Iterative Revelation Mechanisms[*]

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Abstract

This paper considers dynamic resource allocation processes, called iterative revelation mechanisms, with quasi-linear dichotomous utilities and complete information. The social planner gradually identifies the state of the world through a sequence of binary questions, and monetary transfer is determined on a "pay-as-bid" basis. Although incentive compatibility is not guaranteed, an allocation rule is implemented in a subgame perfect Nash equilibrium if it satisfies the strong monotonicity. In particular, the efficient allocation rule is implemented regardless of details of a binary-question process. We also show that if a mechanism is ex post incentive compatible, it is an ascending-price mechanism. In a single-object allocation problem, the English auction is a unique mechanism satisfying efficiency, ex post incentive compatibility, and pay-as-bid transfer.

Keywords: iterative revelation mechanism, ascending price, English auction, strong monotonicity, binary question

JEL codes: D44, D82, D71, H41

1 Introduction

This paper considers dynamic indirect mechanisms that mimic real collective decision-making processes and allocation rules in practical situations, such as public goods and auctions. Although we often focus on direct revelation mechanisms by the Revelation Principle for such problems, they are sometimes not practical or applicable; however, various types of indirect mechanisms are only available or actually used. In particular, typical mechanisms have small message spaces so that the social planner can collect only partial information about agents' types at once. In such a case, a collective decision-making process may need multiple rounds to collect enough information to determine the desirable outcome.

Suppose a public good provision problem in which the government is going to determine whether or not to build a bridge. The government does not know the valuations of the bridge for agents, so it needs to collect information about them. However, the government often cannot ask each agent his value of the bridge directly because such a resource allocation process is too costly. In a typical allocation process, the government offers a construction plan along with an associated cost sharing (or tax increase), and agents simply vote for or against it. Thus, agents' message space is just binary. In addition, when the current plan is rejected by agents, the government sometimes modifies the plan and proposes again. A final decision may be made after several rounds of votes.

More specifically, let us consider the following example of a discrete public good problem with two agents.

Example 1 The government determines whether or not to build a bridge, which costs 20. There are two agents, A and B, and their values of the bridge are $v_A = 15$ and $v_B = 9$. To find the efficient outcome along with monetary transfer, the government determines an allocation and payments in the following process, similar to a "tatonnement." The government offers a construction with a cost sharing (p_A, p_B) with $p_A + p_B = 20$ to agents. If both agents agree with the current plan, it is im-

plemented. If one agent, say A, agrees but the other does not, then the government modifies the cost sharing and proposes again; the government increases A's share by 1 and decreases B's share by 1. The government continues modifying and re-proposing a plan as long as only one agent agrees. If both agents are against a plan, then the government determines to not build.

Suppose that both agents respond truthfully and sincerely and that the government starts the collective decision-making process from the equal cost sharing $(p_A, p_B) = (10, 10)$. Agent A agrees with the initial plan, whereas agent B does not. The government proposes $(11, 9)$ in the second round, then both agents agree with it. The bridge is built with payments 11 by A and 9 by B.

Indirect mechanism design using such binary questions is investigated in the growing literature of mechanism design with communication complexity, such as Van Zandt (2007), Fadel and Segal (2009), and Mookherjee and Tsumagari (2014). Among them, Fadel and Segal (2009) clarify the difference between communication cost for determining an allocation rule and for implementing it. That is, they show that additional cost exists for computing incentive-compatible monetary transfers other than the communication cost of finding the desirable allocation. This implies that the social planner may have to ask questions that do not influence the allocation but determine monetary transfer. However, with communication complexity, the planner may not afford to ask such questions that do not affect allocation decisions.

In Example 1, the government successfully identifies the efficient outcome under truthful behavior, but the monetary transfer rule does not induce incentive compatibility. Agent B clearly has an incentive to disagree with the payment of 9. However, the government does not have information from the observed responses enough to impose (ex post) incentive compatibility. In addition, when the government deals with budget constraints, it may employ a decision-making process like above, knowing it is not incentive compatible.

Motivated by the above example, we consider a class of dynamic resource allocation processes comprising *threshold binary questions* and a simple monetary transfer

rule on the *pay-as-bid* basis, called *iterative revelation mechanisms (IRMs)*. We consider an allocation problem in which state of the world is characterized by a vector of values for agents. Discrete public good problem such as Example 1 and an auction problem are included. In each round of a process, the social planner offers prices to some agents. Agents make a binary response of *yes* (to accept paying the offered price) or *no* (to reject the offered price). The planner iteratively identifies agents' values through price questions and determines an outcome according to the information elicited. Agents pay the maximum price for which they responded yes in the process.

We show that under complete information, an allocation rule is implemented in a subgame perfect Nash equilibrium (SPNE) if it satisfies the *strong monotonicity*. An allocation rule is said to be strongly monotone if, when an allocation under a state is desirable for agent i, the allocation does not change by increasing his value. Because of the fixed monetary transfer rule, incentive compatibility is not guaranteed. Although agents strategically misreport their valuations, the equilibrium allocation rule coincides with that obtained by truthful behavior. In particular, the efficient allocation rule is strongly monotone. The efficient outcome is implemented in SPNE regardless of offer-price principles. In addition, in whatever manner the social planner offers prices to agents, there exists the same threshold strategy that constitutes an SPNE.

We then turn to a mechanism design question and consider ex post incentive compatibility of IRMs. Under the specified equilibrium, agents strategically tell lies to the social planner, using the values and observed actions by the others. Our equilibrium analysis critically relies on complete information. We show that if we impose ex post incentive compatibility on an IRM while keeping the pay-as-bid transfers, then the IRM must be an ascending-price mechanism. In a single-object auction environment with two buyers, the unique efficient and ex post incentive-compatible IRM is an English auction.

Although an ascending-price mechanism is necessary for ex post incentive com-

patibility, it is not sufficient. We can borrow knowledge from the literature of ascending-price auctions, and an ascending-price mechanism is efficient and ex post incentive compatible only in a limited environment such as a single-object auction.

1.1 Contribution of the Paper

The contribution of this paper is twofold. First, we provide an equilibrium for dynamic binary question mechanisms that mimic real resource allocation processes. From the mechanism design point of view, Fadel and Segal (2009) show that even if an IRM identifies the desirable outcome under truth telling, the social planner may not be able to calculate proper monetary transfer that induces ex post incentive compatibility. Our first result complements Fadel and Segal (2009), and shows that even if an IRM cannot calculate ex post incentive compatible transfer, the desirable allocation rule is implemented in SPNE with a simple pay-as-bid transfer rule. When communication is limited and the social planner can ask only a limited number of questions, the communication cost for imposing incentive compatibility is a constraint to increase the social welfare. Our result indicates that an allocation rule designed without incentive compatibility is implemented in SPNE as long as it is strongly monotone. Hence, we are able to investigate the welfare-maximizing algorithm under limited communication, ignoring the incentive compatibility constraint, which is an algorithmic issue and beyond this research.

Second, we give a fundamental support for focusing on ascending-price mechanisms among various dynamic indirect mechanisms. The so-called Vickrey–Clarke–Groves (VCG) mechanism is known as a unique direct mechanism that is efficient, dominant strategy incentive compatible, and individually rational. In a single-object auction, the VCG mechanism is a second-price auction, and an English auction is known as a strategically equivalent dynamic mechanism in a private value setting. Although many studies investigate ascending-price auctions for multiple objects corresponding to the VCG mechanism (Ausubel, 2004; Ausubel and Milgrom, 2002; de Vries et al., 2007), there has been no clear argument for considering an ascending-

price format only. We provide an axiomatic characterization of an English auction.

