Incentives to form the grand coalition versus no incentive to split off from the grand coalition

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Abstract. The *core* is a solution concept for coalitional games that require no coalition to break away from the grand coalition and take a joint action that makes all of them better off. In other words, no player has an incentive to split from the grand coalition under core allocations; more specifically, under core allocations, the grand coalition is *maintained* and stable. In this study, we propose and discuss a new solution concept (i.e., set of allocations) under which each player has some incentive to form the grand coalition. We answer whether there is any relation between the *core* and the *solution concept* proposed in this study, i.e., between the notions of *maintaining* and *forming* the grand coalition. Further, we describe the situations in which these two actions coincide.

1 Introduction

The *core* is a solution concept for coalitional games that require no coalition (i.e., group of players) to break away and take a joint action that makes all participants better off. In other words, no player has an *incentive to split from the grand coalition* under core allocations. Note that this does not imply that each player has an *incentive to form the grand coalition* under core allocations. More specifically, some players may have no incentive to form the grand coalition even though core allocations are proposed. Consider the following coalitional game and its core allocations. Let $N = \{1, 2, 3, 4\}$ and

$$v(S) = \begin{cases} 2 & \text{if } |S| = 2, \\ 3 & \text{if } |S| = 3, \\ 6 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we can see that both payoff vectors $\boldsymbol{x} = (1.5, 1.5, 1.5, 1.5)$ and $\boldsymbol{y} = (0, 2, 2, 2)$ are in the core of (N, v). Under the core allocation $\boldsymbol{x} = (1.5, 1.5, 1.5, 1.5)$, the grand coalition N could be formed via the following coalition formation process. For the one-person subgame $(\{i\}, v^{\{i\}})$, $i \in N$, the unique allocation (i.e., imputation) for player i is $\boldsymbol{x}_i^{\{i\}} = 0$ since $v(\{i\}) = 0$. Further, for the two-person subgame $(\{i, j\}, v^{\{i,j\}})$, allocation $\boldsymbol{x}^{\{i,j\}} = (1, 1)$ is acceptable for each $i, j \in N$ since $v(\{i, j\}) = 2$ and the allocation improves both smaller (i.e., one player) subgames $(\{i\}, v^{\{i\}})$ and $(\{j\}, v^{\{j\}})$ (i.e., $\boldsymbol{x}_k^{\{i,j\}} = 1 > 0 = \boldsymbol{x}_i^{\{k\}}$, $k \in \{i, j\}$). Therefore, each player has an incentive to form the larger coalition $\{i, j\}$ in seeking an increase or non-decrease, in a weak sense, in its allocation. Similarly, $\boldsymbol{x}^{\{i,j,k\}} = (1, 1, 1)$ is acceptable for each $i, j, k \in N$ and each player has an incentive (in a weak sense) to form the larger coalition $\{i, j, k\}$. Finally, $\boldsymbol{x} = (1.5, 1.5, 1.5, 1.5)$ is also acceptable for each player in N because $\boldsymbol{x}$ improves all allocations $\boldsymbol{x}_i^{\{i\}} = 0$, $\boldsymbol{x}^{\{i,j\}} = (1, 1)$, and $\boldsymbol{x}^{\{i,j,k\}} = (1, 1, 1)$ for each player. In other words, the payoff vector $\boldsymbol{x} = (1.5, 1.5, 1.5, 1.5)$ can be considered a target in any coalition formation process (e.g., $\{1\} \to \{1, 2\} \to \{1, 2, 3\} \to N$) toward the grand coalition in the game (N, v).

Meanwhile, under the core allocation $\boldsymbol{y} = (0, 2, 2, 2)$, it might be impossible for the grand coalition N to be formed or reached because the allocation $\boldsymbol{y}$ would be unacceptable to some players in some coalition formation processes. As an example, in a three-person subgame $(\{i, j, k\}, v^{\{i,j,k\}})$, an imputation $\boldsymbol{z}^{\{i,j,k\}} := (\boldsymbol{z}_i^{\{i,j,k\}}, \boldsymbol{z}_j^{\{i,j,k\}}, \boldsymbol{z}_k^{\{i,j,k\}})$ such as $\boldsymbol{z}_i^{\{i,j,k\}} < \boldsymbol{z}_j^{\{i,j,k\}} \leq \boldsymbol{z}_k^{\{i,j,k\}}$ is unacceptable to player i. Indeed, since $v(\{i, j, k\}) = 3$, it follows that $\boldsymbol{z}_i^{\{i,j,k\}} < 1$ and $\boldsymbol{z}_j^{\{i,j,k\}} \leq (3 - \boldsymbol{z}_i^{\{i,j,k\}})/2$; therefore, $\boldsymbol{z}_i^{\{i,j,k\}} + \boldsymbol{z}_j^{\{i,j,k\}} < 2$. Further, any imputation $\boldsymbol{z}^{\{i,j\}}$ of $(\{i, j\}, v^{\{i,j\}})$ is not (Pareto-)improved by $\boldsymbol{z}^{\{i,j,k\}}$ since $\boldsymbol{z}_i^{\{i,j\}} + \boldsymbol{z}_j^{\{i,j\}} = 2 > \boldsymbol{z}_i^{\{i,j,k\}} + \boldsymbol{z}_j^{\{i,j,k\}}$. And at least one of the players i and j has no incentive to form larger coalition $\{i, j, k\}$ to obtain payoff $\boldsymbol{z}^{\{i,j,k\}}$. Given that game (N, v) is symmetric and the discussion above, $\boldsymbol{z}^{\{i,j,k\}} = (1, 1, 1)$ is the unique acceptable allocation in subgame $(\{i, j, k\}, v^{\{i,j,k\}})$. When three-person coalitions containing player 1 (e.g., $\{1, 2, 3\}$) have been formed under allocation $(1, 1, 1)$, it is trivial that player 1 never forms a grand coalition if allocation $\boldsymbol{y} = (0, 2, 2, 2)$ is proposed. When coalition $\{2, 3, 4\}$ has been formed under allocation $(1, 1, 1)$, if player 1 does not participate in the grand coalition (i.e., subgame $(\{2, 3, 4\}, v^{\{2,3,4\}})$ is played), then the payoff for each player 2, 3, and 4 remains as one. Therefore, all players in coalition $\{1, 2, 3\}$ would have incentive to accept any payoff greater than one. In other words, if $\boldsymbol{y} = (0, 2, 2, 2)$ is proposed to player 1, he will refuse this proposal and submit an appropriate re-proposal: e.g., , $(0.3, 1.9, 1.9, 1.9)$ or $(2.7, 1.1, 1.1, 1.1)$. Note that under core allocations, no coalition can improve its payoff by itself; however, also under core allocations, there is no rule that states that a coalition can make others worse off. Indeed, in this case, player 1 can use such a *threat*, though this is rather *incredible threat*. Hence, in both cases, player 1 has no incentive to form the grand coalition if unacceptable allocation $\boldsymbol{y} = (0, 2, 2, 2)$ is proposed.

