \int_{1-\xi}^1 L_t^b(i)di = \xi L_t^b.$$

Aggregating production across firms yields

$$\int_0^1 Y_t(f)df = \int_0^1 A_t L_t^b(f)^{\xi} L_t^s(f)^{1-\xi} df. \tag{25}$$

As discussed in the previous section, the ratio of hours worked of different types is independent of firm-specific characteristics. Therefore, using the labour market equilibrium conditions, we can rewrite the right-hand side of the previous expression as

$$\int_0^1 A_t L_t^b(f)^{\xi} L_t^s(f)^{1-\xi} df = A_t(\xi L_t^b)^{\xi}((1-\xi)L_t^s)^{1-\xi} = \xi^{\xi}(1-\xi)^{1-\xi} A_t L_t,$$

where aggregate labour is

$$L_t \equiv (L_t^b)^{\xi}(L_t^s)^{1-\xi}. \tag{26}$$

Using the demand for firm f's product, the left-hand side of (25) can also be rewritten in terms of aggregate variables only as

$$\int_0^1 Y_t(f) df = \Delta_t Y_t,$$

where Δ_t is an index of price dispersion, defined as

$$\Delta_t \equiv \int_0^1 \left[\frac{P_t(f)}{P_t}\right]^{-\varepsilon} df.$$

Given the definition of the price index and the assumption of staggered price setting, the index of price dispersion evolves according to

$$\Delta_t = \lambda \Delta_{t-1} \Pi_t^{\varepsilon} + (1-\lambda)\left(\frac{1-\lambda \Pi_t^{\varepsilon-1}}{1-\lambda}\right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

Therefore, in the aggregate, production is described by

$$\Delta_t Y_t = \xi^{\xi}(1-\xi)^{1-\xi} A_t L_t.$$

The last step of the aggregation is the derivation of the law of motion of debt. To obtain this equation, we start from the flow budget constraint of a generic borrower

$$P_t C_t^b(i) - D_t^b(i) + Q_t H_t^b(i) = W_t^b L_t^b(i) - R_{t-1}^b D_{t-1}^b(i) + Q_t H_{t-1}^b(i) + \Omega_t^b(i) - T_t^b(i).$$

We assume that each household $i \in [0,1]$ receives an equal share of aggregate value added

$$\Omega_t^j(i) = P_t Y_t - W_t L_t,$$

for $j = \{b, s\}$. From the first order conditions of intermediate goods producers we have

$$W_t^b L_t^b(f) = \xi M_t Y_t(f) \qquad \text{and} \qquad W_t^s L_t^s(f) = (1-\xi) M_t Y_t(f).$$

Integrating over firms, we obtain

$$W_t^b L_t^b = W_t^s L_t^s = W_t L_t = M_t \Delta_t Y_t,$$

where we have used the labour market equilibrium conditions, the definition of the wage and labour indexes, and the definition of the price dispersion index.

Aggregating the borrowers' individual budget constraints, we can then write

$$C_t^b - \frac{D_t^b}{P_t} + \frac{Q_t}{P_t} H_t^b = Y_t - \frac{R_{t-1}^b}{\Pi_t} \frac{D_{t-1}^b}{P_{t-1}} + \frac{Q_t}{P_t} H_{t-1}^b - \mathcal{T}^b, \tag{27}$$

where $\mathcal{T}^b$ is a steady state net tax/subsidy, which includes the borrowers' contribution to the firms' subsidy that make steady state output efficient, and the subsidy borrowers receive to obtain an efficient allocation. We can rewrite the last expression to capture the law of motion of debt

$$\frac{D_t^b}{P_t} = \frac{R_{t-1}^b}{\Pi_t} \frac{D_{t-1}^b}{P_{t-1}} + C_t^b - Y_t + \frac{Q_t}{P_t}(H_t^b - H_{t-1}^b) + \mathcal{T}^b.$$

B Efficient Steady State

This section first establishes the conditions under which a zero inflation ($\Pi = 1$) steady state is efficient, and then discusses how we can obtain efficiency of the steady state allocation in the decentralised equilibrium.[28]

Consider a social planner who maximises a weighted average of borrowers and savers' per-period welfare

$$\mathbb{U} \equiv \tilde{\xi} U(C^b, H^b, L^b) + (1 - \tilde{\xi}) U(C^s, H^s, L^s), \tag{28}$$

for some Pareto weights $\tilde{\xi} \in [0, 1]$, where $U(C^j, H^j, L^j)$ is the per-period utility function of type $j = \{b, s\}$. The social planner chooses allocations subject to the constraints imposed by the aggregate production function and the market clearing conditions for goods, housing, and labour. Importantly, the planner is not subject to the borrowing constraint.

In steady state, there is no price dispersion ($\Delta = 1$). We can further normalise steady state productivity A to one and combine the production function with the goods and labour market constraints to yield

$$(L^b)^{\xi} (L^s)^{1-\xi} = \xi C^b + (1 - \xi) C^s.$$

Let μ_1 be the Lagrange multiplier on this constraint and μ_2 be the multiplier on the housing resource constraint. The first-order conditions for an efficient steady state are

$$\begin{aligned}
\tilde{\xi} U'_{C^b} &= \mu_1 \xi \\
(1 - \tilde{\xi}) U'_{C^s} &= \mu_1 (1 - \xi) \\
\tilde{\xi} U'_{H^b} &= \mu_2 \xi \\
(1 - \tilde{\xi}) U'_{H^s} &= \mu_2 (1 - \xi) \\
\tilde{\xi} U'_{L^b} &= \mu_1 \xi Y / L^b \\
(1 - \tilde{\xi}) U'_{L^s} &= \mu_1 (1 - \xi) Y / L^s
\end{aligned}$$

[28] Without loss of generality, we normalize the price level to one so that all variables can be thought of as expressed in real terms.

If the Pareto weights coincide with the population weights ($\tilde{\xi} = \xi$), the marginal utility of consumption and housing are equal across types

$$U'_{C^b} = U'_{C^s} = \mu_1 \qquad \text{and} \qquad U'_{H^b} = U'_{H^s} = \mu_2,$$

and so are their levels. In addition, if the disutility of labour has a constant elasticity of substitution, as we assumed, hours supplied by borrowers and savers are proportional to each other depending on the disutility parameters χ^s and χ^b.

For a given type of household, we also obtain

$$\frac{U'_{C^j}}{U'_{H^j}} = \frac{\mu_1}{\mu_2}.$$

The ratio of the marginal utilities of consumption and housing for the two types are the same. The efficient steady state also implies the usual optimality conditions that equates the marginal rate of substitution between consumption and leisure to the marginal rate of transformation between labour and output

$$\frac{U'_{L^j}}{U'_{C^j}} = \frac{Y}{L^j}.$$

Assuming the subsidy τ^p is set as to remove the distortions from monopolistic competition in steady state ($M = 1$), the labour market equilibrium implies

$$\left[\frac{\chi_L^b (L^b)^\varphi}{z\exp(-zC^b)}\right]^\xi \left[\frac{\chi_L^s (L^s)^\varphi}{z\exp(-zC^s)}\right]^{1-\xi} = \frac{Y}{L}.$$

Using the goods and labour market clearing conditions, and replacing output with labour from the production function, equilibrium hours solve

$$L^\varphi \exp(zL) = \frac{z}{\left(\chi_L^b\right)^\xi \left(\chi_L^s\right)^{1-\xi}}.$$

We can choose the labour supply disutility parameters χ_L^j to deliver a desired target for hours worked by each group (e.g. 2/3 of the households' time endowments). Given this result, the production function pins down the equilibrium level of output. Therefore, importantly, the steady state efficient level of output and hours is independent of the distribution of wealth/debt across household types.

The next step is to find conditions under which the steady state allocation of the decentralised economy is efficient. In particular, we seek the taxes that achieve this objective. In the steady state of the decentralised economy, the savers' discount rate pins down the real rate of interest

$$R^d = \frac{1}{\beta_s}.$$

Since the ratio between equity and deposits is at its desired level, the spread between the return on equity and the return on deposits is zero, and so is the spread between loan and deposit rates

$$R^b = R^e = R^d.$$

In what follows, we drop the superscripts from returns and simply call the steady state gross real interest rate R. From the Euler equation for borrowers, we can obtain the value of the Lagrange multiplier on the collateral constraint

$$\tilde{\mu} = 1 - \beta_b R,$$

which is positive as long as our initial assumption $\beta_b < 1/R = \beta_s$ is satisfied (that is, borrowers are relatively impatient).[29] With a positive multiplier, the constraint binds, and so equilibrium debt is

$$D^b = \Theta Q H^b.$$

Finally, we turn to the housing block. Starting from the law of motion of debt in steady state, we can write

$$C^b = Y - \mathcal{T}^b - (R^b - 1)D^b.$$

In an efficient steady state, the level of consumption must be equal ($C^b = C^s$). Therefore, from the resource constraint, we have that $C^b = C^s = Y = C$. Substituting into the previous condition yields

$$\tau^b = -\frac{1 - \beta_s}{\beta_s}\eta,$$

where $\eta \equiv D^b/Y$ is the ratio of debt to GDP and $\tau^b \equiv \mathcal{T}^b/Y$ is subsidy to borrowers (net of their contribution to the production subsidy) that equalises consumption across types.

The last element that we need to determine is the housing tax τ^h. In steady state, the housing demand equation for borrowers is

$$(1 - \Theta\tilde{\mu} - \beta_b)Q = \frac{\chi_H^b (H^b)^{-\sigma_h}}{e^{-zC}},$$

while for savers we have

$$(1 + \tau^h - \beta_s)Q = \frac{\chi_H^s (H^s)^{-\sigma_h}}{e^{-zC}},$$

where we have used the equality of consumption across types. For the steady state housing allocation to be efficient, we must have that the numerator of the right-hand side of the last two expressions (the marginal utility of housing) is equal across types.[30] Therefore, the steady state housing tax must be

$$\tau^h = (\beta_s - \beta_b)\left(1 - \frac{\Theta}{\beta_s}\right),$$

[29] Alternatively, we could write the value of the Lagrange multiplier on the borrowing constraint as $\tilde{\mu} = (\beta_s - \beta_b)/\beta_s > 0$, as long as $\beta_s > \beta_b$, which again corresponds to the initial assumption on the individual discount factors.

[30] In addition, if the housing preference parameters are the same across households ($\chi^b = \chi^s = \chi$), then also the actual level of housing services consumed is the same ($H^b = H^s = H$).

where we used the expression for the steady state Lagrange multiplier $\tilde{\mu} = 1 - \beta_b/\beta_s$, which we obtained by combining the Euler equations for credit and deposits of the two types. Note that, the steady state tax on housing is zero if either $\Theta = \beta_s$ or in the limit $\beta_b \to \beta_s$.

B.1 Macro-Prudential Policy in the Efficient Equilibrium

This section shows that, in a flexible-price efficient equilibrium, macro-prudential policy carries distributional consequences but has no impact on the level of aggregate activity.

As derived in Appendix A, labour supply for type j's satisfies

$$W_t^j = \frac{\chi_L^j (L_t^j)^{\varphi}}{z \exp(-z C_t^j)}.$$

Weighting the labour supply of each type by their respective shares, using the definition of the wage index, and equating with labour demand gives

$$\left[\frac{\chi_L^b (L_t^b)^{\varphi}}{z \exp(-z C_t^b)}\right]^{\xi} \left[\frac{\chi_L^s (L_t^s)^{\varphi}}{z \exp(-z C_t^s)}\right]^{1-\xi} = \frac{Y_t}{L_t}$$

Using the definition of the labor aggregator and the resource constraint, we can simplify the previous expression to

$$\frac{\left(\chi_L^b\right)^{\xi} \left(\chi_L^s\right)^{1-\xi} L_t^{1+\varphi}}{z \exp[-z(Y_t - \Gamma_t)]} = Y_t,$$

where Γ_t is the portfolio adjustment cost term. In a flexible-price efficient equilibrium, the aggregate production function is simply $Y_t = A_t L_t$. Therefore, we can express the last condition in terms of the efficient level of output Y_t^* as

$$\frac{\left(\chi_L^b\right)^{\xi} \left(\chi_L^s\right)^{1-\xi}}{z \exp(-z Y_t^*)} \left(\frac{Y_t^*}{A_t}\right)^{1+\varphi} = Y_t^*. \tag{29}$$

In principle, portfolio adjustment costs associated with savers' debt-equity choice do affect output under flexible prices, but this effect is second order because Γ_t is quadratic. As a result, to a first order approximation, the efficient level of output only depends on technology and preference parameters.

In spite of no first-order effects on aggregate supply, macro-prudential measures retain distributional consequences even in an efficient equilibrium. The macro-prudential authority will not be indifferent between different levels of LTV ratios or capital requirements. We return to this point in the next section after deriving a linear-quadratic approximation of the model that allows us to study the optimal joint conduct of monetary and macro-prudential policy. The flexible-price efficient equilibrium will be a useful starting point for our analysis.

C Derivation of the Loss Function

We define the welfare objective for the policymaker $\mathbb{W}_0$ as the present discounted value of the per-period utility of the two types, weighted by arbitrary weights $\tilde{\xi}$, and we assume the policymaker discounts the future at rate β_s

$$\mathbb{W}_0 \equiv \mathbb{E}_0 \left(\sum_{t=0}^{\infty} \beta_s^t U_t \right), \tag{30}$$

where

$$U_t \equiv \tilde{\xi} U^b(C_t^b, H_t^b, L_t^b) + (1-\tilde{\xi}) U^s(C_t^s, H_t^s, L_t^s). \tag{31}$$

In order to derive a quadratic welfare objective, we take a second-order approximation of (31) around the efficient steady state in which $C^b = C^s = C = Y$, $H^b = H^s = H$, and $L^b = L^s = L$.

Ignoring terms of order three and higher, we get:

$$\begin{aligned} U_t - U \simeq\ & \tilde{\xi}[U_c^b(C_t^b - C^b) + \frac{1}{2}U_{cc}^b(C_t^b - C^b)^2] + (1-\tilde{\xi})[U_c^s(C_t^s - C^s) + \frac{1}{2}U_{cc}^s(C_t^s - C^s)^2] \\ & + \tilde{\xi}[U_h^b(H_t^b - H^b) + \frac{1}{2}U_{hh}^b(H_t^b - H^b)^2] + (1-\tilde{\xi})[U_h^s(H_t^s - H^s) + \frac{1}{2}U_{hh}^s(H_t^s - H^s)^2] \\ & + \tilde{\xi}[U_l^b(L_t^b - L^b) + \frac{1}{2}U_{ll}^b(L_t^b - L^b)^2] + (1-\tilde{\xi})[U_l^s(L_t^s - L^s) + \frac{1}{2}U_{ll}^s(L_t^s - L^s)^2]. \end{aligned}$$

Next, we factor out the marginal utility of consumption, housing, and the marginal disutility of labour for each group to obtain

$$\begin{aligned} U_t - U \simeq\ & \tilde{\xi}U_c^b[(C_t^b - C^b) + \frac{1}{2}\frac{U_{cc}^b}{U_c^b}(C_t^b - C^b)^2] + (1-\tilde{\xi})U_c^s[(C_t^s - C^s) + \frac{1}{2}\frac{U_{cc}^s}{U_c^s}(C_t^s - C^s)^2] \\ & + \tilde{\xi}U_h^b[(H_t^b - H^b) + \frac{1}{2}\frac{U_{hh}^b}{U_h^b}(H_t^b - H^b)^2] + (1-\tilde{\xi})U_h^s[(H_t^s - H^s) + \frac{1}{2}\frac{U_{hh}^s}{U_h^s}(H_t^s - H^s)^2] \\ & + \tilde{\xi}U_l^b[(L_t^b - L^b) + \frac{1}{2}\frac{U_{ll}^b}{U_l^b}(L_t^b - L^b)^2] + (1-\tilde{\xi})U_l^s[(L_t^s - L^s) + \frac{1}{2}\frac{U_{ll}^s}{U_l^s}(L_t^s - L^s)^2]. \end{aligned}$$

Using the first-order conditions associated with the efficient steady state, we get

$$\begin{aligned} U_t - U \simeq\ & \mu_1\xi[(C_t^b - C^b) + \frac{1}{2}\frac{U_{cc}^b}{U_c^b}(C_t^b - C^b)^2] + \mu_1(1-\xi)[(C_t^s - C^s) + \frac{1}{2}\frac{U_{cc}^s}{U_c^s}(C_t^s - C^s)^2] \\ & + \mu_2\xi[(H_t^b - H^b) + \frac{1}{2}\frac{U_{hh}^b}{U_h^b}(H_t^b - H^b)^2] + \mu_2(1-\xi)[(H_t^s - H^s) + \frac{1}{2}\frac{U_{hh}^s}{U_h^s}(H_t^s - H^s)^2] \\ & - \mu_1\xi\frac{Y}{L^b}[(L_t^b - L^b) + \frac{1}{2}\frac{U_{ll}^b}{U_l^b}(L_t^b - L^b)^2] - \mu_1(1-\xi)\frac{Y}{L^s}[(L_t^s - L^s) + \frac{1}{2}\frac{U_{ll}^s}{U_l^s}(L_t^s - L^s)^2]. \end{aligned}$$