1.2 Related Literature

This study is related to a growing literature on mechanism design with communication complexity. Static single-object auction design under restricted message space is studied by Blumrosen et al. (2007) and Kos (2012). Blumrosen and Feldman (2013) consider implementability of information-theoretically optimal allocation rules in several settings. Dynamic mechanisms reduce communication. Mookherjee and Tsumagari (2014) provide a necessary and sufficient condition for Bayesian incentive compatibility. Kos (2014) considers an IRM design in a single-object auction with limited communication.

For the characterization of an English auction, Li (2015) introduces a notion of obvious strategy-proofness to explain the superiority of an English auction over a second-price auction in experimental studies.

In this paper, we consider the environment in which agents have quasi-linear utilities and one-dimensional type. Each agent has dichotomous preferences, in which outcomes are classified into just two categories: good or bad outcomes. Each agent makes the same value for every outcome in the same category. Hence, each agent's type is represented by a single value for a good outcome, by normalizing zero-value for a bad outcome (Babaioff et al., 2005; Mishra and Roy, 2013). Our model includes discrete public good, single-object auctions, and double auctions with single-unit demand and supply. In addition, even a multiple-object auction problem is also included if bidders are "single-minded," and interested in particular packages of goods. Auctions with single-minded bidders are studied by Lehmann et al. (2002), Sano (2011), and Milgrom and Segal (2013) among others. A companion paper of mine (Sano, 2015) examines ascending-price package auctions with single-minded bidders, but allows a flexible pricing rule. It shows that efficient allocation is achievable in an SPNE for single-minded bidders, but that the efficient allocation may not be achievable when a bidder is not single-minded.

In multiple-object auctions, preference elicitation is an issue because bidders have a complex valuation function that evaluates each package of goods. Ascending auctions are typical mechanisms that reduce the amount of communication compared with direct revelation mechanisms. Ausubel (2004, 2006) provides incentive-compatible ascending or dynamic auctions for multiple homogeneous or heterogeneous goods. Ausubel and Milgrom (2002) and Bikhchandani and Ostroy (2002) provide relationships between the VCG outcome and core of the auction game. The VCG outcome is in the core if goods are substitutes and they propose ascending auction mechanisms that converge to the VCG outcome. These mechanisms are ex post incentive compatible. Nisan and Segal (2006), however, show that the communication required for finding the efficient allocation is exponential in number of goods.

In a discrete public good game, the efficient provision of a public good is achievable in a complete information Nash equilibrium (Palfrey and Rosenthal, 1984). Admati and Perry (1991) study dynamic or sequential contribution mechanisms to public good or joint project. They show that the efficient outcome is achieved in an SPNE when past contributions are refundable.

Bernheim and Whinston (1986) formulate a generalized first price auction, called a "menu auction," and show that the efficient (and core) outcome is achieved in a Nash equilibrium. Bergemann and Valimaki (2003), in a framework of common agency, extend Bernheim and Whinston to a different type of a dynamic allocation process, called "agenda game." A set of admissible outcomes is chosen by an auction in the first round and the final outcome is auctioned from the selected admissible outcomes in the second round.

2   Model

Let $N \equiv \{0, 1, 2, \ldots, n\}$ be the set of all individuals. The benevolent social planner is denoted by 0 and $I = \{1, \ldots, n\}$ is the set of agents. The social planner chooses an outcome x from a finite set of alternatives X along with monetary transfer.

Each agent has a quasi-linear utility function with integer-valued valuation function $u_i : X \to \mathbb{Z}$.[1] Agent i's utility is denoted by $\pi_i = u_i(x) - p_i$, where p_i denotes the monetary transfer to the planner. Each agent has dichotomous preferences; i.e., he has a non-empty set of *interests* and has the same value for each outcome of his interest. Let $X_i \subset X$ be the set of i's interests. Agent i's valuation function takes the form of

$$u_i(x) = \begin{cases} v_i & \text{if } x \in X_i \\ 0 & \text{if } x \notin X_i \end{cases}. \tag{1}$$

For simplicity of notation, a set of consecutive integers from a to b is denoted by $[a, b]$. Domain of values is bounded and denoted by $V_i \equiv \{\underline{v}_i, \underline{v}_i + 1, \ldots, \bar{v}_i\} = [\underline{v}_i, \bar{v}_i]$. It is possible for agents to have a negative value or cost $v_i < 0$ for outcomes of interest. The social planner's personal utility function is given by $\pi_0 = u_0(x) + \sum_I p_i$, where u_0 is the planner's personal valuation function, which represents a cost function in a public good environment or valuation of the seller in an auction environment. We do not assume that u_0 is dichotomous, however, u_0 is commonly known to each other.

We assume that each agent's interests X_i and domain of values V_i are all commonly known to each other including the planner. This is natural in such problems as public goods and auctions. The state of the world is simply the vector of values, $v = (v_1, \ldots, v_n) \in V \equiv \times_{i \in I} V_i \subset \mathbb{Z}^n$. Moreover, we assume that a state v is commonly known to each agent but only the social planner does not know the state.

2.1 Iterative Revelation Mechanisms

We consider an environment in which the planner is not able to directly ask each agent his value. The social planner gradually collects information about the state of the world by a sequence of threshold binary questions to find the efficient outcome. The planner offers prices (or subsidies) for achieving an outcome of their interests at once. When the planner offers p_i to agent i and he is willing to pay p_i (or receive $-p_i$

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[1]Integer valuation is not crucial in our analysis. It guarantees that a game has finite horizon, especially in case where the social planner finds the efficient outcome.

when $p_i < 0$) for $x \in X_i$, it implies $v_i \geq p_i$. Conversely, when agent i responds that he is not willing to pay p_i, it implies $v_i < p_i$.[2] By offering multiple different prices, the planner iteratively identifies v_i. For example, if agent i accepts a payment $p_i = 5$ and rejects $p_i = 12$, then the social planner considers i's value to be between 5 and 11.

Formally, from the responses by agents up to round t of a resource allocation process, the planner identifies i's value as in a set $V_i(h^t)$, where h^t indicates a history at the end of t. Any value $\tilde{v}_i \in V_i(h^t)$ is consistent with all the responses from agent i. We call $V_i(h^t)$ the *revealed value set* of i at round t. Note that $V_i(h^t)$ should have the form of $V_i(h^t) = [\underline{v}_i(h^t), \bar{v}_i(h^t)]$ and $V_i(\emptyset) = V_i$, where $h^t = \emptyset$ is a null history[3]. Let $V(h^t) \equiv \times_{i \in I} V_i(h^t)$, called the *revealed state set at t*, which indicates the set of possible states consistent with a history up to t. Notice that $V_i(h^t) \subseteq V_i(h^s)$ and $V(h^t) \subseteq V(h^s)$ for all $t > s$ and that $V(h^t) \neq \emptyset$ as long as any irrational behavior is ruled out. As the social planner makes a question to an agent, his revealed value set shrinks and, accordingly, the revealed state set also does.

An *iterative revelation mechanism (IRM)* is defined by $\Gamma = (\{J^t, p^t\}_t, g, p)$ as follows. For each round $t = 1, 2, \ldots$, $J^t : H^{t-1} \rightarrow 2^I$ determines a set of agents whom the planner offers a price, where $h^t \in H^t$ indicates a history at the end of round t with $H^0 = \{\emptyset\}$. The price the planner offers to agent $i \in J^t(h^{t-1})$ is determined by $p_i^t : H^{t-1} \rightarrow \mathbb{Z}$. Without loss of generality, we assume that for every t, every $h^{t-1} \in H^{t-1}$, and every $i \in J^t(h^{t-1})$, $p_i^t \in V_i(h^{t-1}) \setminus \{\underline{v}_i(h^{t-1})\}$.[4] Each agent $i \in J^t(h^{t-1})$ at t makes a report $a_i^t \in \{yes, no\}$. The mechanism terminates at T when $J^{T+1}(h^T) = \emptyset$.[5] An entire history is denoted by $h \in H$, where H is

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[2]We assume that a negative response against a payment of p_i implies a strict preference in order that the social planner certainly identifies v_i.