From the above discussion, the payoff vector $\boldsymbol{x} = (1.5, 1.5, 1.5, 1.5)$ has the potential to be a target or goal in a coalition formation process toward the grand coalition N, but $\boldsymbol{y} = (0, 2, 2, 2)$ can never be a goal. In this paper, we propose and investigate allocation set $A(N, v)$ whose elements have the potential to be targets in some coalition formation process toward the grand coalition N. Note that $A(N, v)$ will be defined mathematically in Definition 4 in Section 2. If $A(N, v) = \emptyset$, then some players have no incentive to form the grand coalition agreement in the game (N, v) under any allocations. Therefore, the grand coalition would not be formed at all. Conversely, if $A(N, v) \neq \emptyset$, there are some coalition formation paths toward a potential target $\boldsymbol{x} \in A(N, v)$; therefore, it is possible for the grand coalition to be formed or reached.

In addition to this introductory section, our paper is organized as follows. In Section 2, we provide some necessary preliminaries. Next in Section 3, we show several relations between the allocation set $A(N, v)$ and the core $\mathcal{C}(N, v)$ of games. In Section 4, we present several properties of $A(N, v)$ of convex games. In Section 5, through numerical examples, we describe several relations among $A(N, v)$, the core, the kernel, and the nucleolus. Finally, in Section 6, we present our conclusions and describe avenues for future work.

2 Preliminaries

Let $N := \{1, \ldots, n\}$ $(n > 0)$ be a set of *players*. A *coalition* is a subset of N. An n-person *cooperative game with side-payments* or *transferable utility game* (i.e., a *coalitional game* or simply *game* for short) is a pair (N, v), where v is a function $v : 2^N \to \mathbb{R}$ from the set of coalitions to the set of real numbers $\mathbb{R}$ with $v(\emptyset) = 0$, which is called a *characteristic function*. The value $v(S)$ for a coalition $S \subseteq N$ may be thought of as the *worth* won by S if all members in S agree to cooperate, and the other ones do not. A game (N, v) is said to be *super additive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$, and to be *convex* if

$$v(S) + v(T) \leq v(S \cup T) - v(S \cap T)$$

for any $S, T \subseteq N$. Given a game (N, v) and a coalition $S \subseteq N$, the *subgame* (S, v^S) is obtained by restricting v to 2^S, i.e., $v^S(T) = v(T) \quad \forall T \subseteq S$. We use shorthand notation here and write i for $\{i\}$, $S \cup i$ for $S \cup \{i\}$, $S \setminus i$ for $S \setminus \{i\}$, ij for $\{i, j\}$, and so on.

In a game (N, v), a payoff vector $\boldsymbol{x} \in \mathbb{R}^N$ is called *efficient* if the total worth $v(N)$ is allocated to all the members in N, i.e., $\sum_{i \in N} \boldsymbol{x}_i = v(N)$, where $\boldsymbol{x}_i$ is the i-th component of $\boldsymbol{x}$ (i.e., $\boldsymbol{x}_i$ is the allocation to the player i). The set of all efficient payoff vectors of a game (N, v) is called the *pre-imputation* set of (N, v) and is denoted by $I^*(N, v)$. The *imputation* set $I(N, v)$ of a game (N, v) is defined by

$$I(N, v) = \{\boldsymbol{x} \in I^*(N, v) \mid \boldsymbol{x}_i \geq v(i) \quad \forall i \in N\}.$$

Further, an imputation $\boldsymbol{x} \in I(N, v)$ is said to be *dominated* by another imputation $\boldsymbol{y} \in I(N, v)$ if there exists a coalition $S \subseteq N$ such that each player $i \in S$ prefers $\boldsymbol{y}$ to $\boldsymbol{x}$ (i.e., $\boldsymbol{y}_i \geq \boldsymbol{x}_i$ for any $i \in S$ and $\boldsymbol{y}_j > \boldsymbol{x}_j$ for some $j \in S$) and S can enforce $\boldsymbol{y}$ (i.e., $\sum_{i \in S} \boldsymbol{y}_i \leq v(S)$). The *core* $\mathcal{C}(N, v)$ of a game (N, v) is the set of all imputations that are not dominated by any other imputations (i.e., $\forall \boldsymbol{x} \in \mathcal{C}(N, v), \neg\exists\ [\boldsymbol{y} \in I(N, v)$ and $S \subseteq N]$ s.t. $\boldsymbol{y}_i > \boldsymbol{x}_i\ \forall i \in S$ and $\boldsymbol{y}(S) \leq v(S)$, where $\boldsymbol{y}(S) := \sum_{i \in S} \boldsymbol{y}_i$). The constraints imposed on the core $\mathcal{C}(N, v)$ ensure that no coalition would have an incentive to split from the grand coalition N and do better on its own. Then, the core $\mathcal{C}(N, v)$ of a game (N, v) is represented by

$$\mathcal{C}(N, v) := \{\boldsymbol{x} \in I(V, N) \mid \boldsymbol{x}(S) \geq v(S) \quad \forall S \subseteq N\}. \tag{1}$$

For a game (N, v), the *excess* of $S \subseteq N$ at a payoff vector $\boldsymbol{x} \in \mathbb{R}^N$ is defined as

$$e(S, \boldsymbol{x}, v) := v(S) - \boldsymbol{x}(S).$$

Note here that positive excess $e(S, \boldsymbol{x}, v)$ may be interpreted as the *dissatisfaction* of the coalition S when faced with the proposal $\boldsymbol{x}$, because, $e(S, \boldsymbol{x}, v)$ represents the total gain that members of S will have if they depart from $\boldsymbol{x}$ and form their own coalition S.