Given the assumed preferences, we have

$$\frac{U^j_{cc}}{U^j_c} = \frac{-z^2\exp(-zC)}{z\exp(-zC)} = -z = -\frac{\sigma}{Y}$$
$$\frac{U^j_{hh}}{U^j_h} = \frac{\chi^j_H \sigma_h (H^j)^{-\sigma_h - 1}}{\chi^j_H (H^j)^{-\sigma_h}} = -\frac{\sigma_h}{H^j}$$
$$\frac{U^j_{ll}}{U^j_l} = \frac{\chi^j_L \varphi (L^j)^{\varphi-1}}{\chi^j_L (L^j)^{\varphi}} = \frac{\varphi}{L^j}$$

After substituting for these semi-elasticities, we collect the linear terms in consumption and housing to get

$$\begin{aligned} U_t - U \simeq{}& \mu_1[\xi(C^b_t - C^b) + (1-\xi)(C^s_t - C^s)] + \mu_2[\xi(H^b_t - H^b) + (1-\xi)(H^s_t - H^s)] \\ &- \frac{1}{2}\mu_1 z \left[\xi(C^b_t - C^b)^2 + (1-\xi)(C^s_t - C^s)^2\right] \\ &- \frac{1}{2}\mu_2 \frac{\sigma_h}{H} \left[\xi(H^b_t - H^b)^2 + (1-\xi)(H^s_t - H^s)^2\right] \\ &- \mu_1 \xi \frac{Y}{L^b}[(L^b_t - L^b) + \frac{1}{2}\frac{\varphi}{L^b}(L^b_t - L^b)^2] - \mu_1(1-\xi)\frac{Y}{L^s}[(L^s_t - L^s) + \frac{1}{2}\frac{\varphi}{L^s}(L^s_t - L^s)^2]. \end{aligned} \quad (32)$$

To eliminate first-order terms, we first make use of the identity

$$Z_t \equiv e^{\ln Z_t},$$

for any variable Z_t. We approximate Z_t around $\ln Z$ to the second order. Let $y_t \equiv \ln Z_t$. Then, we have

$$e^{y_t} \simeq e^y + e^y(y_t - y) + \frac{1}{2}e^y(y_t - y)^2,$$

where we have ignored terms of order three and higher. Applying the transformation back, we get

$$Z_t \simeq Z + Z(\ln Z_t - \ln Z) + \frac{1}{2}(\ln Z_t - \ln Z)^2,$$

which implies

$$\frac{Z_t - Z}{Z} \simeq z_t + \frac{1}{2}z_t^2,$$

or

$$Z_t \simeq Z(1 + z_t + \frac{1}{2}z_t^2),$$

where we have defined $z_t \equiv \ln(Z_t/Z)$.

Now, we apply the approximation to the goods market resource constraint

$$Y_t = \xi C^b_t + (1-\xi)C^s_t + \Gamma_t,$$

where

$$\Gamma_t = \frac{\Psi}{2}\left(\frac{\tilde{\kappa}_t}{\tilde{\kappa}} - 1\right)^2 \tilde{\kappa}\xi D_t^b.$$

Note that in steady state Γ_t is equal to zero, and that all first and second derivatives are zero except for the second derivative with respect to $\tilde{\kappa}_t$. Therefore, up to a second order approximation, we have

$$\Gamma_t \simeq \frac{\Psi\tilde{\kappa}\xi D^b}{2}\kappa_t^2,$$

where $\kappa_t = (\tilde{\kappa}_t - \tilde{\kappa})/\tilde{\kappa}$. Consequently, up to the second order, the resource constraint becomes

$$(Y_t - Y) - \frac{\Psi\tilde{\kappa}\xi D^b}{2}\kappa_t^2 = \xi(C_t^b - C^b) + (1-\xi)(C_t^s - C^s),$$

which gives us a second order approximation of aggregate consumption in terms of aggregate output and adjustment costs

$$\xi(C_t^b - C^b) + (1-\xi)(C_t^s - C^s) = Y(y_t + \frac{1}{2}y_t^2) - \frac{\Psi\tilde{\kappa}\xi D^b}{2}\kappa_t^2. \tag{33}$$

Similarly, for the housing market resource constraint

$$H = \xi H_t^b + (1-\xi)H_t^s,$$

we have

$$0 = \xi(H_t^b - H^b) + (1-\xi)(H_t^s - H^s), \tag{34}$$

so that the second term of the first line of (32) disappears.

Going back to our approximation, we can rewrite

$$\begin{aligned} U_t - U \simeq\ & \mu_1 Y(y_t + \frac{1}{2}y_t^2) - \mu_1\frac{\Psi\tilde{\kappa}\xi D^b}{2}\kappa_t^2 \\ & - \frac{1}{2}\mu_1 z\left[\xi(C_t^b - C^b)^2 + (1-\xi)(C_t^s - C^s)^2\right] \\ & - \frac{1}{2}\mu_2\frac{\sigma_h}{H}\left[\xi(H_t^b - H^b)^2 + (1-\xi)(H_t^s - H^s)^2\right] \\ & - \mu_1\xi\frac{Y}{L^b}[(L_t^b - L^b) + \frac{1}{2}\frac{\varphi}{L^b}(L_t^b - L^b)^2] - \mu_1(1-\xi)\frac{Y}{L^s}[(L_t^s - L^s) + \frac{1}{2}\frac{\varphi}{L^s}(L_t^s - L^s)^2]. \end{aligned} \tag{35}$$

Using the coefficients defined above, we can further rearrange the previous expression as

$$\begin{aligned} U_t - U \simeq\ & \mu_1 Y[y_t - \xi l_t^b - (1-\xi)l_t^s] \\ & + \frac{1}{2}\mu_1 Y y_t^2 - \frac{1}{2}\mu_1\Psi\tilde{\kappa}\xi D^b\kappa_t^2 \\ & - \frac{1}{2}\mu_1\sigma Y\left[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2\right] - \frac{1}{2}\mu_2\sigma_h H\left[\xi(h_t^b)^2 + (1-\xi)(h_t^s)^2\right] \\ & - \frac{1}{2}\mu_1\xi Y(1+\varphi)(l_t^b)^2 - \frac{1}{2}\mu_1(1-\xi)Y(1+\varphi)(l_t^s)^2. \end{aligned} \tag{36}$$

Next, we focus on eliminating the first-order terms left in the approximation. From the aggregate production function derived in section A.4 we have

$$\hat{\Delta}_t + y_t = a_t + \xi l_t^b + (1-\xi) l_t^s,$$

where $\hat{\Delta}_t \equiv \Delta_t - 1$, since there is no price dispersion in steady state. Replacing from this equation for the difference between output and the weighted average of two types' labour supply, we can write

$$\begin{aligned} U_t - U \simeq\ & \frac{1}{2}\mu_1 Y y_t^2 - \frac{1}{2}\mu_1 \Psi \tilde{\kappa} \xi D^b \kappa_t^2 - \mu_1 Y \hat{\Delta}_t \\ & - \frac{1}{2}\mu_1 \sigma Y \left[\xi (c_t^b)^2 + (1-\xi)(c_t^s)^2\right] - \frac{1}{2}\mu_2 \sigma_h H \left[\xi (h_t^b)^2 + (1-\xi)(h_t^s)^2\right] \\ & - \frac{1}{2}\mu_1 \xi Y (1+\varphi)(l_t^b)^2 - \frac{1}{2}\mu_1 (1-\xi) Y (1+\varphi)(l_t^s)^2, \end{aligned}$$

where we have dropped the term in productivity coming from the production function because it is independent of policy.

At this point, the welfare objective is fully quadratic.[31] However, we can further manipulate the approximation to obtain terms that have a more meaningful economic interpretation. To this end, we combine the terms in output, consumption

$$\begin{aligned} U_t - U \simeq & -\frac{1}{2}\mu_1 Y \left\{\sigma[\xi (c_t^b)^2 + (1-\xi)(c_t^s)^2] - y_t^2 + (1+\varphi)[\xi (l_t^b)^2 + (1-\xi)(l_t^s)^2]\right\} \\ & - \frac{1}{2}\mu_2 \sigma_h H \left[\xi (h_t^b)^2 + (1-\xi)(h_t^s)^2\right] \\ & - \mu_1 Y \hat{\Delta}_t - \frac{1}{2}\mu_1 \Psi \tilde{\kappa} \xi D^b \kappa_t^2. \end{aligned} \tag{37}$$

Let us focus on the first line of the right-hand side of (37). Adding and subtracting $(\varphi+\sigma) y_t^2$, we can write

$$\begin{aligned} & \sigma[\xi (c_t^b)^2 + (1-\xi)(c_t^s)^2] - y_t^2 + (1+\varphi)[\xi (l_t^b)^2 + (1-\xi)(l_t^s)^2] = \\ & \qquad \sigma[\xi (c_t^b)^2 + (1-\xi)(c_t^s)^2] - y_t^2 + (\varphi+\sigma) y_t^2 - (\varphi+\sigma) y_t^2 + (1+\varphi)[\xi (l_t^b)^2 + (1-\xi)(l_t^s)^2]. \end{aligned}$$

We can take σy_t^2 inside the consumption terms and $(1+\varphi) y_t^2$ inside the labour terms to write

$$\begin{aligned} & \sigma[\xi (c_t^b)^2 + (1-\xi)(c_t^s)^2] - y_t^2 + (1+\varphi)[\xi (l_t^b)^2 + (1-\xi)(l_t^s)^2] = \\ & \qquad (\varphi+\sigma) y_t^2 + \sigma[\xi (c_t^b)^2 + (1-\xi)(c_t^s)^2 - y_t^2] + (1+\varphi)[\xi (l_t^b)^2 + (1-\xi)(l_t^s)^2 - y_t^2]. \end{aligned} \tag{38}$$

[31] As we will show formally below, Δ_t is a price dispersion index, hence it is a term of order two.

Now we work with the second term of the right-hand side of (38), which we can write as

$$\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2 - y_t^2 = \xi[(c_t^b)^2 - y_t^2] + (1-\xi)[(c_t^s)^2 - y_t^2]$$
$$= \xi(c_t^b + y_t)(c_t^b - y_t) + (1-\xi)(c_t^s + y_t)(c_t^s - y_t).$$

We use again the resource constraint to replace the difference between each type's consumption and output

$$\begin{aligned}\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2 - y_t^2 &= \xi(c_t^b + y_t)(1-\xi)(c_t^b - c_t^s) - (1-\xi)(c_t^s + y_t)\xi(c_t^b - c_t^s)\\ &= \xi(1-\xi)(c_t^b - c_t^s)[(c_t^b + y_t) - (c_t^s + y_t)]\\ &= \xi(1-\xi)(c_t^b - c_t^s)^2. \qquad (39)\end{aligned}$$

Before moving on, we need to take an approximation of the labour supply conditions

$$\frac{\chi_L^b (L_t^b)^{1+\varphi}}{ze^{-zC_t^b}} = W_t^b L_t^b \qquad \text{and} \qquad \frac{\chi_L^s (L_t^s)^{1+\varphi}}{ze^{-zC_t^s}} = W_t^s L_t^s.$$

Taking a geometric average of the above two conditions, with weights reflecting the two types' shares, we get an aggregate labour supply condition of the form

$$\frac{\chi_L (L_t)^{1+\varphi}}{ze^{-z(Y_t - \Gamma_t)}} = W_t L_t.$$

Substituting on the left-hand side for aggregate employment from the aggregate production function, we can write

$$\frac{\chi_L}{\xi^\xi (1-\xi)^{1-\xi}} \left(\frac{\Delta_t Y_t}{A_t}\right)^{1+\varphi} \frac{1}{ze^{-z(Y_t - \Gamma_t)}} = W_t L_t.$$

As we proved in section A.4, $W_t^j L_t^j = W_t L_t$ for $j = \{b, s\}$. Therefore, we can write

$$\frac{\chi_L}{\xi^\xi (1-\xi)^{1-\xi}} \left(\frac{\Delta_t Y_t}{A_t}\right)^{1+\varphi} \frac{1}{ze^{-z(Y_t - \Gamma_t)}} = \frac{\chi_L^b (L_t^b)^{1+\varphi}}{ze^{-zC_t^b}} = \frac{\chi_L^s (L_t^s)^{1+\varphi}}{ze^{-zC_t^s}}.$$

Approximating these two conditions and solving for each type's labour supply, we obtain

$$\begin{aligned} l_t^b &= \hat{\Delta}_t + y_t - a_t - \frac{\sigma}{1+\varphi}(c_t^b - y_t) \\ l_t^s &= \hat{\Delta}_t + y_t - a_t - \frac{\sigma}{1+\varphi}(c_t^s - y_t). \end{aligned}$$

Using the first order approximation of the resource constraint, we can rewrite the two conditions above as

$$\begin{aligned} l_t^b &= \hat{\Delta}_t + y_t - a_t - \frac{\sigma}{1+\varphi}(1-\xi)(c_t^b - c_t^s) \\ l_t^s &= \hat{\Delta}_t + y_t - a_t + \frac{\sigma}{1+\varphi}\xi(c_t^b - c_t^s). \end{aligned}$$

Now we can move on to the third term of the right-hand side of (38) substituting out labour supply of the two types.

$$\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2 - y_t^2 = \\ \xi\left[\hat{\Delta}_t + y_t - a_t - \frac{\sigma}{1+\varphi}(1-\xi)(c_t^b - c_t^s)\right]^2 + (1-\xi)\left[\hat{\Delta}_t + y_t - a_t + \frac{\sigma}{1+\varphi}\xi(c_t^b - c_t^s)\right]^2 - y_t^2.$$

We expand the two squared terms on the right-hand side of the last equation isolating the terms in the consumption gap

$$\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2 - y_t^2 = \\ \xi(y_t - a_t)^2 + \xi\left[\frac{\sigma}{1+\varphi}(1-\xi)(c_t^b - c_t^s)\right]^2 - 2\xi(y_t - a_t)\frac{\sigma}{1+\varphi}(1-\xi)(c_t^b - c_t^s) \\ + (1-\xi)(y_t - a_t)^2 + (1-\xi)\left[\frac{\sigma}{1+\varphi}\xi(c_t^b - c_t^s)\right]^2 + 2(1-\xi)(y_t - a_t)\frac{\sigma}{1+\varphi}\xi(c_t^b - c_t^s) - y_t^2,$$

where the term $\hat{\Delta}_t$ drops out because it is of order two, hence its square and its product with first order terms is irrelevant for welfare up to the second order.

We can now combine terms and simplify to obtain

$$\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2 - y_t^2 = (y_t - a_t)^2 + \xi(1-\xi)\left[\frac{\sigma}{1+\varphi}(c_t^b - c_t^s)\right]^2 - y_t^2, \tag{40}$$

We replace (39) and (40) into (38)

$$\sigma[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2] - y_t^2 + (1+\varphi)[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2] \\ = (\varphi+\sigma)y_t^2 + \sigma\xi(1-\xi)(c_t^b - c_t^s)^2 + (1+\varphi)\left\{(y_t - a_t)^2 + \xi(1-\xi)\left[\frac{\sigma}{1+\varphi}(c_t^b - c_t^s)\right]^2 - y_t^2\right\} \\ = (\sigma-1)y_t^2 + \xi(1-\xi)\sigma\left(\frac{1+\sigma+\varphi}{1+\varphi}\right)(c_t^b - c_t^s)^2 + (1+\varphi)(y_t - a_t)^2.$$

Expanding the last term on the right-hand side and combining it with the first, we can write

$$\sigma[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2] - y_t^2 + (1+\varphi)[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2] \\ = (\sigma+\varphi)y_t^2 - 2(1+\varphi)a_t y_t + \xi(1-\xi)\sigma\left(\frac{1+\sigma+\varphi}{1+\varphi}\right)(c_t^b - c_t^s)^2.$$

A first-order approximation of (29) gives us an expression for efficient output in terms of productivity

$$y_t^* = \frac{1+\varphi}{\sigma+\varphi}a_t.$$

Replacing on the right-hand side of the equation above we have

$$\sigma[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2] - y_t^2 + (1+\varphi)[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2] \\ = (\sigma+\varphi)(y_t^2 - 2y_t^* y_t) + \xi(1-\xi)\sigma\left(\frac{1+\sigma+\varphi}{1+\varphi}\right)(c_t^b - c_t^s)^2.$$

Since efficient output is independent of policy, we can add and subtract it from the right-hand side to obtain

$$\begin{aligned}\sigma[\xi(c_t^b)^2 + (1-\xi)(c_t^s)^2] - y_t^2 + (1+\varphi)[\xi(l_t^b)^2 + (1-\xi)(l_t^s)^2] \\ &= (\sigma+\varphi)(y_t - y_t^*)^2 + \xi(1-\xi)\sigma\left(\frac{1+\sigma+\varphi}{1+\varphi}\right)(c_t^b - c_t^s)^2 \\ &= (\sigma+\varphi)x_t^2 + \xi(1-\xi)\sigma\left(\frac{1+\sigma+\varphi}{1+\varphi}\right)(c_t^b - c_t^s)^2,\end{aligned}$$

where $x_t \equiv y_t - y_t^*$ is the efficient output gap.