[3]Formally, $\underline{v}_i(h^t) = \max_{\tau \leq t}\{p_i^\tau | a_i^\tau = yes\}$ and $\bar{v}_i(h^t) = \min_{\tau \leq t}\{p_i^\tau | a_i^\tau = no\} - 1$, where a_i^τ indicates a response at round τ.

[4]By revealed preferences, agent i must accept any $p_i \leq \underline{v}_i(h^{t-1})$ and reject any $p_i > \bar{v}_i(h^{t-1})$. Any irrational behavior is ruled out and $V_i(h^t) \neq \emptyset$.

[5]We do not impose a specific termination condition. By definition and assumption, the process terminates in finitely many rounds.

a set of all entire histories. For each entire history $h \in H$, an outcome $g(h) \in X$ and monetary transfers to the planner $p(h) \in \mathbb{Z}^n$ are determined. In other words, a process $\{J^t, p^t\}_t$ describes a game tree in which each node has two successive nodes. A history h^t is an information set, and an entire history h is a terminal node of the game tree. Functions g and p are a mapping from each terminal node to an outcome and monetary transfer. A decision rule g along with an offer-price scheme $\{J^t, p^t\}_t$ is sometimes called a *protocol*.

We do not impose any specification on an offer-price process $\{J^t, p^t\}_t$. The social planner can employ various offer-price principles such as ascending-price, descending-price, and others. For example, in an ascending-price principle, the planner starts with offering a sufficiently low price. The planner gradually increases the price as long as an agent responds yes. Another principle is bisection method, in which the planner offers the median value in an agent's revealed value set in each round. Bisection method can minimize the number of questions for uniquely specifying i's value.

Although we allow an arbitrary offer-price principle, we focus on a particular monetary transfer rule, called the *pay-as-bid* rule.

Assumption 1 (Pay-as-bid transfer) The payment rule is determined by

$$
p_i(h) = \begin{cases} \underline{v}_i(h) & \text{if } g(h) \in X_i \\ 0 & \text{if } g(h) \notin X_i \end{cases}.
$$

If the final outcome is of i's interest, then he needs to pay the minimum value in his revealed set. Equivalently, each agent pays the maximum amount that he said *yes* to in the mechanism. Thus, this payment rule is a dynamic version of a "pay-as-bid" mechanism and can be viewed as an extension of Bernheim and Whinston's (1986) menu auction.

Assumption 1 is restrictive from the mechanism design point of view, which considers payment schemes that incentivize agents. A justification for this assumption is that the social planner has to minimize communication to find the efficient outcome.

Fadel and Segal (2009) show that there exists additional communication to calculate ex post incentive-compatible transfer. If the communication cost may be large, then the social planner might give up collecting additional information after the efficient outcome is identified.

Agents observe all the past information. A (pure) strategy $\sigma_i \in \Sigma_i$ of agent i in an IRM Γ is a profile of actions $a_i(z) \in \{yes, no\}$ at every decision node $z = (h^{t-1}, p_i^t)$ such that $i \in J^t(h^{t-1})$. An agent i behaves *sincerely* when his strategy is such that for every i's decision node $z = (h^{t-1}, p_i^t)$, $a_i(z) = yes$ if and only if $p_i^t \leq v_i$. The equilibrium concept is a subgame perfect Nash equilibrium (SPNE).

2.2 Direct Allocation Rule

For a protocol $(\{J^t, p^t\}, g)$, let $\phi : V \to H$ be a mapping from a state to the associated history, assuming sincere reporting by agents. In other words, ϕ is an inverse mapping of $V(h)$. We define a *direct allocation rule* $f : V \to X$ associated with $(\{J^t, p^t\}, g)$ as $f = g \circ \phi$. A direct allocation rule is a mapping from each state $v \in V$ to a corresponding allocation $x \in X$ via history generated by sincere behavior.

The *efficient allocation rule* (with respect to reports) is denoted by f^* and

$$f^*(v) \in \arg\max_X \sum_{i \in N} u_i(x). \tag{2}$$

A direct allocation rule f is said to be *monotone* if for all $i \in I$, all $v \in V$, and all $\tilde{v}_i > v_i$,

$$f(v) \in X_i \Rightarrow f(\tilde{v}_i, v_{-i}) \in X_i.$$

Moreover, a direct allocation rule f is said to be *strongly monotone* if for all $i \in I$, all $v \in V$, and all $\tilde{v}_i > v_i$,

$$f(v) \in X_i \Rightarrow f(\tilde{v}_i, v_{-i}) = f(v). \tag{3}$$

Strong monotonicity is clearly equivalent to monotonicity if each agent's interests set is singleton as in the case of a discrete public good problem. Notice that the efficient allocation rule f^* is also strongly monotone.

Definition 1 An IRM Γ is said to be *(strongly) monotone* if the associated direct allocation rule f is (strongly) monotone. An IRM Γ is said to be *efficient* if the associated direct allocation rule is efficient and f^*.

Given a (strongly) monotone allocation rule f, the *critical value* $c_i^f(v_{-i})$ is defined by

$$c_i^f(v_{-i}) = \begin{cases} \min\{\tilde{v}_i \in V_i | f(\tilde{v}_i, v_{-i}) \in X_i\} & \text{if exists} \\ \infty & \text{otherwise} \end{cases}. \tag{4}$$

Given the other agents' values, a critical value of agent i is the minimum value that the associated allocation is of i's interest.

2.3 Examples

2.3.1 Discrete Public Good

The discrete public good problem illustrated in Example 1 is restated using the formal notations as follows. The government determines whether to build a bridge ($x = 1$) or not ($x = 0$); $X = \{0, 1\}$. There are two agents, A and B, whose interests are $X_A = X_B = \{1\}$. Agents' values take an integer from $V = [0, 20]$, and their values are assumed to be $v_A = 15$ and $v_B = 9$. The government's cost function is given by $u_0(0) = 0$ and $u_0(1) = -20$.

In an IRM, the government offers a cost sharing (p_A^t, p_B^t) simultaneously or sequentially, keeping $p_A^t + p_B^t = 20$. If both agents agree with a plan, it implies $v_i \geq p_i^t$ for $i = A, B$, which means the bridge is built. If an agent is against the plan, it implies $v_i < p_i^t$, so that the government modifies the cost sharing. If both agents disagree with a plan, it implies $v_A + v_B < 20$ and the bridge is not built.

2.3.2 Single-Object Auctions

Another example is an auction problem. In a single-object auction, the social planner allocates a single unit of an object to potential buyers. The winner of an auction earns a value v_i.

A typical efficient IRM is an English auction. The planner as the auctioneer offers all agents an initial price p that is sufficiently low. The planneer gradually increases the price as long as two or more agents are active and respond yes for the current price. Once only one agent responds yes for a price, then the auction ends with that price.

A Dutch auction is also a typical IRM. Grigorieva et al. (2007) propose another iterative auction named the "bisection auction," in which the planner offers the median value of the revealed value set of a potential winner in each round[6].

2.3.3 Package Auction with Single-Minded Bidders

Even in a multiple-object auction problem, so-called single-minded bidders are an example of dichotomous preferences. Consider a hypothetical spectrum license auction with two goods and three bidders. There are two spectrum licenses. One of the two, denoted by E, is a license covering the eastern area, whereas the other, denoted by W, covers the western area. Bidders 1 and 2 are local telecommunications firms. Bidder 1 operates in the eastern area and wants license E only. Bidder 2 operates in the western area and wants license W only. Bidder 3 is a global firm that operates in both areas and wants both E and W.