Definition 1 (ϵ-core [12]**, least-core** [7]**).** Let ϵ be a real number. The *ϵ-core* of a game (N, v), denoted $\mathcal{C}_\epsilon(N, v)$, is defined by

$$\mathcal{C}_\epsilon(N, v) := \{\boldsymbol{x} \in I^*(N, v) \mid e(S, \boldsymbol{x}, v) \leq \epsilon \ \ \forall S \subseteq N (S \neq \emptyset)\}. \tag{2}$$

Clearly, $\mathcal{C}_0(N, v) = \mathcal{C}(N, v)$. Also, $\mathcal{C}_\epsilon(N, v) \neq \emptyset$ if ϵ is large enough. Further, $C_{\epsilon'}(N, v) \subsetneq C_\epsilon(N, v)$ whenever $C_\epsilon(N, v) \neq \emptyset$ and $\epsilon' < \epsilon$. The ϵ-core ttherefore has the following interpretation: it is the set of efficient payoff vectors that cannot be improved upon by any coalition if coalition formation entails a *cost* of ϵ or a *bonus* of $-\epsilon$ if ϵ is negative. In order to be an acceptable substitute for the core,

$\mathcal{C}_\epsilon(N,v)$ must be non-empty and ϵ should be small. The *least-core* of a game (N,v), denoted $\mathcal{LC}(N,v)$, is the intersection of all non-empty ϵ-cores of (N,v), i.e.,

$$\mathcal{LC}(N,v) = \mathcal{C}_{\epsilon_0}(N,v), \quad \text{where} \;\; \epsilon_0 := \min_{x\in I^*(N,v)} \max_{\emptyset\neq S\subseteq N} e(S,\boldsymbol{x},v).$$

If a payoff vector $\boldsymbol{x}$ has been proposed in a game (N,v), player $i \in N$ can compare his position with that of player $j \in N (i \neq j)$ by considering *maximum surplus* $s^v_{ij}(\boldsymbol{x})$ of i against j with respect to $\boldsymbol{x}$, which is defined as

$$s^v_{ij}(\boldsymbol{x}) := \max_{S:S\ni i, S\not\ni j} e(S,\boldsymbol{x},v).$$

Here, the maximum surplus of i against j with respect to $\boldsymbol{x}$ can be regarded as the highest payoff that player i can gain (or the maximum amount that i can lose if $s^v_{ij}(\boldsymbol{x})$ is negative) without the cooperation of player j. Player i can do this by forming a coalition without j, assuming that the other members of the coalition are satisfied with $\boldsymbol{x}$. Therefore, $s^v_{ij}(\boldsymbol{x})$ can be regarded as a measure of the possible threat of i against j (i.e., the strength of i against j at $\boldsymbol{x}$). Further, if $\boldsymbol{x}$ is an imputation, then player j cannot be threatened by i or any other player when $\boldsymbol{x}_j = v(j)$ since j can obtain or win the amount $\boldsymbol{x}_j$ by going it alone. If $s^v_{ij}(\boldsymbol{x}) > s^v_{ji}(\boldsymbol{x})$ and $\boldsymbol{x}_j > v(j)$ (i.e., at $\boldsymbol{x}$, the strength of i against j is greater than that of j against i and the player j cannot obtain the amount $\boldsymbol{x}_j$ by going it alone), then we say that player i *outweighs* j at $\boldsymbol{x}$. From this, the *kernel*, introduced in [3], consists of those imputations for which no player outweighs another one.

Definition 2 (kernel, pre-kernel [3]**).** The *kernel* $\mathcal{K}(N,v)$ of a game (N,v) is defined by

$$\mathcal{K}(N,v) := \{\boldsymbol{x} \in I(N,v) \mid s^v_{ij}(\boldsymbol{x}) \leq s^v_{ji}(\boldsymbol{x}) \text{ or } \boldsymbol{x}_j = v(j) \;\; \forall i,j \in N\}, \qquad (3)$$

and the *pre-kernel* $\mathcal{PK}(N,v)$ of a game (N,v) is defined by

$$\mathcal{PK}(N,v) := \{\boldsymbol{x} \in I^*(N,v) \mid s^v_{ij}(\boldsymbol{x}) = s^v_{ji}(\boldsymbol{x}) \;\; \forall i,j \in N\}. \qquad (4)$$

Here, the kernel and the pre-kernel are always non-empty and $\mathcal{K}(N,v)\cap\mathcal{C}(N,v) = \mathcal{PK}(N,v)\cap\mathcal{C}(N,v)$ if $\mathcal{C}(N,v) \neq \emptyset$.

Since $e(S,\boldsymbol{x},v)$ can be regarded as a measure of the dissatisfaction of S at $\boldsymbol{x}$, we next attempt to identify a payoff vector that minimizes the maximum excess. We construct a vector $\boldsymbol{\theta}(\boldsymbol{x}) := (\theta_1(\boldsymbol{x}),\ldots,\theta_{2^n-1}(\boldsymbol{x}))$ by arranging the excesses of the 2^n-1 non-empty subsets (i.e., all non-empty subsets) of N in decreasing order; more specifically, we arrange the non-empty subsets $S_i \subseteq N$ in an order such that $\theta_i(\boldsymbol{x}) := e(S_i,\boldsymbol{x},v)$ and $\theta_i(\boldsymbol{x}) \geq \theta_j(\boldsymbol{x})$ whenever $1 \leq i \leq j \leq 2^n-1$. We say that $\boldsymbol{\theta}(\boldsymbol{x})$ is *lexicographically smaller* than $\boldsymbol{\theta}(\boldsymbol{y})$ if there exists a positive integer $k \leq 2^n-1$ such that $\theta_i(\boldsymbol{x}) = \theta_i(\boldsymbol{y})$ whenever $i < k$ and $\theta_k(\boldsymbol{x}) < \theta_k(\boldsymbol{y})$. We denote $\boldsymbol{\theta}(\boldsymbol{x}) <_L \boldsymbol{\theta}(\boldsymbol{y})$, while $\boldsymbol{\theta}(\boldsymbol{x}) \leq_L \boldsymbol{\theta}(\boldsymbol{y})$ will be used to indicate that either $\boldsymbol{\theta}(\boldsymbol{x}) <_L \boldsymbol{\theta}(\boldsymbol{y})$ or $\boldsymbol{\theta}(\boldsymbol{x}) = \boldsymbol{\theta}(\boldsymbol{y})$.