We now go back to (37) and substitute the result we have just derived to obtain

$$\begin{aligned}U_t - U \simeq & -\frac{1}{2}\mu_1 Y\left[(\sigma+\varphi)x_t^2 + \xi(1-\xi)\sigma\left(\frac{1+\sigma+\varphi}{1+\varphi}\right)(c_t^b - c_t^s)^2\right] \\ & -\frac{1}{2}\mu_2\sigma_h H\left[\xi(h_t^b)^2 + (1-\xi)(h_t^s)^2\right] \\ & -\mu_1 Y\hat{\Delta}_t - \frac{1}{2}\mu_1\Psi\tilde{\kappa}\xi D^b\kappa_t^2. \qquad (41)\end{aligned}$$

Next, we work with the second line of (41). From the linear approximation of the housing market clearing condition, we have

$$\begin{aligned}h_t^b &= -(1-\xi)(h_t^b - h_t^s) \\ h_t^s &= \xi(h_t^b - h_t^s).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\xi(h_t^b)^2 + (1-\xi)(h_t^s)^2 &= \xi(1-\xi)^2(h_t^b - h_t^s)^2 + (1-\xi)\xi^2(h_t^b - h_t^s)^2 \\ &= \xi(1-\xi)(h_t^b - h_t^s)^2.\end{aligned}$$

Notice also that the coefficient multiplying the housing term is $\mu_2 H$. Using the conditions for the efficient steady state, we can rewrite

$$\mu_2 H = \mu_1 Y\frac{\mu_2 H}{\mu_1 Y} = \mu_1 Y\frac{U_h^j}{U_c^j}\frac{H}{Y}.$$

We can choose the housing utility parameters χ_H^j so that

$$\frac{U_h^j}{U_c^j}\frac{H}{Y} = 1.$$

Substituting back into the welfare approximation, we arrive at

$$\begin{aligned} U_t - U \simeq & -\frac{1}{2}\mu_1 Y \left[(\sigma+\varphi)x_t^2 + \xi(1-\xi)\sigma\left(\frac{1+\sigma+\varphi}{1+\varphi}\right)(c_t^b - c_t^s)^2 + \xi(1-\xi)\sigma_h (h_t^b - h_t^s)^2 \right] \\ & - \mu_1 Y \hat{\Delta}_t - \frac{1}{2}\mu_1 \Psi \tilde{\kappa} \xi D^b \kappa_t^2. \end{aligned} \tag{42}$$

Lastly, we take a second order approximation of the price dispersion index, which yields

$$\hat{\Delta}_t = \lambda \hat{\Delta}_{t-1} + \frac{1}{2}\frac{\lambda\varepsilon}{1-\lambda}\pi_t^2.$$

Solving the previous difference equation backward, we have

$$\hat{\Delta}_t = \lambda \hat{\Delta}_{-1} + \frac{1}{2}\frac{\lambda\varepsilon}{1-\lambda}\sum_{j=0}^{t}\lambda^{t-j}\pi_j^2,$$

for some initial level of price dispersion $\hat{\Delta}_{-1}$. We are interested in the present discounted value of the previous expression, that is

$$\sum_{t=0}^{\infty}\beta_s^t \hat{\Delta}_t = \frac{1}{2}\frac{\lambda\varepsilon}{1-\lambda}\sum_{t=0}^{\infty}\beta_s^t \sum_{j=0}^{t}\lambda^{t-j}\pi_j^2,$$

where we have dropped the initial level of price dispersion as it is independent of policy. Let us now focus on the double sum on the right-hand side of the last expression, which we can expand to obtain

$$\begin{aligned} \sum_{t=0}^{\infty}\beta_s^t \sum_{j=0}^{t}\lambda^{t-j}\pi_j^2 &= \pi_0^2 + \beta_s(\lambda\pi_0^2 + \pi_1^2) + \beta_s^2(\lambda^2\pi_0^2 + \lambda\pi_1^2 + \pi_2^2) + ... \\ &= \sum_{j=0}^{\infty}(\beta_s\lambda)^j\pi_0^2 + \beta_s\sum_{j=0}^{\infty}(\beta_s\lambda)^j\pi_1^2 + \beta_s^2\sum_{j=0}^{\infty}(\beta_s\lambda)^j\pi_2^2 + ... \\ &= \pi_0^2\sum_{j=0}^{\infty}(\beta_s\lambda)^j + \beta_s\pi_1^2\sum_{j=0}^{\infty}(\beta_s\lambda)^j + \beta_s^2\pi_2^2\sum_{j=0}^{\infty}(\beta_s\lambda)^j + ... \\ &= \sum_{t=0}^{\infty}\beta_s^t\pi_t^2\sum_{j=0}^{\infty}(\beta_s\lambda)^j = \frac{1}{1-\beta_s\lambda}\sum_{t=0}^{\infty}\beta_s^t\pi_t^2 \end{aligned}$$

Therefore, we can write

$$\sum_{t=0}^{\infty}\beta_s^t \hat{\Delta}_t = \frac{1}{2}\frac{\lambda\varepsilon}{(1-\lambda)(1-\beta_s\lambda)}\sum_{t=0}^{\infty}\beta_s^t\pi_t^2.$$

Substituting back into the approximation of utility, we obtain

$$U_t - U \simeq -\frac{1}{2}\mu_1 Y \left[(\sigma+\varphi)x_t^2 + \xi(1-\xi)\sigma\left(\frac{1+\sigma+\varphi}{1+\varphi}\right)(c_t^b - c_t^s)^2 + \xi(1-\xi)\sigma_h(h_t^b - h_t^s)^2 \right. \\ \left. + \frac{\lambda\varepsilon}{(1-\lambda)(1-\beta_s\lambda)}\pi_t^2 + \psi\eta\kappa_t^2\right], \tag{43}$$

where $\psi \equiv \Psi\tilde{\kappa}$ is the semi-elasticity of the borrowing rate to capital requirements and $\eta \equiv \xi D^b/Y$ is the aggregate debt-to-GDP ratio.

Therefore, up to the second order and ignoring terms independent of policy, we can rewrite the welfare objective as

$$\mathbb{W}_0 \simeq -\frac{\Omega}{2}\mathbb{E}_0\sum_{t=0}^{\infty}\beta_s^t\left(x_t^2 + \lambda_\pi\pi_t^2 + \lambda_\kappa\kappa_t^2 + \lambda_c\tilde{c}_t^2 + \lambda_h\tilde{h}_t^2\right) \tag{44}$$

where $\tilde{c}_t \equiv c_t^b - c_t^s$ and $\tilde{h}_t \equiv h_t^b - h_t^s$ are the consumption and housing gaps, respectively. The composite parameters in the loss function are

$$\begin{aligned} \Omega &\equiv \mu_1 Y(\sigma+\varphi) \\ \lambda_\pi &\equiv \frac{\lambda\varepsilon}{(1-\lambda)(1-\beta_s\lambda)(\sigma+\varphi)} \\ \lambda_\kappa &\equiv \frac{\psi\eta}{\sigma+\varphi} \\ \lambda_c &\equiv \frac{\xi(1-\xi)\sigma(1+\sigma+\varphi)}{(1+\varphi)(\sigma+\varphi)} \\ \lambda_h &\equiv \frac{\xi(1-\xi)\sigma_h}{\sigma+\varphi}. \end{aligned}$$

Observe that the higher is σ, the greater the weight on the output and consumption gap terms, though the weight on the consumption gap grows quadratically in σ, whereas the weight on the output gap is linear in σ; the higher is φ, the greater the weight on the aggregate output gap, and the smaller the weight on the consumption gap.

To give a rough idea of magnitudes, take $\sigma = \varphi = 1$ as a baseline case. Then, the relative weight on the consumption gap is $\lambda_c = 3\xi(1-\xi)/4$. Since $\xi(1-\xi)$ reaches a maximum of $1/4$, the maximum relative weight on the consumption gap is $3/16$. In general, the policymaker will attribute more weight to the volatility of aggregate output than to the volatility of relative consumption. Nevertheless, for a given policy, the latter may be large, thus becoming a significant source of welfare costs. The ability to use multiple policy instruments to deal with different tradeoffs should mitigate these costs.

D Linearised Constraints

In this section, we derive a first-order approximation of the equilibrium conditions that constitute the constraints for the optimal policy problem in our linear-quadratic setting. Unless otherwise stated, lower-case variables denote log-deviations from steady state, that is, for a generic variable X_t with steady state value Z, $Z_t \equiv \ln(Z_t/Z)$.

D.1 Savers

The Euler equation for savers is

$$c_t^s = -\sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + \mathbb{E}_t c_{t+1}^s + u_t^c, \tag{45}$$

where $i_t \equiv \ln R_t^d$ is the net nominal interest rate on deposits, $\sigma \equiv zY$ is the inverse of the elasticity of intertemporal substitution, and $u_t^c = \rho_c u_{t-1}^c + \epsilon_t^c$ is an aggregate demand shock that also affects the borrowers' Euler equation, with $\rho_c \in (0,1)$ and $\epsilon_t^c \sim \mathcal{N}(0, \sigma_c^2)$. A similar condition applies to equity investment, taking into account portfolio adjustment costs. Up to the first order, no arbitrage therefore implies

$$i_t^e = i_t + \Psi \kappa_t. \tag{46}$$

The labour supply condition for savers is

$$w_t^s = \varphi l_t^s + \sigma c_t^s,$$

where w_t^s is the log-deviation of the savers' real wage.

Demand for housing by savers is

$$q_t = \frac{1 + \tau^h - \beta_s}{1 + \tau^h}(u_t^h - \sigma_h h_t^s + \sigma c_t^s) + \frac{\beta_s}{1 + \tau^h}\mathbb{E}_t(-\sigma c_{t+1}^s + \sigma c_t^s + q_{t+1}), \tag{47}$$

where $\tau^h = \beta_s - \beta_b - \tilde{\mu}\Theta$ (with $\tilde{\mu} = 1 - \beta_b/\beta_s$) is a tax that makes the steady state allocation of housing efficient, q_t is the log-deviation of the real price of housing from its steady state value, and $u_t^h = \rho_h u_{t-1}^h + \epsilon_t^h$ is a housing demand shock, with $\rho^h \in (0,1)$ and $\epsilon_t^h \sim \mathcal{N}(0, \sigma_h^2)$.

D.2 Borrowers

The Euler equation for borrowers takes into account the effect of the collateral constraint

$$c_t^b = -\sigma^{-1}\left(i_t^b - \mathbb{E}_t \pi_{t+1} + \frac{\tilde{\mu}}{1 - \tilde{\mu}}\mu_t\right) + \mathbb{E}_t c_{t+1}^b + u_t^c, \tag{48}$$

where $i_t^b \equiv \ln R_t^b$ is the net nominal interest rate faced by borrowers. Note that, everything else equal, a tightening of the collateral constraint ($\mu_t > 0$) tends to raise the cost of borrowing for impatient households.

Labour supply for borrowers is

$$w_t^b = \varphi l_t^b + \sigma c_t^b,$$

where w_t^b is the log-deviation of the borrowers' real wage.

Borrowers' demand for housing is

$$q_t = \frac{\tilde{\mu}\Theta}{1-\tilde{\mu}\Theta}(\mu_t + \theta_t) + \frac{1-\tilde{\mu}\Theta-\beta_b}{1-\tilde{\mu}\Theta}(u_t^h - \sigma_h h_t^b + \sigma c_t^b) + \frac{\beta_b}{1-\tilde{\mu}\Theta}\mathbb{E}_t(-\sigma c_{t+1}^b + \sigma c_t^b + q_{t+1}), \quad (49)$$

where θ_t, which can be either a shock or a macro-prudential policy instrument, is the log-deviation of the collateral constraint parameter (the LTV ratio) from its steady state value.

The linearized borrowing constraint at equality is

$$d_t^b = \theta_t + q_t + h_t^b, \quad (50)$$

where d_t^b denotes the log-deviation of the real quantity of debt from its steady state value.

Finally, from the borrowers' budget constraint, we can derive the law of motion for debt as

$$d_t^b = R^b(i_{t-1}^b + d_{t-1}^b - \pi_t) + \frac{1}{\Theta}(h_t^b - h_{t-1}^b) + \frac{1}{\eta}(c_t^b - y_t), \quad (51)$$

where $\eta \equiv (D^b/P)/Y$ represents the steady state real household debt-to-GDP ratio.

D.3 Banks

Banks price loans as a weighted average between the return on equity and the deposit rate

$$i_t^b = \tilde{\kappa} i_t^e + (1-\tilde{\kappa}) i_t.$$

Using the no arbitrage condition between return on equity and on deposits from the savers' problem (46), we obtain an expression for the spread of the the loan rate on the deposit rate

$$i_t^b = i_t + \psi \kappa_t, \quad (52)$$

where $\psi \equiv \xi\tilde{\kappa}\Psi/(1-\xi)$ and κ_t is the log-deviation of the capital requirement from its steady state value, which, like θ_t, can be either a shock or a macro-prudential policy instrument.

D.4 Production

Up to a linear approximation, the production function is

$$y_t = a_t + l_t.$$

The labour demand equation is

$$w_t = m_t + y_t - l_t,$$

where m_t is the real marginal cost. The wage bill must be equal across types

$$w_t^s + l_t^s = w_t^b + l_t^b,$$

where the wage index is

$$w_t = \xi w_t^b + (1-\xi) w_t^s$$

and labour market clearing requires

$$l_t = \xi l_t^b + (1-\xi) l_t^s.$$

Finally, the Phillips Curve is

$$\pi_t = \frac{(1-\lambda)(1-\beta\lambda)}{\lambda} m_t + \beta \mathbb{E}_t \pi_{t+1} + u_t^m,$$

where $u_t^m = \rho_m u_{t-1}^m + \epsilon_t^m$ is a mark-up shock, with $\rho_m \in (0,1)$ and $\epsilon_t^m \sim \mathcal{N}(0, \sigma_m^2)$.

D.5 Market Clearing

Goods market clearing entails

$$y_t = \xi c_t^b + (1-\xi) c_t^s, \tag{53}$$

while housing market clearing requires

$$\xi h_t^b + (1-\xi) h_t^s = 0. \tag{54}$$

The market clearing conditions complete the description of the linearised model.

D.6 Gaps and Aggregate Variables

In what follows, we combine the equilibrium relations to obtain a parsimonious set of constraints for the optimal policy problem. To simplify the derivations, we assume $\Theta = 1$ (a 100% LTV ratio). We return to the case $\Theta < 1$ in the quantitative analysis.

On the supply side, we can rewrite the Phillips curve in terms of the efficient output gap by noting that, with flexible prices and no markup shocks, $m_t = 0$. Weighting the labour supply equations by population shares, we can write the equilibrium in the labour market as

$$l_t^* + \sigma c_t^* = a_t,$$

where an * represents variables in the efficient equilibrium. Using the production function and the resource constraint, we can solve for the efficient level of output

$$y_t^* = \frac{1+\varphi}{\sigma+\varphi} a_t.$$

With sticky prices, the labour market equilibrium condition, expressed in terms of output, is

$$m_t = (\sigma + \varphi)y_t - (1 + \varphi)a_t = (\sigma + \varphi)(y_t - y_t^*).$$

Replacing into the Phillips curve gives

$$\pi_t = \gamma x_t + \beta \mathbb{E}_t \pi_{t+1} + u_t^m, \tag{55}$$

where $\gamma \equiv (\sigma + \varphi)(1 - \lambda)(1 - \beta\lambda)/\lambda$ and $x_t \equiv y_t - y_t^*$ is the efficient output gap.