An ascending-price auction, called the combinatorial clock auction, proceeds as follows (Porter et al., 2003). The initial price vector for licenses are set $(p_E^1, p_W^1) = (0, 0)$. Bidders report their demand under the current price vector. Each price increases as long as there is an excess demand. The combinatorial clock auction is interpreted as an IRM. In each round of the auction, bidders are simultaneously offered a price $(p_1^t, p_2^t, p_3^t) = (p_E^t, p_W^t, p_E^t + p_W^t)$. The combinatorial clock auction terminates when the efficient allocation is identified. Winners pay the terminal price of the good(s) that they demand.

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[6]Grigorieva et al. (2007) do not impose the pay-as-bid transfer rule but design an ex post incentive compatible payment rule.

3 Main Result

To simplify the analysis, we mainly focus on a special case of *perfect information IRMs*, in which for all t and all h^{t-1}, $|J^t(h^{t-1})| = 1$. It is easy to arrange the result to the case of imperfect information, which is described later.

To state the result, we introduce a notion of *target values*.

Definition 2 The *target value* v_i^t of i at round t is defined by

$$v_i^t \equiv \begin{cases} v_i & \text{if } v_i \in V_i(h^t) \\ \bar{v}_i(h^t) & \text{if } v_i > \bar{v}_i(h^t) \\ \underline{v}_i(h^t) & \text{if } v_i < \underline{v}_i(h^t) \end{cases} \cdot \qquad (5)$$

Since agents observe all the past actions, they know the target value of each other. Notice that $v_i^t \in V_i(h^t)$.

Using target values, we specify an SPNE for every strongly monotone IRM. The following theorem is the first main result, which states that every strongly monotone IRM has an SPNE that implements the associated allocation rule f.

Theorem 1 *Suppose an IRM is perfect information and strongly monotone. Then there exists an SPNE in which each agent i takes the following strategy:*

$$a_i(h^{t-1}, p_i^t) = \begin{cases} yes & \text{if } p_i^t \leq \min\{v_i, c_i^f(v_{-i}^{t-1})\} \\ no & \text{otherwise} \end{cases}, \qquad (6)$$

where f is the associated direct allocation rule. The SPNE implements f.

Proofs are given in the Appendix.

An intuition of Theorem 1 is given as follows. By definition of target values, each agent j is willing to pay at most v_j^{t-1} given the observed actions. It is clear that every agent responds no sincerely for any price $p_i^t > v_i$ and a non-negative payoff is guaranteed. Suppose that agents foresee that the equilibrium outcome is $f(v^{t-1})$, and that agent i makes an action at round t. Suppose $f(v^{t-1}) \in X_i$ and $p_i^t < v_i$. If $f(v^t) \in X_i$ after i's response and his forecast is true, $f(v^t)$ would be the final

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outcome and he would obtain a non-negative payoff. If $f(v^t) \notin X_i$ after i's response and his forecast is true, the final outcome would be given by $f(v^t)$ and i's payoff would be zero. Therefore, each agent should keep $f(v^t) \in X_i$ to maximize his payoff. By definition of the critical value, each agent responds yes for any price below the critical value to keep $f(v^t) = f(v^{t-1})$. He responds no for any price above the critical value to reduce his payment.

Agents tell lies in equilibrium. They respond no for a price above $c_i^f(v_{-i}^{t-1})$ to reduce their payments.

Because the efficient allocation rule is strongly monotone, every efficient IRM achieves the efficient allocation in equilibrium.

Corollary 1 *For every efficient IRM, there exists an SPNE that implements the efficient allocation.*

In efficient IRMs, the critical value is defined by the famous "Vickrey-Clarke-Groves" price:

$$c_i^{f^*}(v_{-i}^{t-1}) = v_i - \max_X \sum_{j \in N} u_j^{t-1}(x) + \max_X \sum_{j \neq i} u_j^{t-1}(x),$$

where

$$u_j^{t-1}(x) = v_j^{t-1} \mathbf{1}_{\{x \in X_j\}}.$$

Remark 1 Multiple equilibria generally exist in the sense that every agent such that $f(v^{t-1}) \notin X_i$ is indifferent to responses as long as $f(v^t) \notin X_i$ holds. However, the equilibrium strategy (6) is unique for agents such that $f(v) \in X_i$ if we assume that agents strictly prefer an outcome $x \in X_i$ with paying the true value v_i to an outcome $x' \notin X_i$ with no transfer. This is straightforward from the incentive to minimize payments. Multiple equilibria would be eliminated by additionally imposing trembling-hand perfection, but we do not formally confirm it. The uniqueness of SPNE depends on perfect information and is not a robust property.

3.1 Illustrations

In this subsection, we describe illustrations of SPNE of Theorem 1 for some efficient IRMs.

3.1.1 Discrete Public Good

An illustration of the equilibrium path is given for Example 1 in the Introduction. The values of the bridge for agents A and B are supposed to be $v_A = 15$ and $v_B = 9$, respectively. The construction cost is $u_0(1) = -20$. The marginal contribution of each agent is $M_A = M_B = 4$. Hence, the threshold prices of agents in the specified equilibrium are $v_A - M_A = 11$ and $v_B - M_B = 5$ in the initial round.

In the example, the government first offers the equal cost sharing $(p_A^1, p_B^1) = (10, 10)$. The equilibrium action is yes for agent A and no for agent B. In the second round, the government offers $(11, 9)$. Then, in the equilibrium, agent B responds no, whereas agent A responds yes. B's action in the second round decreases his target value to $v_B^2 = 8$, which makes A's marginal contribution $M_A^2 = 3$. The government offers $(12, 8)$ in the third round. Because the change in B's target value increases A's threshold price $v_A - M_A^2 = 12$, agent A responds yes and agent B responds no. Then, B's target value decreases again and $v_B^3 = 7$, which makes A's marginal contribution $M_A^3 = 2$ and increases his threshold price $v_A - M_A^3 = 13$, and so on. Eventually, the government offers $(15, 5)$, and both agents respond yes. The bridge is built with payments $p_A = 15$ and $p_B = 5$, which achieves the efficient outcome.

3.1.2 Single-Object Auctions

To see the independence of offer-price principles for the equilibrium strategy, let us consider a single-object auction next. Suppose that there are two bidders A and B, who have a value $v_A = 10$ and $v_B = 13$, respectively. Hence, bidder B wins the object in the efficient outcome. The marginal contribution of bidder B is 3, whereas that of A is zero. Hence, under the equilibrium strategy, both A and B have the same threshold price of 10. Assume $V_A = V_B = [0, 15]$ and that the planner always

chooses bidder B in tie cases.

In an English auction, the planner gradually increases a price from $p^1 = 1$. In the SPNE, both A and B continue to respond yes until $p^{10} = 10$, and they respond no at $p^{11} = 11$. By assumption, bidder B wins with the payment of 10, which is the same as the truthfully-bidding equilibrium of an English auction.

In a Dutch auction, the planner gradually decreases a price from $p^1 = 15$. In the SPNE both A and B continue to respond no until $p^5 = 11$, and they respond yes at $p^6 = 10$. Similarly by assumption, bidder B wins with the payment of 10 as in the English auction.

In the bisection auction of Grigorieva et al. (2007), the planner offers the median value of a revealed value set. At the initial round, the planner offers $p^1 = 8$, which is median of $[0, 15]$ to both bidders. Because they respond yes for 8, the planner next offers $p^2 = 12$. Because they respond no for 12, the planner next offers $p^3 = 10$, where both bidders respond yes. Finally, the planner offers $p^4 = 11$, both bidders respond no, and the auction ends. Bidder B wins with the payment of 10 as in the English and Dutch auction.