Definition 3 (nucleolus [8]**).** The *nucleolus* $\mathcal{N}(N,v)$ of a game (N,v) is defined by

$$\mathcal{N}(N,v) := \{\boldsymbol{x} \in I(N,v) \mid \boldsymbol{\theta}(\boldsymbol{x}) \leq_L \boldsymbol{\theta}(\boldsymbol{y}) \ \ \forall \boldsymbol{y} \in I(N,v)\}, \tag{5}$$

and the *pre-nucleolus* of a game (N,v) is defined by

$$\mathcal{PN}(N,v) := \{\boldsymbol{x} \in I^*(N,v) \mid \boldsymbol{\theta}(\boldsymbol{x}) \leq_L \boldsymbol{\theta}(\boldsymbol{y}) \ \ \forall \boldsymbol{y} \in I^*(N,v)\}.$$

Here, the nucleolus always consists of one point that is an element of the kernel and is in the core whenever the core is non-empty, i.e., $|\mathcal{N}(N,v)| = 1$ and $\mathcal{N}(N,v) \subseteq \mathcal{K}(N,v) \cap \mathcal{C}(N,v)$ if $\mathcal{C}(N,v) \neq \emptyset$. For any convex game (N,v), we have $\mathcal{PK}(N,v) = \mathcal{K}(N,v) = \mathcal{N}(N,v) = \mathcal{PN}(N,v)$.

Next, we introduce a new solution concept of a game (N,v), *the allocation set* $A(N,v)$, from the viewpoint of *coalition formation* and *Pareto improvement.* To do so, consider the following scenario in a game (N,v). For each one-person subgame (i,v^i), the unique imputation $\boldsymbol{x}^i$ for the player $i \in N$ of the subgame (i,v^i) is $\boldsymbol{x}^i = (v(i))$. If there exists an imputation $\boldsymbol{x}^{\{i,j\}}$ of a subgame (ij,v^{ij}) that improves both the allocations $\boldsymbol{x}^i$ and $\boldsymbol{x}^j$ (i.e., $\boldsymbol{x}^{ij}_i \geq \boldsymbol{x}^i_i$ and $\boldsymbol{x}^{ij}_j \geq \boldsymbol{x}^j_j$), then the player i and j form the coalition ij and play the game (ij,v^{ij}) to seek an increase in their allocations, i.e., in order to obtain an imputation $\boldsymbol{x}^{ij}$. Such an imputation $\boldsymbol{x}^{ij}$ seems to have the potential to be a target in a coalition formation process toward the coalition ij or a short-term goal of a coalition formation process toward the grand coalition N. Further, if there are no allocations of (ij,v^{ij}) that improve both the allocations $\boldsymbol{x}^i$ and $\boldsymbol{x}^j$ but there is an imputation $\boldsymbol{x}^{ijk}$ of (ijk,v^{ijk}) that improves both the allocations $\boldsymbol{x}^i$ and $\boldsymbol{x}^j$, then players i and j would like to form the coalition ijk and play the game (ijk,v^{ijk}) to obtain imputation $\boldsymbol{x}^{ijk}$; however, if the imputation $\boldsymbol{x}^{ijk}$ cannot improve any allocations for the player k of subgame (T,v^T) for some $T(\ni k) \subsetneq \{ijk\}$ (i.e., there is no imputation $\boldsymbol{z}^T$ of (T,v^T) for some $T(\ni k) \subsetneq \{ijk\}$ such that $\boldsymbol{x}^{ijk}_k \geq \boldsymbol{z}^T_k$), then, the player k has no incentive to form the coalition ijk and obtain the allocation $\boldsymbol{x}^{ijk}_k$. More specifically, the allocation $\boldsymbol{x}^{ijk}$ cannot be a target for the player k in any coalition formation processes toward the coalition ijk via T. For the imputation $\boldsymbol{x}^{ijk}$ of (ijk,v^{ijk}) to be acceptable to all the players i, j, and k (i.e., the imputation $\boldsymbol{x}^{ijk}$ is a potential target in some coalition formation process toward the coalition ijk,) for each $T(\neq \emptyset) \subsetneq ijk$, there should exist an imputation $\boldsymbol{x}^T$ whenever there are acceptable allocations of the game (T,v^T), such that $\boldsymbol{x}^{ijk}_l \geq \boldsymbol{x}^T_l \ \forall l \in T$. Given this, we define the allocation set $A(ijk,v^{ijk})$ as the set of all imputations that are potential targets in some coalition formation processes toward the coalition ijk. We can extend $A(S,v^S)$, $0 < |S| \leq 3$, to the case in which $|S| \geq 4$ in the same manner discussed above using the definition that follows:

Definition 4. Given a game (N,v), the allocation set $A(N,v)$ of a game (N,v) is a set of imputations, i.e., $A(N,v) \subseteq I(N,v)$, defined recursively as follows:

$$A(i,v^i) := (v(i)) \quad \forall i \in N,$$

$$A(S, v^S) := \bigcap_{T \in \mathscr{A}(S,v^S)} \bigcup_{\boldsymbol{y}^T \in A(T,v^T)} \{\boldsymbol{x} \in I(S, v^S) \mid \boldsymbol{x}_i \geq \boldsymbol{y}_i^T \quad \forall i \in T\}$$

for any $S \subseteq N$ ($|S| \geq 2$), where $\mathscr{A}(S, v^S) := \{T \subsetneq S \mid A(T, v^T) \neq \emptyset, T \neq \emptyset\}$.