On the demand side, we start from the savers' Euler equation (45) and replace savers' consumption from the resource constraint (53) to obtain

$$y_t - \xi \tilde{c}_t = -\sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + \mathbb{E}_t(y_{t+1} - \xi \tilde{c}_{t+1}) + u_t^c.$$

We can express the last equation in terms of output gap

$$x_t - \xi \tilde{c}_t = -\sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + \mathbb{E}_t(x_{t+1} - \tilde{c}_{t+1}) + \nu_t^c, \tag{56}$$

where

$$\nu_t^c \equiv u_t^c + \mathbb{E}_t y_{t+1}^* - y_t^*.$$

The second condition that characterises aggregate demand comes from substituting housing demand for borrowers from the market clearing condition (54) into the borrowing constraint equality (50)

$$d_t^b = \theta_t + q_t + (1 - \xi)\tilde{h}_t. \tag{57}$$

Finally, we can replace the goods and housing market resource constraints, and the banking condition (52) into the borrowers' budget constraint (51) to obtain an equation for the law of motion of debt

$$d_t^b = \frac{1}{\beta_s}(i_{t-1} + \psi\kappa_{t-1} + d_{t-1}^b - \pi_t) + (1 - \xi)(\tilde{h}_t - \tilde{h}_{t-1}) + \frac{1 - \xi}{\eta}\tilde{c}_t. \tag{58}$$

For the housing gap, we take the difference of the two housing demand functions and eliminate the multiplier on the borrowing constraint from borrowers' Euler equation to obtain

$$\begin{aligned}(1 - \tilde{\mu} - \beta_b)\sigma_h \tilde{h}_t = {} & -\sigma(1 - \tilde{\mu})\mathbb{E}_t(c_t^b - c_{t+1}^b) - (1 - \tilde{\mu})(i_t^{b*} - \mathbb{E}_t \pi_{t+1} + u_t^c) + \tilde{\mu}\theta_t \\ & + (1 - \tilde{\mu} - \beta_b)\sigma\tilde{c}_t + (\beta_s - \beta_b)(q_t - \mathbb{E}_t q_{t+1}) + \beta_b \mathbb{E}_t \sigma(c_t^b - c_{t+1}^b) - \beta_s \mathbb{E}_t \sigma(c_t^s - c_{t+1}^s).\end{aligned}$$

We can use the savers' Euler equation to eliminate the last term of the previous expression, as well as the banks' zero profit condition to get rid of the borrowing rate. After substituting for borrowers' consumption from the resource constraint and using again the savers' Euler equation, some manipulations allow us to write

$$\tilde{h}_t = -\frac{\omega - \xi(\beta_s - \beta_b)}{\sigma_h \xi \omega}(i_t - \mathbb{E}_t \pi_{t+1}) + \frac{\beta_s - \beta_b}{\sigma_h \omega}(q_t - \mathbb{E}_t q_{t+1}) - \frac{\sigma}{\sigma_h \xi}(x_t - \mathbb{E}_t x_{t+1}) + \frac{\sigma}{\sigma_h}\tilde{c}_t + \frac{\tilde{\mu}}{\sigma_h \omega}\theta_t - \frac{1-\tilde{\mu}}{\sigma_h \omega}\psi \kappa_t + \nu_t^h, \quad (59)$$

where

$$\nu_t^h \equiv -\frac{\sigma}{\sigma_h \xi}(y_t^* - \mathbb{E}_t y_{t+1}^*) + \frac{\omega - \xi(\beta_s - \beta_b)}{\sigma_h \xi \omega}\sigma u_t^c,$$

and $\omega \equiv 1 - \tilde{\mu} - \beta_b = \beta_b(1/\beta_s - 1) > 0$.

To complete the description of the demand side, we start by taking a population-weighted average of the two housing demand equations. We then use the goods and housing market equilibrium conditions to get

$$(\omega+\beta)q_t = \xi\tilde{\mu}(\mu_t+\theta_t)+\omega\left(\sigma y_t + u_t^h\right)+\xi\beta_b[\sigma(c_t^b-\mathbb{E}_t c_{t+1}^b)+\mathbb{E}_t q_{t+1}]+(1-\xi)\beta_s[\sigma(c_t^s-\mathbb{E}_t c_{t+1}^s)+\mathbb{E}_t q_{t+1}].$$

If we use the borrowers' Euler equation to eliminate the Lagrange multiplier, replace borrowers' consumption from the goods market equilibrium, savers' consumption from their Euler equation, and the borrowing rate from the banks' zero profit condition, we obtain an aggregate house price equation that reads as

$$q_t = -(i_t - \mathbb{E}_t \pi_{t+1}) + \frac{\sigma\omega}{\omega+\beta}\mathbb{E}_t x_{t+1} + \frac{\xi\tilde{\mu}}{\omega+\beta}\theta_t - \frac{\xi(1-\tilde{\mu})}{\omega+\beta}\psi\kappa_t + \frac{\beta}{\omega+\beta}\mathbb{E}_t q_{t+1} + \nu_t^q \quad (60)$$

where

$$\nu_t^q \equiv \frac{\omega}{\omega+\beta}u_t^h + \sigma u_t^c + \frac{\sigma\omega}{\omega+\beta}\mathbb{E}_t y_{t+1}^*.$$

Given a specification of monetary and macro-prudential policy $\{i_t, \theta_t, \kappa_t\}$, equations (55)-(60), constitute a system of six equations in six unknowns (x_t, π_t, $c_t^b - c_t^s$, $h_t^b - h_t^s$, q_t, and d_t^b) that characterises the equilibrium.

E Optimal Macro-Prudential Policy

In this section, we study optimal macro-prudential policy under discretion in the linear-quadratic approximation that we have derived so far.

The definition of "optimal discretion" that we use requires the following protocol:

- At the start of each period nature reveals the values of all shocks.
- Today's policymaker acts as a Stackelberg leader with respect to (a) private agents today; (b) future policymakers and private agents. The policymaker acts after shocks are revealed, so takes them as given.

- Today's policymaker is constrained by the behaviour of private agents today: outcomes today must be compatible with the decisions of private agents. The policymaker is not constrained by the decisions of private agents in the future (that is a problem for future policymakers).

- Today's policymaker recognises that future policymakers will act according to a feedback rule that determines outcomes (for the endogenous variables). That feedback rule is Markovian in the sense that it determines outcomes as a function of the minimum number of variables that can determine equilibrium in period t: the state vector plus the value of today's shocks. Today's policymaker takes the feedback rule as given.

- We use the first order conditions of the policymaker's problem to solve for a fixed point: the future policymaker's feedback rule and today's decisions have the same form.

This definition of discretion is consistent with the implementation of the numerical simulations.

E.1 Flexible Prices

We begin with the case of flexible prices and real debt. In an efficient equilibrium, flexible prices ($\lambda \to 0$) and no markup shocks ($u_t^m = 0,\ \forall t$) imply, from the Phillips curve, that we have $x_t = 0$, that is, output is solely determined by productivity. Moreover, inflation disappears from the loss function because $\lambda_\pi = 0$. In addition, the nominal interest rate i_t is replaced by the real interest rate r_t, and inflation π_t is also irrelevant for the equilibrium conditions.

The Lagrangian for the optimal policy problem under discretion is

$$
\begin{aligned}
\mathcal{L}(S_{t-1}, \mathbf{v}_t) = & \min_{\{\kappa_t, \tilde{c}_t, \tilde{h}_t, r_t, \theta_t, q_t, S_t\}} \frac{1}{2}(\lambda_\kappa \kappa_t^2 + \lambda_c \tilde{c}_t^2 + \lambda_h \tilde{h}_t^2) + \beta_s \mathbb{E}_t \mathcal{L}(S_t, \mathbf{v}_{t+1}) \\
& - \delta_{xt}(-\xi \tilde{c}_t + \sigma^{-1} r_t + \xi \mathbb{E}_t \tilde{c}_{t+1} - \nu_t^c) \\
& - \delta_{ct} \left[S_t - \beta_s^{-1} S_{t-1} - (1-\xi)(1-\beta_s) \tilde{h}_t - \frac{1-\xi}{\eta} \tilde{c}_t - r_t - \psi \kappa_t \right] \\
& - \delta_{St} \left[S_t - \theta_t - q_t - (1-\xi)(1-\beta_s) \tilde{h}_t - r_t - \psi \kappa_t \right] \\
& - \delta_{qt} \left[q_t + r_t - \frac{\xi \tilde{\mu}}{\omega + \beta} \theta_t + \frac{\xi(1-\tilde{\mu})}{\omega + \beta} \psi \kappa_t - \frac{\beta}{\omega + \beta} \mathbb{E}_t q_{t+1} - \nu_t^q \right] \\
& - \delta_{ht} \left[\tilde{h}_t + \frac{\omega - \xi(\beta_s - \beta_b)}{\sigma_h \xi \omega} r_t - \frac{\beta_s - \beta_b}{\sigma_h \omega} (q_t - \mathbb{E}_t q_{t+1}) - \frac{\sigma}{\sigma_h} \tilde{c}_t - \frac{\tilde{\mu}}{\sigma_h \omega} \theta_t + \frac{1-\tilde{\mu}}{\sigma_h \omega} \psi \kappa_t - \nu_t^h \right],
\end{aligned}
$$

where $\mathbf{v}_t$ is the vector of exogenous shocks. Under discretion, the planner takes future policies as given. With state variables, optimal policy under discretion corresponds to taking expectations of future control variables (in our case, consumption in the Euler equation, and house prices in the house price and housing gap equations) and losses as given, while internalising the effects of current decisions on future losses and expected future variables through the state variable.

The first order conditions for this problem are:

$$0 = \lambda_\kappa \kappa_t + \psi\delta_{ct} + \psi\delta_{St} - \frac{\xi(1-\tilde{\mu})}{\omega+\beta}\psi\delta_{qt} - \frac{1-\tilde{\mu}}{\sigma_h\omega}\psi\delta_{ht} \tag{61}$$

$$0 = \lambda_c \tilde{c}_t + \xi\delta_{xt} + \frac{1-\xi}{\eta}\delta_{ct} + \frac{\sigma}{\sigma_h}\delta_{ht} \tag{62}$$

$$0 = \lambda_h \tilde{h}_t + (1-\beta_s)(1-\xi)\delta_{ct} + (1-\beta_s)(1-\xi)\delta_{St} - \delta_{ht} \tag{63}$$

$$0 = -\sigma^{-1}\delta_{xt} + \delta_{ct} + \delta_{St} - \delta_{qt} - \frac{\omega - \xi(\beta_s - \beta_b)}{\sigma_h\xi\omega}\delta_{ht} \tag{64}$$

$$0 = \delta_{St} + \frac{\xi\tilde{\mu}}{\omega+\beta}\delta_{qt} + \frac{\tilde{\mu}}{\sigma_h\omega}\delta_{ht} \tag{65}$$

$$0 = \delta_{St} - \delta_{qt} + \frac{\beta_s - \beta_b}{\sigma_h\omega}\delta_{ht} \tag{66}$$

$$0 = \beta_s\frac{\partial \mathbb{E}_t\mathcal{L}_{t+1}}{\partial S_t} - \xi\frac{\partial \mathbb{E}_t\tilde{c}_{t+1}}{\partial S_t}\delta_{xt} - \delta_{St} - \delta_{ct} + \frac{\beta}{\omega+\beta}\frac{\partial \mathbb{E}_t q_{t+1}}{\partial S_t}\delta_{qt} - \frac{\beta_s - \beta_b}{\sigma_h\omega}\frac{\partial \mathbb{E}_t q_{t+1}}{\partial S_t}\delta_{ht} \tag{67}$$

where $\mathcal{L}_t \equiv \mathcal{L}(S_{t-1}, \mathbf{v}_t)$. The envelope theorem gives us

$$\frac{\partial \mathcal{L}_t}{\partial S_{t-1}} = \frac{1}{\beta_s}\delta_{ct}. \tag{68}$$

We seek to characterise the optimal policy plan under discretion in terms of a pair of targeting rules. We can think of these two rules as determining, explicitly or implicitly, the optimal level of capital requirements and LTV ratios.

Notice that equations (61)-(66) are static. We begin by manipulating those expressions to derive the first targeting rule. In particular, subtracting (66) from (65) and rearranging, we have

$$\delta_{qt} = \frac{(\omega+\beta)(\beta_s - \beta_b - \tilde{\mu})}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}\delta_{ht}.$$

Replacing the result back into (66) gives

$$\delta_{St} = \left[\frac{(\omega+\beta)(\beta_s - \beta_b - \tilde{\mu})}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})} - \frac{\beta_s - \beta_b}{\sigma_h\omega}\right]\delta_{ht}.$$

Recalling that $\beta = \xi\beta_b + (1-\xi)\beta_s$, we can simplify the coefficient in parenthesis above to obtain

$$\delta_{St} = -\frac{\tilde{\mu}(\omega+\beta_s)}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}\delta_{ht}.$$

Consider now the first order condition (63). Replacing the results for δ_{qt} and δ_{St} obtained above, we can write

$$\delta_{ct} = \left[\frac{1}{(1-\beta_s)(1-\xi)} + \frac{\tilde{\mu}(\omega+\beta_s)}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}\right]\delta_{ht} - \frac{\lambda_h}{(1-\beta_s)(1-\xi)}\tilde{h}_t,$$

or

$$\delta_{ht} = \frac{1}{\zeta_h}\left[\delta_{ct} + \frac{\lambda_h}{(1-\beta_s)(1-\xi)}\tilde{h}_t\right],$$

where

$$\zeta_h \equiv \frac{1}{(1-\beta_s)(1-\xi)} + \frac{\tilde{\mu}(\omega+\beta_s)}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}.$$

Next, we can write (62) as

$$-\delta_{xt} = \frac{\lambda_c}{\xi}\tilde{c}_t + \frac{1-\xi}{\xi\eta}\delta_{ct} + \frac{\sigma}{\sigma_h\xi}\delta_{ht},$$

and (64) as

$$\delta_{xt} = \sigma\delta_{ct} + \sigma\delta_{St} - \sigma\delta_{qt} - \frac{\sigma}{\sigma_h}\frac{\omega - \xi\left(\beta_s - \beta_b\right)}{\xi\omega}\delta_{ht}.$$

Adding the last two equations to each other yields

$$0 = \frac{\lambda_c}{\xi}\tilde{c}_t + \left(\frac{1-\xi}{\xi\eta} + \sigma\right)\delta_{ct} + \frac{\sigma}{\sigma_h\xi}\left[1 - \frac{\omega - \xi\left(\beta_s - \beta_b\right)}{\omega}\right]\delta_{ht} + \sigma\delta_{St} - \sigma\delta_{qt}.$$

The last three terms drop out because of (66). Hence, we can write

$$0 = \frac{\lambda_c}{\xi}\tilde{c}_t + \left(\frac{1-\xi}{\xi\eta} + \sigma\right)\delta_{ct}. \tag{69}$$

Consider now (61). We can replace for δ_{qt} and δ_{St} as a function of δ_{ht} from the results derived above to obtain

$$\frac{\lambda_\kappa}{\psi}\kappa_t + \delta_{ct} - \left[\frac{\tilde{\mu}(\omega+\beta_s)}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})} + \frac{\xi(1-\tilde{\mu})(\beta_s-\beta_b-\tilde{\mu})}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})} + \frac{1-\tilde{\mu}}{\sigma_h\omega}\right]\delta_{ht} = 0.$$

Simplifying the coefficient on δ_{ht} yields

$$\frac{\lambda_\kappa}{\psi}\kappa_t + \delta_{ct} - \frac{\omega+\beta_s}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}\delta_{ht} = 0.$$

Using the equation that relates δ_{ht} to δ_{ct} and $\tilde{h}_t$ derived above, we can rewrite the last expression as

$$\frac{\lambda_\kappa}{\psi}\kappa_t + \delta_{ct} - \frac{\omega+\beta_s}{\zeta_h\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}\left[\delta_{ct} + \frac{1}{(1-\beta_s)(1-\xi)}\tilde{h}_t\right] = 0.$$

Factoring the terms in δ_{ct} and using the expression for ζ_h, we can write

$$\frac{\lambda_\kappa}{\psi}\kappa_t + \tilde{\Phi}_c\delta_{ct} - \tilde{\Phi}_h\tilde{h}_t = 0,$$

where

$$\tilde{\Phi}_c \equiv 1 - \frac{1}{\sigma_h \omega \zeta_h}$$
$$\tilde{\Phi}_h \equiv \frac{\lambda_h(\omega + \beta_s)}{\zeta_h \sigma_h \omega (1-\beta_s)(1-\xi)(\omega + \beta + \xi\tilde{\mu})}.$$

Finally, we can substitute from the relation between δ_{ct} and $\tilde{c}_t$ to obtain

$$\kappa_t = \Phi_c \tilde{c}_t + \Phi_h \tilde{h}_t, \tag{70}$$

where

$$\Phi_c \equiv \frac{\psi}{\lambda_\kappa}\left[\frac{\eta\lambda_c}{1+\xi(\sigma\eta - 1)}\right]\tilde{\Phi}_c,$$

and

$$\Phi_h \equiv \frac{\psi}{\lambda_\kappa}\tilde{\Phi}_h.$$

It is straightforward to see that $\Phi_h > 0$. In order for $\Phi_c > 0$, we can simply check the sign of $\tilde{\Phi}_c$, which corresponds to ensure that its second component is less than one, or alternatively

$$\frac{\sigma_h \omega}{1-\beta_s}(\omega + \beta + \xi\tilde{\mu}) > (1-\tilde{\mu})(1-\xi)(\omega + \beta_s).$$

If $\sigma_h = (1-\beta_s)/\omega = \beta_s/\beta_b \approx 1$, it is easy to check that the inequality above is satisfied. In our baseline calibration for the quantitative experiments, we assume $\sigma_h = 25$. More generally, Φ_c will be positive for most reasonable parameter configurations.

So far we have derived a targeting rule in terms of contemporaneous endogenous variables. To obtain the second targeting rule, we work with (67), which contains forward looking terms.