Thus, in every offer-price principle, the proposed threshold strategy forms an SPNE and results in the same outcome and monetary transfer. However, in general, the resulting monetary transfer depends on offer-price principle.

3.2   Efficient IRMs and Core

In the above example of public good, the government successfully achieves the efficient provision of the bridge along with budget balance. To be precise, the equilibrium outcome of an efficient IRM is in the core with respect to the true state of the world. To consider the core of the allocation problem, we formulate the coalition value function. We assume that a coalition value generated by agents only is zero. This is justified if we impose the excludability on the feasible outcomes X.

For example, let $X(J) \subset X$ be a set of feasible outcomes achievable for a coalition $J \cup \{0\}$. A coalition $J \cup \{0\}$ selects an outcome from $X(J)$. A set X of alternatives

satisfies the *excludability* if for each $i \notin J$, $X_i \cap X(J) = \emptyset$. The excludability holds for various environment such as an auction problem. Although pure public good problems do not necessarily satisfy the excludability, it is often assumed in similar problems. The excludability implies that any coalition including the social planner can exclude agents outside the coalition. The coalition value of J is defined as the maximum social welfare generated by J. The coalition value function (or characteristic function) $\omega : 2^N \times V \to \mathbb{R}$ is defined by

$$\omega(J; v) = \begin{cases} \max_{X(J_{-0})} \sum_{j \in J} u_j(x) & \text{if } 0 \in J \\ 0 & \text{if } 0 \notin J \end{cases}. \tag{7}$$

A payoff profile is in the core if it is efficient, individually rational, and not blocked by any coalition. The core given a state of the world is denoted by

$$C(\omega, v) = \{\pi \in \mathbb{R}^{n+1} | (\forall i \in N)\pi_i \geq 0, (\forall J \subseteq N) \sum_{j \in J} \pi_j \geq \omega(J, v)\}.$$

The following theorem states that the payoff profile in every efficient IRM is in the core. In the above definition of core, the social planner is also included as a player. This implies that in a public good problem, for example, the government's revenue from agents covers the provision cost. Similarly, in a double auction with single-unit demand and supply, a Walrasian auctioneer has no budget deficit in equilibrium.

Theorem 2 *Suppose that the coalition value function ω is defined as equation (7). The equilibrium payoff profile of every efficient IRM, denoted by π^*, is in the core with respect to the true state of the world: $\pi^* \in C(\omega, v)$.*

3.3 When $|J^t| \geq 2$

The above result is easily arranged to the case where the social planner can simultaneously offer prices to many agents. The equilibrium strategy for such cases is basically the same as Theorem 1, but some coordination of responses is necessary to avoid a tragedy, in which many agents simultaneously say no and their contributions

fall short to the desirable outcome. Thus, a coordination problem should be solved in each round.

Corollary 2 *Suppose $|J^t| \geq 1$. In every strongly monotone IRM, there exists a following SPNE: for any history h^{t-1} and (J^t, p^t), find a maximal set $J_n^t \subseteq \bar{J}^t \equiv \{i \in J^t | c_i^f(v_{-i}^{t-1}) < p_i^t \leq v_i\}$ satisfying the following conditions:*

1. *Agent i says no if $i \in J_n^t$ or if $p_i^t > v_i$,*

2. *Agent i says yes if $i \in \bar{J}^t \setminus J_n^t$ or if $p_i^t \leq c_i^f(v_{-i}^{t-1})$, and*

3. *$f(v^t) = f(v^{t-1})$.*

It is obvious that our result is not affected even when the social planner can offer multiple prices to an agent. For example, we have the same equilibrium when the planner offer a set of prices $\{p_{i1}^t, \ldots, p_{ik}^t\}$ to agent i in a round. Agent i responds with the maximum price that he accepts (or says yes) among the offered prices. Note that a direct revelation of a value is equivalent to offering a set of prices $\{\underline{v}_i, \underline{v}_i + 1, \ldots, \bar{v}_i\}$. Then, Corollary 2 shows a direct revelation pay-as-bid mechanism implements every strongly monotone allocation rule in a Nash equilibrium. Bernheim and Whinston (1986) show the special case for the efficient IRM.

Corollary 3 *In a direct mechanism with a strongly monotone allocation rule f and pay-as-bid transfers, there exists a Nash equilibrium, in which each agent submits a bid of $b_i = c_i^f(b_{-i})$ and $f(b) = f(v)$ holds.*

3.4 Limited Communication

Although we do not explicitly consider, our analysis so far incorporates a nature of limited communication. For example, the planner may be able to ask only a limited number of questions of each agent, or in total. Alternatively, it may take a cost to ask an agent a question. In the presence of communication constraints, the planner may not or cannot specify the efficient outcome, but carefully designs a protocol that finds an approximately efficient outcome. In a single-object auction problem,

Blumrosen et al. (2007) and Kos (2012, 2014) examine an informationally efficient protocol that maximizes the social welfare given a limited number of actions. It is challenging to answer the question as to how we can maximize social welfare in a more general allocation problem with limited communication.

Consider a public good problem of Example 1 again. Suppose that the government is able to ask each agent at most one question because it is costly to offer a price. Under a heuristic resource allocation rule, the government offers a price p to each agent simultaneously or sequentially. The government builds the bridge if an agent or both agree and report *yes*. Even in the simple example above, it is not obvious whether the informationally efficient resource allocation rule is implementable or even the definition of such a rule[7].

As Fadel and Segal (2009) show, under limited communication, the ex post incentive compatibility constraint generates efficiency loss because the social planner needs to ask questions just for determining proper transfers. Theorem 1 implies that under complete information, an allocation rule is implemented in SPNE even without incentive compatibility as long as it is strongly monotone. Hence, we can investigate an informationally efficient protocol ignoring the incentive constraint, which is beyond this research.

4 Incentive Compatibility

Theorem 1 show that every strongly monotone IRM achieves the desired allocation rule in an SPNE but each agent strategically misreports his type. Moreover, the specified strategy needs the others' true values and observed information. The analysis critically relies on complete information. Hence, we are naturally interested in an incentive-compatible mechanism.

Definition 3 An IRM is *ex post incentive compatible* if for any state v, sincere reporting is an ex post equilibrium.

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[7]See Blumrosen and Feldman (2013) for the informationally efficient allocation rule for this environment.

We show that an IRM is ex post incentive compatible if it employs an ascending-price principle. In addition to the ex post incentive compatibility, we also impose a weak condition called weak tightness.

Definition 4 An IRM is *weakly tight* if $i \in J^t(h^{t-1})$ implies that there exists a state $\tilde{v} \in V(h^{t-1})$ and $f(\tilde{v}) \in X_i$.

Weak tightness requires that if agent i is a mover of a node, then there is a path such that an outcome of i's interest is realized in that subgame. If it does not hold, then i never realizes any outcome of his interest, regardless of the current and future responses. Thus, agent i may be redundant to make a serious response. If the social planner wants to reduce communication, then the weak tightness would be required.[8]

Now we state the second main result. When we consider an ex post incentive-compatible IRM, we need to limit attention to ascending-price IRMs. This result crucially relies on Assumption 1.

Theorem 3 *If an IRM is monotone, ex post incentive compatible, and weakly tight, then it is an ascending-price mechanism: $p_i^s < p_i^t$ for all i and all $s < t$.*

4.1 Characterizing an English Auction

As a corollary of Theorem 3, an English auction is essentially a unique iterative mechanism that is efficient and ex post incentive compatible in a single-object auction. In this subsection, we focus on a single-object auction environment with two bidders, 1 and 2. A value of each bidder is drawn from $V = [0, \bar{v}]$. Formally, an English auction is defined as follows. For notation, we denote $p_i^t = p_i^{t-1}$ if agent i is not a mover at round t.