We observe here that from the above definition, $A(S, v^S) = \mathcal{C}(S, v^S)$ if $1 \leq |S| \leq 2$. Indeed, $A(i, v^i) = \{v(i)\} = \mathcal{C}(i, v^i)$ and

$$\begin{aligned}
A(ij, v^{ij}) &= \bigcap_{T \in \mathscr{A}(ij,v^{ij})} \bigcup_{\boldsymbol{y}^T \in A(T,v^T)} \{\boldsymbol{x} \in I(S, v^S) \mid \boldsymbol{x}_i \geq \boldsymbol{y}_i^T \quad \forall i \in T\} \\
&= \bigcap_{T \in \{\{i\},\{j\}\}} \bigcup_{\boldsymbol{y}^T \in C(T,v^T)} \{\boldsymbol{x} \in I(S, v^S) \mid \boldsymbol{x}_i \geq \boldsymbol{y}_i^T \quad \forall i \in T\} \\
&= \{(x_i, x_j) \mid x_i + x_j = v(ij),\ x_i \geq v(i)\} \\
&\quad \cap \quad \{(x_i, x_j) \mid x_i + x_j = v(ij),\ x_j \geq v(j)\} \\
&= \{(x_i, x_j) \mid x_i + x_j = v(ij),\ x_i \geq v(i),\ x_j \geq v(j)\} = \mathcal{C}(ij, v^{ij}).
\end{aligned}$$

Here, the allocation set $A(N, v)$ is the set of all imputations $\boldsymbol{x} \in I(N, v)$ for which there exists an allocation $\boldsymbol{y}^T \in A(T, v^T)$ of each subgame (T, v^T), $T \in \mathscr{A}(N, v)$ such that $\boldsymbol{x}_i \geq \boldsymbol{y}_i^T$ $\forall i \in T$ (i.e., $\boldsymbol{x}$ is a *Pareto improvement* of some allocation $\boldsymbol{y}^T$ in every subgame (T, v^T), $T \in \mathscr{A}(N, v)$). Therefore, such an allocation is acceptable for all players in the grand coalition N. For an imputation $\boldsymbol{z} \notin A(N, v)$, the imputation cannot improve any imputations for some players in a subgame (T, v^T), $T \in \mathscr{A}(N, v)$ (i.e., $\forall \boldsymbol{x}^T \in A(T, v^T)$ $\exists i \in T$ s.t. $\boldsymbol{z}_i < \boldsymbol{x}_i$, for some $T \in \mathscr{A}(N, v)$). Given this, players in T never want to participate in the coalition N if the allocation $\boldsymbol{z}$ is proposed. Therefore, if $A(N, v) = \emptyset$, then the grand coalition would not be formed even though the core $\mathcal{C}(N, v)$ is non-empty (see Proposition 5).

3 Relations between $\mathcal{C}(N, v)$ and $A(N, v)$

Recall that the core is a set of imputations under which no group of players would break away and take any joint action that would improve all of them. Here, the allocation set $A(N, v)$ is the set of all imputations that are potential targets in some coalition formation process toward the grand coalition. In this section, we focus on investigating the relations between the core $\mathcal{C}(N, v)$ and $A(N, v)$ (i.e., between the two notions of having no incentive to split off from the grand coalition and having the incentive to form the grand coalition).

Proposition 1 (lemma for Theorem 1). *For any two non-empty coalitions $T \subsetneq S$, if $A(T, v^T) = \emptyset$ and $A(S, v^S) \neq \emptyset$, then*

$$\boldsymbol{x}(T) > v(T) \quad \forall \boldsymbol{x} \in A(S, v^S),$$

where $\boldsymbol{x}(T) := \sum_{i \in T} \boldsymbol{x}_i$.

Corollary 1. *If $A(S, v^S) \neq \emptyset$, then, for any non-empty coalition $T \subseteq S$,*

$$\boldsymbol{x}(T) \geq v(T) \quad \forall \boldsymbol{x} \in A(S, v^S),$$

i.e., $\boldsymbol{x}$ satisfies coalitional rationality.

Proposition 2 (lemma for Theorem 1). *Given a game (N, v), if $A(S, v^S) \neq \emptyset$, then $A(S, v^S) \cap C(S, v^S) \neq \emptyset$.*

Corollary 2. *Given a game (N, v), if $C(S, v^S) = \emptyset$, then $A(S, v^S) = \emptyset$.*

Theorem 1. *Given a game (N, v), $A(S, v^S) \subseteq C(S, v^S) \quad \forall S \subseteq N$.*

More specifically, the allocation set $A(N, v)$ is a more stringent solution concept than the core $\mathcal{C}(N, v)$ of coalitional games, i.e., the allocation set $A(N, v)$ can be thought of as a refinement of the cocept to core $\mathcal{C}(N, v)$.

Proposition 3 (lemma for Proposition 4). *Given a game (N, v), if $|S| = 3$,*

$$A(S, v^S) \supseteq C(S, v^S).$$

Proposition 4. *Given a game (N, v), $A(S, v^S) = C(S, v^S)$ if $1 \leq |S| \leq 3$.*

More specifically, at most three-person games, the allocation set $A(N, v)$ coincides with the core $\mathcal{C}(N, v)$ of a game (N, v). This coincidence of $A(N, v)$ and $\mathcal{C}(N, v)$ does not, in general, extend to games with $|N| \geq 4$. Note that in [3], the kernel $\mathcal{K}(N, v)$ is a singleton for a game (N, v) with $|N| \leq 3$; however, this property does not, in general, extend to games with $|N| \geq 4$.

Proposition 5. *For any non-empty set N ($|N| \geq 4$), there exists a game (N, v) such that $\emptyset = A(N, v) \subsetneq \mathcal{C}(N, v)$.*

Note that we present such a game (N, v) in Example 2 of section 5 below.