In our linear-quadratic setting, the Markov-perfect decision rules, and their conditional expectations, are linear functions of the states, which implies

$$\frac{\partial \mathbb{E}_t Z_{t+1}}{\partial S_t} = B_z,$$

where, in our case, $Z_t = \{\tilde{c}_t, q_t\}$, and where B_z denotes the coefficients of the linear relationship. This consideration implies that we can write equation (67) as

$$0 = \mathbb{E}_t \delta_{ct+1} - \xi B_{\tilde{c}} \delta_{xt} - \delta_{St} - \delta_{ct} + \frac{\beta}{\omega + \beta} B_q \delta_{qt} - \frac{\beta_s - \beta_b}{\sigma_h \omega} B_q \delta_{ht},$$

where we have used the envelope condition to substitute out for the value function. Next, we replace for δ_{qt}, δ_{xt} and δ_{St} from the expressions derived above to obtain

$$\delta_{ct} = \mathbb{E}_t\delta_{ct+1} + \left[\frac{\beta(\beta_s-\beta_b-\tilde{\mu})}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})} - \frac{\beta_s-\beta_b}{\sigma_h\omega}\right] B_q\delta_{ht} + \xi B_{\tilde{c}}\left(\frac{\lambda_c}{\xi}\tilde{c}_t + \frac{1-\xi}{\xi\eta}\delta_{ct} + \frac{\sigma}{\sigma_h\xi}\delta_{ht}\right) + \frac{\tilde{\mu}(\omega+\beta_s)}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}\delta_{ht}.$$

We can simplify the coefficient in the square bracket and factor terms together as to obtain

$$\left(1-\frac{1-\xi}{\eta}\right)\delta_{ct} \;=\; \mathbb{E}_t\delta_{ct+1} + \frac{1}{\sigma_h\omega}\left[\sigma\omega B_{\tilde{c}} + \frac{\tilde{\mu}(\omega+\beta_s)}{\omega+\beta+\xi\tilde{\mu}} - \frac{(1+\omega)(\beta_s-\beta_b)}{\omega+\beta+\xi\tilde{\mu}}B_q\right]\delta_{ht} + B_{\tilde{c}}\lambda_c\tilde{c}_t.$$

Finally, we can use the solution for δ_{ht} and δ_{ct} obtained above to write

$$\tilde{c}_t + \Omega_h\tilde{h}_t = \Omega_c\mathbb{E}_t\tilde{c}_{t+1}, \tag{71}$$

where

$$\begin{aligned}
\Omega_h &\equiv \frac{\Xi_h\lambda_h[1+\xi(\sigma\eta-1)]}{\lambda_c[\zeta_h\eta\sigma_h\omega - \Xi_h - (1-\xi)B_{\tilde{c}}]} \\
\Omega_c &\equiv \frac{\zeta_h\eta\sigma_h\omega}{\zeta_h\eta\sigma_h\omega - \Xi_h - (1-\xi)B_{\tilde{c}}} \\
\Xi_h &\equiv \sigma\omega B_{\tilde{c}} + \frac{\tilde{\mu}(\omega+\beta_s)}{\omega+\beta+\xi\tilde{\mu}} - \frac{\tilde{\mu}\beta_s+\omega(\beta_s-\beta_b)}{\omega+\beta+\xi\tilde{\mu}}B_q.
\end{aligned}$$

Note that the three coefficients above depend on $B_{\tilde{c}}$ and B_q, which are unknown. We solve for these coefficients numerically using the method of undetermined coefficients.

E.2 Sticky Prices

With sticky prices, all terms in the loss function (44) are back in place. The Lagrangian for the optimal policy problem under discretion becomes

$$
\begin{aligned}
\mathcal{L}(S_{t-1}, \mathbf{v}_t) = & \min_{\{x_t,\pi_t,\kappa_t,\tilde{c}_t,\tilde{h}_t,r_t,\theta_t,q_t,S_t\}} \frac{1}{2}\left(x_t^2 + \lambda_\pi \pi_t^2 + \lambda_\kappa \kappa_t^2 + \lambda_c \tilde{c}_t^2 + \lambda_h \tilde{h}_t^2\right) + \beta_s \mathbb{E}_t \mathcal{L}(S_t, \mathbf{v}_{t+1}) \\
& - \delta_{xt}\left[x_t - \xi\tilde{c}_t + \sigma^{-1}(i_t - \mathbb{E}_t\pi_{t+1}) - \mathbb{E}_t\left(x_{t+1} - \xi\tilde{c}_{t+1}\right) - \nu_t^c\right] \\
& - \delta_{ct}\left[S_t - \beta_s^{-1}S_{t-1} + \beta_s^{-1}\pi_t - (1-\xi)(1-\beta_s)\tilde{h}_t - \frac{1-\xi}{\eta}\tilde{c}_t - i_t - \psi\kappa_t\right] \\
& - \delta_{St}\left[S_t - \theta_t - q_t - i_t - \psi\kappa_t - (1-\beta_s)(1-\xi)\tilde{h}_t\right] \\
& - \delta_{qt}\left[q_t + (i_t - \mathbb{E}_t\pi_{t+1}) - \frac{\sigma\omega}{\omega+\beta}\mathbb{E}_t x_{t+1} - \frac{\xi\tilde{\mu}}{\omega+\beta}\theta_t + \frac{\xi(1-\tilde{\mu})}{\omega+\beta}\psi\kappa_t - \frac{\beta}{\omega+\beta}\mathbb{E}_t q_{t+1} - \nu_t^q\right] \\
& - \delta_{ht}\left[\tilde{h}_t + \frac{\omega - \xi(\beta_s-\beta_b)}{\sigma_h\xi\omega}(i_t - \mathbb{E}_t\pi_{t+1}) - \frac{\beta_s-\beta_b}{\sigma_h\omega}(q_t - \mathbb{E}_t q_{t+1}) + \frac{\sigma}{\sigma_h\xi}(x_t - \mathbb{E}_t x_{t+1})\right. \\
& \qquad \left. - \frac{\sigma}{\sigma_h}\tilde{c}_t - \frac{\tilde{\mu}}{\sigma_h\omega}\theta_t + \frac{1-\tilde{\mu}}{\sigma_h\omega}\psi\kappa_t - \nu_t^h\right] \\
& - \delta_{\pi t}\left[\pi_t - \gamma x_t - \beta\mathbb{E}_t\pi_{t+1} - \nu_t^\pi\right].
\end{aligned}
$$

The main differences with respect to the case of flexible prices and real debt are the presence of the output gap and the inflation rate in the loss function, and of the nominal interest rate and the inflation rate in the structural equations.

The first order conditions are:

$$0 = x_t - \delta_{xt} - \frac{\sigma}{\sigma_h \xi}\delta_{ht} + \gamma\delta_{\pi t} \tag{72}$$

$$0 = \lambda_\pi \pi_t - \beta_s^{-1}\delta_{ct} - \delta_{\pi t} \tag{73}$$

$$0 = \lambda_\kappa \kappa_t + \psi\delta_{ct} + \psi\delta_{St} - \frac{\xi\left(1-\tilde{\mu}\right)}{\omega+\beta}\psi\delta_{qt} - \frac{1-\tilde{\mu}}{\sigma_h\omega}\psi\delta_{ht} \tag{74}$$

$$0 = \lambda_c \tilde{c}_t + \xi\delta_{xt} + \frac{1-\xi}{\eta}\delta_{ct} + \frac{\sigma}{\sigma_h}\delta_{ht} \tag{75}$$

$$0 = -\sigma^{-1}\delta_{xt} + \delta_{ct} + \delta_{St} - \delta_{qt} - \frac{\omega - \xi\left(\beta_s - \beta_b\right)}{\sigma_h\xi\omega}\delta_{ht} \tag{76}$$

$$0 = \delta_{St} + \frac{\xi\tilde{\mu}}{\omega+\beta}\delta_{qt} + \frac{\tilde{\mu}}{\sigma_h\omega}\delta_{ht} = 0 \tag{77}$$

$$0 = \delta_{St} - \delta_{qt} + \frac{\beta_s - \beta_b}{\sigma_h\omega}\delta_{ht} \tag{78}$$

$$0 = \lambda_h \tilde{h}_t + \left(1-\xi\right)\left(1-\beta_s\right)\delta_{ct} + \left(1-\beta_s\right)\left(1-\xi\right)\delta_{St} - \delta_{ht} \tag{79}$$

$$\begin{aligned} 0 = {} & \beta\frac{\partial \mathbb{E}_t \mathcal{L}_{t+1}}{\partial S_t} + \left(\sigma^{-1}\frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial S_t} + \frac{\partial \mathbb{E}_t x_{t+1}}{\partial S_t} - \xi\frac{\partial \mathbb{E}_t \tilde{c}_{t+1}}{\partial S_t}\right)\delta_{xt} - \delta_{St} - \delta_{ct} \\ & + \left(\frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial S_t} + \frac{\sigma\omega}{\omega+\beta}\frac{\partial \mathbb{E}_t x_{t+1}}{\partial S_t} + \frac{\beta}{\omega+\beta}\frac{\partial \mathbb{E}_t q_{t+1}}{\partial S_t}\right)\delta_{qt} \\ & + \left[\frac{\omega - \xi\left(\beta_s-\beta_b\right)}{\sigma_h\xi\omega}\frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial S_t} - \frac{\beta_s-\beta_b}{\sigma_h\omega}\frac{\partial \mathbb{E}_t q_{t+1}}{\partial S_t} + \frac{\sigma}{\sigma_h\xi}\frac{\partial \mathbb{E}_t x_{t+1}}{\partial S_t}\right]\delta_{ht} + \beta\frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial S_t}\delta_{\pi t} \end{aligned} \tag{80}$$

Like in the case of the efficient equilibrium, the envelope theorem requires

$$\frac{\partial \mathcal{L}_t}{\partial S_{t-1}} = \frac{1}{\beta_s}\delta_{ct}. \tag{81}$$

In order to derive the targeting rules for optimal policy under discretion in the case of sticky prices, we start by noting that the first order conditions (74)-(79) are identical to the case with flexible prices and real debt. Therefore, following the same steps as in the previous section, we can derive a static macro-prudential rule identical to (70).

For the second targeting rule, we start from equation (72),which implies that

$$\delta_{xt} = x_t - \frac{\sigma}{\sigma_h\xi}\delta_{ht} + \gamma\delta_{\pi t}.$$

Substituting this expression into (75) gives

$$\lambda_c\tilde{c}_t + \xi x_t + \frac{1-\xi}{\eta}\delta_{ct} + \xi\gamma\delta_{\pi t} = 0.$$

We can then use (73) to eliminate $\delta_{\pi t}$

$$\lambda_c\tilde{c}_t + \xi x_t + \left[\frac{1-\xi}{\eta} - \frac{\xi\gamma}{\beta_s}\right]\delta_{ct} + \xi\gamma\lambda_\pi\pi_t = 0$$

Since the derivations of the static macro-prudential rule correspond to the flexible-price case, we can use the solution for δ_{ct} in (69) to obtain

$$x_t + \gamma\lambda_\pi \pi_t + \Lambda_c \tilde{c}_t = 0, \tag{82}$$

where

$$\Lambda_c \equiv \frac{\eta\lambda_c(\beta_s\sigma - \gamma)}{\beta_s[1 + \xi(\sigma\eta - 1)]}.$$

For standard calibrations, β_s is close to one, σ bigger or equal than one, and γ is quite small (of the order of 0.1). Therefore, Λ_c will generally be positive.

Equation (82) resembles the standard New Keynesian monetary targeting rule under discretion, but also includes an adjustment for the consumption gap. Since $\Lambda_c > 0$, monetary policy will typically "lean against the wind". Everything else equal, a shock that opens a positive consumption gap would require a negative combination of output and inflation (appropriately weighted), contrary to the standard case in which the same combination should be set equal to zero. Therefore, the monetary authority contributes to financial stability with its interest rate policy.

As in the case of flexible prices, we derive the final targeting rule starting from the first order condition with respect to S_t, which contains forward looking terms. We use again the envelope theorem and the linearity argument to rewrite (80) as

$$\begin{aligned}\delta_{ct} = \mathbb{E}_t\delta_{ct+1} + \left(\sigma^{-1}C_\pi + C_x - \xi C_{\tilde{c}}\right)\delta_{xt} - \delta_{St} + \left(C_\pi + \frac{\sigma\omega}{\omega+\beta}C_x + \frac{\beta}{\omega+\beta}C_q\right)\delta_{qt} \\ + \left[\frac{\omega - \xi\left(\beta_s - \beta_b\right)}{\sigma_h\xi\omega}C_\pi - \frac{\beta_s - \beta_b}{\sigma_h\omega}C_q + \frac{\sigma}{\sigma_h\xi}C_x\right]\delta_{ht} + \beta C_\pi\delta_{\pi t},\end{aligned}$$

where the C_i coefficients (for $i = \{\pi, x, \tilde{c}, q\}$) denote the loading of the endogenous variables on the endogenous state S_t in the first-order Markov-perfect solution.

From here, we can proceed as in the case of flexible prices. In particular, we can rewrite the previous expression as

$$\begin{aligned}\delta_{ct} = \mathbb{E}_t\delta_{ct+1} + \left(\sigma^{-1}C_\pi + C_x - \xi C_{\tilde{c}}\right)\delta_{xt} - \delta_{St} + \left(C_\pi + \frac{\sigma\omega}{\omega+\beta}C_x + \frac{\beta}{\omega+\beta}C_q\right)\delta_{qt} \\ + \left[\frac{\omega - \xi\left(\beta_s - \beta_b\right)}{\sigma_h\xi\omega}C_\pi - \frac{\beta_s - \beta_b}{\sigma_h\omega}C_q + \frac{\sigma}{\sigma_h\xi}C_x\right]\delta_{ht} + \beta C_\pi\delta_{\pi t}.\end{aligned}$$

We can solve for δ_{xt} and $\delta_{\pi t}$ from (72) and (73), respectively, to obtain

$$\delta_{xt} = x_t - \frac{\sigma}{\sigma_h\xi}\delta_{ht} + \gamma\delta_{\pi t},$$

and

$$\delta_{\pi t} = \lambda_\pi\pi_t - \beta_s^{-1}\delta_{ct}.$$

Replacing these two expressions into the first order condition for S_t, we have

$$\delta_{ct} = \mathbb{E}_t\delta_{ct+1} + \left(\sigma^{-1}C_\pi + C_x - \xi C_{\tilde{c}}\right)\left[x_t - \frac{\sigma}{\sigma_h\xi}\delta_{ht} + \gamma(\lambda_\pi\pi_t - \beta_s^{-1}\delta_{ct})\right] - \delta_{St}$$
$$+\left(C_\pi + \frac{\sigma\omega}{\omega+\beta}C_x + \frac{\beta}{\omega+\beta}C_q\right)\delta_{qt} + \left[\frac{\omega - \xi\left(\beta_s - \beta_b\right)}{\sigma_h\xi\omega}C_\pi - \frac{\beta_s - \beta_b}{\sigma_h\omega}C_q + \frac{\sigma}{\sigma_h\xi}C_x\right]\delta_{ht} + \beta C_\pi\left(\lambda_\pi\pi_t - \beta_s^{-1}\delta_{ct}\right).$$

We can now use the results for δ_{St} and δ_{qt} derived in the case of flexible prices (as mentioned, these conditions do not change with flexible prices) to obtain

$$\delta_{ct} = \mathbb{E}_t\delta_{ct+1} + \left(\sigma^{-1}C_\pi + C_x - \xi C_{\tilde{c}}\right)\left[x_t - \frac{\sigma}{\sigma_h\xi}\delta_{ht} + \gamma(\lambda_\pi\pi_t - \beta_s^{-1}\delta_{ct})\right] + \frac{\tilde{\mu}(\omega+\beta_s)}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}\delta_{ht}$$
$$+\left(C_\pi + \frac{\sigma\omega}{\omega+\beta}C_x + \frac{\beta}{\omega+\beta}C_q\right)\frac{(\omega+\beta)(\beta_s-\beta_b-\tilde{\mu})}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}\delta_{ht} + \left[\frac{\omega - \xi\left(\beta_s - \beta_b\right)}{\sigma_h\xi\omega}C_\pi - \frac{\beta_s - \beta_b}{\sigma_h\omega}C_q + \frac{\sigma}{\sigma_h\xi}C_x\right]\delta_{ht}$$
$$+ \beta C_\pi\left(\lambda_\pi\pi_t - \beta_s^{-1}\delta_{ct}\right).$$

Collecting terms, we can rewrite

$$\left(1 + \frac{\Sigma_\pi}{\beta_s}\right)\delta_{ct} = \mathbb{E}_t\delta_{ct+1} + (\sigma^{-1}C_\pi + C_x - \xi C_{\tilde{c}})x_t + \Sigma_\pi\lambda_\pi\pi_t + \Sigma_h\delta_{ht}$$

where

$$\Sigma_\pi \equiv (\beta + \gamma\sigma^{-1})C_\pi + C_x - \xi C_{\tilde{c}}$$

and

$$\Sigma_{ht} \equiv \frac{\tilde{\mu}(\omega+\beta_s)}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})} + \left(1 - \frac{\beta_s - \beta_b}{\sigma_h\omega}\right)C_\pi + \frac{\sigma\omega}{\omega+\beta}C_x + \frac{\sigma}{\sigma_h}C_{\tilde{c}} - \frac{\tilde{\mu}\beta_s + \omega(\beta_s - \beta_b)}{\sigma_h\omega(\omega+\beta+\xi\tilde{\mu})}C_q.$$

Finally, using the results for δ_{ht} and δ_{ct} derived in the case of flexible prices, we obtain the dynamic targeting rule in the case of sticky prices

$$\tilde{c}_t + \Upsilon_x x_t + \Upsilon_\pi\pi_t + \Upsilon_h\tilde{h}_t = \Upsilon_c\mathbb{E}_t\tilde{c}_{t+1}, \tag{83}$$

where

$$\begin{aligned}
\Upsilon_x &\equiv \frac{\sigma^{-1}C_\pi + C_x - \xi C_{\tilde{c}}}{\Sigma_c} \\
\Upsilon_\pi &\equiv \frac{\Sigma_\pi\lambda_\pi}{\Sigma_c} \\
\Upsilon_h &\equiv \frac{\Sigma_h\lambda_h}{\zeta_h(1-\beta_s)(1-\xi)\Sigma_c} \\
\Upsilon_c &\equiv \left(1 + \frac{\Sigma_\pi}{\beta_s} - \frac{\Sigma_h}{\zeta_h}\right)^{-1},
\end{aligned}$$

and

$$\Sigma_c \equiv \left(1 + \frac{\Sigma_\pi}{\beta_s} - \frac{\Sigma_h}{\zeta_h}\right) \frac{\eta \lambda_c}{1 + \xi(\sigma\eta - 1)}.$$

Equation (83) is our third targeting rule. The parameters of this targeting rule depend on the unknown C coefficients. As in the case of flexible prices, we solve for these coefficients numerically using the method of undetermined coefficients.