Definition 5 An IRM for two agents is an *English auction* if it is ascending-price starting from $p_i^1 \geq p_i^0 \equiv 0$, it holds $|p_1^t - p_2^t| \leq 1$ for all $t \geq 1$, and if the agent who first responds no is always the loser. The winner pays the price at the termination.

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[8]One might consider a stronger notion of tightness: An IRM is *tight* if for all t and all h^t, it holds that $i \notin J^{t+1}(h^t)$ if $\forall \tilde{v} \in V(h^t)$, $f(\tilde{v}) \in X_i$ or if $\forall \tilde{v} \in V(h^t), f(\tilde{v}) \notin X_i$. Notice that if $\forall \tilde{v} \in V(h^t)$, $f(\tilde{v}) \in X_i$ or $f(\tilde{v}) \notin X_i$, a question to agent i is meaningless to determine an allocation.

The above definition includes two standard forms of English auctions. A model of an English auction is a so-called clock auction, in which the auctioneer announces a price in each round and raises it until only a single bidder remains. Hence, the prices of bidders increase with keeping $p_1^t = p_2^t$.[9] In another model, each bidder places a new bid in each round. A bidder needs to submit a strictly higher bid than the tentatively highest bid, which means $p_i^t = p_j^{t-1} + 1$ in each round.

In the following theorem, we employ a predetermined tie-breaking rule, which is denoted by $\succ_0^v$. In the efficient allocation rule f^*, bidder 1 (bidder 2, respectively) always wins if $v_1 > v_2$ ($v_2 > v_1$, respectively). In a tie case with $v_1 = v_2 = v$, it is broken by the preference order $\succ_0^v$ over bidders. Thus, bidder 1 (bidder 2, respectively) is chosen if $v_1 = v_2 = v$ and $1 \succ_0^v 2$ ($2 \succ_0^v 1$, respectively). That means preferences over bidders in tie cases can depend on their value.

Theorem 4 *Consider a single-object auction environment with two bidders and perfect information IRMs. An IRM is efficient, ex post incentive compatible, weakly tight, and has the pay-as-bid monetary transfer if and only if it is an English auction.*

In a single-object auction with $n \geq 3$ bidders, there exists another IRM that is efficient and ex post incentive compatible and pay-as-bid in the sense that $|p_i^t - p_j^t| \geq 2$ for some bidders i and j, and some round t. Such a mechanism is constructed as a "tournament auction"; an English auction is conducted among $n - 1$ bidders and its winner and the remaining bidder participate in an English auction again.

4.2 Corollary from Literature of Ascending-Price Auctions

Theorem 3 provides several findings along with knowledge from ascending-price auctions theory. When we focus on "ascending-price auctions," the following result is known.

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[9]To be precise, $p_1^t \neq p_2^t$ even for a clock auction because we consider sequential move and perfect information.

Proposition 1 (Ausubel and Milgrom, 2002; de Vries et al., 2007) *There exists an efficient and incentive-compatible ascending-price mechanism if for any state v, the coalition value function is submodular: for any $J' \subseteq J$,*

$$\omega(J' \cup i; v) - \omega(J'; v) \geq \omega(J \cup i; v) - \omega(J; v).$$

It is easily verified that in many environments of dichotomous preferences, the coalition value function is not submodular. Roughly speaking, the submodular condition corresponds to a substitutes condition, whereas dichotomous preferences typically exhibit complementarity. In a public good problem, for example, each agent's marginal contribution to the economy $\omega(J) - \omega(J_{-i})$ is non-decreasing in the size of the economy J. When a bidder has a non-substitute valuation, there is no ascending-price auction that terminates with the VCG outcome (de Vries et al., 2007). Thus, there exists no incentive-compatible IRM, in general.

The negative result depends on both the pay-as-bid transfer rule and tightness. Fadel and Segal (2009) show that even without Assumption 1, an efficient protocol may not be implemented in ex post equilibrium. This is because an ex post incentive-compatible transfer rule is equivalent to that of the VCG mechanism, whereas an efficient protocol may not collect enough information to calculate the VCG outcome. However, Fadel and Segal also show that every efficient allocation rule (protocol) is implemented in Bayesian Nash equilibrium using a payment rule similar to the "AGV mechanism." In a multiple-object auction model, Mishra and Parkes (2007) propose an ex post incentive-compatible ascending auction for general valuations, which does not satisfy pay-as-bid transfer or tightness.

5   Conclusion

IRMs are a class of dynamic indirect mechanisms in which the social planner iteratively asks binary questions and identifies the state of the world after a sequence of questions and responses. With complete information and dichotomous preferences, every strongly monotone allocation rule is implemented in an SPNE although

the "pay-as-bid" monetary transfer does not guarantee incentive compatibility. In particular, the efficient outcome is achievable in an SPNE in every efficient IRM, regardless of offer-price principles. However, agents strategically misreport their values contingent on the others' past responses. To impose ex post incentive compatibility, the planner has to employ an ascending-price principle. In a single-object allocation problem, the English auction is a unique mechanism satisfying efficiency, ex post incentive compatibility, weak tightness, and pay-as-bid transfer. However, an incentive-compatible IRM often fails to exist in general.

We have assumed that the planner has knowledge about agents' interests. It is an open question when the planner does not know agents' interests and needs to ask their interests too. Another extension is relaxing the assumption of dichotomous preferences. Knowledge from package auctions tells us that there exist incentive-compatible ascending package auctions when goods are substitutes. Unfortunately, however, Sano (2015) shows that in the presence of complementarities, an ascending package auction may not achieve the efficiency in SPNE.

A Proofs

We use the following notations in this appendix. The set of agents whose interests include x is denoted by $N(x) \equiv \{i \in I | x \in X_i\}$. For each subgame starting from a decision node $z = (h^{t-1}, p_i^t)$, let $I(z)$ be the set of all agents who make actions in the subgame.

A.1 Proof of Theorem 1

Step 1. Suppose $|I(z)| = 1$. Agent i is a unique agent making actions at z and all the subsequent nodes.

Step 1.1. Suppose $f(v^{t-1}) \notin X_i$. If $\forall \tilde{v}_i \in V_i(h^{t-1})$, $f(\tilde{v}_i, v_{-i}^{t-1}) \notin X_i$, then agent i earns zero payoff regardless of his responses, and any strategy is indifferent and optimal. Hence, suppose $\exists \tilde{v}_i \in V_i(h^{t-1})$, $f(\tilde{v}_i, v_{-i}^{t-1}) \in X_i$. Because $f(v^{t-1}) \notin X_i$,

we have $v_i^{t-1} < c_i^f(v_{-i}^{t-1})$ and $c_i^f(v_{-i}^{t-1}) \in V_i(h^{t-1})$. Because $v_i^{t-1} \neq \bar{v}_i(h^{t-1})$, we have $v_i^{t-1} = \max\{v_i, \underline{v}_i(h^{t-1})\} \geq v_i$. Under the proposed strategy, agent i responses sincerely regarding v_i (equivalently v_i^{t-1}), so that the final allocation must be $f(v^{t-1})$ and agent i earns zero payoff. If agent i deviates and takes another strategy achieving some $x \in X_i$, then the social planner identifies i's revealed value as $\tilde{v}_i \geq c_i^f(v_{-i}^{t-1})$. This indicates that agent i says yes for a price $p_i^s \geq c_i^f(v_{-i}^{t-1})$ at some round $s \geq t$, and that i's payoff is negative: $v_i - p_i(h) \leq v_i - p_i^s < 0$. Hence, the proposed strategy is optimal.