Proposition 6. *For any non-empty set N ($|N| \geq 4$), there exists a game (N, v) such that $\emptyset \neq A(N, v) \subsetneq \mathcal{C}(N, v)$.*

We present such a game (N, v) in Example 3 of section 5) below.

The above propositions show that there is, in general, some imputation $\boldsymbol{x} \in \mathcal{C}(N, v) \setminus A(N, v)$. Such an $\boldsymbol{x}$ is an interesting and/or strange allocation in the following sense. If a core allocation $\boldsymbol{x} \in \mathcal{C}(N, v) \setminus A(N, v)$ is imposed on the players in N, no coalition (i.e., no group of players) would have any incentive to split off from the grand coalition; however, such an $\boldsymbol{x}$ cannot be a target in some coalition formation processes toward the grand coalition N since $\boldsymbol{x} \notin A(N, v)$. More specifically, in the coalition formation processes, if such an allocation $\boldsymbol{x} \in \mathcal{C}(N, v) \setminus A(N, v)$ is proposed to the players, then the grand coalition would not be formed.

4 Properties on $A(N, v)$ for convex games

Recall that for any convex game (N, v), the bargaining set [3] and the core of the game coincide, as so do the kernel and nucleolus [7]. Further, the kernel is a subset of the core for any convex game [7]. In this section, we show several properties on the allocation set $A(N, v)$ for convex games.

As a corollary of Shapley [11] and Ichiishi [5], we start with the following result.

Proposition 7 (lemma for Theorem 2). *Let $N := \{1, \ldots, n\}$ and (N, v) be a game. For a given permutation σ on N, consider the corresponding marginal vector $\boldsymbol{m}^\sigma(N, v) \in \mathbb{R}^N$ defined as*

$$\boldsymbol{m}^\sigma(N, v)_{\sigma(k)} := v({S^\sigma}_k) - v({S^\sigma}_{k-1})$$

for each $k \in N$, where ${S^\sigma}_k$ is defined as

$${S^\sigma}_k := \begin{cases} \{\sigma(j) \mid j \leq k\} & \text{if } k \neq 0, \\ \emptyset & \text{if } k = 0 \end{cases}$$

for each $k \in N$. Then, (N, v) is convex if and only if its core $\mathcal{C}(N, v)$ is the convex hull of all marginal vectors $\{\boldsymbol{m}^\sigma(N, v)\}_{\sigma \in \Pi(N)}$, where $\Pi(N)$ is the set of all permutations on N.

Proposition 8 (lemma for Theorem 2). *If (N, v) is convex, then $\boldsymbol{m}^\sigma(N, v) \in A(N, v)$ for any $\sigma \in \Pi(N)$.*

Proposition 9 (lemma for Theorem 2). *Let (N, v) be a game with $A(N, v) \neq \emptyset$. Then, for any $\boldsymbol{x}, \boldsymbol{y} \in A(N, v)$,*

$$\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in A(N, v) \quad \forall \lambda \in [0, 1].$$

That is, $A(N, v)$ is convex as with the core $\mathcal{C}(N, v)$.

Theorem 2. *If (N, v) is convex, then $A(N, v) = \mathcal{C}(N, v)$.*

Corollary 3. *If (N, v) is convex, $A(S, v^S) \neq \emptyset \;\; \forall S(\neq \emptyset) \subseteq N$.*

More specifically, in any convex game (N, v), every player has the incentive to form every coalition, and any core allocation $\boldsymbol{x} \in \mathcal{C}(N, v)$ has the potential to be a target in some coalition formation processes toward the grand coalition.

Corollary 4. *If (N, v) is convex, then*

$$\mathcal{N}(N, v) = \mathcal{K}(N, v) \subseteq \mathcal{LC}(N, v) \subseteq A(N, v) = \mathcal{C}(N, v).$$

5 Numerical examples of $\mathcal{N}$, $\mathcal{K}$, $\mathcal{LC}$, $\mathcal{C}$, and A

In this section, we present several numerical examples involving the nucleolus $\mathcal{N}$, kernel $\mathcal{K}$, least core $\mathcal{LC}$, core $\mathcal{C}$ and allocation set A of games.

Example 1 ($A = \mathcal{C} = \mathcal{LC} \supsetneq \mathcal{K} = \mathcal{N}$). Let $N = \{1, 2, 3, 4\}$ and

$$v(S) = \begin{cases} 1 & \text{if } S = 12 \text{ or } 34, \\ 2 & \text{if } S = 13 \text{ or } 24, \\ 3 & \text{if } S = 14 \text{ or } 23, \\ 4 & \text{if } |S| = 3, \\ 6 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

Here, (N, v) is super additive (i.e., $\mathcal{PN} = \mathcal{N}$ and $\mathcal{PK} = \mathcal{K}$) but not convex. Indeed, $v(123) < v(13) + v(23) - v(3)$. Here, the nucleolus $\mathcal{N}(N, v)$, kernel $\mathcal{K}(N, v)$, least-core $\mathcal{LC}(N, v)$, core $\mathcal{C}(N, v)$, and allocation set $A(N, v)$ are represented as

$$\mathcal{N}(N, v) = \mathcal{K}(N, v) = \mathcal{CH}(\{(1.5, 1.5, 1.5, 1.5)\}) = \{(1.5, 1.5, 1.5, 1.5)\},$$

$$A(N, v) = \mathcal{LC}(N, v) = \mathcal{C}(N, v) = \mathcal{CH}(\{(1, 1, 2, 2), (2, 1, 2, 1), (2, 2, 1, 1), (1, 2, 1, 2)\}),$$

where $\mathcal{CH}(\boldsymbol{S})$ is the convex hull of the set of payoff vectors $\boldsymbol{S}$. These solutions are represented by the form $(\alpha, \beta, 3 - \beta, 3 - \alpha)$ as shown in Fig. 1.