F Quantitative Experiment Details

In this Appendix, we provide more details of the approach used to compute the simulations in Section 4.

F.1 Changes to model equations

Here we show the effects of adding a slow moving debt limit to the linearised model. We focus on the equations that change. Because we wish to incorporate the effect of occasionally binding constraints, we also need to work with equations that include the mulitplier on the borrowing constraint, μ, rather than the compact version of the model considered in Appendix E in which the borrowing constraint is assumed to always bind.

Debt limit

Incorporating the slow-moving debt limit used by Guerrieri and Iacoviello (2017) gives:

$$d_t^b \leq (1 - \gamma_d) \left[\theta_t + q_t + (1 - \xi)\, \tilde{h}_t\right] + \gamma_d \left(d_{t-1}^b - \pi_t\right) \tag{84}$$

where (as before) the housing gap is defined as:

$$\tilde{h}_t = h_t^b - h_t^s$$

Housing demand and house prices

We start from the saver's housing demand equation, (47):

$$q_t = \frac{1 + \tau^h - \beta_s}{1 + \tau^h} \left(\sigma c_t^s - \sigma_h h_t^s + u_t^h\right) + \frac{\beta_s}{1 + \tau^h} \mathbb{E}_t \left(\sigma c_t^s - \sigma c_{t+1}^s + q_{t+1}\right)$$

Repeating the logic of the steady-state analysis for the simple model, but incorporating the slow-moving debt limit, shows that the housing tax required to implement the efficient steady state is:

$$\tau^h = \beta_s - \tilde{\mu}\, (1 - \gamma_d)\, \Theta - \beta_b$$

which collapses to the expression in Appendix B when $\gamma_d = 0$.

Note also that with a slow moving debt limit, the steady state multiplier on the borrowing constraint is:

$$\tilde{\mu} = \frac{1 - \beta_s^{-1}\beta_b}{1 - \gamma_d \beta_b}$$

which collapses to $1 - \beta_s^{-1}\beta_b$, as previously derived, when $\gamma_d = 0$.

We note that goods and housing market clearing imply, respectively, that:

$$\begin{aligned} c_t^s =& x_t - \xi \tilde{c}_t + y_t^* \\ h_t^s =& -\xi \tilde{h}_t \end{aligned}$$

where $\tilde{c}$ is the consumption gap.

Using these definitions and rearranging gives:

$$q_t = \frac{1 + \tau^h - \beta_s}{1 + \tau^h}\left(\sigma_h \xi \tilde{h}_t + u_t^h\right) + \sigma x_t - \sigma \xi \tilde{c}_t + \frac{\beta_s}{1 + \tau^h}\mathbb{E}_t\left(\sigma \xi \tilde{c}_{t+1} - \sigma x_{t+1} + q_{t+1}\right) + \nu_t^{qs} \tag{85}$$

where the composite shock

$$\nu_t^{qs} \equiv \frac{1 + \tau^h - \beta_s}{1 + \tau^h}\sigma y_t^* + \frac{\beta_s}{1 + \tau^h}\mathbb{E}_t\left[\sigma y_t^* - \sigma y_{t+1}^*\right]$$

is independent of the housing preference shock.

We can repeat the process for the borrower's housing demand equation:

$$\begin{aligned} q_t = & \frac{(1-\gamma_d)\tilde{\mu}\Theta}{1 - (1-\gamma_d)\tilde{\mu}\Theta}(\mu_t + \theta_t) + \frac{1 - (1-\gamma_d)\tilde{\mu}\Theta - \beta_b}{1 - (1-\gamma_d)\tilde{\mu}\Theta}\left(\sigma c_t^b - \sigma_h h_t^b + u_t^h\right) \\ & + \frac{\beta_b}{1 - (1-\gamma_d)\tilde{\mu}\Theta}\mathbb{E}_t\left(\sigma c_t^b - \sigma c_{t+1}^b + q_{t+1}\right) \end{aligned}$$

which collapses to (49) when $\gamma_d = 0$.

The market clearing conditions imply:

$$\begin{aligned} c_t^b &= x_t + (1-\xi)\tilde{c}_t + y_t^* \\ h_t^b &= (1-\xi)\tilde{h}_t \end{aligned}$$

Using these results and rearranging gives:

$$\begin{aligned} q_t = & \frac{(1-\gamma_d)\tilde{\mu}\Theta}{1 - (1-\gamma_d)\tilde{\mu}\Theta}(\mu_t + \theta_t) - \frac{1 - (1-\gamma_d)\tilde{\mu}\Theta - \beta_b}{1 - (1-\gamma_d)\tilde{\mu}\Theta}\left[\sigma_h(1-\xi)\tilde{h}_t - u_t^h\right] \\ & + \sigma(1-\xi)\tilde{c}_t + \sigma x_t + \frac{\beta_b}{1 - (1-\gamma_d)\tilde{\mu}\Theta}\mathbb{E}_t\left(q_{t+1} - \sigma(1-\xi)\tilde{c}_{t+1} - \sigma x_{t+1}\right) + \nu_t^{qb} \end{aligned} \tag{86}$$

where

$$\nu_t^{qb} \equiv \frac{1 - (1-\gamma_d)\tilde{\mu}\Theta - \beta_b}{1 - (1-\gamma_d)\tilde{\mu}\Theta}\sigma y_t^* + \frac{\beta_b}{1 - (1-\gamma_d)\tilde{\mu}\Theta}\mathbb{E}_t\left(\sigma y_t^* - \sigma y_{t+1}^*\right)$$

The multiplier on the borrowing constraint

With a slow-moving debt limit, the borrower's first order conditions for consumption and debt are:

$$U^b_{c,t}(i) = \lambda_t(i)$$
$$-\frac{\hat{\mu}_t(i)}{P_t} + \frac{\lambda_t(i)}{P_t} = \beta_b \mathbb{E}_t \left[-\gamma_d \frac{\hat{\mu}_{t+1}(i)}{P_{t+1}} + \frac{R^b_t \lambda_{t+1}(i)}{P_{t+1}} \right]$$

Combining the first order conditions gives:

$$\left(U^b_{ct}(i) - \hat{\mu}_t(i) \right) \frac{1}{P_t} - \mathbb{E}_t \beta_b \left[U^b_{ct+1}(i) \frac{R^b_t}{P_{t+1}} - \gamma_d \frac{\hat{\mu}_{t+1}(i)}{P_{t+1}} \right] = 0,$$

rearranging to

$$1 - \tilde{\mu}_t(i) = \mathbb{E}_t \beta_b \frac{U^b_{ct+1}(i)}{U^b_{ct}(i)} \frac{R^b_t - \gamma_d \tilde{\mu}_{t+1}(i)}{\Pi_{t+1}}, \tag{87}$$

where

$$\tilde{\mu}_t(i) \equiv \frac{\hat{\mu}_t(i)}{U^b_{ct}(i)}.$$

Aggregating and log-linearizing gives:

$$c^b_t = \mathbb{E}_t c^b_{t+1} + \sigma^{-1} \mathbb{E}_t \pi_{t+1} - \frac{\tilde{\mu}}{\sigma(1-\tilde{\mu})} \mu_t - \frac{\beta_b}{\sigma(1-\tilde{\mu})\beta_s} (i_t + \psi \kappa_t) + \frac{\beta_b \gamma_d \tilde{\mu}}{\sigma(1-\tilde{\mu})} \mathbb{E}_t \mu_{t+1}$$

Once again, we can use the goods market clearing condition to write c^b in terms of the consumption gap and the output gap. This gives us:

$$\begin{aligned} x_t + (1-\xi)\tilde{c}_t = {} & \mathbb{E}_t (x_{t+1} + (1-\xi)\tilde{c}_{t+1}) + \sigma^{-1} \mathbb{E}_t \pi_{t+1} - \frac{\tilde{\mu}}{\sigma(1-\tilde{\mu})} \mu_t \\ & - \frac{\beta_b}{\sigma(1-\tilde{\mu})\beta_s} (i_t + \psi \kappa_t) + \frac{\beta_b \gamma_d \tilde{\mu}}{\sigma(1-\tilde{\mu})} \mathbb{E}_t \mu_{t+1} + \nu^\mu_t \end{aligned}$$

where

$$\nu^\mu_t \equiv -y^* + \mathbb{E}_t y^*_{t+1} + u^c_t$$

is a composite shock that is independent of the housing preference shock.

The composite state variable, S

As before, we define a composite state variable:

$$S_t \equiv i^b_t + d^b_t - \frac{\beta_s}{\Theta} h^b_t$$

In equilibrium (ie imposing housing market equilibrium and the borrowing rate equation), we have:

$$S_t = i_t + \psi \kappa_t + d^b_t - \frac{\beta_s(1-\xi)}{\Theta} \tilde{h}_t \tag{88}$$

We can also write the borrower's budget constraint in terms of the composite state variable:

$$d_t^b = \beta_s^{-1}\left(S_{t-1} - \pi_t\right) + \frac{1-\xi}{\Theta}\tilde{h}_t + \frac{1-\xi}{\eta}\tilde{c}_t \tag{89}$$

F.2 Optimal policy

The analysis of optimal discretionary policy mirrors the analytical approach for the simple model (which assumes $\gamma_d = 0$, $\Theta = 1$ and that the borrowing constraint always binds). However, we also include multipliers on the constraints that the nominal interest rate and bank capital instruments must be positive (recognising that both are bounded).

F.2.1 The policy problem

We have:

$$\begin{aligned}
\min\ & \frac{1}{2}\left[x_t^2 + \lambda_\pi \pi_t^2 + \lambda_\kappa \kappa_t^2 + \lambda_c \tilde{c}_t^2 + \lambda_h \tilde{h}_t^2\right] + \beta_p \mathbb{E}_t \mathcal{L}_{t+1} \\
& - \delta_{\pi,t}\left[\pi_t - \gamma x_t - \beta \mathbb{E}_t \pi_{t+1} - u_t^m\right] \\
& - \delta_{x,t}\left[x_t - \xi\tilde{c}_t + \sigma^{-1}(i_t - \mathbb{E}_t\pi_{t+1}) - \mathbb{E}_t\left(x_{t+1} - \xi\tilde{c}_{t+1}\right) - \nu_t^c\right] \\
& - \delta_{d,t}\left[d_t^b - (1-\gamma_d)\left[\theta_t + q_t + (1-\xi)\tilde{h}_t\right] - \gamma_d\left(d_{t-1}^b - \pi_t\right) + \delta_t^b\right] \\
& - \delta_{c,t}\left[d_t^b - \beta_s^{-1}S_{t-1} + \beta_s^{-1}\pi_t - \frac{1-\xi}{\Theta}\tilde{h}_t - \frac{1-\xi}{\eta}\tilde{c}_t\right] \\
& - \delta_{S,t}\left[S_t - d_t^b - i_t - \psi\kappa_t + \beta_s\frac{1-\xi}{\Theta}\tilde{h}_t\right] \\
& - \delta_{q,t}\left[\begin{array}{l} q_t - \frac{(1-\gamma_d)\tilde{\mu}\Theta}{1-(1-\gamma_d)\tilde{\mu}\Theta}\left(\mu_t + \theta_t\right) + \frac{1-(1-\gamma_d)\tilde{\mu}\Theta-\beta_b}{1-(1-\gamma_d)\tilde{\mu}\Theta}\left[\sigma_h\left(1-\xi\right)\tilde{h}_t - u_t^h\right] \\ -\sigma\left(1-\xi\right)\tilde{c}_t - \sigma x_t - \frac{\beta_b}{1-(1-\gamma_d)\tilde{\mu}\Theta}\mathbb{E}_t\left(q_{t+1} - \sigma\left(1-\xi\right)\tilde{c}_{t+1} - \sigma x_{t+1}\right) - \nu_t^{qd} \end{array}\right] \\
& - \delta_{h,t}\left[\ q_t - \frac{1+\tau^h-\beta_s}{1+\tau^h}\left[\sigma_h\xi\tilde{h}_t + u_t^h\right] - \sigma x_t + \sigma\xi\tilde{c}_t - \frac{\beta_s}{1+\tau^h}\mathbb{E}_t\left(\sigma\xi\tilde{c}_{t+1} - \sigma x_{t+1} + q_{t+1}\right) - \nu_t^{qs}\ \right] \\
& - \delta_{\mu,t}\left[\begin{array}{l} x_t + (1-\xi)\tilde{c}_t - \mathbb{E}_t\left(x_{t+1} + (1-\xi)\tilde{c}_{t+1}\right) - \sigma^{-1}\mathbb{E}_t\pi_{t+1} + \frac{\tilde{\mu}}{\sigma(1-\tilde{\mu})}\mu_t \\ +\frac{\beta_b}{\sigma(1-\tilde{\mu})\beta_s}\left(i_t + \psi\kappa_t\right) - \frac{\beta_b\gamma_d\tilde{\mu}}{\sigma(1-\tilde{\mu})}\mathbb{E}_t\mu_{t+1} - \nu_t^\mu \end{array}\right] \\
& - \delta_t^i\left[i_t - \mathcal{B}\right] - \delta_t^\kappa\left[\kappa_t + \kappa^{ss}\right]
\end{aligned}$$

where the zero bound is $\mathcal{B}$ and the steady-state value of the bank capital ratio is κ^{ss}. We use β_p to denote the discount factor of the policymaker (and assume $\beta_p = \beta_s$ in our experiments).

F.2.2 First order conditions

In this variant of the model, there are two state variables, d^b and S. To simplify the representation of the first order conditions, we use a similar notation to Appendix E to capture the effects of the choice of the current state variable on expected future variables. Specifically, let:

$$B_{Y,Z} \equiv \frac{\partial \mathbb{E}_t Y_{t+1}}{\partial Z_t}$$

denote the effect of state variable Z on the expectation of variable Y. Similarly, let:

$$V_{Z,t} \equiv \frac{\partial \mathcal{L}_{t+1}}{\partial Z_t}$$

be the effect of the state variable Z on the marginal future loss.