Step 1.2. Suppose $f(v^{t-1}) \in X_i$. If $v_i^{t-1} = \underline{v}_i(h^{t-1}) \geq v_i$, it implies for any $\tilde{v}_i \in V_i(h^{t-1})$, $f(\tilde{v}_i, v_{-i}^{t-1}) \in X_i$. For payment minimization, it is optimal to say no for all prices, which is consistent with the proposed strategy because $c_i^f(v_{-i}^{t-1}) \leq \underline{v}_i(h^{t-1})$.

Suppose $v_i > \underline{v}_i(h^{t-1})$ and $v_i^{t-1} = \min\{v_i, \bar{v}_i(h^{t-1})\}$. If $c_i^f(v_{-i}^{t-1}) \leq \underline{v}_i(h^{t-1})$, then the proposed strategy is clearly optimal as in the previous paragraph. Hence, suppose $c_i^f(v_{-i}^{t-1}) > \underline{v}_i(h^{t-1})$. By $f(v^{t-1}) \in X_i$, it holds that $c_i^f(v_{-i}^{t-1}) \leq v_i^{t-1} \leq v_i$. If agent i reports as if his value is $\tilde{v}_i < c_i^f(v_{-i}^{t-1})$, then his resulting payoff is zero. If agent i takes the proposed strategy, it holds that $c_i^f(v_{-i}^{t-1}) \in V_i(h)$ regardless of the offer-price process. Hence, we have

$$g(h) = g\big(\phi(c_i^f(v_{-i}^{t-1}), v_{-i}^{t-1})\big) = f\big(c_i^f(v_{-i}^{t-1}), v_{-i}^{t-1}\big) = f(v^{t-1}) \in X_i. \qquad (8)$$

The first equality is from $v_j^{t-1} \in V_j(h)$ for all $j \neq i$. The third equality is from strong monotonicity. In addition, agent i says no for any $p_i^s > c_i^f(v_{-i}^{t-1})$ for all $s \geq t$, his payoff is at least $v_i - c_i^f(v_{-i}^{t-1}) \geq 0$. It is clearly suboptimal for i to say yes for $p_i^s > c_i^f(v_{-i}^{t-1})$ for some $s \geq t$.

Therefore, we have shown that the proposed strategy constitutes an SPNE and $g(h) = f(v^{t-1})$ when $|I(z)| = 1$.

Step 2. Now we consider any decision node z of round t and the corresponding mover i. We impose the following induction hypothesis; for every subsequent node z' of round $t' > t$ after z, the proposed strategy is an SPNE and $f(v^{t'}) = f(v^{t'-1})$ for all $t' > t$. Hence, $f(v^t)$ is chosen in the SPNE by the hypothesis.

Step 2.1. Suppose $f(v^{t-1}) \notin X_i$ and $p_i^t \in V_i(h^{t-1}) \backslash \{\underline{v}_i(h^{t-1})\}$. If $c_i^f(v_{-i}^{t-1}) > \bar{v}_i(h^{t-1})$, we have $f(v^t) \notin X_i$ regardless of i's action.[10] By the induction hypothesis, i's resulting payoff is zero, and the proposed strategy is optimal.

Suppose $c_i^f(v_{-i}^{t-1}) \leq \bar{v}_i(h^{t-1})$. Because $v_i^{t-1} < c_i^f(v_{-i}^{t-1}) \leq \bar{v}_i(h^{t-1})$, we have $v_i^{t-1} = \max\{v_i, \underline{v}_i(h^{t-1})\}$. If $p_i^t < c_i^f(v_{-i}^{t-1})$, then $v_i^t \in \{v_i^{t-1}, p_i^t, p_i^t - 1\}$ after i's response. Hence, $f(v^t) \notin X_i$ regardless of i's response. Hence, the proposed strategy is optimal. Under the strategy, we have $v_i^t = v_i^{t-1}$, which induces $f(v^t) = f(v^{t-1})$.

Suppose $p_i^t \geq c_i^f(v_{-i}^{t-1})$. If agent i says no by following the proposed strategy, then $v_i^t = v_i^{t-1}$ and $f(v^t) = f(v^{t-1})$. If he deviates and says yes, then $v_i^t = p_i^t$ and $f(v^t) = f(p_i^t, v_{-i}^{t-1}) \in X_i$. By induction hypothesis, the final allocation is $f(v^t) \in X_i$. Because i's payoff is at most $v_i - p_i^t < 0$, the deviation is not profitable.

Step 2.2. Suppose $f(v^{t-1}) \in X_i$ and $p_i^t \in V_i(h^{t-1}) \backslash \{\underline{v}_i(h^{t-1})\}$. If $c_i^f(v_{-i}^{t-1}) < \underline{v}_i(h^{t-1})$, we have $f(v^t) = f(v^{t-1}) \in X_i$ regardless of i's action by strong monotonicity. Hence, for payment minimization, the proposed strategy in which agent i says no for all $p_i^t \in V_i(h^{t-1})$ is optimal.

Suppose $c_i^f(v_{-i}^{t-1}) > \underline{v}_i(h^{t-1})$. Because $v_i^{t-1} \geq c_i^f(v_{-i}^{t-1})$, we have $v_i^{t-1} = \min\{v_i, \bar{v}_i(h^{t-1})\}$. If $p_i^t \leq c_i^f(v_{-i}^{t-1})$, agent i says yes under the proposed strategy, and we have $v_i^t = v_i^{t-1}$ (because $p_i^t \leq v_i^{t-1}$). Hence, $f(v^t) = f(v^{t-1}) \in X_i$ and by induction hypothesis, $g(h) = f(v^t)$. Because agent i says no for any price $p_i^s > v_i$ for any subsequent node, his payoff must be nonnegative $v_i - p_i(h) \geq 0$. If agent i deviates and says yes, then $v_i^t = p_i^t - 1 < c_i^f(v_{-i}^{t-1})$ and $f(v^t) \notin X_i$. By induction hypothesis, i's payoff is zero, so that the proposed strategy is optimal and $f(v^t) = f(v^{t-1})$ holds.

If $p_i^t > c_i^f(v_{-i}^{t-1})$, agent i says no under the proposed strategy, and we have $v_i^t = \min\{v_i, p_i^t - 1\}$.[11] Because $p_i^t - 1 \geq c_i^f(v_{-i}^{t-1})$, we have $f(v^t) = f(v^{t-1}) \in X_i$ by strong monotonicity. By induction hypothesis, $g(h) = f(v^t)$. Agent i's payment is at most $p_i^t - 1$, so that his resulting payoff is at least $v_i - p_i^t + 1$. If agent i deviates and says yes, then $v_i^t = \max\{v_i^{t-1}, p_i^t\} \geq v_i^{t-1}$ and $f(v^t) = f(v^{t-1}) \in X_i$. By induction

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[10]It is not guaranteed that $f(v^t) = f(v^{t-1})$.

[11]Remember that $v_i^{t-1} = \min\{v_i, \bar{v}_i(h^{t-1})\}$ and $p_i^t \leq \bar{v}_i(h^{t-1})$.

hypothesis, $g(h) = f(v^t)$. Agent i's payment is at least p_i^t, so that his resulting payoff is at most $v_i - p_i^t$. Therefore, the proposed strategy is optimal and $f(v^t) = f(v^{t-1})$ holds.