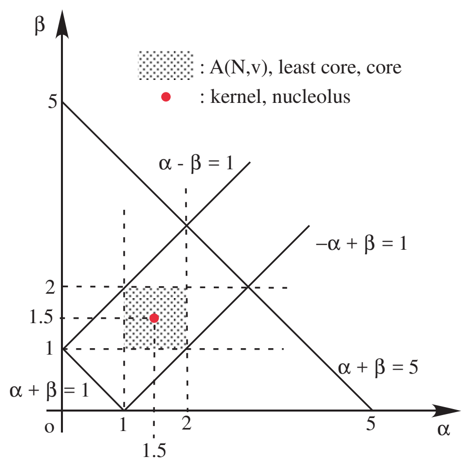

Fig. 1. Illustrating how allocation set $A(N, v)$ coincides with the core $\mathcal{C}(N, v)$

Example 2 $(A = \emptyset,\ \mathcal{N} \subsetneq \mathcal{K} \subsetneq \mathcal{LC} = \mathcal{C})$. Let $N = \{1, 2, 3, 4\}$ and

$$v(S) = \begin{cases} 1 & \text{if } S = 12 \text{ or } 34, \\ 2 & \text{if } S = 13 \text{ or } 24, \\ 3 & \text{if } S = 14 \text{ or } 23, \\ 3 & \text{if } |S| = 3, \\ 6 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

Here, (N, v) is super additive but not convex. The nucleolus $\mathcal{N}(N, v)$, kernel $\mathcal{K}(N, v)$, least-core $\mathcal{LC}(N, v)$, and core $\mathcal{C}(N, v)$ are then represented as

$$\mathcal{N}(N, v) = \mathcal{CH}(\{(1.5, 1.5, 1.5, 1.5)\}) = \{(1.5, 1.5, 1.5, 1.5)\},$$

$$\mathcal{K}(N, v) = \mathcal{CH}(\{(1, 1, 2, 2), (2, 2, 1, 1)\}),$$

$$\mathcal{LC}(N, v) = \mathcal{C}(N, v) = \mathcal{CH}(\{(1, 0, 3, 2), (3, 2, 1, 0), (2, 3, 0, 1), (0, 1, 2, 3)\}),$$

where $\mathcal{CH}(\boldsymbol{S})$ is the convex hull of the set of payoff vectors $\boldsymbol{S}$. These solutions are represented by the form $(\alpha, \beta, 3 - \beta, 3 - \alpha)$ as shown in Fig. 2. Here, $A(N, v) = \emptyset$. Indeed, it follows from Proposition 4 that $A(123, v^{123}) = \mathcal{C}(123, v^{123}) = \{(0, 1, 2)\}$, $A(124, v^{124}) = \mathcal{C}(124, v^{124}) = \{(1, 0, 2)\}$, $A(134, v^{134}) = \mathcal{C}(134, v^{134}) = \{(2, 0, 1)\}$, and $A(234, v^{234}) = \mathcal{C}(234, v^{234}) = \{(2, 1, 0)\}$. Therefore, if $\boldsymbol{x} \in A(N, v)$, then, from the definition of $A(N, v)$, $x_i \geq 2$ for any $i \in N$. This contradicts constraint $6 = v(N) = \boldsymbol{x}(N)$.

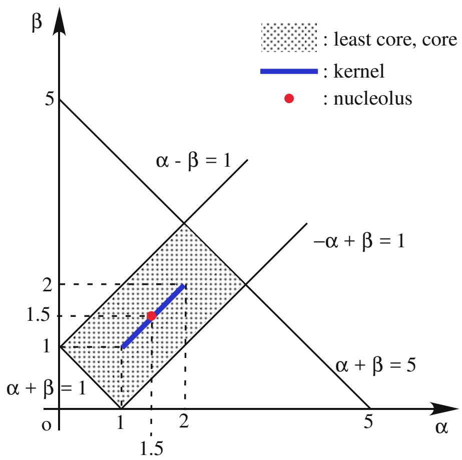

Fig. 2. $A(N, v) = \emptyset$, $\mathcal{C}(N, v) \neq \emptyset$

Example 3 ($v_1 : A = \mathcal{N} = \mathcal{K},\ A \subsetneq \mathcal{LC} \subsetneq \mathcal{C},\ v_2 : A = \mathcal{C} \supsetneq \mathcal{LC} \supsetneq \mathcal{K} = \mathcal{N}$). Let $N = \{1, 2, 3, 4\}$ and

$$v_1(S) = \begin{cases} 1 & \text{if } S = 12 \text{ or } 34, \\ 2 & \text{if } S = 13 \text{ or } 24, \\ 3 & \text{if } S = 14 \text{ or } 23, \\ 3 & \text{if } |S| = 3, \\ 8 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}, \quad v_2(S) = \begin{cases} 1 & \text{if } S = 12 \text{ or } 34, \\ 2 & \text{if } S = 13 \text{ or } 24, \\ 3 & \text{if } S = 14 \text{ or } 23, \\ 4 & \text{if } |S| = 3, \\ 8 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

Here, (N, v_1) is super additive but not convex; futher (N, v_2) is convex (i.e., $A(N, v_2) = \mathcal{C}(N, v_2)$ from Theorem 2). Nucleoli $\mathcal{N}(N, v_1)$ and $\mathcal{N}(N, v_2)$, kernels $\mathcal{K}(N, v_1)$ and $\mathcal{K}(N, v_2)$, least-cores $\mathcal{LC}(N, v_1)$ and $\mathcal{LC}(N, v_2)$, and cores $\mathcal{C}(N, v_1)$ and $\mathcal{C}(N, v_2)$ are represented as

$$\mathcal{N}(N, v_1) = \mathcal{N}(N, v_2) = \mathcal{K}(N, v_1) = \mathcal{K}(N, v_2) = \mathcal{CH}(\{(2, 2, 2, 2)\}) = \{(2, 2, 2, 2)\},$$

$$\mathcal{LC}(N, v_1) = \mathcal{LC}(N, v_2)$$

$$= \mathcal{CH}(\{(1, 1, 3, 3), (2, 1, 3, 2), (3, 2, 2, 1), (3, 3, 1, 1), (2, 3, 1, 2), (1, 2, 2, 3)\}),$$