Using this notation, the first order conditions to the problem are:

$$0 = x_t + \gamma\delta_{\pi,t} - \delta_{x,t} + \sigma\delta_{q,t} + \sigma\delta_{h,t} - \delta_{\mu,t} \tag{FoC: x}$$

$$0 = \lambda_\pi \pi_t - \delta_{\pi,t} - \beta_s^{-1}\delta_{c,t} - \gamma_d\delta_{d,t} \tag{FoC: π}$$

$$0 = \lambda_c \tilde{c}_t + \xi\delta_{x,t} + \frac{1-\xi}{\eta}\delta_{c,t} + \sigma(1-\xi)\delta_{q,t} - \sigma\xi\delta_{h,t} - (1-\xi)\delta_{\mu,t} \tag{FoC: $\tilde{c}$}$$

$$\begin{aligned} 0 = {} & \lambda_h \tilde{h}_t + (1-\gamma_d)(1-\xi)\delta_{d,t} + \frac{1-\xi}{\Theta}\delta_{c,t} - \beta_s\frac{1-\xi}{\Theta}\delta_{S,t} \\ & - \frac{1-(1-\gamma_d)\tilde{\mu}\Theta - \beta_b}{1-(1-\gamma_d)\tilde{\mu}\Theta}\sigma_h(1-\xi)\delta_{q,t} + \frac{1+\tau^h-\beta_s}{1+\tau^h}\sigma_h\xi\delta_{h,t} \end{aligned} \tag{FoC: $\tilde{h}$}$$

$$0 = (1-\gamma_d)\delta_{d,t} - \delta_{q,t} - \delta_{h,t} \tag{FoC: q}$$

$$0 = \frac{(1-\gamma_d)\tilde{\mu}\Theta}{1-(1-\gamma_d)\tilde{\mu}\Theta}\delta_{q,t} - \sigma^{-1}\frac{\tilde{\mu}}{1-\tilde{\mu}}\delta_{\mu,t} \tag{FoC: μ}$$

$$0 = \lambda_\kappa \kappa_t + \psi\delta_{S,t} - \sigma^{-1}\frac{\beta_b}{\beta_s(1-\tilde{\mu})}\psi\delta_{\mu,t} - \delta_t^\kappa \tag{FoC: κ}$$

$$0 = (1-\gamma_d)\delta_{d,t} + \frac{(1-\gamma_d)\tilde{\mu}\Theta}{1-(1-\gamma_d)\tilde{\mu}\Theta}\delta_{q,t} \tag{FoC: θ}$$

$$0 = -\sigma^{-1}\delta_{x,t} + \delta_{S,t} - \sigma^{-1}\frac{\beta_b}{\beta_s(1-\tilde{\mu})}\delta_{\mu,t} - \delta_t^i \tag{FoC: i}$$

$$\begin{aligned} 0 = {} & \beta_p V_{S,t} + \beta B_{\pi,S}\delta_{\pi,t} + \left[\sigma^{-1}B_{\pi,S} + B_{x,S} - \xi B_{\tilde{c},S}\right]\delta_{x,t} - \delta_{S,t} \\ & + \frac{\beta_b}{1-(1-\gamma_d)\tilde{\mu}\Theta}\left[B_{q,S} - \sigma(1-\xi)B_{\tilde{c},S} - \sigma B_{x,S}\right]\delta_{q,t} \\ & + \frac{\beta_s}{1+\tau^h}\left[\xi\sigma B_{\tilde{c},S} - \sigma B_{x,S} + B_{q,S}\right]\delta_{h,t} \\ & + \left[\sigma^{-1}B_{\pi,S} + B_{x,S} + (1-\xi)B_{\tilde{c},S} + \frac{\beta_b}{\sigma}\frac{\gamma_d\tilde{\mu}}{(1-\tilde{\mu})}B_{\mu,S}\right]\delta_{\mu,t} \end{aligned} \tag{FoC: S}$$

$$\begin{aligned} 0 = {} & \beta_p V_{d^b,t} + \beta B_{\pi,d^b}\delta_{\pi,t} + \left[\sigma^{-1}B_{\pi,d^b} + B_{x,d^b} - \xi B_{\tilde{c},d^b}\right]\delta_{x,t} - \delta_{d,t} - \delta_{c,t} + \delta_{S,t} \\ & + \frac{\beta_b}{1-(1-\gamma_d)\tilde{\mu}\Theta}\left[B_{q,d^b} - \sigma(1-\xi)B_{\tilde{c},d^b} - \sigma B_{x,d^b}\right]\delta_{q,t} \\ & + \frac{\beta_s}{1+\tau^h}\left[\xi\sigma B_{\tilde{c},d^b} - \sigma B_{x,d^b} + B_{q,d^b}\right]\delta_{h,t} \\ & + \left[\sigma^{-1}B_{\pi,d^b} + B_{x,d^b} + (1-\xi)B_{\tilde{c},d^b} + \frac{\beta_b}{\sigma}\frac{\gamma_d\tilde{\mu}}{(1-\tilde{\mu})}B_{\mu,d^b}\right]\delta_{\mu,t} \end{aligned} \tag{FoC: d^b}$$

As in the analysis of the simple model, we note that the marginal effect of changes in current states on future losses will be equivalent to the effect on the current policymaker of a change in the state they inherit. So:

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial S_{t-1}} &= \beta_s^{-1}\delta_{c,t} \\
\frac{\partial \mathcal{L}_t}{\partial d_{t-1}^b} &= \gamma_d \delta_{d,t}
\end{aligned}$$

which implies that:

$$\begin{aligned}
V_{S,t} &= \beta_s^{-1}\mathbb{E}_t\delta_{c,t+1} \\
V_{d^b,t} &= \gamma_d \mathbb{E}_t \delta_{d,t+1}
\end{aligned} \tag{90}$$

This implies that the first order conditions with respect to S and d^b can be written as:

$$\begin{aligned}
0 = {} & \beta_p\beta_s^{-1}\mathbb{E}_t\delta_{c,t+1} + \beta B_{\pi,S}\delta_{\pi,t} + \left[\sigma^{-1}B_{\pi,S} + B_{x,S} - \xi B_{\tilde{c},S}\right]\delta_{x,t} - \delta_{S,t} \\
& + \frac{\beta_b}{1-(1-\gamma_d)\tilde{\mu}\Theta}\left[B_{q,S} - \sigma(1-\xi)B_{\tilde{c},S} - \sigma B_{x,S}\right]\delta_{q,t} \\
& + \frac{\beta_s}{1+\tau^h}\left[\xi\sigma B_{\tilde{c},S} - \sigma B_{x,S} + B_{q,S}\right]\delta_{h,t} \\
& + \left[\sigma^{-1}B_{\pi,S} + B_{x,S} + (1-\xi)B_{\tilde{c},S} + \frac{\beta_b}{\sigma}\frac{\gamma_d\tilde{\mu}}{(1-\tilde{\mu})}B_{\mu,S}\right]\delta_{\mu,t} && (91) \\
0 = {} & \beta_p\gamma_d\mathbb{E}_t\delta_{d,t+1} + \beta B_{\pi,d^b}\delta_{\pi,t} + \left[\sigma^{-1}B_{\pi,d^b} + B_{x,d^b} - \xi B_{\tilde{c},d^b}\right]\delta_{x,t} - \delta_{d,t} - \delta_{c,t} + \delta_{S,t} \\
& + \frac{\beta_b}{1-(1-\gamma_d)\tilde{\mu}\Theta}\left[B_{q,d^b} - \sigma(1-\xi)B_{\tilde{c},d^b} - \sigma B_{x,d^b}\right]\delta_{q,t} \\
& + \frac{\beta_s}{1+\tau^h}\left[\xi\sigma B_{\tilde{c},d^b} - \sigma B_{x,d^b} + B_{q,d^b}\right]\delta_{h,t} \\
& + \left[\sigma^{-1}B_{\pi,d^b} + B_{x,d^b} + (1-\xi)B_{\tilde{c},d^b} + \frac{\beta_b}{\sigma}\frac{\gamma_d\tilde{\mu}}{(1-\tilde{\mu})}B_{\mu,d^b}\right]\delta_{\mu,t} && (92)
\end{aligned}$$

F.3 The role of ψ

In this section we consider how the value of ψ affects macro-prudential policy. Note that κ_t only enters the following equations:

$$\begin{aligned}
x_t + (1-\xi)\tilde{c}_t = {} & \mathbb{E}_t\left(x_{t+1} + (1-\xi)\tilde{c}_{t+1}\right) - \sigma^{-1}\frac{\beta_b}{\beta_s(1-\tilde{\mu})}\left(i_t + \psi\kappa_t - \mathbb{E}_t\pi_{t+1}\right) \\
& - \sigma^{-1}\frac{\tilde{\mu}}{1-\tilde{\mu}}\mu_t + \frac{\beta_b}{\sigma}\frac{\gamma_d\beta_b\tilde{\mu}}{\beta_s(1-\tilde{\mu})}\mathbb{E}_t\mu_{t+1} + \nu_t^{\mu} \\
S_t = {} & i_t + \psi\kappa_t + d_t^b - \beta_s(1-\xi)\tilde{h}_t
\end{aligned}$$

If we define:

$$\hat{\kappa}_t \equiv \psi\kappa_t \;\Leftrightarrow\; \kappa_t = \psi^{-1}\hat{\kappa}_t$$

then these equations can be written as:

$$\begin{aligned}x_t + (1-\xi)\,\tilde{c}_t = \mathbb{E}_t\left(x_{t+1} + (1-\xi)\,\tilde{c}_{t+1}\right) - \sigma^{-1}\frac{\beta_b}{\beta_s\,(1-\tilde{\mu})}\left(i_t + \hat{\kappa}_t - \mathbb{E}_t\pi_{t+1}\right)\\ - \sigma^{-1}\frac{\tilde{\mu}}{1-\tilde{\mu}}\mu_t + \frac{\beta_b}{\sigma}\frac{\gamma_d\beta_b\tilde{\mu}}{\beta_s\,(1-\tilde{\mu})}\mathbb{E}_t\mu_{t+1} + \nu_t^{\mu}\\ S_t = i_t + \hat{\kappa}_t + d_t^b - \beta_s\,(1-\xi)\,\tilde{h}_t\end{aligned}$$

This change of variables writes the model in 'spread space' since $\hat{\kappa}$ is equiproportionate to i in the model. Under this change of variables, the term in the loss function capturing the effects of changes in the capital ratio is given by:

$$\lambda_\kappa\kappa_t^2 = \lambda_\kappa\psi^{-2}\hat{\kappa}_t^2 = \hat{\lambda}_\kappa\hat{\kappa}_t^2$$

where

$$\hat{\lambda}_\kappa = \frac{\eta}{\psi\,(\sigma+\phi)}$$

Notice that when $\psi \to 0$, $\hat{\lambda}_\kappa \to \infty$ and the cost of using $\hat{\kappa}$ becomes very large. So capital ratios will be used relatively little when ψ is very low. In particular, the first order condition for $\hat{\kappa}$ can be written as:

$$\hat{\kappa}_t = \frac{\psi\,(\sigma+\phi)}{\eta}\left[\sigma^{-1}\frac{\beta_b}{\beta_s\,(1-\tilde{\mu})}\delta_{\mu,t} - \delta_{S,t} + \delta_t^{\kappa}\right]$$

which implies that it collapses to $\hat{\kappa}_t = 0$ as $\psi \to 0$.

F.4 Dealing with Occasionally Binding Constraints

The approach was set out by Holden and Paetz (2012) and is convenient in our case.

F.4.1 The Model and the Rational Expectations Solution

The set of first order conditions derived above can be stacked with the constraints (the equations describing private sector behavior) in the form:

$$H_F\mathbb{E}_t x_{t+1} + H_C x_t + H_B x_{t-1} = \Psi\epsilon_t + \Psi_\delta\delta_t \tag{93}$$

The new component to the model is a vector of 'shocks' δ that are introduced in order to impose the occasionally binding constraints. These shocks are added to the model equations which do not hold when the occasionally binding constraints are binding. In our case, $\delta_t \equiv \left[\delta_t^b, \delta_t^i, \delta_t^\kappa\right]'$

For example, consider the zero bound. The first order condition for the nominal interest rate, (FoC: i), includes the 'shock', δ_t^i. When the zero bound does not bind, $\delta_t^i = 0$. When the zero bound binds, δ_t^i is chosen so that the nominal interest rate is equal to the lower bound.

A similar approach is used to account for the fact that the borrowing limit may not bind. The evolution of the debt limit includes the 'shock' δ_t^d. When the borrowing constraint binds, $\delta_t^d = 0$

so that $d_t^b = (1-\gamma_d)\left[\theta_t + q_t + (1-\xi)\,\tilde{h}_t\right] + \gamma_d\left(d_{t-1}^b - \pi_t\right)$ and the level of debt is determined by the borrowing constraint. When the borrowing constraint is slack, $\delta_t^d > 0$ is chosen so that the multiplier on the constraint is zero ($\tilde{\mu}_t = -\tilde{\mu}^{ss}$). In that case, $d_t^b < (1-\gamma_d)\left[\theta_t + q_t + (1-\xi)\,\tilde{h}_t\right] + \gamma_d\left(d_{t-1}^b - \pi_t\right)$ and the level of debt is less than the borrowing constraint.

Our approach shares some similarities with the 'OccBin' approach developed by Guerrieri and Iacoviello (2015). For example, when the ZLB binds, the targeting criterion (24) does not form part of the model. Instead, the shock $\delta_{i,t}$ is chosen to enforce that the interest rate satisfies the zero bound: in effect, we replace (24) with the equation $i_t = i^{ZLB} < 0$ where i^{ZLB} is the lower bound. This is analogous to the OccBin approach of defining different sets of model equations that apply when constraints are or are not binding. One advantage of our approach is that it scales easily as the number of occasionally binding constraints grows.[32] As we show below, our approach also allows us to check for the uniqueness of the solution.

Our approach requires us to solve for the values of δ_t (with $t = 1, \dots$) that impose the occasionally binding constraints. To do that, we will use the rational expectations solution of the model (93) which the Anderson and Moore (1985) algorithm delivers as:

$$x_t = Bx_{t-1} + \Phi\epsilon_t + \sum_{i=0}^{\infty} F^i \Phi_\delta \mathbb{E}_t \delta_{t+i} \tag{94}$$

where B, F, Φ and Φ_δ are functions of the coefficient matrices H_F, H_C, H_B, Ψ and Ψ_δ in (93). The solution in (94) is valid for any expected shock sequence $\{\delta_{t+i}\}_{i=0}^{\infty}$.[33] This is important because our solution must cope with the fact that the equilibrium in period t may be affected by the expectation that occasionally binding constraints are binding in period(s) $s > t$.

F.4.2 The Baseline Simulation

To simulate the model we first assume that none of the occasionally binding constraints binds. This is our 'baseline simulation'. To produce it we set $\delta_t = 0, \forall t$ and then from a given initial condition x_0 and a realization of the shocks ϵ_1 we compute $x_t = Bx_{t-1} + \Phi\epsilon_t$ for $t = 1, \dots, H$ for some simulation horizon H.

With the baseline simulation in hand, we then check whether it violates the assumption that the constraints never bind. So, for example, we check whether the implied trajectory of the multiplier on the borrowing constraint is always positive ($\tilde{\mu}_t > -\tilde{\mu}^{ss}, \forall t$) and whether the path of the policy rate is always positive ($i_t > i^{ZLB}, \forall t$). If we find that any of these assumptions is violated in the baseline, then we need to invoke a quadratic programming procedure to ensure that the occasionally binding constraints are enforced.

[32] Incorporating N occasionally binding constraints using OccBin requires specifying 2^N alternative sets of model equations, whereas in our approach we need to add N 'shocks' (and possibly up to N auxiliary equations/variables such as d^{gap}).

[33] As long the shocks do not increase at a rate faster than (the inverse of) the maximum eigenvalue of F.

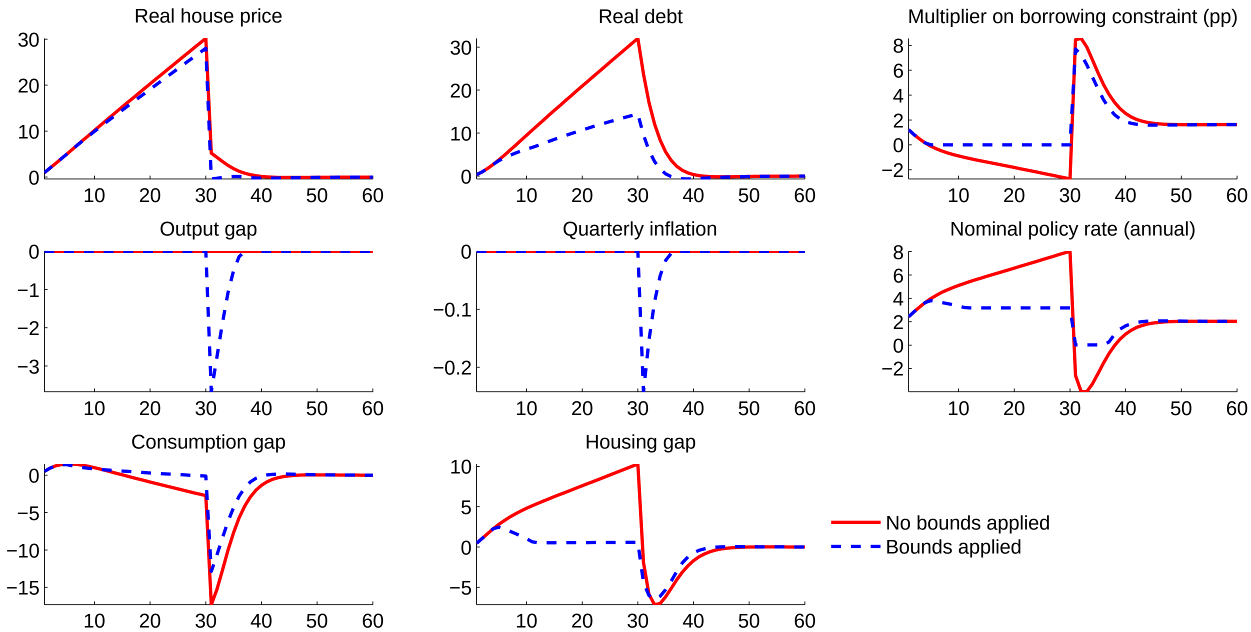

Figure F.7: Effects of occasionally binding constraints on housing boom/bust under 'flexible inflation targeting'

F.4.3 Implications of Occasionally Binding Constraints

To illustrate the effects of the occasionally binding constraints, Figure F.7 shows the outcome of our housing boom/bust simulation when the policymaker pursues flexible inflation targeting. Two variants of the simulation are shown. The solid red lines show the case in which the occasionally binding constraints are ignored. In this case, the multiplier μ on the borrowing constraint is permitted to take negative values as is the nominal policy rate. The dashed blue lines show the case in which the simulation respects the occasionally binding constraints.

The results demonstrate the importance of applying the occasionally binding constraints. When disregarded, the policymaker is able to fully stabilize the output gap and inflation. However, achieving that stabilization requires quite large fluctuations in the nominal interest rate. Indeed, the collapse in housing demand generates a long period in which the nominal interest rate is negative. Moreover, during the period of increasing house prices, nominal interest rates rise by around two percentage points. This reaction is required in order to stabilize aggregate demand, which is supported by increased consumption demand by borrowers, given that the multiplier on the borrowing constraint enters negative territory.