Finally, every subgame or every decision node z preceding a terminal node is such that $|I(z)| = 1$. From Step 1, the induction hypothesis is true for every final round of the IRM, and we complete the proof. ∎

A.2 Proof of Theorem 2

Let π^* be the equilibrium payoff profile in an efficient IRM. Because the equilibrium allocation x^* is efficient, for each feasible allocation $x \in X$, we have

$$u_0(x^*) + \sum_{N(x^*)} \underline{v}_j(h) \geq u_0(x) + \sum_{N(x) \cap N(x^*)} \underline{v}_j(h) + \sum_{N(x) \backslash N(x^*)} \bar{v}_j(h). \qquad (9)$$

Note that the social planner's equilibrium payoff is

$$\pi_0^* = u_0(x^*) + \sum_{N(x^*)} \underline{v}_j(h).$$

Consider any coalition J including the planner. Obviously, it suffices to consider the case of $J = N(x) \cup \{0\}$ for any feasible outcome $x \in X$. Then,

$$
\begin{aligned}
\sum_J \pi_j^* &= u_0(x^*) + \sum_{N(x^*)} \underline{v}_j(h) + \sum_{N(x)} \pi_j^* \\
&\geq u_0(x) + \sum_{N(x^*) \cap N(x)} \underline{v}_j(h) + \sum_{N(x) \backslash N(x^*)} \bar{v}_j(h) + \sum_{N(x)} \pi_j^* \\
&= u_0(x) + \sum_{N(x^*) \cap N(x)} v_j + \sum_{N(x) \backslash N(x^*)} \bar{v}_j(h) \qquad (10) \\
&\geq u_0(x) + \sum_{N(x^*) \cap N(x)} v_j + \sum_{N(x) \backslash N(x^*)} v_j \\
&= \omega(J; v).
\end{aligned}
$$

The third line comes from the fact that $\pi_j^* = v_j - \underline{v}_j(h)$ for every $j \in N(x^*)$ and $\pi_j^* = 0$ for $j \notin N(x^*)$. Note that in the SPNE, $c_i^{f^*}(v_{-i}^t) \geq v_i$ for all $i \notin N(x^*)$ and all t. Hence, $\bar{v}_j(h) \geq v_j$ for each $i \notin N(x^*)$, which induces the fourth line. ∎

A.3 Proof of Theorem 3

Suppose for contradiction there exists agent i and some rounds s and t with $s < t$, and $p_i^s > p_i^t$. By definition of an IRM, $a_i^s = no$. Consider the revealed information at round $t-1$. By weak tightness, there exists $v \in V(h^{t-1})$ and $f(v) \in X_i$. In addition, $v \in V(h^{s-1})$ because $V_j(h^{t-1}) \subseteq V_j(h^{s-1})$ for each agent. Suppose that the true state is $(\hat{v}_i, v_{-i})$ with $\hat{v}_i \in V_i(h^{s-1})$ and $\hat{v}_i > p_i^s$. By monotonicity, $f(\hat{v}_i, v_{-i}) \in X_i$. Under the sincere reporting, agent i responds yes at round s with p_i^s, and he earns a payoff $\hat{v}_i - p_i \leq \hat{v}_i - p_i^s$. If agent i responds no at p_i^s and pretends to have v_i, then the final outcome is $f(v) \in X_i$ and he earns a payoff $\hat{v}_i - p_i > \hat{v}_i - p_i^s$, which is a contradiction. ∎

A.4 Proof of Theorem 4

Because *if* part is obvious, we will show *only if* part.

Suppose an IRM is efficient, ex post incentive compatible, weakly tight, and has the pay-as-bid transfer rule. By Theorem 3, it is an ascending-price mechanism because the efficient allocation rule is clearly monotone; we have $p_i^s \leq p_i^t$ for each $i = 1, 2$ and all $s < t$. In what follows, we show that $|p_1^t - p_2^t| \leq 1$ for all $t \geq 1$. Let $a_i(p_i^t)$ be an action by bidder i when the price is p_i^t.[12] Notice that in any ascending-price mechanism, any agent who has responded *no* does not become a mover in any subsequent round because $p_i^t \in V_i(h^{t-1})$.

Suppose there exists some round $t \geq 1$ and $|p_1^t - p_2^t| \geq 2$. Let t be the smallest of such rounds (i.e., the earliest node). Without loss of generality, let $p_1^t > p_2^t \equiv p$. Then, bidder 1 is the mover at round t. This is because if round t is bidder 2's node, it implies $p_2^{t-1} < p_2^t$ and $p_1^{t-1} = p_1^t$, so that $p_1^{t-1} - p_2^{t-1} > 2$, which is a contradiction. Hence, $p_2^{t-1} = p_2^t = p$ and $p_1^{t-1} \in \{p-1, p, p+1\}$.

Claim. The revealed value set of bidder 2 at $t-1$ must be $V_2(h^{t-1}) = [p, \bar{v}]$; i.e., $a_2(p) = yes$.

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[12] Remember that $a_i(p_i^t)$ does not mean an action of i at node t because p_i^t may be offered at an earlier node.

Proof of Claim. Suppose not; $a_2(p) = no$ and $\bar{v}_2(h^{t-1}) = \bar{v}_2(h^t) = p - 1$. Because $p_1^{t-1} < p_1^t$, $a_1(p_1^{t-1}) = yes$ and $\underline{v}_1(h^{t-1}) \geq p - 1$. When the tie-breaking rule determines $1 \succ_0^{p-1} 2$, bidder 1 should be the winner for any state in $V(h^{t-1})$. Hence, it is dominant strategy for bidder 1 to take $a_1(p_1^t) = no$ at t, which contradicts ex post incentive compatibility. If the tie-breaking rule determines $2 \succ_0^{p-1} 1$, the social planner cannot identify the allocation if $a_1(p_1^t) = no$. This is because the planner cannot make any further question once both bidders respond no and $V(h^t)$ contains both states such that bidders 1 and 2 win. This contradicts efficiency. $\square$

Case 1. Suppose $p_1^t \geq p + 3$.

Suppose $a_1(p_1^t) = no$, which implies the social planner no longer makes a question to bidder 1. Then, the revealed value sets at t are such that $V_1(h^t) \supseteq \{p+1, p+2\}$ and $V_2(h^t) = \{p, p+1, \ldots, \bar{v}\}$. If $V_1(h^t)$ contains at least three elements, then the planner cannot identify the winner regardless of tie-breaking rule, which contradicts efficiency. Hence, suppose $V_1(h^t) = \{p+1, p+2\}$. If $2 \succ_0^{p+1} 1$ and if $v_2 = p + 1$, then the planner cannot identify the winner. If $1 \succ_0^{p+2} 2$ and if $v_2 = p + 2$, then the planner cannot identify the winner. Therefore, $1 \succ_0^{p+1} 2$ and $2 \succ_0^{p+2} 1$. Now suppose $v_1 > p_1^t$ and $v_2 = p + 1$. Under the sincere reporting, bidder 1 wins and his payoff is at most $v_1 - p_1^t$. If bidder 1 deviates and $a_1(p_1^t) = no$, bidder 1 still wins[13] and his payoff is $v_1 - p_1^{t-1} > v_1 - p_1^t$, which contradicts ex post incentive compatibility.

Case 2. Suppose $p_1^t = p + 2$.

Suppose $a_1(p_1^t) = no$, which implies the social planner no longer makes a question to bidder 1. Consider the revealed value set of bidder 1. If $V_1(h^t)$ contains at least two elements (i.e., $V_1(h^t) \supseteq \{p, p+1\}$), we have a contradiction by an argument similar wtih Case 1. Hence, suppose $V_1(h^t) = \{p+1\}$. Now suppose $v_1 > p + 2$ and $v_2 = p$. By the same argument as Case 1, bidder 1 is better off by deviating at t and taking $a_1(p_1^t) = no$. Therefore, we have shown $p_1^t - p_2^t \leq 1$.

Finally, it is clear that the agent who first responds no is the loser. If not and

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[13]This is because the social planner identifies the winner at the termination with $V_1(h) = \{p+1, p+2\}$. Because $v_2 = p + 1$, the planner must identify at least $\bar{v}_2(h) = p + 1$.

a bidder, say bidder 1, first responds no but wins in a history h. Then, bidder 1 is better off by deviating to h when $v_1 > \bar{v}_1(h)$ and $v_2 \in V_2(h)$. ∎

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