$$\mathcal{C}(N, v_i) = \mathcal{CH}(\{\boldsymbol{m}^\sigma(N, v_i)\}_{\sigma \in \Pi(N)}), \quad i = 1, 2,$$

where $\mathcal{CH}(\boldsymbol{S})$ is the convex hull of the set of payoff vectors $\boldsymbol{S}$. Here, the least core $\mathcal{LC}(N, v_i)$, $i = 1, 2$ coincides with the ϵ-core $\mathcal{C}_\epsilon(N, v_i)$ with $\epsilon = 1$ (i.e., $\mathcal{LC}(N, v_i) \subsetneq \mathcal{C}(N, v_i)$, $i = 1, 2$). These solutions are represented by the form $(\alpha, \beta, 3 - \beta, 3 - \alpha)$ as shown in Fig. 3. Further, we have the allocation sets $A(N, v_1) = \mathcal{N}(N, v_1) = \mathcal{K}(N, v_1)$ and $A(N, v_2) = \mathcal{C}(N, v_2)$. Interestingly, $A(N, v_1) \subsetneq \mathcal{LC}(N, v_1)$ and $A(N, v_2) \supsetneq \mathcal{LC}(N, v_2)$; i.e., the allocation set $A(N, v)$ and the least-core $\mathcal{LC}(N, v)$ are considered as two different types of refinements of the core $\mathcal{C}(N, v)$.

6 Conclusions

In this paper, we proposed a new solution concept $A(N, v)$ from the viewpoint of coalition formation and Pareto improvement. More specifically, if $\boldsymbol{x} \in A(N, v)$, then $\boldsymbol{x}$ is an improvement of some acceptable imputation $\boldsymbol{y}^T$ for each subgame (T, v^T), $T \subseteq N$. Then, all players in the coalition T have incentives to form the grand coalition N in seeking increases to their allocations. Further, if $A(N, v) = \emptyset$, then there exists some coalition $T \subseteq N$ such that every acceptable allocation of (T, v^T) cannot be improved by any imputation of (N, v) (i.e., at least one player in T has no incentive to form the grand coalition, thus the grand coalition will not be formed). In our paper, we showed that the allocation set $A(N, v)$ is a subset of the core $\mathcal{C}(N, v)$ in any game (N, v). The existence or non-emptiness of

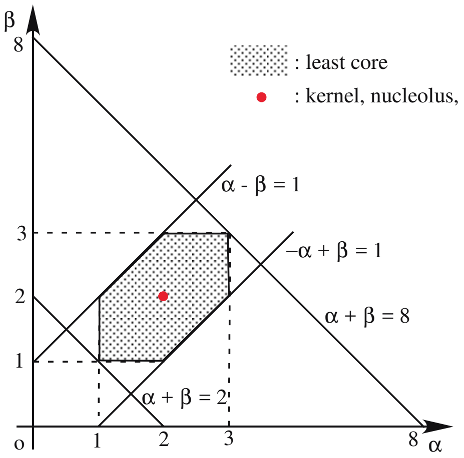

Fig. 3. Illustrating $A(N, v_1) \neq A(N, v_2)$ but other solution concepts of (N, v_1) and (N, v_2) coincide

$\mathcal{C}(N, v) \setminus A(N, v)$ suggests that "the notion having incentives to form the grand coalition" is stronger than "the notion of having no incentive to split off from the grand coalition". When $|N| \leq 3$ or (N, v) is convex, $A(N, v)$ coincides with $\mathcal{C}(N, v)$ (i.e., the above two notions regarding incentives coincide). Sometimes the least core $\mathcal{LC}(N, v)$ contains the allocation set $A(N, v)$, while at other times the allocation set $A(N, v)$ contains the least core $\mathcal{LC}(N, v)$, as shown in Example 3. More specifically, the allocation set $A(N, v)$ and the least-core $\mathcal{LC}(N, v)$ are considered to be two different refinements of the core $\mathcal{C}(N, v)$. As future work, we put forth the following conjecture.

Conjecture 1. Let (N, v) be a game. If $A(N, v) \neq \emptyset$ then $A(N, v) \supseteq \mathcal{K}(N, v)$ Therefore, $\mathcal{N}(N, v) \subseteq A(N, v)$ whenever $A(N, v) \neq \emptyset$.

References

1. Aumann, R.J., Hart, S. (Eds): Handbook of game theory with economic applications Vol. 1, North Holland (1992)
2. Curiel, I.: Cooperative Game Theory and Applications: Cooperative Games Arising from Combinatorial Optimization Problems, Kluwer Academic Publishers (1997)
3. Davis, M., Maschler, M.: The kernel of a cooperative game, Naval Research Logistics Quarterly, 12, 223-259 (1965)
4. Driessen, T.: Cooperative Games, Solutions and Applications, Kluwer Academic Publishers (1988)
5. Ichiishi, T.: Supermodularity: applications to convex games and to the greedy algorithm for LP, Journal of Economic Theory, 25, 283–286 (1981)
6. Kahan, J.P., Rapoport, A.: Theories of Coalition Formation, Psychology Press (1984)
7. Maschler, M., Peleg, B., Shapley, L.S.: Geometric properties of the kernel, nucleolus, and related solution concepts, Mathematics of Operations Research, 4(4), 303-338 (1979)

8. Schmeidler, D.: The nucleolus of a characteristic function game, Siam Journal of Applied Mathematics, 17, 1163–1170 (1969)
9. Osborne, M.J., Rubinstein, A.: A Course in Game Theory, The MIT Press (1994)
10. Peleg, B., Sudhölter, P.: Introduction to The Theory of Cooperative Games, Kluwer Academic Pub. (2004)
11. Shapley, L.S.: Cores of convex games, International Journal of Game Theory, 1, 11–26 (1971)
12. Shapley, L.S., Shubik, M.: The core of an economy with nonconvex preferences, RM-3518, The Rand Corporation, Santa Monica, CA. (1963)
13. Tamir, A.: On the core of cost allocation games defined on location problems, Transportation Science, 27(1), 81–86 (1992)