When the occasionally binding constraints are imposed, it is no longer possible for the monetary policymaker to stabilize the output gap and inflation when house prices fall. There is a recession and a sharp decline in inflation while the nominal interest rate is constrained by the zero lower bound. The occasionally binding constraints are also important during the house price boom. The expectation of strong real house prices causes the borrowing constraint to go slack (so that the multiplier μ equals zero for a number of periods). Relative to the case in which the constraints are not applied (solid red lines), there is a smaller increase in debt and the consumption gap is also smaller. The more moderate spending behavior of borrowers puts less pressure on aggregate demand so that (before the house price collapse) aggregate demand and inflation are stabilized with a relatively modest increase in the nominal interest rate.

F.4.4 Imposing the Occasionally Binding Constraints (OBC)

The OBC can be represented as inequality constraints on a set of 'target variables' which we will denote as τ. In our case, τ would include the policy rate and the multiplier on the borrowing constraint. To impose the OBC, we will solve for a set of shocks $\{\delta_t\}_{t=1}^{H}$ that impose the OBC. The simulation horizon H can be chosen to be arbitrarily large.

The approach is based on the insight that the effect of the δ shocks can be simply added to the baseline simulation, given the linearity of the model. Inspection of (94) reveals that the effect of the fundamental and δ shocks enter linearly. So to find the set of δ shocks that ensure that the target variables satisfy the OBC, we solve for a set of shocks that, when added to the baseline simulation will achieve this. To do so, we need to be able to record the impact of δ shocks at all horizons $t = 1, \ldots, H$ on the target variables in all periods $t = 1, \ldots, H$.

Let S_τ be a selector matrix that selects the target variables from the vector of endogenous variables. Thus:

$$\tau_t = S_\tau x_t \tag{95}$$

Consider now the effects of the δ shocks $\{\delta_t\}_{t=1}^{H}$ on the endogenous variables in period 1 of the simulation. This is given by:

$$\widehat{x}_1 = \sum_{i=0}^{H-1} F^i \Phi_\delta \delta_{1+i}, \tag{96}$$

which captures the fact that in period 1 all of the shocks occur in (present and) future periods. The effects on the target variables are given by $\widehat{\tau}_1 = S_\tau \widehat{x}_1$.

In period 2, we can use the RE solution to note that the effects on endogenous variables are:

$$\widehat{x}_2 = B\widehat{x}_1 + \sum_{i=0}^{H-2} F^i \Phi_\delta \delta_{2+i}, \tag{97}$$

and from the expression for $\widehat{x}_1$, we can write:

$$\widehat{x}_2 = B \sum_{i=0}^{H-1} F^i \Phi_\delta \delta_{1+i} + \sum_{i=0}^{H-2} F^i \Phi_\delta \delta_{2+i}. \tag{98}$$

This step provides a recursive scheme for building a matrix that maps the effects of shocks to the dummy shocks in periods $t = 1, \ldots, H$ to the target variables in each period. The first (block) row of this matrix can be found by expanding (96):

$$\widehat{\tau}_1 = \begin{bmatrix} S_\tau \Phi_\delta & \dots & S_\tau F^{k-1} \Phi_\delta & \dots & S_\tau F^{H-1} \Phi_\delta \end{bmatrix} \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_k \\ \vdots \\ \delta_H \end{bmatrix}. \tag{99}$$

The second row is built by using equation (98) to multiply the coefficients in the first row by B and then adding the coefficients on shocks that arrive from period 2 onwards:

$$\widehat{\tau}_2 = \begin{bmatrix} S_\tau B \Phi_\delta & \dots & S_\tau B F^{k-1} \Phi_\delta + S_\tau F^{k-2} \Phi_\delta & \dots & S_\tau B F^{H-1} \Phi_\delta + S_\tau F^{H-2} \Phi_\delta \end{bmatrix} \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_k \\ \vdots \\ \delta_H \end{bmatrix},$$

and this can be applied for each row in turn.

This scheme implies that we can write the mapping from the dummy shocks to the target variables as:

$$\mathcal{T} = \mathcal{M}\mathcal{D}, \tag{100}$$

where

$$\mathcal{T} = \begin{bmatrix} \widehat{\tau}_1 \\ \vdots \\ \widehat{\tau}_k \\ \vdots \\ \widehat{\tau}_H \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_k \\ \vdots \\ \delta_H \end{bmatrix}, \tag{101}$$

and the rows of $\mathcal{M}$ are built using the recursive scheme described above.

Notice that if the number of OBCs is n, then τ and δ are $n \times 1$ vectors so that $\mathcal{T}$ and $\mathcal{D}$ are $(nH) \times 1$. The mapping we have derived records the effects of all δ shocks on all target variables. This reflects the fact that there may be interactions between the different OBCs. For example, if the borrowing constraint happens to be slack in a period of weak growth, economic outcomes may be better than otherwise and so the ZLB becomes non-binding. To correctly capture these types of effects we need to incorporate the effects of shocks that implement each OBC on all target variables.

To incorporate the bounds on the OBCs, we compute the vector $\widehat{\mathcal{T}}$ as the deviation of the target variables from their constraint values. This is just a normalization, but it is useful in setting up the quadratic programming problem (because it allows us to incorporate a contemporary slackness condition easily). To do this, we simply record the relevant rows of the baseline simulation $\{x_t\}_{t=1}^{H}$ and subtract the value of the constraints. This normalization implies that if the baseline simulation implied $\widehat{\mathcal{T}} > 0$, then the baseline solution, which assumes that the OBCs never bind, would be correct.

We can now set up a quadratic progamming problem to solve for $\mathcal{D}$:

$$\min \quad \frac{1}{2}\mathcal{D}' \left(\mathcal{M} + \mathcal{M}'\right)\mathcal{D} + \widehat{\mathcal{T}}'\mathcal{D} \tag{102}$$

$$\text{subject to: } \widehat{\mathcal{T}} + \mathcal{M}\mathcal{D} \geq 0 \tag{103}$$

$$\mathcal{D} \geq 0 \tag{104}$$

The problem in equations (102)–(104) can be understood as follows. The constraint (103) ensures that the OBCs are respected. $\widehat{\mathcal{T}}$ is the baseline simulation for the target variables, measured relative to the constraint values. $\mathcal{M}\mathcal{D} = \mathcal{T}$ is the marginal effect of the δ shocks $\mathcal{D}$ on the target variables. So $\widehat{\mathcal{T}} + \mathcal{M}\mathcal{D}$ is the path of the target variables measured relative to their constraints after the δ shocks have been applied: requiring this to be non-negative implies that the constraints are respected.

The constraint (104) requires that the δ shock values used to impose the constraints are positive. This requirement ensures that the OBCs are truly binding. To see why this is important, suppose for a moment that monetary policy is determined by a Taylor rule including a δ shock to enforce the ZLB. Consider a simulation in which there is an initial negative shock to demand that causes the Taylor rule to prescribe a negative value for the policy rate in the first few periods of the baseline simulation. Now suppose that we seek δ shocks to the Taylor rule to ensure that the ZLB is respected. One solution could be to apply *negative* future shocks to the policy rule that push future rates lower than the baseline simulation, but still above the zero bound. This could be sufficient to stimulate demand in the near term such that the ZLB never binds (and so constraint (103) is respected as a strict inequality).

Finally, note that the minimand (102) can be expanded as follows:

$$\begin{aligned}\frac{1}{2}\mathcal{D}' \left(\mathcal{M} + \mathcal{M}'\right)\mathcal{D} + \widehat{\mathcal{T}}'\mathcal{D} &= \frac{1}{2}\mathcal{D}'\mathcal{M}\mathcal{D} + \frac{1}{2}\mathcal{D}'\mathcal{M}'\mathcal{D} + \frac{1}{2}\widehat{\mathcal{T}}'\mathcal{D} + \frac{1}{2}\mathcal{D}'\widehat{\mathcal{T}} \\ &= \frac{1}{2}\mathcal{D}'\left(\mathcal{M}\mathcal{D} + \widehat{\mathcal{T}}\right) + \frac{1}{2}\left(\mathcal{M}\mathcal{D} + \widehat{\mathcal{T}}\right)'\mathcal{D},\end{aligned}$$

where the first line exploits the fact that $\widehat{\mathcal{T}}'\mathcal{D}$ is a scalar and the second line collects terms. The minimand is therefore analogous to a contemporary slackness condition: it achieves a minimum of zero when $\mathcal{D} = 0$ or $\widehat{\mathcal{T}} + \mathcal{M}\mathcal{D} = 0$.

The above discussion assumed that the final constraint $\mathcal{D} > 0$ is economically sensible given the model at hand. For this to be true we require that an anticipated positive δ shock that arrives j periods ahead will be expected to increase the bounded variable in period j. This seems like it should be automatically satisfied, but the interaction of lead/lag relationships in models with inertia means that it need not be satisfied. One example is the 'reversed sign' responses of some

DSGE models to monetary policy shocks in the distant future.[34]

It is straightforward to check whether the model suffers from this problem by inspecting the signs of the diagonal elements of the $\mathcal{M}$ matrix. If they are all positive, then we can apply the algorithm as presented above. If some are negative, we need to amend the $\mathcal{D} \geq 0$ constraint to flip the sign applied to the relevant elements of $\mathcal{D}$.

Another issue is that the quadratic programming problem has a unique solution only if the matrix $(\mathcal{M} + \mathcal{M}')$ is positive semi-definite. A sufficient condition for the matrix to be positive semi-definite is for its eigenvalues to be non-negative which can be easily checked.

Finally, there is no guarantee that a solution exists. For very large shocks, the overarching assumption that the model returns to 'normal' in a finite period of time may be violated (for example, the model may get stuck in a deflation trap). Non-existence is likely to be a problem when there is a strong feedback between the OBCs. Again, in practice this can be checked by ensuring that the δ shocks are zero at the end of the simulation horizon H.

F.5 Incorporating instrument bounds under discretion

The approach for dealing with occasionally binding constraints described in the previous subsection works well for cases in which the behavior of the model when the constraint binds is equivalent to its behavior when a time-varying shock is appended to a time-invariant equation. So, a slack borrowing constraint (so that the non-negativity constraint on the multiplier of that constraint is binding) can be captured by appending a 'shock' to the equation describing evolution of the debt limit. Similarly, a 'shock' can be appended to the 'flexible inflation targeting' criterion (24) and used to impose the zero bound on the nominal interest rate.

This approach works because the equations to which the 'shocks' are appended are ones in which the coefficients are constant. More generally, discretionary solutions with instrument bounds will give rise to first order conditions with time varying coefficients. In particular, the 'B_{yx}' coefficients, that capture the marginal effects of x_t today on $\mathbb{E}_t y_{t+1}$ will not be constants during a period in which the policy instruments are constrained. That is because the instrument constraints alter the ability of the policymaker to affect the current state of the economy and hence expectations. The first order conditions derived in Appendix F.2 assume that bounds on instruments never bind, so that the marginal effects of allocations at date t on expected outcomes at date $t+1$ incorporate (in equilibrium) policy responses that are a linear function of the state vector.[35]

To deal with this issue, Brendon et al. (2011) develop an algorithm to solve for the equilibrium

[34] In this case a positive shock to the monetary policy rule in the distant future causes a contraction today because of forward looking behavior. The monetary policy reaction function prescribes a near-term loosening in response to the contractionary effect of the future policy tightening. If the variables that enter the policy rule and/or the policy rate itself are sufficiently inertial, we may observe an equilibrium in which the policy rate is lower in period j because the negative effects on the arguments of the rule outweigh the positive effects of the shock.

[35] For example, consider the marginal effect of debt on expected inflation, $B_{\pi,d}$. When unconstrained by the zero bound, changes in the short-term nominal interest rate affect debt via borrowers' budget constraints and hence future allocations. But when constrained by the zero bound, the marginal effect of changes in current debt on expected inflation do not include any response by the policymaker.

allocations of a linear model subject to instrument constraints under perfect foresight. That algorithm casts the problem into a discrete time dynamic programming problem, creating a set of first order conditions that account for the number of periods that the instrument bound(s) are expected to bind. This approach therefore generates 'B_{yx}' coefficients that vary during the period over which the instruments are constrained. We use this algorithm to compute the equilibrium in our model in the relevant cases.

As Brendon et al. (2011) note, their algorithm "unfortunately requires some guesswork" because it is based on a 'guess and verify' procedure. To provide an initial guess for the periods in which the instrument bounds are binding, we first solve the model using the first order conditions in Appendix F.2 (that is, with time-invariant 'B_{yx}' coefficients) and impose instrument bounds using the approach described in Appendix F.4.4. This provides a good starting guess which is then used to initialize the algorithm in Brendon et al. (2011).[36]

[36] We are grateful to Matt Waldron for helpful discussions on these issues and for sharing his code to implement the Brendon et al. (2011) algorithm.

G Additional figures

Here we present selected paths from the 'recursive simulation' analyzed in Section 4.

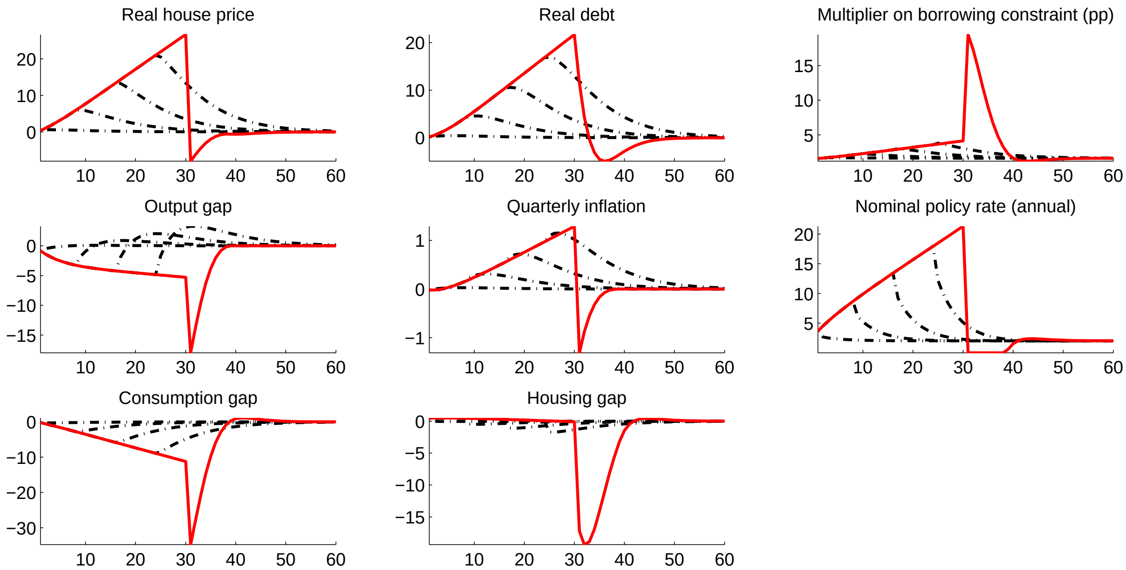

Figure G.1: Recursive simulation outcomes for 'leaning against the wind' policy assumption

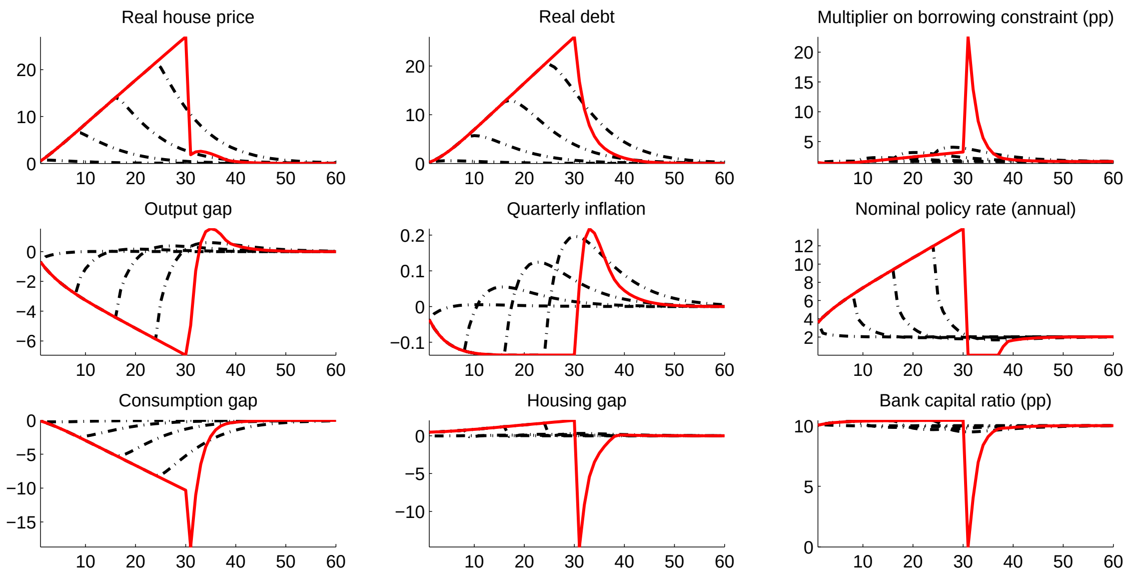

Figure G.2: Recursive simulation outcomes for 'full coordination' policy assumption